Some of What Mathematicians Do

Martin H. Krieger

Whether it be at a party or at a tavern or while being examined by a physician, on announcing that you are a mathematician, you are likely to be greeted with comments about your companion's failure in high school math, or a request for a brief account of the proof of Fermat's Last Theorem, or perhaps an offer of a counterexample to the Four Color Theorem. Your parents, your friends and relatives, airplane seatmates, or your dean or provost are not likely to be mathematicians, and they too would like to know what you do, preferably in bite-sized pieces.

Might we provide an *everyday* description that has sufficient technical detail so that a mathematician would recognize the work as real research mathematics? I suggest that if we think of mathematical work as showing that what might seem arbitrary is actually necessary, as analyzing everyday notions, as calculation, and as analogizing—using rich examples of mathematical work itself, we might be able to say a bit more about *some* of what mathematicians do. None of these descriptions are easy, but I think they connect better with the work of other people, so that they might see our work and their own as having some shared features.

Conventions

Mathematicians make certain notions conventional. What might seem arbitrary is shown to be in effect necessary, at least within a wide enough range of situations. For example, means and variances were once taken merely as ways of "combining observations", to use a term of art of two hundred years ago. There were other ways, including medians and average absolute deviations ($\Sigma|x_i - x|/N$). But through the central limit theorem, for example, the variance became entrenched as a good measure of the width of a distribution for various different kinds of more or less identically distributed independent random variables. Moreover, it was easy to depict such statistics in a Euclidean space of observations, the various formulas being Pythagorean theorems with Euclidean distances. And if one used a large electromechanical calculator, it was not hard to set up the calculation so that one could calculate a sum of the squares of x_i and y_i and a sum of $x_i y_i$. In the law of the iterated logarithm, Khinchin provided an estimate of fluctuations that would not be readily accounted for by gaussian behavior, so even exceptional behavior fit under this regimen.

Variances turned out to be good measures of the kinds of noise and dissipation physicists encountered, and Einstein's work on fluctuations (1905, 1917) entrenched variances as the measure of choice. It also turned out that variances were good measures of the risk involved in financial markets, and the calculus of Lévy and Itô (where, in effect, dx is replaced by $\sqrt{dx}$) became the bread and butter of finance professors.

As for exceptions to means and variances, Lévy showed that the crucial fact was the asymptotic norming constant, the $\sqrt{N}$ that appears in the central limit theorem: that is, $N^{1/\alpha}$, here $\alpha = 2$. For α need not be 2 but could be other numbers for other

Martin H. Krieger is professor of planning at the University of Southern California. His email address is `krieger@usc.edu`.

distributions ("distributions without variance", that is, with infinite variance), which still scaled asymptotically, such as the world of fractals. However, if the variance is finite, then the only game in town is the gaussian. The deep idea turns out to be asymptotic approximation and scaling, that $N^{1/\alpha}$. And this is seen in modern results related to random matrices and prime number distributions, where the norming constant can be $N^{1/6}$, for example.

What is made conventional here is the gaussian, characterized by its mean and variance, and its being the asymptotic limit of sums of nice random variables. And that is made clear by the description of its exceptions. Although means and variances might well be arbitrary, they are demonstrably the right statistics ("necessary") for a wide range of cases.

Nowadays, statisticians are realizing that for actual data sets, often infected by wild and outlying data, one needs statistical methods that are "robust" and "resistant", not a strong point of means and variances. For a wide range of new cases, means and variances will no longer be conventions, and presumably new statistics are proven to be "necessary" and become the reigning conventions.

Mathematicians affirm that these conventions are not arbitrary. They are well grounded in mathematical practice and theory.

Analyzing Everyday Notions

Mathematicians formally analyze everyday notions. Topology developed as a way of understanding nearbyness, connectivity, and networks. It turned out that the key idea was continuity of mappings and how that continuity was affected by other transformations. For continuity preserved nearbyness, connectivity, and networks. Of course, this demanded a number of conceptual and mathematical discoveries. One great discovery was the subtleties of continuity, uniform vs. pointwise, for example. A second discovery was the fact that one might represent continuity and neighborhoods in terms of mappings: if the neighborhood of a point was mapped into an open set, that neighborhood itself was open, *if* the mapping was continuous. A third discovery was that networks could be characterized in terms of how they decomposed into simpler networks and that characterization would be preserved under continuous mappings. Moreover, a space might well be approximated by a skeletal framework, and a study of that framework would tell us about the space. A fourth discovery was that that decomposition sequence had a natural algebraic analog in commutative algebra. And a fifth discovery was that the algebraic decomposition had a natural analog with derivatives and second derivatives (Stokes's and Green's theorems and Gibbs's vector calculus), again the world of continuity.

As a consequence of this analysis, it was realized that there are many different kinds of nearbyness and many different topologies for a space, yet they might share important features. Functions came to be understood as mappings, in terms of what they did. And the transcendental realm turned out to be deeply involved with the algebraic realm. That analysis of everyday notions led to powerful technologies for analyzing connectivity and networks, techniques vital to current society. Those technologies are grounded in the formal mathematical analysis.

Calculation

Perhaps "proofs should be driven not by calculation but solely by ideas", as Hilbert averred in what he called Riemann's Principle. But some of the time, if not often, mathematicians have to calculate—doggedly and lengthily—in order to get interesting results. In some future time, knowing the solution, other mathematicians may well be able to provide a one-line proof driven solely by ideas, plus a great deal of mathematical superstructure built up in the intervening period of time. Or, in fact, lengthy proof and calculation are unavoidable, and delicate arguments involving hairy technology are the only way to go. The mathematician's achievement is, first of all, to actually follow through on that long and complex calculation and come to a useful conclusion, and, second, to present that calculation so that it is mildly illuminating. As we shall see, such a presentation involves matters of structure, organizing the whole; strategy, being able to tell a story about how it all holds together; and tactics, being able to do what needs to be done to get on with the next main step of the proof.

The first proof, by Dyson and Lenard (1967–1968), of the stability of matter—that bulk matter, held together by electrical forces of electrons and nuclei, won't collapse (then to explode)—is considered one of these long and elaborate calculations. What one has to prove is that the binding energy of bulk matter per nucleus is *bounded from below* by a negative constant, $-E^*$. The proof begins with an idea: an insight by Onsager (1939) about how to incorporate the screening of positively charged nuclei by negatively charged electrons. But the actual calculation would seem to involve a number of preliminary theorems and a goodly number of lemmas, all of which might seem a bit distant from the main problem. Actually, many of the preliminary theorems motivate the proof and indicate what is needed if a proof is to go through. And the lemmas might be seen as lemmas hanging from a tree of theorems or troops lined up to do particular work. As in many such calculations, the result almost miraculously appears at the end. And in this case the proportionality constant is about 10^{14} larger in absolute value than it need be.

A few years later, Lieb and Thirring (1975) were able to figure out how to efficiently use the crucial physics of the problem (Onsager's screening, and also that the electrons are fermions and are represented by antisymmetric wave functions). As a consequence, the proof was now about ideas, involved comparatively little calculation, and could be readily seen in outline, and the proportionality constant was about 10 rather than 10^{14}. Their crucial move was to employ the Thomas-Fermi model of an atom: the many electrons in an atom exist in a field due to their own charges (as well as that of the nucleus), and hence one seeks a self-consistent field.

Dyson and Lenard had all these ideas except for Thomas-Fermi. But in their pioneering proof, getting to the endpoint was avowedly more important than efficiency or controlling the size of the proportionality constant, $-E^*$. Theirs was a first proof of a fundamental fact of our world. By the way, in retrospect, the Dyson-Lenard proof is rather less long than it once appeared, its various manipulations along the way rather more rich with meaning.

Over the next decades a variety of rigorous proofs were provided of various fundamental facts about our world, many of which proofs are lengthy and complex and involve much calculation.

(1) *Thermodynamics.* One would like to be able to estimate the binding energy of bulk matter, the energy required to break it up into isolated atoms, as being proportional to the number of atoms. Such an estimate justifies thermodynamics, with its separation of intensive variables (such as temperature) and extensive variables (such as volume or number of particles). In a remarkable and lengthy proof, Lebowitz and Lieb (1972) provided a calculation of the asymptotic form of the binding energy of bulk matter, $E \approx -AN$, where N is the number of atoms—just the required form. Along the way, they employed the Dyson-Lenard result.

In all of these calculations, one technical problem is to figure out how to break up space into balls or boxes, fitting the atoms into those containers ("balls into boxes"). For example, Lebowitz and Lieb develop a Swiss-cheese decomposition: smaller balls fit into the interstices between larger balls.

(2) *A gas of atoms.* One would like to prove that at a suitable temperature and pressure, atoms form, and one has a gas of such atoms. Charles Fefferman (1983–1986) provides the proof with all of its "gruesome details", as he refers to the latter endeavor. First, he employs a technology he developed for solving partial differential equations—what he called "the uncertainty principle", the idea that the phase space of x and d/dx might be divided into suitably shaped boxes on which the differential equation is trivial—and then fill balls of phase space with these boxes, fitting "boxes into balls". Along the way, he redoes the Lieb-Thirring proof.

What is notable is his technical definition of an atom and, later, of a gas of atoms, a mathematically precise way of describing a physical state, one that would allow him to make mathematical progress on the problem. What is remarkable, and this is true for much of Fefferman's work, is his capacity to push through a lengthy calculation.

In order to complete the proof of "the atomic nature of matter" (that a gas of atoms forms), Fefferman then needs an even better estimate for the proportionality constant for the stability of matter than was provided by Lieb and Thirring, and with de la Llave and Trotter he provides a lengthy proof and an exact numerical calculation for E^*. (Lieb and his followers have provided another route to such better constants.) So far, it should be noted, the calculated E^* is still about two times too big for Fefferman's purposes and given what we expect.

(3) *An isolated atom.* Finally, one would like to estimate the ground state energy of a large *isolated* atom. The hydrogen atom's proverbial 13.6 electron-volts is the only calculation one might make in closed form (one of the first calculations in a quantum mechanics course). For larger atoms one must use approximations in which the errors are not in general rigorously known. In a series of calculations, some rigorous, some merely physical, by Lieb and Simon, Scott, Dirac, and Schwinger, a good idea of the asymptotic formula for the ground state energy in terms of Z, the atomic charge, is given in terms of a series in $Z^{1/3}$: $Z^{7/3}, Z^{6/3}, Z^{5/3}$. What Fefferman and Seco (1990–1996) provide in something like 800 pages of proof is a rigorous derivation of this formula with a rigorous estimate of its error, $O(Z^{5/3-1/a})$. Whole new technologies for partial differential equations are developed along the way, and even the paper that brings these all together is almost two hundred pages in length. Their achievement is again the ability to divide up the problem into tractable parts, to orchestrate the parts so that they work together, and to be able to tell a story of the proof (in this case, in fourteen pages). There have been subsequent simplifications for parts of the Fefferman-Seco derivation, but much of the calculation remains lengthy and complicated. And Córdoba, Fefferman, and Seco have found the next term in the asymptotic expansion.

Lengthy calculation demands enormous technical skill, courage, and insight and usually demands herculean inventions along the way. But sometimes it is the only way to make progress on a problem. I have chosen examples in which the lengthy calculations also lead to analyses of everyday notions, such as a gas of atoms.

Analogy

Some time ago, Pólya showed that analogy plays a vital role in mathematical work. Sometimes those analogies are provably true, such as the analogy between ideals and varieties: polynomials and their properties, considered as algebraic objects, and the graphs of those polynomials and their properties, considered as geometric objects. At other times, the analogies are not provable but provide for ongoing research programs for hundreds of years. Here I want to describe a syzygy, an analogy of analogies, between mathematical work and work in mathematical physics. What the physicists find, the mathematicians would expect, although the mathematicians could never have predicted such an analogy in the physical realm without the physicists' work.

For the mathematicians, I am thinking of the Riemann-Dedekind/Weber-Weil-Langlands analogy of analysis, algebra, and arithmetic. I will call it the Dedekind-Weil analogy, for short. Dedekind and Weber tried to derive Riemann's results concerning the transcendental realm (that is, referring to the realm of the continuous)—think here of Riemann surfaces and the Riemann-Roch theorem—using rigorous algebraic methods with no intuitions about continuity. Again, could there be a useful analogy between curves or surfaces and algebra? They were guided by what was known algebraically about numbers (number theory); in fact, they were able to translate those concepts and results to the realm of polynomials, and so were able to algebraicize Riemann's transcendental point of view. Subsequently, Hilbert and Weil and others extended the analogy.

André Weil describes the analogy in a particularly poignant way in a long letter he wrote from prison to his sister, Simone, in 1940. It is a remarkable document, combining a rich mixture of mathematics, a notional history of the analogy, reflections on how Weil himself does mathematics, and analogies of the interchange among the moments of the analogy to incest and war. I urge the reader to get hold of it (either in the original French in the first volume of Weil's *Collected Papers*, or in English translation in my *Doing Mathematics*).

Weil refers to three columns, in analogy with the Rosetta Stone's three languages and their arrangement, and the task is to "learn to read Riemannian". Given an ability to read one column, can you find its translation in the other columns? In the first column are Riemann's transcendental results and, more generally, work in analysis and geometry. In the second column is algebra, say polynomials with coefficients in the complex numbers or in a finite field. And in the third column is arithmetic or number theory and combinatorial properties. So, for example: (Column 3) Arithmetically, the zeta function packages the prime numbers. (2) Algebraically, its Mellin transform (a Fourier-like transform) is the theta function, originally found by Fourier in solving the heat equation. Theta has wonderful algebraic properties, such as automorphy (transformations of the function, that is, of its argument, can be expressed in terms of the function itself) and a functional equation that defines it. And (1), analytically, the spectrum of the zeta function (its zeros) is rich with information about the prime numbers. A simple example of the threefold analogy is found in the sine function: its series expansion packages the factorials of the odd numbers; sin Mx is expressible in terms of the trigonometric functions themselves (say, sin x and cos x); and the periodicity of the sine function (its spectrum) more or less defines it. Weil points out that the analogy continues to be productive, his later having proved the Riemann hypothesis in the algebraic column being a case in point.

In the twentieth century, mathematicians discovered that attaching group representations (or systems of matrices) to objects would often lead to progress in understanding those objects. Langlands's very great contribution (1960s ff) was to suggest, following Emil Artin, that attaching a group representation to the algebraic or automorphy column would turn out to be very productive for understanding the arithmetic column. The idea is to extend the analogy of theta functions to zeta functions into a much more complicated realm. Moreover, what might be impossibly difficult to prove from the point of view of one column is readily built in in another, much as theta's automorphy and functional equation leads to zeta's functional equation.

While the mathematicians worked at their analogy, physicists were solving a simple classical model of a ferromagnet using statistical mechanics: the Ising model in two dimensions, up-down spins arranged on a, say, rectangular lattice. The spins' interaction is local and simplified. The first exact solution was provided by Onsager in 1944, using a combination of Clifford or quaternion algebra and elliptic functions. Over the subsequent sixty years, physicists have provided many different solutions of the Ising model. (One solution refers to itself as the "399th solution".) Of course, they all get the same result for the partition function (in effect, the zeta function for this problem). When we examine the solutions, we discover that we might group the solutions into those that are arithmetic and combinatorial, those that are algebraic and automorphic, and those that are analytic or transcendental concerned with the zeros of the partition function. Moreover, from the initial solutions of the Ising model by Kramers and Wannier and by Montroll (1941), matrices played a crucial role in many of the solutions. They were in fact group representations, although they were not taken as such. They were taken to be matrices that conveniently did the combinatorics, and it was the algebraic properties of those matrices that allowed for the Onsager solution. No one worried much about what those matrices were a group representation *of*, although Onsager surely had many insights. The trace of those transfer matrices was the partition function of interest. Moreover, once again, there were functional equations that allowed for the solution for the partition function, and there were the scaling symmetries and automorphies characteristic of theta or elliptic functions. The latter were eventually canonized in the renormalization group techniques of Wilson (1960s, 1970s).

Parenthetically, I should note that Onsager's original paper might well be another candidate for a lengthy calculation. Subsequent calculations of asymptotic properties of the Ising model by Wu and McCoy (1966 ff) and collaborators are impressive for their length and complexity and for the courage needed to carry them through. What is striking is that at the end of one such calculation, the Painlevé transcendents appear, and that appearance has since become significant for much of contemporary mathematics and mathematical physics.

It would seem that there are two analogies here. The Dedekind-Weil analogy has been worked on as an analogy for 150+ years, most recently in its connection with representation theory in the Langlands Program. The physicists have been exactly solving the Ising model in two dimensions for more than sixty years and have produced a wide variety of solutions, employing what are in effect group representations from the beginning. Those various solutions would seem to be naturally described and classified using the categories provided by the mathematicians. The analogy the mathematicians seek to develop generically is developed and proven in its particular realm as a matter of course by the everyday work of the physicists. What the mathematicians seek, the physicists by the way provide an example of. The multiplicity of the physicists' solutions is given meaning and order by the mathematicians' hard-won concepts. I am unsure whether the physicists' analogy is provably the same as the mathematicians'. But surely the Dedekind-Weil analogy provides a way of thinking of diverse phenomena as being naturally connected, rather than their being merely many ways of solving a problem.

These analogies and the analogy between them (the syzygy) organize an enormous amount of information, suggest facts in one realm that might be true in another, and illuminate concepts among the columns and the analogies.

What Do Mathematicians Do?

Words such as convention, analyzing everyday notions, calculation, and analogy might be used to describe activities other than mathematics. And it is just in this sense that we might give outsiders a sense of what mathematicians do. At the same time, those notions have very specific meanings for mathematical work. And it is just in this latter sense that we might describe mathematics to ourselves. The shared set of terms allows us to connect our highly technical and often esoteric work with the work of others. Mathematicians show why some ways of thinking of the world are the right ways, they explore our everyday intuitions and make them rather more precise, they do long and tortuous calculations in order to reveal the consequences of their theories, and they explore analogies of one theory with others in order to find out the truths of the mathematical world.

I would also claim that, in a very specific sense, mathematical work is a form of philosophical analysis. The mathematicians and mathematical physicists find out through their rigorous proofs just which features of the world are necessary if we are to have the kind of world we do have. For example, if there is to be stability of matter, electrons must be fermions. The mathematicians show just what we mean by everyday notions such as an average or nearbyness. And mathematics connects diverse phenomena through encompassing theories and speculative analogies.

So when you are asked, What do mathematicians do?, you can say: I like to think we are just like lawyers or philosophers who explore the meanings of our everyday concepts, we are like inventors who employ analogies to solve problems, and we are like marketers who try to convince others they ought to think "Kodak" when they hear "photography" (or the competition, who try to convince them that they ought to think "Fuji"). Moreover, some of the time, our work is not unlike solving a two-thousand-piece jigsaw puzzle, all in one color. That surely involves lots of scut work, but also ingenuity along the way in dividing up the work, sorting the pieces, and knowing that it often makes sense to build the border first.

Sources

The material in this article is drawn from Martin H. Krieger, *Constitutions of Matter* (Chicago: University of Chicago Press, 1996) and *Doing Mathematics* (Singapore: World Scientific, 2003). See, especially, R.P. Langlands, "Representation theory: Its rise and role in number theory", which originally appeared in *Proceedings of the Gibbs Symposium* (Providence: AMS, 1990), but is also available at `http://www.sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/gibbs-ps.pdf`.