Olga Alexandrovna Ladyzhenskaya (1922–2004)

Susan Friedlander, Peter Lax, Cathleen Morawetz, Louis Nirenberg, Gregory Seregin, Nina Ural'tseva, and Mark Vishik

Olga Alexandrovna Ladyzhenskaya died in her sleep on January 12th, 2004, in St. Petersburg, Russia, at the age of eighty-one. She left a wonderful legacy for mathematics in terms of her fundamental results connected with partial differential equations and her "school" of students, collaborators, and colleagues in Russia. In a life dedicated to mathematics she overcame personal tragedy arising from the cataclysmic events of twentieth century Russia to become one of that country's leading mathematicians. Denied a place as an undergraduate at university, she was an exceptionally gifted young girl, but one whose father disappeared in Stalin's gulag. She eventually became a leading member of the Steklov Institute (POMI) and was elected to the Russian Academy of Science. Her mathematical achievements were honored in many countries. She was a foreign member of numerous academies, including the Leopoldina, the oldest German academy. Among other offices, she was president of the Mathematical Society of St. Petersburg and, as such, a successor of Euler.

Ladyzhenskaya made deep and important contributions to the whole spectrum of partial differential equations and worked on topics that ranged from uniqueness of solutions of PDE to convergence of Fourier series and finite difference approximation of solutions. She used functional analytic techniques to treat nonlinear problems using Leray-Schauder degree theory and pioneered the theory of attractors for dissipative equations. Developing ideas of De Giorgi and Nash, Ladyzhenskaya and her coauthors gave the complete answer to Hilbert's nineteenth problem concerning the dependence of the regularity of the solution on the regularity of the data for a large class of second-order elliptic and parabolic PDE. She published more than 250 articles and authored or coauthored seven monographs and textbooks. Her very influential book *The Mathematical Theory of Viscous Incompressible Flow*, which was published in 1961, has become a classic in the field. Her main mathematical "love" was the PDE of fluid dynamics, particularly the Navier-Stokes equation. This equation has a long and glorious history but remains extremely challenging: for example, the issue of existence of physically reasonable solutions to the Navier-Stokes equations in three dimensions was chosen as one of the seven millenium "million dollar" prize problems of the Clay Mathematical Institute. (The CMI website gives a description by Fefferman of the prize problem.) The three-dimensional problem remains open to this day, although it was in the 1950s that Ladyzhenskaya obtained the key result of global unique solvability of the initial boundary problem for the two-dimensional Navier-Stokes equation. She continued to obtain influential results and to raise stimulating issues in fluid dynamics, even up to the days before her death.

This memorial article contains brief surveys of some of Ladyzhenskaya's significant mathematical achievements by two of her collaborators in St. Petersburg, Gregory Seregin and Nina Ural'tseva. These are followed by "memories" of Olga Alexandrovna by other distinguished mathematicians who have known her and her work for many years, namely Peter Lax, Cathleen Morawetz, Louis Nirenberg, and Mark Vishik.

Olga Alexandrovna was a woman of great charm and beauty. We are very grateful to Tamara Rozhkovskaya for presenting some of the lovely photographs of Olga that appear in the two volumes in honor of the eightieth birthday of Ladyzhenskaya edited by Rozhkovskaya, who is a "mathematical grandchild" of Olga and editor and publisher of the International Mathematical Series in which these volumes appear. At the conclusion of her appreciation article in the second of these two volumes, Nina Ural'tseva (a "mathematical child" and close friend and collaborator of Olga) writes "In the Science Museum in Boston there is an exhibition devoted to mathematics. The names of the most influential mathematicians of the 20th century are carved on a large marble desk…and Olga Ladyzhenskaya is among them".

—Susan Friedlander

Selected Honors of Olga Ladyzhenskaya

1969 The State Prize of the USSR
1985 Elected a foreign member of the Deutsche Akademie Leopoldina
1989 Elected a member of the Accademia Nazionale dei Lincei
1990 Elected a full member of the Russian Academy of Science
2002 Awarded the Great Gold Lomonosov Medal of the Russian Academy
2002 Doctoris Honoris Causa, University of Bonn

Left to right, Nina Ural'tseva, Olga Ladyzhenskaya, V. Smirnov.

Gregory Seregin and Nina Ural'tseva

Olga Alexandrovna Ladyzhenskaya passed away quite unexpectedly on January 12 this year. She was in good spirits two days before that. She had just sketched a paper on some computational aspects in hydrodynamics and planned to finish it in Florida. Olga was fond of sunlight; therefore, every winter, the darkest period in our St. Petersburg, she longed for the South…. In her last years she had serious problems with her eyes, and she suffered enormously because of the winter darkness. She resisted such problems by all means, for example, using special pencils when writing. She was to leave for Florida on January 12. On the eve of January 11 she went to rest before her long trip and did not wake up. God gave her an easy death.

Olga Ladyzhenskaya was a brilliant mathematician and an outstanding person who was admired by distinguished scientists, writers, artists, and musicians, often becoming a source of inspiration for them. The eminent mathematician V. Smirnov; the great pianist and professor of the Academy of Music, N. Golubovskaja; the distinguished world-famous poets A. Akhmatova and J. Brodsky; and the famous writer A. Solzhenitsyn are found among them.

Her influence on twentieth century mathematics is highly valued by the world scientific community. It was not only Olga's scientific results, though truly deep and fundamental, but also her personal integrity and energy that played an especial role in her contribution to mathematics.

Olga's focus in life was not limited to mathematics and science. She was deeply interested in arts and intellectual life in general. She loved animals, was a passionate mushroom lover, and enjoyed flowers. She was an enthusiastic traveler and had the wonderful skill of a storyteller when sharing her impressions with friends.

There were few things that did not touch her; she reacted keenly to any injustice, to the misfortunes of others; and she helped lone and feeble people. She took this very personally.

She expressed openly her views on social matters, even in the years of the totalitarian political regime, often neglecting her own safety.

Olga grew up during very hard times in Russia. She was born on March 7, 1922, in the tiny town of Kologriv in the north part of Russia. Her father, Alexander Ivanovich, taught mathematics in a high school. Her mother, Anna Mikhailovna, kept house

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Olga as a baby, with her father, mother, and two sisters.

and looked after her husband and three daughters. Olga's "grandfather"[1], Gennady Ladyzhensky, was a famous painter. There were many books, including books on history and fine arts, in their house. Books were almost the only source of cultural education, because Kologriv was too far from cultural centers. The town was situated among wild forests, near the picturesque river Unzha. All her life Olga carefully kept beautiful landscape paintings by her grandfather, some of them depicting fine views of the Unzha.

It was in the summer of 1930 that Alexander Ivanovich decided to teach mathematics to his own daughters. After he made some explanation of the basic notions of geometry, he formulated a theorem and suggested that his daughters prove it themselves. It turned out that the youngest one, Olga, was the best of his students. She loved to discuss mathematics with her father, and soon they studied calculus together.

In October 1937 Alexander Ivanovich was arrested and soon killed by NKVD, the forerunner of the KGB. It was a great shock for the family. Later, in 1956, he was officially exonerated "due to the absence of a corpus delicti". His death put the family in a very distressed situation. The mother and older daughters had to go to work. Olga's mother had grown up in a small village in Estonia and could do craft work. She made dresses, shoes, soap, and many other things. That was the way the family survived in those days.

In 1939 Olga graduated with honors from Kologriv High School and went to Leningrad to continue her education. However, it was forbidden for a person whose father was considered an "enemy of his Nation" to enter Leningrad State University.[2] She was accepted at Pokrovskii Pedagogical Institute, and finished two years of studies there in June 1941. The war forced her to leave Leningrad. In the fall of 1941 she worked as a teacher in the town of Gorodets, and in the spring of 1942 she returned to Kologriv, where she took her father's position as a high school mathematics teacher. In October 1943 she finally became a student at Moscow State University and graduated in 1947. During these years, I. G. Petrovskii was her adviser. At that time she was strongly influenced by Gelfand's seminar.

In 1947 Olga married A. A. Kiselev, a Leningrad resident, so she moved there, with a recommendation from Moscow State University to the graduate school of Leningrad State University (LGU). At LGU S. L. Sobolev was appointed to be her scientific adviser. From the time she came to LGU, a very close friendship developed between Olga and V. I. Smirnov and his family. In the fall of 1947 Smirnov, at Ladyzhenskaya's request, became head of a seminar on mathematical physics, which has been active ever since. The seminar brought together many mathematicians of the city working in the area of partial differential equations and their applications. Until her last days Ladyzhenskaya was one of the leading participants.

Finite-Difference Method and Hyperbolic Equations

Even the first results that Ladyzhenskaya obtained in the late 1940s and in the early 1950s were a breakthrough in the theory of PDEs. In her Ph.D. thesis, defended at LGU in 1949, she proposed a difference analog of Fourier expansions for periodic functions on grids and investigated their convergence as the grid step goes to zero in the difference analogues of the space W_2^k. Using these expansions, she investigated various difference schemes for model equations, distinguished those that are stable for certain ratios of grid steps in the space and time variables, and then proved the stability of these schemes for equations with variable coefficients. In particular, she gave a simpler proof of unique local solvability of the Cauchy problem for hyperbolic quasilinear systems than the proof that Petrovskii had given.

After defending her Ph.D. thesis, Ladyzhenskaya decided to study initial boundary-value problems for linear hyperbolic equations of second order. She started with justification of the Fourier method. In her publications of 1950–1952, Ladyzhenskaya gave exhaustive answers concerning series expansions of functions in $W_2^k(\Omega)$ in the eigenfunctions of arbitrary symmetric second-order elliptic

[1] *Olga's actual grandfather was Ivan whose brother was the painter Gennady Ladyzhensky. Gennady lived with Ivan and his family. Olga and her sisters always called Gennady "Dedushka" which means grandfather.*

[2] *She had passed the entrance exams to this university which at the time was considered the best in the Soviet Union.*

operators in a bounded domain $\Omega \in \mathbb{R}^n$, under any of the classical boundary conditions on $\partial\Omega$. Moreover, in [1] she found a solution to the problem of describing the domain of the closure in $L_2(\Omega)$ of an elliptic operator $\mathcal{L}$ with the Dirichlet boundary condition. The solution is based on the inequality

$$(1) \qquad \|u\|_{W_2^2(\Omega)} \le C(\Omega)(\|\mathcal{L}u\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}),$$

proved by Ladyzhenskaya. Here, $\mathcal{L}$ is a general second-order elliptic operator with smooth coefficients in front of the second derivatives, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and u is an arbitrary function in $W_2^2(\Omega)$ that vanishes on the boundary or satisfies a nondegenerate homogeneous boundary condition of the first order. The significance of this result for the theory of differential operators, including spectral theory, can hardly be overestimated.

Most of the results mentioned above were included in her first monograph [2], published in 1953. Unfortunately, this book was not translated into European languages.

Thanks to this work, the notion of a weak solution to an initial boundary-value problem becomes an important concept in mathematical physics. Systematic investigation of the entire scale of weak solutions in various function spaces, masterly analytic techniques for obtaining estimates for the integral norms, together with general arguments of functional analysis, provided Ladyzhenskaya with a strong background that led to success in the study of the solvability of the boundary-value problems and initial boundary-value problems for linear PDEs of classical type.

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Quasilinear PDEs of Elliptic and Parabolic Types

An extensive series of joint papers by Ladyzhenskaya and Nina Ural'tseva was devoted to the investigation of quasilinear elliptic and parabolic equations of the second order. At the beginning of the last century S. N. Bernstein proposed an approach to the study of the classical solvability of boundary-value problems for such equations based on a priori estimates for solutions. He also described conditions that are in a certain sense necessary for such solvability. These conditions, which are usually called natural growth restrictions, are formulated in terms of growth orders of functions entering the equations with respect to gradients of solutions. Due to works by Leray and Shauder, investigation of the classical solvability of the Dirichlet problem was reduced to obtaining a priori bounds for its solutions in $C^{1+\alpha}$ norm. Up to the mid-1950s the program had been realized for the Dirichlet problem for two-dimensional elliptic equations, but even in those cases some unnatural conditions were supposed.

The conception of a generalized solution first showed up in the calculus of variations. The efforts of many outstanding mathematicians such as Hilbert, Tonelli, Morrey, and A. G. Sigalov led to the construction of direct methods for proving the existence of solutions to the Euler equations of the corresponding variational integrals. In contrast to the existence of minimizers, the number of independent variables plays a significant role in the study of their differentiability properties. By the beginning of the 1940s Morrey was investigating the regularity for the case $n = 2$.

The attack on higher-dimensional problems started in the mid-1950s. In 1955, at the Department of Physics of LGU, Ladyzhenskaya delivered a special course of lectures about her new results on quasilinear parabolic equations, later published in [1] and [2]. These papers contained a rather general form of the method of auxiliary functions, which goes back to Bernstein's work. Ladyzhenskaya set herself the task of unraveling the secret of finding auxiliary functions for the evaluation of the maximum modulus of the gradient of the solution in the multidimensional case. She reduced it to solving a nonlinear differential inequality for functions of one variable and proved its solvability under a certain condition of smallness. This made it possible to estimate the spatial gradient for any solution in terms of its modulus of continuity under natural growth conditions. In papers [1] and [2], she applied her result to a narrower class of quasilinear elliptic and parabolic equations that does not require a preliminary estimate for the oscillations of the solution. Almost simultaneously with [1], in papers by J. Nash and E. de Giorgi, a bound for the Hölder norm of solutions was established for the simplest case of linear parabolic and elliptic equations having a divergence form with bounded measurable coefficients.

De Giorgi's ideas were further developed in [3]–[8]. The results of these investigations were the main content of monographs [9]–[11] on the theory of quasilinear equations of elliptic and

parabolic types (the latter written jointly with Solonnikov). The books presented a rather complete theory for quasilinear equations of divergence form and, in particular, global solvability of the classical boundary-value problems under natural growth restrictions. Moreover, the methods developed by the authors in [9] enabled them to analyze the dependence of smoothness of generalized solutions on the smoothness of data for equations of divergence form. It was proved that under natural growth restrictions, the regularity of weak extremals for the higher-dimensional variational is completely determined by the smoothness of the integrand. In a sense this concluded the investigation of Hilbert's nineteenth and twentieth problems for second-order equations. Also, classes of quasilinear systems with diagonal principal part and a special structure of the terms quadratic in gradients were singled out there, and it was shown that they share properties that are characteristic of scalar divergence equations. Such systems later became the object of detailed studies in research on harmonic maps of manifolds.

The books [9]–[11] also presented a number of deep results in the theory of quasilinear equations of general nondivergence form. However, for such equations it was only later, at the beginning of the 1980s, that unnatural restrictions were completely removed and the results brought to the same level of generality as in the divergence form case. This required techniques developed by N. V. Krylov and M. V. Safonov for investigating linear equations of nondivergence form with bounded measurable coefficients. In [12]–[14], the conditions on the data were made even weaker, the presence of integrable singularities with respect to the independent variables was allowed, and the solvability of the Dirichlet problem was studied in Sobolev spaces.

The methods developed in the monographs [9]–[11] turned out to be effective also in the study of broader classes of equations that contain non-uniformly elliptic operators like mean curvature operator [15]. The joint work of Ladyzhenskaya with N. M. Ivochkina was devoted to geometric topics as well. In 1994–1997, they published a series of papers investigating the global classical solvability of the first initial boundary-value problem for nonlinear equations of parabolic type that describe the flows generated by the symmetric functions of the Hessian of the unknown surface or by its principal curvatures.

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Hydrodynamics

The mathematical theory of viscous incompressible fluids was the favorite topic of Olga Ladyzhenskaya. She was involved with this activity from the middle of the 1950s till the very end of her life. Her personal contributions to mathematical hydrodynamics were deep and very important. But what was perhaps even more important is that she was a great source of new problems for others.

In the middle of the 1950s, Ladyzhenskaya began with the simplest case, the stationary Stokes system:

$$(0.1)\qquad \left.\begin{aligned}\mu\Delta u - \nabla p &= -f\\ \operatorname{div} u &= 0\end{aligned}\right\}\ \text{in } \Omega,\qquad u|_{\partial\Omega} = 0$$

Here, Ω is a domain in $\mathbb{R}^n (n = 2, 3)$, $f \in L_2(\Omega;\mathbb{R}^n)$ is a given force, μ is constant viscosity, u and p are unknown velocity field and pressure. If we define by $H(\Omega)$ and $H^1(\Omega)$ the closures of the set

$$\overset{\circ}{C}{}_0^\infty(\Omega;\mathbb{R}^n) = \{v \in C_0^\infty(\Omega;\mathbb{R}^n) \parallel \operatorname{div} v = 0\}$$

in $L_2(\Omega;\mathbb{R}^n)$ and $W_2^1(\Omega;\mathbb{R}^n)$, respectively, the velocity can be determined from the variational identity

$$(0.2)\quad \mu\int_\Omega \nabla u : \nabla v\, dx = \int_\Omega f\cdot v\, dx,\qquad \forall v \in H^1(\Omega).$$

At that time, Ladyzhenskaya had already been able to prove that the solution u to (0.2) has the second derivatives which are locally square summable in Ω. In order to recover the pressure, she proved the following fundamental fact:

Theorem 0.1. *Let $w \in L_2(\Omega;\mathbb{R}^n)$ be orthogonal to all $v \in \overset{\circ}{C}{}_0^\infty(\Omega;\mathbb{R}^n)$, then $w = \nabla p$ for some $p \in L_{2,\mathrm{loc}}(\Omega)$ with $\nabla p \in L_2(\Omega;\mathbb{R}^n)$.*

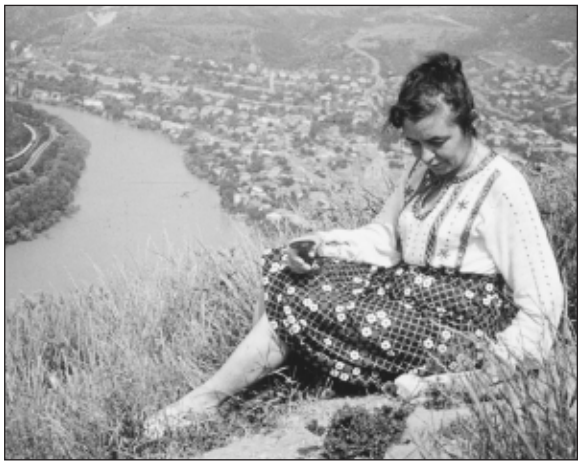
The theorem was proved in the celebrated monograph [1], which contains the main results obtained by Ladyzhenskaya in the 1950s on Stokes and Navier-Stokes equations.

In 1957, the joint paper [2] by Ladyzhenskaya and Kiselev was published, in which they considered the first initial boundary value problem for the Navier-Stokes equations:

$$(0.3)\qquad \left.\begin{aligned}\partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p &= f\\ \operatorname{div} v &= 0\end{aligned}\right\}\text{in } Q_T = \Omega\times]0,T[$$

$$v|_{\partial\Omega\times[0,T]} = 0,\qquad v|_{t=0} = a$$

and proved unique solvability of it in an arbitrary three-dimensional domain. At that time, the only one result of such kind was the pioneering result [3] of J. Leray on the Cauchy problem. All statements in [2] were proved without any special representation of solutions but allowed freedom in the choice for a class of generalized solutions in which the uniqueness theorem is preserved. In particular, they worked in the class $W_2^1(Q_T)\cap L_\infty(0,T;L_4(\Omega))$ for v, with $\partial_t\nabla v \in L_2(Q_T)$, and showed that unique solvability in the class on a nonempty interval $]0,T[$, where T depends on W_2^2-norm of a and on L_2-norms of the known functions f and $\partial_t f$. If these norms are sufficiently small, then $T = +\infty$. In this paper, they cited Hopf's work [5], saying that the class of weak solutions introduced by E. Hopf is too wide to prove uniqueness.



Ladyzhenskaya near the River Dnieper in the Ukraine, while on a break from a math conference.

One of the most remarkable results, proved by Ladyzhenskaya at the end of the 1950s and published in [4], was the global unique solvability of the initial boundary problem for the 2D Navier-Stokes equations. For the Cauchy problem, it was established by J. Leray. The proof was based on ideas developed in [2] and on new types of inequalities, which nowadays are called multiplicative inequalities. In particular, Ladyzhenskaya proved that, for any $v \in C_0^\infty(\Omega)$, it holds:

$$(0.4)\qquad \|v\|^2_{L_4(\Omega)} \le C\|\nabla v\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}.$$

Here, Ω is an arbitrary domain in $\mathbb{R}^2$ and C is a universal constant. The inequality (0.4) now bears her name. At almost the same time, Lions and Prodi proved in [6] the global unique solvability of the two-dimensional problem in a different way but with the help of Ladyzhenskaya's inequality (0.4).

As to the three-dimensional case, Ladyzhenskaya did not pay special attention to additional conditions that provide uniqueness of the so-called weak Leray-Hopf solutions. A function u is called a weak Leray-Hopf solution if it has the finite energy, satisfies a certain variational identity in which test functions are divergence free and compactly supported in the space-time cylinder Q_T, is continuous in time with values in $L_2(\Omega)$ equipped with the weak topology, and satisfies the energy inequality for all moments of time and the initial condition in L_2-sense. It was mentioned in the first edition of her monograph [1] that the additional condition $v \in L_{4,8}(Q_T) \equiv L_8(0,T;L_4(\Omega))$ gives uniqueness in the class of weak Leray-Hopf solutions. This condition is a particular case of the following theorem proved essentially by Prodi in [7] and by Serrin in [8].

Ladyzhenskaya, age 79, in her St. Petersburg apartment.

Theorem 0.2. *Assume that v is a weak Leray-Hopf solution and*

$$v \in L_{s,l}(Q_T) \tag{0.5}$$

for positive numbers s and l satisfying the condition

$$\frac{3}{s} + \frac{2}{l} \le 1, \qquad s > 3. \tag{0.6}$$

Then any other weak Leray-Hopf solution to the same initial boundary-value problem coincides with v.

Ladyzhenskaya later included results of this type in the second Russian edition [9] of her monograph.

In 1967, Ladyzhenskaya proved in [10] that in fact weak Leray-Hopf solutions satisfying conditions (0.5) and (0.6) are smooth.

Theorem 0.3. *Assume that all conditions of Theorem 0.2 are fulfilled. Let, in addition, $f \in L_2(Q_T)$, $a \in H^1(\Omega)$, and Ω is the bounded domain in $\mathbb{R}^3$ of class C^2. Then $\partial_t v$, $\nabla^2 v$, and ∇p are in $L_2(Q_T)$.*

As it was indicated in one of the last papers by Ladyzhenskaya [11], this theorem might be a good guideline for those who would like to solve the main problem of mathematical hydrodynamics. By the way, Ladyzhenskaya formulated it as follows: One should look for the right functional class in which global unique solvability for initial boundary value problems for the Navier-Stokes takes place.

Up to now, very few qualitatively new results on global unique solvability results for the three-dimensional nonstationary problem are known. The most impressive among them is Ladyzhenskaya's result for the Cauchy problem in the class of radially asymmetric flows with zero angular velocity. This was proved in [12].

As we mentioned above, Ladyzhenskaya believed that in the 3D case the Navier-Stokes equations, even with very smooth initial data, do not provide uniqueness of their solutions on an arbitrary time interval. Hopf and later other mathematicians, Ladyzhenskaya among them, tried unsuccessfully to state a reasonable principle of choice for determinacy. Giving up these attempts, Ladyzhenskaya introduced the so-called modified Navier-Stokes equations, which are now known as Ladyzhenskaya's model. These equations differ from the classical ones only for large-velocity gradients. For them, she proved global unique solvability under reasonably wide assumptions on the data of the problem. These results were the main content of her report to the International Congress of Mathematicians in 1966. For a long time, problems of this type were not very popular due to their difficulties, and it is only in the last decade that they have attracted the attention of many mathematicians trying to improve on Ladyzhenskaya's old results.

At the end of this section, we would like to mention the remarkable results Ladyzhenskaya obtained on the solvability of stationary problems for the Navier-Stokes equations. The first attempts in this direction were made by Odqvist and Leray. However, the question of global solvability (for all values of Reynolds numbers) had remained open till the end of the 1950s. Ladyzhenskaya's approach was based on her conception of weak solution and a good choice of the energy space. This turned out to be the space $H^1(\Omega)$, the closure of $\mathring{C}_0^\infty(\Omega)$ in the Dirichlet integral norm. Using known a priori estimates for the velocity gradient in $L_2(\Omega)$, she proved in [13] and [14] the existence of at least one solution in the energy space $H^1(\Omega)$. Later, in the 1970s, Ladyzhenskaya, together with Solonnikov, successfully studied stationary problems in domains with noncompact boundaries.

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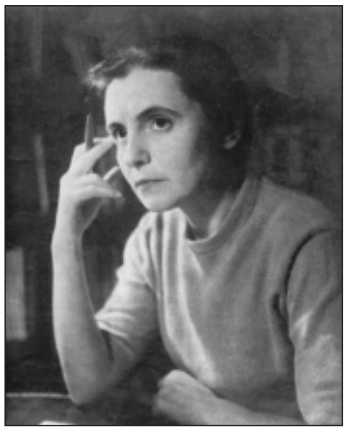
Attractors for PDEs

In [1], Ladyzhenskaya considered a semigroup generated by the two-dimensional initial boundary-value problem for the Navier-Stokes equations. Solution operators were defined as $V_t(a) = v(t)$, and the phase space was $H(\Omega)$. It was assumed that the force f in (0.3) is independent of time and belongs to $L_2(\Omega)$. It was known that solution operators are continuous in (a, t) on $H(\Omega) \times \mathbb{R}^1_+$ and each bounded set $\mathcal{B} \subset H(\Omega)$ is pulled into the ball $\mathcal{B}(R_0) = \{u \mid \|u\|_{L_2(\Omega)} < R_0\} \subset H(\Omega)$ of radius $R_0 > (\lambda\mu)^{-1} \|f\|_{L_2(\Omega)}$ in a finite time, where λ is the first eigenvalue of the Stokes operator for the domain Ω, and $\mathcal{B}$ stays there for all large t. She showed that solution operators are compact on $H(\Omega)$ for each fixed positive t. For finite-dimensional phase spaces this result is immediate, but in the case of infinite-dimensional phase spaces that was a very important step. Then Ladyzhenskaya took the intersection

$$\mathcal{M} = \bigcap_{t>0} V_t(\mathcal{B}(R_0)) \tag{0.1}$$

and proved that the set $\mathcal{M}$ is nonempty and compact and attracts any bounded subsets of the phase space. She found many other interesting properties of $\mathcal{M}$. It is invariant, each trajectory on $\mathcal{M}$ can be extended to negative times, and in a sense it is finite-dimensional. The latter means that there is a number N with the following property: If for two full trajectories lying in $\mathcal{M}$ the projections on the finite dimensional space spanned by the first N eigenfunctions of the Stokes operator coincide, then the trajectories themselves coincide. In the conclusion of this remarkable paper, it was claimed that the proposed approach is applicable to many other dissipation problems, in particular, to problems of parabolic types. So, [1] opened a chapter in the theory of PDE, namely the theory of stability "in the large". This program was realized in her last monograph [3].



In her survey article [2], she proposed to call the set $\mathcal{M}$ a "global minimal $\mathcal{B}$-attractor". Her explanation of that was as follows. The word "minimal" indicates that no proper subset of $\mathcal{M}$ attracts the whole of $H(\Omega)$, and the letter $\mathcal{B}$ emphasizes that any bounded set in $H(\Omega)$ is uniformly attracted to $\mathcal{M}$. Certainly, this makes sense.

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M. I. Vishik

I first met Olga Alexandrovna Ladyzhenskaya in 1945, and we became friends at once. She was then a third-year student at Moscow State University. We often met at the seminars, and we discussed mathematical problems while walking along the wide corridors of the Department of Mechanics and Mathematics of the university.

Her scientific adviser at Moscow State University was Ivan Georgievich Petrovskiĭ. In 1946 Israel

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Moiseevich Gelfand organized a seminar for three young mathematicians: O. A. Ladyzhenskaya, O. A. Oleĭnik, and myself. As a result each participant of the seminar got an interesting mathematical problem to seriously work on. O. A. Ladyzhenskaya got the problem of describing the domain of an elliptic operator of the second order with Dirichlet boundary conditions and with right-hand side belonging to $L_2(\Omega)$. She proved that if this boundary-value problem has a unique solution, then the elliptic operator is an isomorphism between the Sobolev space $H^2(\Omega) \cap H_0^1(\Omega)$ and $L_2(\Omega)$. She also found an estimate for the norm for the corresponding inverse operator.

In the late forties Olga Alexandrovna moved to Leningrad, where she became a postgraduate student of Vladimir Ivanovich Smirnov. Vladimir Ivanovich admired her talent, and she took an active part in his seminar. Later Olga Alexandrovna became the head of the seminar. From time to time Olga Alexandrovna organized conferences on differential equations and their applications. These conferences were very popular. Many mathematicians from other universities and institutes in the Soviet Union used to take part in them.

Olga Ladyzhenskaya with her husband, Andrey Kiselev.

In Leningrad Olga Alexandrovna married Andrey Alekseevich Kiselev, who looked very much like Pierre Bezukhov, the hero of the novel *War and Peace* by Leo Tolstoy. My wife and I often visited them when we were in Leningrad. We witnessed how tender and loving Andrey Alexeevich was. She literally meant the world to him. However, they separated later. Andrey Alexeevich wanted to have children, but Olga Alexandrovna did not. She justified her decision by her wish to devote her life to mathematics only, to which children might be an obstacle. Thus, Olga Alexandrovna stayed single for the rest of her life.

For many years starting in the late forties, Olga Alexandrovna visited us many times, and sometimes even stayed with us. She was usually interested in what new mathematical ideas I had worked out over the summer, and she told me, in turn, of her own new achievements. We also used to discuss with her new compelling and timely mathematical problems; the connections between functional analysis and the theory of differential equations and many other things. Ladyzhenskaya and I published a paper on these problems in *Uspekhi Matematicheskih Nauk* (see [1]).

In 1953, at Moscow State University, Olga Alexandrovna defended her "habilitation" dissertation devoted to the problem of regularity of solutions of a mixed boundary-value problem for a general hyperbolic equation of the second order. She found rather sharp conditions guaranteeing that the solutions of these equations are the solutions in the classical sense. Olga Alexandrovna justified the Fourier method for the solution of hyperbolic boundary-value problems. Furthermore, she also studied the application of the method of the Laplace transform to these equations. All these problems were expounded in her book [2].

In the fifties and sixties Olga Alexandrovna often gave talks at the seminars of Ivan Georgievich Petrovskiĭ at Moscow State University. I recall how greatly the listeners were impressed by her proof of the uniqueness theorem for the initial boundary-value problem for the two-dimensional Navier-Stokes system. Since the existence theorem for these equations was proved earlier, the well-posedness of the main boundary-value problem for the Navier-Stokes system in two dimensions was established thanks to the remarkable result of Olga Alexandrovna. The uniqueness theorem for the three-dimensional Navier-Stokes system is still not proved. Olga Alexandrovna devoted her book on fluid dynamics to three people she most respected, namely, her father, Vladimir Ivanovich Smirnov, and Jean Leray. In this book she studied in great detail many problems related to the stationary and nonstationary Navier-Stokes system.

In collaboration with Nina Nikolaevna Ural'tseva, Olga Alexandrovna wrote a book [4] on linear and quasilinear elliptic equations. Many fundamental results were obtained in this book, which remains a real encyclopedia on the subject. In the sixties, Olga Alexandrovna, Nina Nikolaevna Uralt'seva, and Vsevolod Alekseevich Solonnikov wrote a book [5] on linear and quasilinear parabolic equations. In section 2 of this memorial article, Nina Nikolaevna writes more about this wonderful monograph. Olga Alexandrovna is also the author of the book [6] on the attractors of semigroups and evolution equations. This book is based on a lecture course that Olga Alexandrovna gave in different universities in Italy, the so-called *Lezioni Lincei*.

Olga Alexandrovna was a very educated and highly cultured person. The famous Russian poet, Anna Akhmatova, who knew Ladyzhenskaya very well, devoted a poem to her.

Olga Alexandrovna was once a member of the city council of people's deputies. She helped many mathematicians in Leningrad to obtain apartments (free of charge) for their families.

Once in the Steklov Mathematical Institute in Leningrad, removing my coat in the cloakroom, I said jokingly to the cloakroom attendant that I was going to take away Olga Alexandrovna to Moscow with me, and I got a serious answer: "We will never give up our Olga Alexandrovna!"

Olga Alexandrovna was a deeply religious woman.

Olga Alexandrovna devoted her life to mathematics, sacrificing her own happiness for it. Now, we can surely say that she has become a "classic figure" of mathematical science.

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Peter Lax, Cathleen Morawetz, and Louis Nirenberg

Young mathematicians today would have a hard time imagining how thoroughly the Iron Curtain isolated the Soviet Union from the rest of the world in the 1940s and 1950s. Mathematics was no exception. We were generally aware who the leading figures were—Gelfand, Kolmogorov, Petrovsky, Pontryagin, Sobolev, Vinogradov—and knew the names of some of the up-and-coming new postwar generation—Faddeev, Arnold, Ladyzhenskaya, Oleinik. We were aware of Gelfand's spectacular work on normed rings and were influenced by Petrovsky's survey article on partial differential equations. But in general we were woefully ignorant—few of us knew Russian, or even the Cyrillic alphabet; personal encounters were highly restricted.

The thaw started only after Stalin died; a key event was Khrushchev's denunciation of Stalin's crimes in 1956, the year the Soviet Government rehabilitated Ladyzhenskaya's father, shot as a traitor in 1937. Khrushchev's speech was meant only for the party faithful, but it was soon disseminated generally thanks to the CIA—credit where credit is due. A few visitors were allowed to enter the Soviet Union and have relatively free access to Soviet scientists; the impressions of these visitors were eagerly sought. Leray reported that in Leningrad he saw the Hermitage, Peterhof, and Ladyzhenskaya.

The Iron Curtain worked in both directions; when Ladyzhenskaya first started to work on the Navier-Stokes equation, she was unaware of the work of Leray and Hopf.

A major sign of a thaw was the large Soviet delegation, Olga among them, at the Edinburgh International Congress of Mathematicians in 1958. For many of us this was the beginning of a mathematical and personal friendship with Olga. After the close of the Congress, by chance I (PDL) ran into a group of Soviet mathematicians at the National Gallery in London. Olga and I started talking and got separated from the group. Unfortunately Olga did not remember the name of the hotel where the delegation was staying, so we were forced to go to the Soviet Consulate; with their help the lost sheep was reunited with her flock.

The trickle of Western visitors turned to a tide in the 1960s. An outstanding event was the Opening Conference of the University and Science City at Novosibirsk in 1963, under the direction of Lavrentiev. Here friendships started at Edinburgh were renewed under much more relaxed conditions. LN recalls that Olga had arranged a sailing

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party on the Ob Sea on a vessel with a red sail, made available by Sobolev, director of the Mathematics Institute at Novosibirsk.

The International Congress of Mathematicians in Moscow in 1966 was another outstanding occasion for Western and Soviet mathematicians to get together.

Western publishers started putting out English translations of important books that originally appeared in Russian, including Olga's *Mathematical Theory of Viscous Incompressible Flow* and her book with Ural'tseva, *Linear and Quasilinear Elliptic Equations*. The AMS performed an immensely useful service with its vast translation program of articles and books, including Olga's book with Ural'tseva and Solonnikov on parabolic equations.

Olga's first work was to use the method of finite differences to solve the Cauchy problem for hyperbolic equations, giving a new proof of Petrovsky's result—Petrovsky's method is incomprehensible.This work of Olga's is little known in the West; Olga herself did not return to it, very likely because she decided (as did Friedrichs) that a priori estimates and the methods of functional analysis are more effective tools. She did, however, devise numerical schemes for solving the Navier-Stokes equation.

Olga consistently used the concept of weak solutions, a point of view championed by Friedrichs, whom Olga admired. She was in the forefront of the upsurge of interest in elliptic equations. In a very early work she showed that second-order elliptic boundary-value problems have square integrable solutions up to the boundary under very general boundary conditions. Her books with Ural'tseva and Solonnikov contain many deep results concerning estimates for solutions of elliptic and parabolic equations. They have greatly extended ideas of Serge Bernstein and techniques of DiGiorgi, Moser, and Nash. These books are basic sources for these subjects.

The work for which Olga will be remembered longest is on the Navier-Stokes equation. This is a technically very difficult field in which every advance, even modest, requires great effort. One of the goals is to decide if the initial value problem in three dimensions has a smooth solution for all time, and if not, whether a generalized solution is uniquely determined by the initial data. But even if at some future time these questions are decided, and the Clay Foundation rewards its solver with one of its million-dollar prizes, the main task of hydrodynamics remains to deduce the laws governing turbulent flow.

In additional to a large body of technical results, Olga boldly proposed a modification of the Navier-Stokes equations in regions where the velocity fluctuates rapidly. She also made important contributions to the theory of finite-dimensional attractors of solutions of the Navier-Stokes equations. Flow-invariant measures on manifolds of such attractors might be a building block of turbulence.

On her last visit abroad, at a conference held in Madeira in June 2003, she gave a spirited personal account of her work on the Navier-Stokes equation.

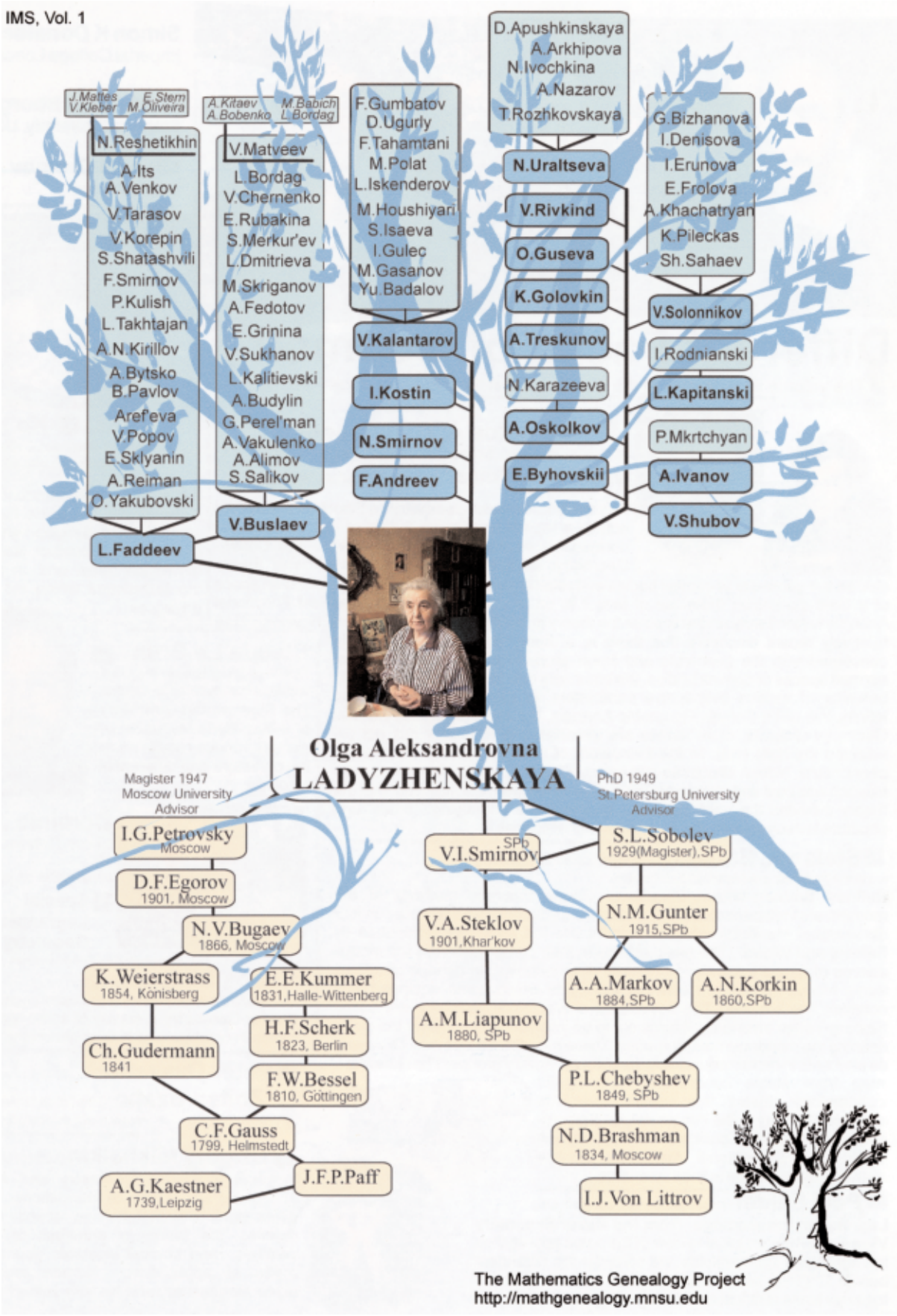
For decades, Olga ran a seminar on partial differential equations that kept up with the development of the subject worldwide; it had a great influence in the Soviet Union.

Olga's career shows that even in the darkest days of Soviet totalitarianism there were courageous academics, Petrovsky and Smirnov in Olga's case, who were willing to defy official proscription of the daughter of a man shot as a traitor, perceive her ability, and ease her path so her talent could bloom and gain recognition.

It is to the great credit of the international mathematical community that at the height of the Cold War, when both sides were piling thousands of nuclear weapons on top of each other, Soviet and Western mathematicians formed an intimate family, where scientific and personal achievements were admired independently of national origin. Olga was among the most admired, for her courage, for overcoming enormous obstacles, for supporting those under attack, for her mathematical achievements, and for her overwhelming beauty.

Acknowledgement

Photographs and the mathematical family tree are reproduced from the first two volumes, *Nonlinear Problems in Mathematical Physics and Related Topics. In Honor of Professor O. A. Ladyzhenskaya I, II*, M. Sh. Birman, S. Hildebrandt, V. A. Solonnikov, N. N. Ural'tseva, eds., of the International Mathematical Series published by Kluwer/Plenum Publishers (English) and by Tamara Rozhkovskaya (publisher, Russian). These photos are reproduced here with permission of the publishers of the International Mathematical Series.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