2005 Conant Prize

The 2005 Levi L. Conant Prize was awarded at the 111th Annual Meeting of the AMS in Atlanta in January 2005.

The Conant Prize is awarded annually to recognize an outstanding expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years. Established in 2000, the prize honors the memory of Levi L. Conant (1857–1916), who was a mathematician at Worcester Polytechnic University. The prize carries a cash award of $1,000.

The Conant Prize is awarded by the AMS Council acting on the recommendation of a selection committee. For the 2005 prize the members of the selection committee were: Anthony W. Knapp, Carl Pomerance (chair), and M. B. Ruskai.

Previous recipients of the Conant Prize are: Carl Pomerance (2001), Elliott Lieb and Jakob Yngvason (2002), Nicholas Katz and Peter Sarnak (2003), and Noam D. Elkies (2004).

The 2005 Conant Prize was awarded to Allen Knutson and Terence Tao. The text that follows presents the committee's citation, brief biographical sketches, and the awardees' responses upon receiving the prize.

Citation

The Levi L. Conant Prize in 2005 is awarded to Allen Knutson and Terence Tao for their stimulating article "Honeycombs and Sums of Hermitian Matrices", *Notices of the AMS* **48** (2001), no. 2, 175–186.

In 1912 Hermann Weyl raised the problem of characterizing the possible sets of eigenvalues of the sum $A+B$ of two Hermitian matrices A, B in terms of the sets of eigenvalues of each of them. This is a very natural problem with applications to many areas, particularly to quantum theory. In particular, it allows one to describe the possible results of measurements of the sum of two observables in terms of those of the individual observables. Yet surprisingly little progress was made until a full solution was found in 1998. Soon after, Knutson and Tao introduced the concept of "honeycombs" and used them to simplify the solution and prove some related conjectures.

In eminently readable and unpretentious style, the authors give an account of their approach to Weyl's problem. After a brief introduction to the 1962 conjecture of Alfred Horn, which recasts the Weyl problem in terms of a conjectured series of inequalities for the eigenvalues of the sum matrix $A+B$, the authors introduce honeycombs, a type of diagram reminiscent of a beehive. Using 15 clearly explained figures that help one to picture various combinatorial nuances, the authors expertly lead the reader through the intricacies of their work. They gently transport us from Weyl's classical problem to a "quantum" analog, involving the Littlewood-Richardson formula for multiplicities of representations of unitary groups within tensor products. They then explain the key "saturation conjecture", which connects the classical and quantum problems to each other and implies the validity of Horn's conjecture. Having shown that the saturation conjecture can be reduced to a problem about honeycombs, they sketch its proof, all the while playing strongly to the reader's intuition. The story that is recounted brushes against symplectic geometry, invariant theory, combinatorics, and computational complexity, but the authors deftly keep the reader from getting overwhelmed by technicalities.

By skillfully combining honeycomb diagrams with a high level of exposition, Knutson and Tao make this fascinating subject accessible to a wide mathematical audience.

Allen Knutson

Biographical Sketch

Allen Knutson did his graduate work in symplectic geometry, overlapping with Terence Tao at Princeton, where their common love of linear algebra brought them together to work on Horn's conjecture. He finished up at the Massachusetts Institute of Technology, his third graduate school, the first being the University of California, Santa Cruz, thus equalling his number of undergraduate institutions: Caltech, New York University, and the Budapest Semesters in Mathematics Program.

In addition to the *Notices* article concerning his and Tao's combinatorial work together, Knutson has another one solely on "The symplectic and algebraic geometry of Horn's problem", *Linear Algebra Appl.* **319** (2000), nos. 1–3, 61–81.

After a National Science Foundation postdoc at Brandeis with Gerald Schwarz, Knutson moved in 1999 to the University of California, Berkeley, where he is now associate professor. His awards include a Clay research summer fellowship, a Sloan Fellowship, and the International Jugglers' Association world record in two-person ball juggling from 1990 to 1995. (The record was for 12 balls; nowadays the record is 13.)

Response

I am extremely honored and gratified to receive the Conant Prize—almost as much as to receive the initial invitation to write the article!

One of the most mysterious aspects of the original conjecture of Horn was a sort of continuous/discrete schizophrenia, in which real eigenvalues were occasionally required to be natural numbers. This already suggested that there should be other related, naturally discrete, mathematical fields in which the "eigenvalues" would be automatically integral. Three of these have come up: dominant weights of representations of $GL(n)$, Schubert classes on Grassmannians, and integral honeycombs.

The work of Totaro and Helmke-Rosenthal, and its more difficult converse by Klyachko, went back and forth between the Hermitian matrices and the Schubert classes. Ours is pretty much entirely in the combinatorial realm, with honeycombs, hives, and puzzles. Belkale's proof is entirely in the Schubert domain and is being given a very pretty generalization by Purbhoo and Sottile, beyond Grassmannians to other "minuscule flag varieties". It still seems amazing that Horn could guess a recursive statement completely within the Hermitian framework!

The saturation problem (as distinct from Horn's conjecture) seems most naturally stated and studied purely within representation theory and has received a solution recently for general groups by Kapovich and Millson, "A path model for geodesics in Euclidean buildings and its applications to representation theory", math.RT/0411182.

Terence Tao

Biographical Sketch

Terence Tao was born in Adelaide, Australia, in 1975. He received his Ph.D. in mathematics from Princeton University in 1996 under the advisorship of Elias Stein. He has been at the University of California, Los Angeles, as a Hedrick assistant professor (1996–98), assistant professor (1999–2000), and professor (2000–). He has also held visiting positions at the Mathematical Sciences Research Institute (1997), the University of New South Wales (1999–2000), and the Australian National University (2001–03).

Allen Knutson

Tao has been supported by grants from the National Science Foundation and fellowships from the Sloan Foundation, Packard Foundation, and the Clay Mathematics Institute. He received the Salem Prize in 2000 and the Bôcher Prize in 2002.

Tao's research concerns a number of areas, including harmonic analysis, geometric and arithmetic combinatorics, analytic number theory, nonlinear evolution equations, and algebraic combinatorics.

Terence Tao

Response

I am deeply touched and honoured, and perhaps even a little surprised, to receive this award. Allen and I were fascinated by this problem of summing Hermitian matrices ever since we were graduate students, and we were always struck by just how much geometric, algebraic, and combinatorial structure underlies this innocuous-sounding problem.

This area of mathematics is highly interdisciplinary, benefitting from ideas in fields as diverse as algebraic geometry, symplectic geometry, representation theory, enumerative combinatorics, linear algebra, and the geometry and combinatorics of convex polytopes; this topic seems to draw the interest of mathematicians from many other fields (for instance, I myself was drawn to this problem despite being primarily in analysis). I hope our article helps popularize this topic further. (An excellent survey of the field can be found in the reference [F2] in the cited article.)

Some further progress has been achieved since the publication of the *Notices* article. For instance, we now understand that while honeycombs (and the closely related Littlewood-Richardson rule) "solve" the Hermitian matrix and $U(n)$ tensor product problems, they are in fact much more tightly connected with the "infinite negative curvature" or "zero temperature" variants of those problems.

Indeed, recent work of Speyer has connected honeycombs to a variant of the Hermitian matrix problem in which the underlying field C is replaced by the field $C\{t\}$ of Pusieux series, while recent work of Henriques and Kamnitzer has also connected honeycombs to GL_n crystal representations. Meanwhile, there have now been several alternative proofs of the Horn and saturation conjectures (in very different settings) which use completely different techniques, such as Derksen and Weyman's proof of the saturation conjecture via quiver representations, Kapovich-Leeb-Millson's proof of the saturation conjecture via the theory of modules over discrete valuation rings, or Belkale's geometric proof of the Horn and saturation conjectures via the transversality analysis of Schubert varieties.

There are clearly some very rich interconnections between very distinct areas of mathematics here, and there is much that is still left to be done; we are nowhere close to uncovering the underlying theory which explains all of these connections. For instance, the situation when the underlying group $U(n)$ (or GL_n) is replaced by another Lie group is still only partially understood. Another completely open question is how these honeycombs "converge" in the large dimensional limit (as n goes to infinity), as there should definitely be some connection with free probability and free convolution.