Teaching Mathematics Graduate Students How to Teach

Solomon Friedberg

True or false: "The primary task of a mathematics graduate student is to learn, and ultimately to create, mathematics." Most graduate school faculty, including this author, would heartily agree. But such an individual, upon graduation, will be asked to teach his or her own course in the academy or to work as part of a group in industry. It is natural to expect such a student to develop the skills necessary to do so while in graduate school. In this article I focus on some novel ways for mathematics departments to promote the development of teaching (and communication) skills, consistent with the primary focus on subject matter mastery noted above.

Before discussing teaching, it seems appropriate to acknowledge that there was a time not long ago when teaching skills were of little importance to many institutions. (See for example the discussion of the Princeton math department in the 1950s in Sylvia Nasar's *A Beautiful Mind*.) Several factors are contributing to a change in this regard. The United States is a nation at risk in terms of precollegiate mathematics education, but if we do not succeed in teaching mathematics to the undergraduate students we get, even if these students are not all highly motivated and not all well prepared, then our nation will not be able to maintain the scientific work force it needs. At the same time, college tuition is large and growing, and consumers rightly expect that the product they purchase will be worth the cost. And it is not a given that as math faculty retire, they will be replaced by new faculty. If we cannot succeed at the teaching of mathematics to undergraduates, then the pressure to have others do so in our place will increase. In the long run, then, mathematics will do better if the next generation of mathematicians on university faculties are excellent teachers. The topic of this article should thus be of genuine importance to the entire math community.

Lofty principles and long-term perspectives are fine, but most of us live day to day. Graduate students serve as teaching assistants (TAs), getting exposure to teaching and helping us do the job of educating our students. TAs do different tasks at different institutions: some run recitation sections, some teach their own class under the supervision of a senior faculty member, some are handed the syllabus their first day on the job and told to go to it. In all cases, good TAs are a benefit to a mathematics department, both in the actual teaching of mathematics to undergraduates and in relations with other departments and the administration. Bad TAs, as measured by student complaints, are a liability. To address this, many institutions offer TA-training (a better phrase might be TA-development) programs. It might surprise some to learn not that this TA-training exists, but that it comes in a great variety of formats at different institutions. Formats include an intensive period during orientation week, a summer course, a required semester course, a voluntary semester course, a one-hour-per-week or per-month

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program during each year, a similar program but only during the first semester of TAing. Some programs involve only new TAs, some involve new and a few experienced TAs, some involve all TAs, some are only for experienced TAs. Besides the diversity in both TA duties and TA training, there is diversity in how these are coupled, with programs spanning the range from minimal supervision and no follow-up as regards TA training to carefully integrated and sustained mentoring.

What is in these TA-training programs? Certainly, to teach effectively one needs to be able to use a blackboard and to speak understandably (a foreign accent is fine; some of the best teachers I know speak heavily accented English). Accordingly, almost all programs include practice in explaining problems at the board, critiqued by the other participants or by the instructor. Some institutions videotape. But after developing teaching skills at this basic level, there is no uniform next topic, no canonical way to proceed.

This is not surprising. After all, there is no one right way of teaching and no one approach to the classroom that is guaranteed to work for all. Good teaching is far from well defined, and my idea of good pedagogy may be different from yours. So how can one possibly teach someone to teach well?

A crucial part of teaching, of course, is what you say and how you say it. In the rest of this article I would like to focus on two factors that contribute to what good teachers say and how they say it: *experience* and *good judgment*. It may seem surprising that one can speed the acquisition of the former and that one can teach the latter, but I will make the argument below that one can do so and that doing so is useful to developing strong teaching in mathematics graduate students.

The first of these factors, experience, is an obvious one. Those who have been in the classroom for a few years have seen the range of student responses to our efforts and have an idea of what works and what doesn't. Graduate students (and some beginning faculty), by contrast, frequently have limited access to experiences with students who are not like themselves—students who study mathematics for different reasons than they did or, even more strongly, students who find mathematics frightening or uninteresting. Experience in teaching such students will come in due time, but the parent paying an enormous tuition bill is no more likely to accept this as an excuse for ineffective teaching than the patient who finds out that his surgeon is doing the operation for the very first time but skipped the practice course.

The second factor, good judgment, is something we are well aware of in mathematics research but perhaps less so in the classroom. In the research setting, the student meets good judgment when the advisor's problem turns out to be solvable and interesting. In the classroom, it is an essential part of excellent teaching. Good judgment manifests itself in the way that the teacher answers for him- or herself and then for the class such questions as: Why is this mathematical concept important? What in the class material is fundamental and what is not? How do we balance conceptual understanding and an appreciation of the big picture with technical details and problem solving? How do we respond in lecture if students are unresponsive and possibly confused? What will motivate the students and engage them intellectually? What assignments will bring out their best? How do we respond to a diversity of levels of preparation? In these and many other questions that we face in the classroom, it is good judgment that makes some people successful and others less effective.[1]

The key point is that these two factors are coupled. *The analysis of experience can contribute to good judgment.* A driver who skids on a slippery road once and thinks about it will drive the road more slowly the next time it rains. In an academic context, in the early twentieth century business schools developed a method of teaching based on the analysis of experience, a method in which key business decisions were described and then analyzed: the case study method. As former Harvard president Lowell stated in the early 1920s, "The case method of business training is deemed the best preparation for business life, because the discussion of questions by the banker, the manufacturer, the merchant or the transporter consists of discerning the essential elements in a situation and applying to them the principles of organization and trade. His most important work consists of solving problems and for this he must have the faculty of rapid analysis and synthesis."[2] The analysis of cases promotes good business judgment.

The use of case studies to promote university teaching was developed extensively by Professor C. Roland Christensen of the Harvard Business School, beginning in the late 1960s and continuing into the 1990s. His cases describe in writing crises in a university classroom, and in Christensen's implementation each crisis is discussed in detail by a group with a discussion leader. The group members need not agree, but they are led to think deeply, in a Socratic-method approach. Christensen's seminar in the Boston area became renowned.[3] His book (joint with Barnes and Hanson) *Teaching and the Case Method* [1] is still

[1] *As these questions illustrate, good judgment has as a foundation a sophisticated understanding of the subject matter being taught.*

[2] *As quoted in [1], p. 41.*

[3] *A videotape giving the flavor of Christensen's seminar,* The Art of Discussion Leading: A Class with Chris Christensen, *is available through the Derek Bok Center for Teaching and Learning, Harvard University.*

a classic. But the issues and experiences in Christensen's cases are far from the ones of immediate concern to mathematics graduate students.

I first learned of case studies in the mid-1990s, when I was present at a case study discussion for middle and high school mathematics teachers led by Katherine Merseth of the Harvard Graduate School of Education. I still remember being deeply impressed by the way that the teachers responded to Merseth's case and the way that in the course of the discussion they visibly reevaluated their ideas about teaching and began to make new judgments about how to handle teaching issues. It seemed to me that this method had the potential to contribute to the preparation of mathematics graduate students in an important way.

During the period 1998–2002 the author led a major effort to develop case studies that would be relevant to mathematics graduate students. The project, dubbed the Boston College Mathematics Case Studies Project,[4] had as its goal the development of case studies—depictions of aspects of teaching math to undergraduates, typically involving a difficulty or an important decision—that would supplement mathematics graduate students' experiences and promote the development of good judgment concerning classroom issues. The resulting materials would be evaluated for their effectiveness.

With some relief, the author can report that the effort has been successful. The development team[5] wrote fourteen case studies, several with multiple parts or mathematical levels. (One of our case

[4] *The project was funded by a grant from the U.S. Department of Education's Fund for the Improvement of Postsecondary Education.*

[5] *Avner Ash, Elizabeth Brown, Solomon Friedberg, Deborah Hughes Hallett, Reva Kasman, Margaret Kenney, Lisa A. Mantini, William McCallum, Jeremy Teitelbaum, and Lee Zia.*

The following case study, Seeking Points, is reprinted from the book *Teaching Mathematics in Colleges and Universities: Case Studies for Today's Classroom* by Solomon Friedberg, Avner Ash, Elizabeth Brown, Deborah Hughes Hallett, Reva Kasman, Margaret Kenney, Lisa A. Mantini, William McCallum, Jeremy Teitelbaum, and Lee Zia, Issues in Mathematics Education, vol. 10, American Mathematical Society, Providence, RI, 2001.

Seeking Points

Daniel sighed as he dumped his books on his office desk. He'd just handed back the first midterm exam from his Calculus I class, and he could tell as he left the classroom that there were a lot of unhappy students. Still, the exam had been just like the practice exam he'd given out, and he was sure it was pretty straightforward. As he sat down to take a look at the paper on duality for fppf sheaves he was supposed to read, he heard a knock on his office door.

"Come in," he called, and he saw Sam, one of his Calculus students, push open the door hesitantly.

"Can I talk to you about my exam?" Sam said.

"I guess this was inevitable," thought Daniel to himself. To Sam, he said "What's up?"

"It's this question number 2," said Sam. "I don't think my answer was graded properly."

"Let me take a look," Daniel replied, "pull up a chair."

Sam sat down and passed his exam booklet over to Daniel. Daniel noticed that Sam had gotten 82 points out of 100 on the exam, which was a high B, but he had missed most of his points on problem 2. Then Daniel looked at the question, which said:

Problem 2 (20 points). Let $f(x) = x^3 - 5$. Use the definition of the derivative to compute the slope of the tangent line to the graph of $f(x)$ at the point where $x = 2$.

Then Daniel turned to Sam's exam paper. Sam had written the following:

Sam's Answer: $f(x) = x^3 - 5$. $f'(x) = 3x^2$. Slope $= f'(2) = 12$.

The grader of the problem had given Sam 5 out of the 20 points.

"Well, Sam," said Daniel, "you see, you didn't do what the question asked. You are supposed to use the definition of the derivative to solve this problem, but you didn't give any method for deriving your answer. How did you do this problem?"

"I used the rule that the derivative of x^n is nx^{n-1}, which makes it really easy," replied Sam.

Daniel felt a little uncomfortable about this. He, like the rest of the Calculus teachers, was emphasizing understanding rather than algorithms for solving problems. He and his fellow instructors had specifically scheduled the first exam after a qualitative discussion of the derivative, and an introduction to the definition, but before discussing the various techniques of differentiation. He hadn't gone over the nx^{n-1} rule in class yet.

"Where did you get that from?" he asked Sam.

"I took Calc in high school, and we learned it there. We learned lots of other methods too. The answer is right, isn't it?"

"Yes, it's correct as far as it goes, but as I said it isn't what we asked for. We wanted you to show that you can use the definition of the derivative."

"You mean that thing with the limit?" said Sam.

"Yes," said Daniel, "exactly, that thing with the limit—the difference quotient. In the review for the test I emphasized that if we asked you to use the definition of the derivative, then we wanted you to use the difference quotient."

"Well," said Sam, "I didn't come to the review session. But it doesn't really seem fair to me that I got so many points off because I did the problem an easy way instead of a hard way."

"It isn't just a question of easy and hard," said Daniel. "We are trying to teach you to understand what the derivative means and where it comes from. We don't want you to just learn a bunch of formulas and how to make them go."

"Look, Professor, I know what the derivative means. It's the slope of the tangent line to the curve at the point, just like you asked. I knew that, because I knew what to calculate once I used my rule. Look at the rest of my exam—I got all the other problems basically right. I think I deserve more points on this problem."

"Sam, before we get into a discussion of points, let me ask you this. Do you know what the difference quotient is? Do you know WHY the formula you used gives you the slope of the tangent line?"

"Yeah, well, you did that in class a while ago, and I understood it then. It has something to do with secant lines and stuff, but I forget right now. I figured it doesn't really matter, 'cause I know these other, easier ways to do the problems. I just feel sorry for the other students who have to do it the hard way. I taught my roommate in another section about the methods I learned and he really appreciated it."

"But, Sam, that's just the point we are trying to get across. It IS just as important to know WHY the formula works as how to use it. The formulas you learned all had to be figured out by someone using the difference quotient. Let's take the problem from the test. What we wanted to see was the following:

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{((2+h)^3-5)-(2^3-5)}{h} \\
&= \lim_{h\to 0} \frac{2^3+3\cdot 2^2\cdot h+3\cdot 2\cdot h^2+h^3-5-2^3+5}{h} \\
&= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} \\
&= \lim_{h\to 0} 12+6h+h^2 \\
&= 12
\end{aligned}
$$

So the derivative is 12, and so is the slope. This calculation shows that the slope, 12, is the limiting value of the slopes of the secant lines."

"Well, maybe it shows that to you, but it looks like a bunch of formulas to me. Just different formulas. You really think all those people who wrote that instead of what I wrote know something I don't? They just went to the review session, which I admit I shoulda done. Look, Prof, I'm not here to argue about all of Mathematics. I promise from now on I'll come to your review sessions and do the problems just the way you want them. I just want 5 more points so I can get an A on this exam."

Daniel's heart sank. It was pretty clear this kid Sam didn't get Daniel's argument about "underlying ideas." And him promising to do whatever Daniel wanted on the next exam just made Daniel feel worse—that made it seem like the kid was just humoring him. As for more points—well, lots of people had made this mistake on the exam, and they'd all gotten five points. So Daniel couldn't really change this kid's point score without changing the others, too, though he did think Sam seemed pretty sharp.

"Sam, I'm afraid I can't give you any more points on this problem. We graded the exam consistently, and we gave everyone who made your mistake 5 points. I appreciate what you're telling me, and I get the impression you are following the course pretty well, so if you continue to do well, you can get your A on the next midterm and the final and you'll get your A in the course."

"So you mean you graded lots of people unfairly, and you don't want to fix it? OK, you're the prof, I guess. And I'll be sure to come to the review session next time so I find out how you want us to do the problems."

Sam picked up his exam and left the room. Daniel stared after him for a minute or two, visibly upset, then took a deep breath and turned back to his desk. He had promised to read this paper before his next meeting with his advisor. Where was he? Oh, yes, he could see that the argument he was reading worked if he used the theorem on flat descent. He remembered sitting in on a lecture during his second year where his professor had described flat descent in detail; he couldn't exactly remember the proof of the theorem, but he did remember that you could apply it in this situation.…

studies, Seeking Points, is included as a sidebar to this article.) The cases were created by an extensive process of writing, feedback from graduate students and faculty, and rewriting. Each of the cases raises a variety of interwoven issues to be explored through group discussion and analysis (though they can also be read independently of such a discussion). The cases give graduate students the chance to analyze complicated realistic teaching situations (perhaps applying general principles they have formulated or discussed); to think in advance about how to handle teaching crises so that they can deal with them when they arise in real life; to formulate their own approach to teaching; and to view teaching as nontrivial and sometimes ambiguous, and as something to talk about. They supplement TA experiences and contribute to the development of good judgment. Discussion of the cases also contributes to listening and communication skills. Our cases were piloted at diverse institutions, public and private, large and small (around twenty in all). The evaluation[6] and extensive feedback from graduate students and faculty colleagues showed that the cases were in fact an effective way of broadening individuals' experience base and of promoting thought and dialogue about teaching.

To give the flavor of this feedback, here are several verbatim comments made by graduate students who had participated in a case discussion: "It helps me get some of these vague ideas I have about teaching, etc., to solidify a little bit and also brings up issues of things I've never considered before." "I had the opportunity to think about some issues that, even though they come up daily as a graduate student, we don't really take the time to think about these issues the way I had time today.… The problems these case studies raise are problems that touch me." "It was a really good situation to sit down and talk with different people who had different experiences. There were differences in the level of experience that people had and that made there be a chance for a lot of new ideas and a lot of seasoned ideas." "It's given me a vocabulary to talk and think about teaching that I wouldn't necessarily have just come up with as I'm worrying about writing my dissertation otherwise."

There were several surprises. We had special concern about the appeal of such a method for foreign graduate students and concern with their ability to be involved in such a discussion. In fact, it turned out that many foreign graduate students found the cases a useful window on American university culture, and many had deep ideas about teaching that were useful to all. We thought that the cases would work the same with all levels of graduate students, but we found that many beginning graduate students did not have the experience base to discuss all cases, since they had not given any thought to the teaching aspect of their designated profession. Fortunately, it also turned out that this could be addressed by having more experienced peers, such as a head TA or two, in the case discussion. Most crucially, we learned that leading a case study discussion requires different skills from lecturing on the part of the faculty leader and that there is a learning curve to leading a successful case study discussion.

To address this last point, we organized two multiday workshops for faculty interested in our case studies while they were under development and added extensive materials to the published faculty edition concerning the use of the cases. Since the conclusion of the development project, the author and Diane Herrmann of the University of Chicago have offered a series of workshops at AMS meetings for faculty interested in learning to lead case discussions (our workshop at the 2005 Joint Meetings was attended by, among others, participants from four non-English-speaking countries).[7] An additional workshop is planned for the 2006 Joint Meetings.[8]

At this point the case studies we have written and the materials we created to guide their use have been published [4], and they are being used in diverse ways in a significant number of institutions. They have been used as part of a first course in teaching and as the basis for a second course in teaching. They have been used as stand-alone materials and coupled with a book giving advice about how to teach, such as [5]. Just as learning mathematics is facilitated by well-thought-out exercises, our materials serve as a comprehensive set of exercises for teaching.

In concluding, let me observe that the ability to teach and to communicate well is of concern throughout science and engineering. Indeed, the National Academy of Sciences's publication *Preparing for the 21st Century: The Education Imperative* reports that "employers do not feel that the current level of education [of Ph.D. graduates in science and engineering] is sufficient in providing skills and abilities…particularly in communications skills (including teaching and mentoring abilities for academic positions),…, [and] teamwork.…" Discipline-based efforts seem most likely to be effective in addressing this concern. Fortunately, an increasing number of individuals are now thinking about how to develop teaching

[6] *Carried out by independent evaluator Mary Sullivan.*

[7] *For a discussion of the use of case studies internationally, see [3].*

[8] *These workshops have been supported by a grant awarded to the AMS by the Calculus Consortium for Higher Education, with additional support provided by the AMS through the efforts of AMS associate director Jim Maxwell.*

strength in mathematics graduate students and investigating how mathematics graduate students learn to teach. Other authors have started to share their own successful materials and programs (for example, [2], [5], [6]). Let us hope that these efforts will lead us to the day when every mathematics graduate student completing a Ph.D. is fully prepared to teach a class independently—and excellently—upon graduation.

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