George B. Dantzig (1914–2005)

Richard Cottle, Ellis Johnson, and Roger Wets

The final test of a theory is its capacity to solve the problems which originated it.

This work is concerned with the theory and solution of linear inequality systems. . . .

The viewpoint of this work is constructive. It reflects the beginning of a theory sufficiently powerful to cope with some of the challenging decision problems upon which it was founded.

So says George B. Dantzig in the preface to his book, *Linear Programming and Extensions*, a now classic work published in 1963, some sixteen years after his formulation of the linear programming problem and discovery of the simplex algorithm for its solution. The three passages quoted above represent essential components of Dantzig's outlook on linear programming and, indeed, on mathematics generally. The first expresses his belief in the importance of real world problems as an inspiration for the development of mathematical theory, not for its own sake, but as a means to solving important practical problems. The second statement is based on the theoretical fact that although a linear programming problem, is, prima facie, concerned with constrained optimization, it is really all about solving a linear inequality system. The third statement reveals Dantzig's conviction that constructive methods (in particular, algorithms) are required to obtain the kinds of solutions called for in practical decision problems.

George Dantzig is best known as the father of linear programming (LP) and the inventor of the simplex method. The practical power of these two contributions is so great that, on these grounds alone, he was arguably one of the most influential mathematicians of the twentieth century. And yet there is much more to the breadth and significance of his work. Our aim in this memorial article is to document the magnitude of George Dantzig's impact on the world by weaving together many strands of his life and professional commitments. In so doing, we bring to light the inadequacy of the "father/inventor" epithet.

The impact we have in mind is of many kinds. The earliest, of course, was on military and industrial planning and production. Dantzig's work greatly impacted economics, mathematics, operations research, computer science, and various fields of applied science and technology. In response to these developments, there emerged the concomitant growth of educational programs. Dantzig himself was a professor for more than half of his professional life and in that capacity had a profound impact on the lives and contributions of his more than fifty doctoral students.

As already suggested, George Dantzig passionately believed in the importance of real world problems as a wellspring of mathematical opportunity. Whether this was a lesson learned or a conviction held since early adulthood is unclear, but it served him very well throughout his long and productive life.

Richard Cottle is professor emeritus of management science and engineering at Stanford University. His email address is rwc@stanford.edu. *Ellis Johnson is professor of industrial and systems engineering at the Georgia Institute of Technology. His email address is* ejohnson@isye.gatech.edu. *Roger Wets is professor of mathematics at the University of California, Davis. His email address is* rjbwets@ucdavis.edu.

Formation

Our knowledge of George Dantzig's childhood is largely derived from part of an interview [1] conducted in November 1984 by Donald Albers. (Much the same article is available in [2].) George was the son of mathematician Tobias Dantzig (1884–1956) and Anja Ourisson who had met while studying mathematics at the Sorbonne. There, Tobias was greatly impressed by Henri Poincaré (1854–1912) and later wrote a book on him [41], though he is best known for his *Number, The Language of Science* [40].

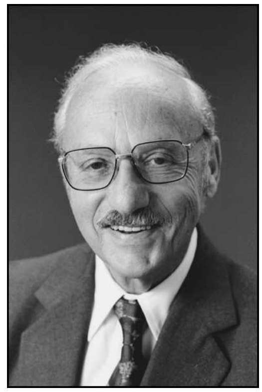
George Dantzig was born in Portland, Oregon, on November 8, 1914. His parents gave him the middle name "Bernard" hoping that he would become a writer like George Bernard Shaw. George's younger brother Henry (1918–1973), who was given the middle name Poincaré, became an applied mathematician working for the Bendix Corporation.

By his own admission, George Dantzig lacked interest in schoolwork until grade seven. He then became keen about science and mathematics although in ninth grade he made a "poor start" in his first algebra course. "To be precise," he said, "I was flunking." Furious with himself, he buckled down and went on to excel in high school mathematics and science courses. He was particularly absorbed by projective geometry and worked "thousands" of such problems given to him by his father, Tobias, who was then on the mathematics department faculty at the University of Maryland.

As an undergraduate, George concentrated in mathematics and physics at the University of Maryland, taking his A.B. degree in 1936. After this he went to the University of Michigan and obtained the M.A. in mathematics in 1938. In Ann Arbor, he took a statistics course from Harry C. Carver (founding editor of the *Annals of Mathematical Statistics* and a founder of the Institute of Mathematical Statistics). He found the rest of the curriculum excessively abstract and decided to get a job after finishing his master's degree program in 1938.

Dantzig's decision to take a job as a statistical clerk at the Bureau of Labor Statistics (BLS) turned out to be a fateful one. In addition to acquiring a knowledge of many practical applications, he was assigned to review a paper written by the eminent mathematical statistician Jerzy Neyman who was then at University College, London, and soon thereafter at the University of California at Berkeley. Excited by Neyman's paper, Dantzig saw in it a logically based approach to statistics rather than a bag of tricks. He wrote to Neyman (at Berkeley) declaring his desire to complete a doctorate under Neyman's supervision, and this ultimately came to pass.

In 1939 Dantzig enrolled in the Ph.D. program of the Berkeley Mathematics Department where Neyman's professorship was located. Dantzig took only two courses from Neyman, but in one of them he had a remarkable experience that was to become a famous legend. Arriving late to one of Neyman's classes, Dantzig saw two problems written on the blackboard and mistook them for a homework assignment. He found them more challenging than usual, but managed to solve them and submitted them directly to Neyman. As it turned out, these problems were actually two open questions in the theory of mathematical statistics. Dantzig's 57-page Ph.D. thesis [12] was composed of his solutions to these two problems. One of these was immediately submitted for publication and appeared in 1940 as [11]. For reasons that are not altogether clear, the other appeared only in 1951 as a joint paper with Abraham Wald [39].



George B. Dantzig

By June 1941, the content of Dantzig's dissertation had been settled: it was to be on the two problems and their solutions. Although he still had various degree requirements to complete, he was eager to contribute to the war effort and joined the U.S. Air Force Office of Statistical Control. He was put in charge of the Combat Analysis Branch where he developed a system through which combat units reported data on missions. Some of this work also involved planning (and hence modeling) that, in light of the primitive computing machinery of the day, was a definite challenge. The War Department recognized his achievements by awarding him its Exceptional Civilian Service Medal in 1944.

Dantzig returned to Berkeley in the spring of 1946 to complete his Ph.D. requirements, mainly a minor thesis and a dissertation defense. Around that time he was offered a position at Berkeley but turned it down in favor of becoming a mathematical advisor at the U.S. Air Force Comptroller's Office. This second fateful decision set him on a path to the discovery of linear programming and the simplex algorithm for its solution in 1947.

Origins

Dantzig's roles in the discovery of LP and the simplex method are intimately linked with the historical circumstances, notably the Cold War and the early days of the Computer Age. The defense efforts undertaken during World War II and its aftermath included very significant contributions from a broad range of scientists, many of whom had fled the horrors of the Nazi regime. This trend heightened the recognition of the power that mathematical modeling and analysis could bring to real-world problem solving. Beginning in 1946, Dantzig's responsibility at the Pentagon involved the "mechanization" of the Air Force's planning procedures to support time-staged deployment of training and supply activities. Dantzig's approach to the mathematization of this practical problem ushered in a new scientific era and led to his fame in an ever-widening circle of disciplines.

In [19], George Dantzig gives a detached, historical account of the origins of—and influences on—linear programming and the simplex method. Almost twenty years later, as he approached the age of seventy, Dantzig began turning out numerous invited articles [21]–[26] on this subject, most of them having an autobiographical tone. From the mathematical standpoint, there is probably none better than [25] which, unlike the rest, explicitly relates part of his doctoral dissertation to the field of linear programming.

Dantzig was not alone in writing about the discovery of linear programming. One of the most informative articles on this subject is that of Robert Dorfman, a professor of economics at Harvard (now deceased). In telling the story, Dorfman [46] clarifies "the roles of the principal contributors". As he goes on to say, "it is not an especially complicated story, as histories of scientific discoveries go, but neither is it entirely straightforward." These opening remarks are meant to suggest that many elements of linear programming had already been come upon prior to their independent discovery by Dantzig in 1947. For the convenience of readers, the present article will revisit a bit of this lore and will endeavor to establish the point made at the outset that Dantzig's greatness rests not only on the discovery of linear programming and the simplex method, but on the depth of his commitment to their development and extensions.

At the outset of Dantzig's work on linear programming, there already existed studies by two of the principal contributors to its discovery. The first of these was done by the mathematician Leonid V. Kantorovich. The second was by the mathematical statistician/economist Tjalling C. Koopmans. For good reasons, these advances were unknown to Dantzig.

As a professor of mathematics and head of the Department of Mathematics at the Institute of Mathematics and Mechanics of Leningrad State University, Kantorovich was consulted by some engineers from the Laboratory of the Veneer Trust who were concerned with the efficient use of machines. From that practical contact sprung his report on linear programming [57], which, though it appeared in 1939, seems not to have been known in the West (or the East) until the late 1950s and was not generally available in English translation [60] until 1960. (This historically important document is a 68-page booklet in the style of a preprint. Kantorovich [61, p. 31] describes it as a "pamphlet".) Two other publications of Kantorovich deserve mention here. In [58], Kantorovich proposed an approach to solving some classes of extremal problems that would include the linear programming problem. A brief discussion of the second paper [59] is given below. It is worth noting here that these two articles—written in English during World War II—were reviewed by H. H. Goldstine [55] and Max Shiffman [75], respectively. Hence they were not altogether unknown in the West.

In 1940 Koopmans emigrated from the Netherlands to the United States. During World War II he was employed by the Combined Shipping Adjustment Board, an agency based in Washington, D.C., that coordinated the merchant fleets of the Allied governments, chiefly the United States and Britain. Koopmans's first paper of a linear programming nature dates from 1942. For security reasons, the paper was classified. It became available as part of a Festschrift in 1970 [63]. Based on this wartime experience, Koopmans's 1947 paper [64] develops what later came to be called the *transportation problem* [14]. These works are not algorithmic; they emphasize the modeling aspect for the particular shipping application that Koopmans had in mind.

As Koopmans was later to discover, the special class of linear programming problems he had written about (namely, the transportation problem) had already been published in 1941 by an MIT algebraist named Frank L. Hitchcock. The title of Hitchcock's paper, "The distribution of a product from several sources to numerous localities" [56], describes what the transportation problem is about, including its important criterion of least cost. (A review of the paper [56] can be found in [62].) Hitchcock makes only the slightest of suggestions on the application of the model and method he advances in his paper.

Just as Hitchcock's paper was unknown to Koopmans (and to Dantzig), so too the work of Kantorovich on the transportation problem was unknown to Hitchcock. Moreover, it appears that in writing his paper [59] on the "translocation of masses", Kantorovich was not familiar with the very much earlier paper [68] of Gustave Monge (1746–1818) on "cutting and filling", a study carried out in conjunction with the French mathematician's work on the moving of soil for building military fortifications. The formulation in terms of continuous mass distributions has come to be called the "Monge-Kantorovich Problem".

A few others contributed to the "pre-history" of linear programming. In 1826 Jean-Baptiste Joseph Fourier announced a method for the solution of linear inequality systems [52]; it has elements in common with the simplex method of LP. Charles de la Vallée Poussin in 1911 gave a method for finding minimum deviation solutions of systems of equations [81]. John von Neumann's famous game theory paper [70] of 1928, and his book [72] written with Oscar Morgenstern and published in

1944, treated finite two-person zero-sum games, a subject that is intimately connected with linear programming; von Neumann's paper [71] is of particular interest in this regard. In addition to these precursors, there remains Theodor Motzkin's scholarly dissertation *Beiträge zur Theorie der Linearen Ungleichungen* accepted in 1933 at the University of Basel and published in 1936 [69]. (For a loose English translation of this work, see [7].) Apart from studying the general question of the existence (or nonexistence) of solutions to linear inequality systems, Motzkin gave an elimination-type solution method resembling the technique used earlier by Fourier (and Dines [43] who seems to have concentrated on *strict* inequalities).

Initially, Dantzig knew nothing of these precedents, yet he did respond to a powerful influence: Wassily Leontief's work [67] on the structure of the American economy. This was brought to Dantzig's attention by a former BLS colleague and friend, Duane Evans. The two apparently discussed this subject at length. In Dantzig's opinion [19, p. 17] "Leontief's great contribution . . . was his construction of a *quantitative model* . . . for the purpose of tracing the impact of government policy and consumer trends upon a large number of industries which were imbedded in a highly complex series of interlocking relationships." Leontief's use of "an empirical model" as distinct from a "purely formal model" greatly impressed Dantzig as did Leontief's organizational talent in acquiring the data and his "marketing" of the results. "These things," Dantzig declares, "are necessary steps for successful applications, and Leontief took them all. That is why in my book he is a hero" [1, p. 303].

Carrying out his assignment at the U.S.A.F. Comptroller's Office, George Dantzig applied himself to the mechanization of the Air Force's planning procedures to support the time-staged deployment of training and supply activities. He created a linear mathematical model representing what supplies were available and what outputs were required over a multi-period time horizon. Such conditions normally lead to an under-determined system, even when the variables are required to assume nonnegative values, which they were in this case, reflecting their interpretation as physical quantitities. To single out a "best" solution, Dantzig introduced a linear *objective function*, that is, a linear minimand or maximand. This was an innovation in planning circles, an achievement in which Dantzig took great pride. As he put it in 1957, "linear programming is an anachronism"; here he was alluding to the work of economists François Quesnay, Léon Walras, and Wassily Leontief as well as mathematician John von Neumann all of whom could (and in his mind, should) have introduced objective functions in their work [42, p. 102].

Dantzig's discovery of the linear programming problem and the simplex algorithm was independent of [52], [81], [57], [56], and [63]. Yet, as he has often related, it was not done in isolation. On the formulation side, he was in contact with Air Force colleagues, particularly Murray Geisler and Marshall K. Wood, and with National Bureau of Standards (NBS) personnel. At the suggestion of Albert Kahn at NBS, Dantzig consulted Koopmans who was by then at the Cowles Commission for Research in Economics (which until 1955 was based at the University of Chicago). The visit took place in June 1947 and got off to a slow start; before long, Koopmans perceived the broad economic significance of Dantzig's general (linear programming) model. This might have prompted Koopmans to disclose information about his 1942 work on the transportation problem and Hitchcock's paper of 1941.

Another key visit took place in October 1947 at the Institute for Advanced Study (IAS) where Dantzig met with John von Neumann. Dantzig's vivid account of the exchange is given in [1, p. 309] where he recalls, "I began by explaining the formulation of the linear programming model . . . I described it to him as I would to an ordinary mortal. He responded in a way which I believe was uncharacteristic of him. 'Get to the point,' he snapped. . . . In less than a minute, I slapped the geometric and algebraic versions of my problem on the blackboard. He stood up and said, 'Oh that.' " Just a few years earlier von Neumann had co-authored and published the landmark monograph [72]. Dantzig goes on, "for the next hour and a half [he] proceeded to give me a lecture on the mathematical theory of linear programs." Dantzig credited von Neumann with edifying him on Farkas's lemma and the duality theorem (of linear programming). On a separate visit to Princeton in June 1948, Dantzig met Albert Tucker who with his students Harold Kuhn and David Gale gave the first rigorous proof of the duality theorem that von Neumann and Dantzig had discussed at their initial meeting.

During the summer of 1947, months before his encounter with von Neumann at the IAS, Dantzig proposed his simplex method (much as described elsewhere in this article); in the process of this discovery he discussed versions of the algorithm with economists Leonid Hurwicz and Tjalling Koopmans. It was recognized that the method amounts to traversing a path of edges on a polyhedron; for that reason he set it aside, expecting it to be too inefficient for practical use. These reservations were eventually overcome when he interpreted the method in what he called the "column geometry", presumably inspired by Part I of his Ph.D. thesis [12], [39], [25, p. 143]. There, the analogue of a so-called convexity constraint is present. With a constraint of the form

$$(1) \qquad x_1 + \cdots + x_n = 1, \quad x_j \geq 0 \quad j = 1, \ldots, n$$

(which is common, but by no means generic) the remaining constraints

$$(2) \qquad A_{\bullet 1}x_1 + \cdots + A_{\bullet n}x_n = b$$

amount to asking for a representation of b as a convex combination of the columns $A_{\bullet 1}, \ldots A_{\bullet n} \in \mathbb{R}^m$. (Of course, the solutions of (1) alone constitute an $n-1$ simplex in $\mathbb{R}^n$, but that is not exactly where the name "simplex method" comes from.) By adjoining the objective function coefficients c_j to the columns $A_{\bullet j}$, thereby forming vectors $(A_{\bullet j}, c_j)$ and adjoining a variable component z to the vector b to obtain (b, z), Dantzig viewed the corresponding linear programming problem as one of finding a positively weighted average of the vectors $(A_{\bullet 1}, c_1), \ldots, (A_{\bullet n}, c_n)$ that equals (b, z) and yields the largest (or smallest) value of z.

It was known that if a linear program (in standard form) has an optimal solution, then it must have an optimal solution that is also an *extreme point* of the *feasible region* (the set of all vectors satisfying the constraints). Furthermore, the extreme points of the feasible region correspond (albeit not necessarily in a bijective way) to *basic feasible solutions* of the constraints expressed as a system of linear equations. Under the reasonable assumption that the system (1), (2) has full rank, one is led to consider nonsingular matrices of the form

$$(3) \qquad B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ A_{\bullet j_1} & A_{\bullet j_2} & \cdots & A_{\bullet j_{m+1}} \end{bmatrix}$$

such that

$$(4) \qquad B^{-1} \begin{bmatrix} 1 \\ b \end{bmatrix} \geq 0.$$

The columns $A_{\bullet j_1}, A_{\bullet j_2}, \ldots, A_{\bullet j_{m+1}}$ viewed as points in $\mathbb{R}^m$ are easily seen to be in *general position*. Accordingly, their convex hull is an m-simplex.

Dantzig conceptualized the n points $A_{\bullet j}$ as lying in the "horizontal" space $\mathbb{R}^m$ and then pictured each $(m+1)$-tuple $(A_{\bullet j}, c_j)$ as a point on the line orthogonal to $\mathbb{R}^m$ and passing through $A_{\bullet j}$ with c_j measuring the vertical distance of the point above or below the horizontal "plane", according to the sign of c_j. The *requirement line* consisting of points (b, z) where $b \in \mathbb{R}^m$ is as above and z is the (variable) value of the objective function is likewise orthogonal to the horizontal plane. For any feasible basis, the requirement line meets σ, the convex hull of the corresponding points $(A_{\bullet j_1}, c_{j_1}), (A_{\bullet j_2}, c_{j_2}), \ldots, (A_{\bullet j_{m+1}}, c_{j_{m+1}})$, in the point (b, z), the ordinate z being the value of the objective function given by the associated basic solution. The $m+1$ vertices of the simplex σ determine a hyperplane in $\mathbb{R}^{m+1}$. The vertical distance from a point $(A_{\bullet j}, c_j)$ to this hyperplane indicates whether this point will improve the objective value of the basic solution that would be obtained if it were to replace one of the basic columns. The convex hull of this new point and σ is an $(m+1)$-simplex τ in $\mathbb{R}^{m+1}$. The requirement line meets the boundary of τ in two points: the previous solution and one other point with a better objective function value. In the (nondegenerate) case where the new point lies in the relative interior of a facet of τ, say ρ, there is a unique (currently basic) column for which the corresponding weight (barycentric coordinate) is zero. This point is opposite the new facet ρ where the requirement line meets the boundary of τ. Note that ρ is an m-simplex and corresponds to a new and improved feasible basis. The process of making the basis change is called *simplex pivoting*. The name can be seen as an apt one when the facets of τ are likened to the (often triangle-shaped) body parts of a hinge.

Beyond the fact that not all linear programming problems include convexity constraints, it was clear that, without significant advances in automatic computing machines, the algorithm—in whatever form it took—was no match for the size of the planning problems for which numerical solutions were needed. One such need is typified by a critical situation that came up less than a year after Dantzig's discoveries: the Berlin Airlift. Lasting 463 days, this program called for the scheduling of aircraft and supply activities, including the training of pilots, on a very large, dynamic scale. During this crisis, Britain, France, and the United States airlifted more than two million tons of food and supplies to the residents of West Berlin whose road, rail, and water contacts with Western Europe had been severed by the USSR.

Initial Impacts

"Mathematical Techniques of Program Planning" is the title of the brief talk in which Dantzig publicly announced his discovery of the simplex method. The presentation took place at a session of the joint annual meeting of the American Statistical Association (ASA) and the Institute of Mathematical Statistics (IMS) on December 29, 1947 [48, p. 134]. As Dorfman reports [46, p. 292], "there is no evidence that Dantzig's paper attracted any particular interest or notice." The paper was not published. The method's next appearance occurred in a session (chaired by von Neumann) at a joint national meeting of the IMS and the Econometric Society on September 9, 1948. Dantzig's talk, titled "Programming in a Linear Structure", aroused more interest than its predecessor. The abstract [13] is remarkable for its visionary scope. In it we find mention of the LP model, the notion of dynamic systems, connections with the theory of games, a reference to computational procedures for "large scale digital computers", and the bold suggestion that the solutions of such problems could be *implemented* and not merely discussed. A lively discussion followed Dantzig's talk; afterwards he participated in a panel discussion beside such distinguished figures as Harold Hotelling, Irving Kaplansky, Samuel Karlin, Lloyd Shapley, and John Tukey. The ball was now rolling.

In an autobiographical piece [66] written around the time he (and Kantorovich) received the 1975 Nobel Prize in Economics, Tjalling Koopmans relates how in 1944 his "work at the Merchant Shipping Mission fizzled" due to a "reshuffling of responsibilities". Renewing his contact with economist Jacob Marschak, Koopmans secured a position at the Cowles Commission in Chicago. He goes on to say "my work on the transportation model broadened out into the study of activity analysis at the Cowles Commission as a result of a brief but important conversation with George Dantzig, probably in early 1947. It was followed by regular contacts and discussions extending over several years thereafter. Some of these discussions included Albert W. Tucker of Princeton who added greatly to my understanding of the mathematical structure of duality."

Oddly, Koopmans's autobiographical note makes no mention of his instrumental role in staging what must be considered one of the most influential events in the development of linear (and nonlinear) programming: the "Conference on Activity Analysis of Production and Allocation" held in Chicago under the auspices of the Cowles Commission for Research in Economics in June 1949. Edited by Koopmans, the proceedings volume [65] of this conference comprises twenty-five papers, four of which were authored—and one co-authored—by Dantzig. The speakers and other participants, approximately fifty in all, constitute an impressive set of individuals representing academia, government agencies, and the military establishment. It is a matter of some interest that the proceedings volume mentions no participants from industry. Exactly the same types of individuals were speakers at the "Symposium on Linear Inequalities and Programming" held in Washington, D.C., June 14–16, 1951. Of the nineteen papers presented, twelve were concerned with mathematical theory and computational methods while the remaining seven dealt with applications. In addition to a paper by George Dantzig and Alex Orden offering "A duality theorem based on the simplex method", the first part of the proceedings contains Merrill Flood's paper "On the Hitchcock distribution problem", that is, the transportation problem. Flood rounds up the pre-existing literature on this subject including the 1942 paper of Kantorovich though not the 1781 paper of Monge. Among the papers in the portion of the proceedings on applications, there appears the abstract (though not the paper) by Abraham Charnes, William Cooper and Bob Mellon (an employee of Gulf Oil Co.) on "Blending aviation gasolines". This—and, more generally, blending in the petrochemical and other industries—was to become an important early application area for linear programming. From this group of papers one can sense the transition of the subject from military applications to its many fruitful applications in the civilian domain. Before long, the writings of Dantzig and others attracted the attention of a wide circle of applied scientists, including many from industry. See [3].

Besides the transportation problem, which continues to have utility in shipping and distribution enterprises, an early direction of applied linear programming is to be found in agriculture, a venerable topic of interest within economics. Our illustration is drawn from a historically important article that provided a natural setting for the LP. In 1945, George J. Stigler [79] published a paper that develops a model for finding a least-cost "diet" that would provide at least a prescribed set of nutritional requirements. [See the companion piece by Gale in this issue for a discussion of the formulation.] Dantzig reports [19, p. 551] how in 1947 the simplex method was tried out on Stigler's nutrition model (or "diet problem" as it is now called). The constraints involved nine equations in seventy-seven unknowns. At the time, this was considered a large problem. In solving the LP by the simplex method, the computations were performed on hand-operated desk calulators. According to Dantzig, this process took "approximately 120 man-days to obtain a solution." Such was the state of computing in those days. In 1953, the problem was solved and printed out by an IBM 701 computer in twelve minutes. Today, the problem would be considered small and solved in less than a second on a personal computer [54]. One may question the palatability of a diet for human consumption based on LP principles. Actually, the main application of this methodology lies in the animal feed industry where it is of great interest.

The Air Force's desire to mechanize planning procedures and George Dantzig's contributions to that effort through his creation of linear programming and the simplex method had a very significant impact on the development of computing machines. The encouraging results obtained by Dantzig and his co-workers convinced the Air Force Comptroller, Lt. Gen. Edwin W. Rawlings, that much could be accomplished with more powerful computers. Accordingly, he transferred the (then large) sum of US$400,000 to the National Bureau of Standards which in turn funded mathematical and electronic computer research (both in-house) as well as the development of several computers such as UNIVAC, IBM, SEAC, and SWAC. Speaking to a military audience in 1956, General Rawlings notes, "I believe it can be fairly said that the Air Force interest in actual financial investments in the development of electronic computers has been one of the important factors in their rapid development in this country" [74]. Dantzig took pride in the part that linear programming played in the development of computers.

The list of other industrial applications of linear programming—even in its early history—is impressive. A quick sense of this can be gleaned from Saul Gass's 1958 textbook [53], one of the first to appear on the subject. In addition to a substantial chapter on applications of linear programming, it contains a bibliography of LP applications of all sorts organized under twelve major headings (selected from the broader work [54]). Under the heading "Industrial Applications", Gass lists publications pertaining to the following industries: chemical, coal, commerical airlines, communications, iron and steel, paper, petroleum, and railroad.

Also published in 1958 was the book *Linear Programming and Economic Analysis* by Dorfman, Samuelson, and Solow [47]. "Intended not as a text but as a general exposition of the relationship of linear programming to standard economic analysis" it had "been successfully used for graduate classes in economics." Its preface proclaims that "linear programming has been one of the most important postwar developments in economic theory." The authors highlight LP's interrelations with von Neumann's theory of games, with welfare economics, and with Walrasian equilibrium. Arrow [4] gives a perspective on Dantzig's role in the development of economic analysis.

Early Extensions

From its inception, the practical significance of linear programming was plain to see. Equally visible to Dantzig and others was a family of related problems that came to be known as *extensions of linear programming.* We turn to these now to convey another sense of the impact of LP and its applications.

In this same early period, important advances were made in the realm of *nonlinear* optimization. H. W. Kuhn and A. W. Tucker presented their work "Nonlinear programming" at the Second Berkeley Symposium on Mathematical Statistics and Probability, the proceedings of which appeared in 1951. This paper gives necessary conditions of optimality for the (possibly) nonlinear inequality constrained minimization of a (possibly) nonlinear objective function. The result is a descendant of the so-called "Method of Lagrange multipliers". A similar theorem had been obtained by F. John and published in the "Courant Anniversary Volume" of 1948. For some time now, these optimality conditions have been called the Karush-Kuhn-Tucker conditions, in recognition of the virtually identical result presented in Wm. Karush's (unpublished) master's thesis at the University of Chicago in 1939. Dorfman's doctoral dissertation *Application of Linear Programming to the Theory of the Firm, Including an Analysis of Monopolistic Firms by Non-linear Programming* appeared in book form [45] in 1951. It might well be the first book that ever used "linear programming" in its title. The "non-linear programming" mentioned therein is what he calls "quadratic programming", the optimization of a quadratic function subject to linear inequality constraints. The first book on linear programming *per se* was [9], *An Introduction to Linear Programming* by A. Charnes, W. W. Cooper, and A. Henderson, published in 1953. Organized in two parts (applications and theory), the material of this slender volume is based on seminar lectures given at Carnegie Institute of Technology (now Carnegie-Mellon University).

Organizations and Journals

Linear and nonlinear programming (collectively subsumed under the name "mathematical programming") played a significant role in the formation of professional organizations. As early as April 1948, the Operational Research Club was founded in London; five years later, it became the Operational Research Society (of the UK). The Operations Research Society of America (ORSA) was founded in 1952, followed the next year, by the Institute of Management Sciences (TIMS). These parallel organizations, each with its own journals, merged in 1995 to form the Institute for Operations Research and the Management Sciences (INFORMS) with a membership today of about 12,000. Worldwide, there are now some forty-eight national OR societies with a combined membership in the vicinity of 25,000. Mathematical programming (or "optimization" as it now tends to be called) was only one of many subjects appearing in the pages of these journals. Indeed, the first volumes of these journals devoted a small portion of their space to mathematical programming. But that would change dramatically with time.

Over the years, many scientific journals have been established to keep pace with this active field of research. In addition to the journal *Mathematical Programming*, there are today about a dozen more whose name includes the word "optimization". These, of course, augment the OR journals of OR societies and other such publications.

Educational Programs

Delivering a summarizing talk at "The Second Symposium in Linear Programming" (Washington, D.C., January 27–29, 1955) Dantzig said "The great body of my talk has been devoted to technical aspects of linear programming. I have discussed simple devices that can make possible the efficient solution of a variety of problems encountered in practice. Interest in this subject has been steadily growing in industrial establishments and in government and some of these ideas may make the difference between interest and use." The notion of "making the difference between *interest and use*" was a deeply held conviction of Dantzig's. He knew how much could be accomplished through the combination of modeling, mathematical analysis, and algorithms like the simplex method. And then he added a prediction. "During the next ten years we will see a great body of important applications; indeed, so rich in content and value to industry and government that mathematics of programming will move to a principal position in college curriculums" [15, p. 685].

By the early 1950s, operations research (OR) courses had begun appearing in university curricula, and along with them came linear programming. We have already mentioned the lectures of Charnes and Cooper at Carnegie Tech in 1953. In that year both OR and LP were offered as graduate courses at Case Institute of Technology (the predecessor of Case Western Reserve University). At Stanford OR was first taught in the Industrial Engineering Program in the 1954–55 academic year, and LP was one of the topics covered. At Cornell instruction in these subjects began in 1955, while at Northwestern it began in 1957. In the decades that followed, the worldwide teaching of OR at post-secondary institutions grew dramatically. In some cases, separate departments of operations research and corresponding degree programs were established; in other cases these words (or alternatives like "management science" or "decision science") were added to the name of an existing department, and there were other arrangements as well. The interdisciplinary nature of the field made for a wide range of names and academic homes for the subject, sometimes even within the same institution, such as Stanford, about which George Dantzig was fond of saying "it's wall-to-wall OR." Wherever operations research gained a strong footing, subjects like linear programming outgrew their place within introductory survey courses; they became many individual courses in their own right.

Along with the development of academic courses came a burst of publishing activity. Textbooks for the classroom and treatises for seminars and researchers on a wide range of OR topics came forth. Books on mathematical programming were a major part of this trend. One of the most important of these was George Dantzig's *Linear Programming and Extensions* [19]. Published in 1963, it succeeded by two years the monograph of Charnes and Cooper [8]. Dantzig's book was so rich in ideas that it soon came to be called "The Bible of Linear Programming". Two of the sections to follow treat Dantzig's direct role in academic programs. For now, we return to the chronology of his professional career.

Dantzig at RAND

In 1952 George Dantzig joined the RAND Corporation in Santa Monica, California, as a research mathematician. The RAND Corporation had come into being in 1948 as an independent, private nonprofit organization after having been established three years earlier as the Air Force's Project RAND, established in 1945 through a special contract with the Douglas Aircraft Company. Despite its separation from Douglas—under which Project RAND reported to Air Force Major General Curtis LeMay, the Deputy Chief of Air Staff for Research and Development—the newly formed RAND Corporation maintained a close connection with the Air Force and was often said to be an Air Force "think tank".

The reason for Dantzig's job change (from mathematical advisor at the Pentagon to research mathematician at RAND) is a subject on which the interviews and memoirs are silent. In [1, p. 311] he warmly describes the environment that existed in the Mathematics Division under its head, John D. Williams. The colleagues were outstanding and the organizational chart was flat. These were reasons enough to like it, but there must have been another motivation: the freedom to conduct research that would advance the subject and to write the book [19].

The 1950s were exciting times for research in the Mathematics Division at RAND. There, Dantzig enjoyed the stimulation of excellent mathematicians and the receptiveness of its sponsors. During his eight years (1952–1960) at RAND, Dantzig wrote many seminal papers and internal research memoranda focusing on game theory, LP and variants of the simplex method, large-scale LP, linear programming under uncertainty, network optimization including the traveling salesman problem, integer linear programming, and a variety of applications. Much of this work appeared first in the periodical literature and then later in [19].

The topic of methods for solving large-scale linear programs, which had drawn Dantzig's attention at the Pentagon, persisted at RAND—and for years to come. Although the diet problem described above was considered "large" at the time, it was puny by comparison with the kinds of problems the Air Force had in mind. Some inkling of this can be obtained from the opening of a talk given by M. K. Wood at the 1951 "Symposium on Linear Inequalities and Programming". He says in its proceedings [73, p. 3], "Just to indicate the general size of the programming problem with which we are faced, I would like to give you a few statistics. We are discussing an organization of over a million people who are classified in about a thousand different occupational skills. These people are organized into about ten thousand distince [sic] organizational units, each with its own staff and functions, located at something over three hundred major operating locations. The organizational units use approximately one million different kinds of items of supplies and equipment, at a total annual cost of something over fifteen billion dollars."

The challenge of solving problems on such a grand scale was one that Dantzig took seriously. Speaking at the "First International Conference on Operational Research" in 1957, he acknowledged, "While the initial proposal was to use a linear programming model to develop Air Force programs, it was recognized at an early date that even the most optimistic estimates of the efficiency of future computing procedures and facilities would not be powerful enough to prepare detailed Air Force programs" [17]. Nevertheless, he steadfastly devoted himself to the cause of improving the capabilities of mathematical programming, especially the simplex method of linear programming.

If one had to characterize Dantzig's work on large-scale systems, more specifically, large-scale linear programs, it could be viewed as the design of variants of the simplex method that require only the use of a "compact" basis, i.e., that simplex iterations will be carried out by relying on a basis of significantly reduced size. Early on, it was observed that given a linear program in standard form, i.e., $\min c \cdot x$ subject to $Ax = b$, $x \geq 0$, the efficiency of the simplex method is closely tied to the number of linear constraints ($Ax = b$) that determine the size of the basis rather than to the number of variables. It is part of the linear programming folklore that "in practice" the number of steps required to solve a linear program is of the order of $3m/2$ where m is the number of linear constraints. Moreover, working with a compact basis (i.e., one of small order), which was essentially going be inverted, would make the method significantly more reliable from a numerical viewpoint.

The first challenging test of the simplex method on a "large-scale" problem had come in dealing with Stigler's diet problem; the introduction of upper bounds on the amounts of various foods greatly enlarged the number of constraints, thereby exacerbating an already difficult problem. The need to have a special version of the simplex method to deal with upper bounds came up again at RAND. Researchers were dissatisfied with the turn-around time for the jobs submitted to their computer center, mostly because certain top-priority projects would absorb all available computing resources for weeks. That is when a more flexible priority scheduling method was devised in which the value assigned to a job decreased as its completion date was delayed. The formulation of this scheduling problem turned out to be a linear program having special (assignment-like) structure and the restriction that the number of hours x_{ij} to be assigned to each project i in week j could not exceed a certain upper bound, say α_{ij}. In linear programming terms, the problem was of the following type,

$$\min c \cdot x \text{ subject to } Ax = b,\ lb \leq x \leq ub,$$

with m linear constraints ($Ax = b$) and, rather than just nonnegativity restrictions on the x-variables, the vectors lb and ub imposed lower and upper bounds on x. Here after an appropriate change of variables and inclusion of the slack variables, the problem would pass from one with m linear constraints to one with at least $m + n$ such constraints. Solving the scheduling problem might have been the source of further delays! That is when Dantzig introduced a new pivoting scheme that would be integrated in any further implementation of the simplex method. It would essentially deal with the problem as one with only m linear constraints requiring just a modicum of additional bookkeeping to keep track of variables at their lower bounds, the basic variables, and those at their upper bound, rather than just with basic and nonbasic variables as in the original version of the simplex method.

Dantzig's work on the transportation problem and certain network problems had made him aware that the simplex method becomes extremely efficient when the basis has either a triangular or a nearly triangular structure. Moreover, as the simplex method was being used to solve more sophisticated models, in particular involving dynamical systems, the need for variants of the simplex method to handle such larger-scale problems became more acute. Although the simplex bases of such problems are not precisely triangular, they have a *block triangular structure*, e.g.,

$$\begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ & & \ddots & \\ A_{T1} & A_{T2} & \dots & A_{TT} \end{bmatrix},$$

that can be exploited to "compactify" the numerical operations required by the simplex method. All present-day, commercial-level, efficient implementations of the simplex method make use of the shortcuts proposed by Dantzig in [16], in particular in the choice of heuristics designed to identify a good starting basis known among professionals as a "crash start".

In the late 1950s, Dantzig and Wolfe proposed the *Decomposition Principle* for linear programs on which they lectured at the RAND Symposium on Mathematical Programming in March 1959. Their approach was inspired by that of Ford and Fulkerson on multistage commodity network problems [51]. Actually, the methodology is based on fundamental results dating back to the pioneering work of Minkowski and Weyl on convex polyhedral sets. By the mid-1950s Dantzig had already written a couple of papers on stochastic programming; he realized that a method was needed to handle really large linear programs, with "large" now taking on its present-day meaning.

Because it illustrates so well how theory can be exploited in a computational environment, we present an abbreviated description of the method. Suppose we have split our system of linear constraints in two groups so that the linear program takes on the form,

$$\min c \cdot x \text{ subject to } Ax = b,\ Tx = d,\ x \geq 0$$

where the constraints have been divided up so that $Ax = b$ consists of a relatively small number, say m, of linking constraints, and the system $Tx = d$ is large. As shall be seen later on, stochastic programs will provide good examples of problems that naturally fall into this pattern. The set $K = \{x \in \mathbb{R}^n_+ \mid Tx = d\}$ is a polyhedral set that in view of the Weyl-Minkowski Theorem admits a "dual" representation as a finitely generated set obtained as the sum of a polytope (bounded polyhedron)

$$\{x = \sum_{k=1}^{r} p^k \lambda_k \mid \sum_{k=1}^{r} \lambda_k = 1, \lambda_k \geq 0, k = 1, \ldots, r\}$$

and a polyhedral cone

$$\{x = \sum_{l=1}^{s} q^l \mu_l \mid \mu \geq 0, l = 1, \ldots, s\}$$

with both r and s finite. Our given problem is thus equivalent to the following linear program, which we shall refer to as the *full master program*,

$$\begin{aligned} \min \quad & \textstyle\sum_{k=1}^r \gamma_k \lambda_k + \sum_{l=1}^s \delta_l \mu_l \\ \text{subject to} \quad & \textstyle\sum_{k=1}^r P^k \lambda_k + \sum_{l=1}^s Q^l \mu_l = b, \\ & \textstyle\sum_{k=1}^r \lambda_k = 1, \\ & \lambda_k \geq 0, k = 1, \ldots, r, \\ & \mu_l \geq 0, l = 1, \ldots, s \end{aligned}$$

where $\gamma_k = c \cdot p^k, P^k = Ap^k$, $\delta_l = c \cdot q^l$, and $Q^l = Aq^l$. Because this linear program involves only $m + 1$ linear constraints, the "efficiency and numerical reliability" of the simplex method would be preserved if, rather than dealing with the large-scale system, we could simply work with this latter problem. That is, if we do not take into account the work of converting the constraints $Tx = d, x \geq 0$ to their dual representation, and this could be a horrendous task that, depending on the structure of these constraints, could greatly exceed that of solving the given problem. What makes the Dantzig-Wolfe decomposition method a viable and attractive approach is that, rather than generating the full master program, it generates only a small, carefully selected subset of its columns, namely, those that potentially could be included in an optimal basis. Let us assume that, via a Phase I type-procedure, a few columns of the (full) master program have been generated, say $k = 1, \ldots, r_\nu < r$, $l = 1, \ldots, s_\nu < s$, such that the resulting (reduced) *master program* is feasible. Clearly, if this master program is unbounded, so is the originally formulated problem. So, let us assume that this problem is bounded, and let (λ^ν, μ^ν) be an optimal solution where $\lambda^\nu = (\lambda_1^\nu, \ldots, \lambda_{r_\nu}^\nu)$ and $\mu_\nu = (\mu_1^\nu, \ldots, \mu_{s_\nu}^\nu)$ with optimal value ζ_ν. Let π^ν be the vector of multipliers attached to the first m linear constraints and θ_ν be the multiplier attached to the $(m + 1)$st constraint $\sum_{k=1}^r \lambda_k = 1$. In sync with the (simplex method) criterion to find an improved solution to the full master problem, one could search for a new column of type $(\gamma_k, P^k, 1)$, or of type $(\delta_l, Q^l, 0)$, with the property that $\gamma_k - \pi^\nu \cdot P^k - \theta_\nu < 0$ or $\delta_l - \pi^\nu \cdot Q^l < 0$. Equivalently, if the linear subprogram

$$\min (c - A^\top \pi^\nu) \cdot x \text{ subject to } Tx = d, x \geq 0,$$

is unbounded, in which case the simplex method would provide a direction of unboundedness q^l, that would, in turn, give us a new column of type $(\delta_l, Q^l, 0)$ to be included in our master program. Or, if this linear sub-program is bounded, the optimal solution p^k generated by the simplex method would be a vertex of the polyhedral set $\{x \in \mathbb{R}^n_+ \mid Tx = d\}$. As long as $\theta_\nu < (c - A^\top \pi^\nu) \cdot p^k$, a new column of type $(\gamma_k, P^k, 1)$ would be included in the master problem. On the other hand, if this last inequality was not satisfied, it would indicate that no column can be found that would enable us to improve the optimal value of the problem, i.e.,

$$x^* = \sum_{k=1}^{r_\nu} p^k \lambda_k^\nu + \sum_{l=1}^{s_\nu} q^k \mu_l^\nu$$

is then the optimal solution of our original problem. Clearly, the method will terminate in a finite number of steps; the number of columns generated could never exceed those in the full master program.

An interesting and particular application of this methodology is to linear programs with variable coefficients. The problem is one of the following form:

$$\min \sum_{j=1}^{n} c_j x_j \text{ subject to } \sum_{j=1}^{n} P^j x_j = Q,\ x \geq 0,$$

where the vectors (c_j, P^j) may be chosen from a closed convex set $C^j \subset \mathbb{R}^{m+1}$. If the sets C^j are polyhedral, we essentially recover the method described earlier, but in general, it leads to a method for nonlinear convex programs that generates "internal" approximations, i.e., the feasible region, say C, of our convex program is approximated by a succession of polyhedral sets contained in C. The strategy is much the same as that described earlier, viz., one works with a *master (problem)* whose columns are generated from the subproblems to improve the succession of master solutions. For all j, these subproblems are of the following type:

$$\min c_j - \pi^\nu \cdot P^j \ \text{ subject to } \ (c_j, P^j) \in C^j.$$

The efficiency of the method depends on how easy or difficult it might be to minimize linear forms on the convex sets C^j; the π^ν correspond to the optimal multipliers associated with the linear constraints of the master problem. Dantzig outlined an elegant application of this method in "Linear control processes and mathematical programming" [20].

In this period, Dantzig co-authored another major contribution: a solution of a large (by the standards of the day) Traveling Salesman Problem (TSP). However, this work was considerably more than just a way to find an optimum solution to a particular problem. It pointed the way to several approaches to combinatorial optimization and integer programming problems.

First, the traveling salesman problem is to find the shortest-distance way to go through a set of cities so as to visit each city once and return to the starting point. This problem was put on the computational mathematical map by the now-famous paper of Dantzig, Fulkerson, and Johnson [28]. It has a footnote on the history of the problem that includes Merrill Flood stimulating interest in the problem, Hassler Whitney apparently lecturing on it, and Harold Kuhn exploring the relation between the TSP and linear programming. The abstract is one sentence: "It is shown that a certain tour of forty-nine cities, one in each of the forty-eight states and Washington, D.C., has the shortest road distance." The distances were taken from a road atlas. The authors reduce the problem to forty-two cities by pointing out that their solution with seven Northeastern cities deleted goes through those seven cities. Thus the solution method is applied to a 42-city problem.

The solution method involves solving the linear program with a 0-1 variable for each link (the terminology they use for undirected arcs) of the complete graph on forty-two nodes. The first set of constraints says the sum of the variables on links incident to a given node must equal two. There are forty-two of these constraints, for the 42-city problem. The second set of constraints is made up of what are called *sub-tour elimination constraints*. These require that the sum of variables over links, with one end in a set S of nodes and the other end not in S, must be greater-than-or-equal to two. There are of order 2^{42} of these constraints. However, they only had to add seven of them to the linear program for the 42-city problem at which point all the rest were satisfied. So despite the fact that there are a huge number of such constraints, only a few were used when they were added on to the linear program as needed.

On several occasions, George Dantzig spoke of a bet he had with Ray Fulkerson. George was convinced that not very many sub-tour elimination constraints would be needed, despite their large number. He was so convinced that he proposed a bet that there would be no more than twelve (the exact number is probably lost forever at this time), but Ray, who was a good poker player, said the bet should be the closest to the actual number and he went lower, saying eleven. In actuality, only seven were needed, plus two more that were presented as being somewhat ad hoc but needed to make the linear programming optimum be integer-valued.

The two additional inequalities are briefly justified and are acknowledged in a footnote, "We are indebted to I. Glicksberg of Rand for pointing out relations of this kind to us." Much of the later research in combinatorial optimization is directed at finding such inequalities that can be quickly identified, when violated. For example, finding a violated sub-tour elimination constraint is equivalent to finding a "minimum cut" in the graph where the weights on the arcs are the primal variables x_{ij}, which may be fractional. The min cut is the one that minimizes that sum of x_{ij} over all edges in the cut; that value is then compared to 2.

The method used is to start with a tour, which can be found by any one of several heuristics. Dual variables on the nodes can then be found, and the paper explains how. Then, non-basic arcs can be priced out and a basis change found. If the solution moves to another tour, then the step is repeated. If it moves to a sub-tour, then a sub-tour elimination constraint is added. If it moves to a fractional solution, then they look for a constraint to add that goes through the current solution vector and cuts off the fractional solution. Thus, it is an integer-primal simplex method, a primal simplex method that stays at integer points by adding cuts as needed.

A device the paper employs is the use of "reduced-cost fixing" of variables. Once a good tour is found and an optimum linear program employing valid cuts, such as sub-tour elimination cuts, the difference between the tour cost and the linear program optimum provides a value that can be used to fix non-basic variables at their current non-basic values. Although the number of variables is not enormous, the 42-city problem has 861 variables, which is a lot if one is solving the linear program by hand, as they did. They were using the revised simplex method so only needed to price out all these columns and enter the best one. Thus, even though the revised simplex method only has to price out over the columns, the process can be onerous when there are 861 variables and computation is being done by hand. For that reason, actually dropping many of these variables from the problem had a real advantage.

The paper also refers to a "combinatorial approach" which depends critically on reducing the number of variables. This approach seems to be related to the reduced-cost fixing referred to above. Although the paper is a bit sketchy about this approach, it seems to be that as the variables are reduced to a small number, some enumeration without re-solving the linear program (but perhaps using the reduced-cost fixing) is required. In this way, the options left open can be evaluated and the optimum solution identified.

According to Bland and Orlin [5], "*Newsweek* magazine ran a story on this 'ingenious application of linear programming' in the 26 July 1954 issue." However, it is highly unlikely that even Dantzig, Fulkerson, and Johnson could have anticipated then the practical impact that this work would ultimately have." This impact has been felt in integer programming generally where strong linear programming formulations are much used. One could say that the general method of finding violated, combinatorially derived cutting planes was introduced in this paper. Dantzig was personally firmly convinced that linear programming was a valuable tool in solving integer programming problems, even hard problems such as the TSP. Practically all of the successful, recent work that has accepted the challenge of solving larger and larger TSPs has been based on the Dantzig, Fulkerson, Johnson approach.

Another contribution of Dantzig in combinatorial optimization is the work based on network flow. Ray Fulkerson and Alan Hoffman were two of his collaborators and co-authors. The crucial observation is that this class of linear programs gives integer basic solutions when the right-hand side and bounds are all integers, so the integer program is solved by the simplex method. In addition, the dual is integer when the costs are integer. In particular the duality theorem provides a proof of several combinatorial optimization results, sometimes referred to as "min-max theorems". An example of a rather complex application of this min-max theorem is the proof of Dilworth's theorem by Dantzig and Hoffman. This theorem says that for a partial order, the minimum number of chains covering all elements is equal to the maximum number of pair-wise unrelated elements.

Of course, George Dantzig was always interested in applications, and developed a rich set of integer programming applications. One of these is what is called today the "fleet assignment model". Current models for this problem are much used in airline planning. As the problems' sizes grow and more details are included in the models, computational challenges abound; nevertheless this model has generally proven to be quite tractable despite being a large, mixed-integer program. In fact, the second of the two papers [50] on this subject was an early example of a stochastic integer programming problem. The problem is to allocate aircraft to routes for a given number of aircraft and routes. The main purpose of the problem is to maximize net revenue by routing the larger aircraft over flights that have more demand or by using more aircraft on such routes. The stochastic version of the problem has simple recourse, i.e., leave empty seats or leave demand unsatisfied depending on the realized demand.

A related problem was posed as finding the minimum number of tankers to cover a schedule. This is used today for example by charter airline companies to cover required flights by the minimum number of aircraft. The schedule of required flights in a charter operation is typically unbalanced requiring ferry flights, or deadhead flights, in order to cover the required flights. Dantzig and Fulkerson [27] modeled this as a network flow problem. With more than the minimum number of planes, the total deadheading distance may be reduced by allowing more than the minimum number of planes.

A notable application paper [34] by G. B. Dantzig and J. H. Ramser introduced what is now called the Vehicle Routing Problem (VRP), an area that has its own extensive literature and solution methods. The problem is a generalization of the TSP having many variations. As put by Toth and Vigo [80, p. xvii], the VRP "calls for the determination of the optimal set of routes to be performed by a fleet of vehicles to serve a given set of customers, and it is one of the most important, and studied, combinatorial optimization problems. More than forty years have elapsed since Dantzig and Ramser introduced the problem in 1959."

In addition to these contributions to network optimization problems, we must also mention Dantzig's article [18] pointing out the rich set of problems that can be modeled as integer programs. This paper motivated computational efforts such as cutting plane methods, enumeration, and branch-and-bound to effectively solve these diverse problems. These efforts have continued up to today.

Dantzig often referred to stochastic programming as the "real problem". He was very well aware that almost all important decision problems are *decision-making problems under uncertainty*, i.e., where at least some of the parameters are at best known in a statistical sense, and sometimes not even reliable statistical information is available. His serious commitment to this class of problems dates from the middle 1950s. As in many other instances, the "spark" probably came from an application, in this case allocating aircraft to routes under uncertain demand. That problem was the concern of Alan Ferguson, one of his colleagues at RAND [49]. After they devised an elegant procedure to solve this particular problem, Dantzig returned to his desk and wrote a fundamental paper that not only introduced the fundamental stochastic programming model, known today as the *stochastic program with recourse*, but also started deriving its basic properties. This certainly reaffirmed the need to deal efficiently with large-scale mathematical programs.

With only a slight reformulation, Dantzig's model was

$$\min c \cdot x + E\{Q(\boldsymbol{\xi}, x)\} \text{ subject to } Ax = b,\ x \geq 0,$$

where $E\{\cdot\}$ denotes the taking of expectation, and

$$Q(\xi, x) = \inf\{q \cdot y \mid Wy = \xi - Tx, y \geq 0\};$$

here $\boldsymbol{\xi}$ is a random vector with values ξ in $\Xi \subset \mathbb{R}^d$; an extension of the model would allow for randomness in the parameters (q, W, T), in addition to just the right-hand sides, defining the function $Q(\cdot, x)$. He proved that in fact this is a well defined convex optimization problem, placing no restrictions on the distribution of the random elements, but requiring that the problem defining Q be solvable for all x and $\xi \in \Xi$, the *complete recourse* condition as it is now called. This means that stochastic programs with recourse, although generally not linear programs, fall into the next "nice" class, viz., convex programs. However, if the random vector $\boldsymbol{\xi}$ has finite support, say $\Xi = \{\xi^l, l = 1, \ldots, L\}$ with $\text{prob}\{\boldsymbol{\xi} = \xi^l\} = p_l$, or the discretization comes about as an approximation to a problem whose random components are continuously distributed, the solution x^* can be found by solving the large-scale linear program:

$$\begin{aligned} \min \quad & c \cdot x + \textstyle\sum_{l=1}^{L} p_l\, q \cdot y^l \\ \text{subject to} \quad & Ax = b \\ & Tx + Wy^l = \xi^l,\ l = 1, \ldots, L \\ & x \geq 0,\ y^l \geq 0,\ l = 1, \ldots, L \end{aligned}$$

"large", of course, depending on the size of L. Just to render this a little bit more tangible, assume that $\boldsymbol{\xi}$ consists of ten independent random variables, each taking on ten possible values; then this linear program comes with at least 10^{11} linear constraints—the 10^{10} possible realizations of the random vector $\boldsymbol{\xi}$ generating 10^{10} systems of 10 linear equations—certainly, a daunting task! To the rescue came (i) the Dantzig-Wolfe decomposition method and (ii) Dantzig's statistical apprenticeship.

It is pretty straightforward that, up to a sign change in the objective, the dual of the preceding problem can be expressed as

$$\begin{aligned} \min \quad & b \cdot \sigma + \textstyle\sum_{l=1}^{L} p_l\, \xi^l \cdot \pi^l \\ \text{subject to} \quad & A^\top \sigma + \textstyle\sum_{l=1}^{L} p_l T^\top \pi^l \leq c, \\ & W^\top \pi^l \leq q,\ l = 1, \ldots; L, \end{aligned}$$

i.e., a problem with a few linking constraints and a subproblem that can, itself, be decomposed into L relatively small (separable) subproblems. The Dantzig-Wolfe decomposition method was perfectly adapted to a structured problem of this type, and this was exploited and explained in [31]. Still, this elegant approach required, at each major iteration, solving (repeatedly) a huge number of "standard" linear programs. And this is how the idea of solving just a sampled number of these subproblems came to the fore. Of course, one could no longer be certain that the absolute optimal solution would be attained, but statisticians know how to deal with this. Dantzig, Glynn, and Infanger [29, 30] relied on the Student-t test to evaluate the reliability of the solutions so obtained. They also proposed a scheme based on importance sampling that reduces sample variance.

It would be misleading to measure the role Dantzig played in this field in terms of just his publications, even taking into account that he was responsible for the seminal work. All along, he encouraged students, associates, colleagues, as well as anybody who would listen, to enter the field, and gave them his full support. He saw the need to deal with the computational challenges, but his major interest seemed to lie in building models that would impact policy-making in areas that would significantly benefit society on a global scale. The "PILOT" project [33] provides an example such an effort. Another is his continued active involvement with the International Institute of Applied Systems Analysis (IIASA) in Laxenburg (Austria) whose mission is to conduct inter-disciplinary scientific studies on environmental, economic, technological, and social issues in the context of human dimensions of global change.

Dantzig at Berkeley

George Dantzig left RAND in 1960 to join the faculty of the University of California in Berkeley; he accepted a post as professor in the Department of Industrial Engineering. Established just four years earlier as a full-fledged department, Industrial Engineering was chaired from 1960 to 1964 by Ronald W. Shephard who, like Dantzig, was born in Portland, Oregon, and had done his doctoral studies under Jerzy Neyman at Berkeley. Asked why he left RAND to return to the academic world, Dantzig told Albers and Reid in their 1984 interview, "My leaving had to do with the way we teamed up to do our research. . . . Each of us got busy doing his own thing. . . . There were no new people being hired to work with us as disciples. . . . My stimulus comes from students and working closely with researchers elsewhere" [1, p. 311]. As it turned out, he had disciples aplenty at Berkeley (and later at Stanford).

By 1960 Operations Research was well on its way to becoming a separate (albeit interdisciplinary) field of study having natural ties to mathematics, statistics, computer science, economics, business administration, and some of the more traditional branches of engineering. Dantzig (and Shephard) founded the Operations Research Center (ORC) which coordinated teaching and research in a range of OR topics. For several years, the ORC was inconveniently situated in a small, shabby wood-frame house at the university's Richmond Field Station, some six miles from the main campus. Nevertheless the ORC managed to attract an enthusiastic group of faculty and students who investigated a range of OR themes. Dantzig's research program was certainly the largest at the ORC—and possibly the largest in the IE Department for that matter.

The research Dantzig began at RAND on compact basis techniques for large-scale linear programs led, at Berkeley, to the development of the *Generalized Upper Bound Technique (GUB)*. The motivation came from a problem that Dantzig encountered while consulting for the Zellerbach Co., now the Crown Zellerbach Co. Rather than simple lower/upper bound constraints, the m linking constraints $Ax = b$ and nonnegativity restrictions are augmented by a collection of L linear constraints with positive right-hand sides and the property that every variable appears at most in one of these constraints and then, with a positive coefficient. In their paper, Dantzig and Van Slyke [38] show that the optimal solution of a problem of this type has the following properties: (i) from any group of variables associated with one of these L constraints, at least one of these variables must be basic and (ii) among these L equations, the number of those ending up with two or more basic variables is at most $m - 1$. These properties were exploited to show that with an appropriate modification of the pivoting rules of the standard simplex method and some creative bookkeeping, one can essentially proceed to solve the problem as if it involved only m linear constraints. Again, a situation when a compact basis will suffice, and this resulting in much improved efficiency and numerical reliability of the method.

Dantzig offered graduate courses in linear and nonlinear programming in addition to a course in network flows. These were given at a time when his *Linear Programming and Extensions* and the classic work *Flows in Networks* by Ford and Fulkerson were still in one stage of incompleteness or another. At best, some galley proofs were available, but photocopy equipment was then in its infancy (and was a wet process at that). A while later, a corps of Dantzig's students were called upon to proofread and learn from *LP&E*. The richness of the subject augmented by Dantzig's vast store of interesting research problems gave his doctoral students tremendous opportunities for dissertation topics and the resources to carry them out. This included exposure to stacks of technical reports, introductions to distinguished visitors at the ORC, and support for travel to important professional meetings.

Over his six-year period at Berkeley, Dantzig supervised eleven doctoral theses including those written by the authors of the present article. The types of mathematical programming in the dissertations supervised by Dantzig include large-scale linear programming, linear programming under uncertainty, integer programming, and nonlinear programming. A development that evolved from the latter subject (a few years later) came to be called *complementarity theory*, that is, the study of complementarity problems. These are fundamental inequality systems of the form

$$(5) \qquad F(x) \geq 0, \quad x \geq 0, \quad x \cdot F(x) = 0.$$

Besides thesis supervision and book writing, Dantzig busied himself with applied research in the life sciences, large-scale LP, and a bit of work on optimal control problems. With all this, he added "educator" to his large reputation.

Dantzig at Stanford

Dantzig left UC Berkeley in 1966 to join the Computer Science Department and the Program in Operations Research at Stanford University. Always reluctant to discuss his motivation for the switch, Dantzig used to quip that OR program chairman, Gerald J. Lieberman, promised him a parking place adjacent to the office. This arrangement did not survive the relocation of the department to other quarters, but Dantzig remained at Stanford anyway.

Since its creation in 1962, the OR program had been authorized to grant the Ph.D. and was an interschool academic unit reporting to three deans and six departments. Fortunately, this cumbersome arrangement changed in 1967, at which time Operations Research became a regular department in the School of Engineering. Dantzig retained his joint appointment the CS Department, but his principal activity was based in OR. Berkeley followed a slightly different route; in 1966, the IE Department became the Department of Industrial Engineering and Operations Research. But with Dantzig now at Stanford, the West Coast center of gravity for mathematical programming shifted southward.

One of those recruited to the OR faculty was R. W. Cottle who had been Dantzig's student at Berkeley. Together, they produced a few papers on the so-called linear complementarity problem, that being the case where the mapping F in (5) is affine. One of these, "Complementary Pivot Theory of Mathematical Programming" had a considerable readership in the OR field and became something of a classic.

George Dantzig had a knack for planning but harbored little interest in organizational politics or administrative detail. Even so, he began his tenure at Stanford by serving as president of The Institute of Management Sciences (TIMS). In 1968 he and A. F. Veinott Jr. co-directed (and co-edited the proceedings of) a very successful five-week "Summer Seminar on the Mathematics of the Decision Sciences" sponsored by the AMS. He followed this by directing in 1971 the "Conference on Applications of Optimization Methods for Large-Scale Resource Allocation Problems" held in Elsinore, Denmark, under the sponsorship of NATO's Scientific Affairs Division. Still later he was program chairman of the "8^{th} International Symposium on Mathematical Programming", which was held at Stanford in 1973. That year he became the first chairman of the newly established Mathematical Programming Society.

On top of these "distractions", Dantzig concentrated on mathematical programming research, the supervision of doctoral students, and the orchestration of proposal writing so essential to securing its external sponsorship.

Dantzig's leadership and the addition of other faculty and staff contributed greatly to the international stature of the Stanford OR Department. Among Dantzig's many ventures during this period, the Systems Optimization Laboratory (SOL) stands out as one of the most influential and enduring. At the 1971 Elsinore Conference mentioned above, Dantzig discoursed "On the Need for a Systems Optimization Laboratory". A year later, he (and a host of co-authors) put forward another version of this concept at an Advanced Seminar on Mathematical Programming at Madison, Wisconsin. As Dantzig put it, the purpose of an SOL was to develop "computational methods and associated computer routines for numerical analysis and optimization of large-scale systems". By 1973 Stanford had its SOL with Dantzig as director. For many years, the SOL was blessed with an outstanding resident research staff consisting of Philip Gill, Walter Murray, Michael Saunders, John Tomlin, and Margaret Wright. The research and software output of the latter group is world famous. Numerous faculty and students rounded out the laboratory's personnel. Over time, Dantzig's concept of an SOL would be emulated at about twenty other institutions.

**Left to right:
T. C. Koopmans, G. B. Dantzig,
L. V. Kantorovitch, 1975**

While bringing the SOL to fruition, Dantzig collaborated with T. L. Saaty on another altogether different sort of project: a book called *Compact City* [35], which proposes a plan for a liveable urban environment. The multi-story city was to be cylindrical in shape (thereby making use of the vertical dimension) and was intended to operate on a 24-hour basis (thereby making better use of facilities). Although the main idea of this publication does not appear to have been implemented anywhere, the book itself has been translated into Japanese.

During the academic year of 1973–74, Dantzig spent his sabbatical leave at the International Institute for Applied Systems Analysis (IIASA). Located in Laxenburg, Austria, IIASA was then about one year old. IIASA scientists worked on problems of energy, ecology, water resources, and methodology. Dantzig headed the Methodology Group and in so doing established a long association with this institute.

Another memorable feature of 1973 was the well-known mideast oil crisis. This event may have been what triggered Dantzig's interest in energy-economic modeling. By 1975 this interest evolved into something he called the PILOT Model, a passion that was to occupy him and a small group of SOL workers until the late 1980s. ("PILOT" is an acronym for "Planning Investment Levels Over Time".) As Dantzig, McAllister, and Stone explain [32], PILOT aims "to assess the impact of old and proposed new technologies on the growth of the U.S. economy, and how the state of the economy and economic policy may affect the pace at which innovation and modernization proceeds." The PILOT project provided a context combining three streams of research that greatly interested Dantzig: modeling of a highly relevant economic issue, large-scale programming methodology, and the computation of optimal solutions or the solutions of economic equilibrium (complementarity) problems.

The year 1975 brought a mix of tidings to Dantzig and the mathematical programming community. The 1975 Nobel Prize in Economics went to Kantorovich and Koopmans "for their contributions to the theory of the optimum allocation of resources." To the shock and dismay of a worldwide body of well-wishers (including the recipients), George Dantzig was not selected for this distinguished award. In their prize speeches, Kantorovich and Koopmans recognized the independent work of Dantzig; Koopmans felt so strongly about the omission that he donated a third of his prize money to IIASA in Dantzig's honor.

But on a happier note, that same year George Dantzig received the prestigious National Medal of Science from President Gerald Ford, the John von Neumann Theory Prize from ORSA and TIMS, and membership in the American Academy of Arts and Sciences. He had, in 1971, already become a member of the National Academy of Sciences and, indeed, would become, in 1985, member of the National Academy of Engineering. Over his lifetime, many other awards and eight honorary doctorates were conferred on Dantzig. Beginining in 1970, he was listed on the editorial boards of twenty-two different journals.

As a colleague and as a mentor, George Dantzig was a remarkable asset. That he had broad knowledge of the mathematical programming field and a wealth of experience with the uses of its methodology on practical problems goes without saying. What also made him so valuable is that he was a very patient and attentive listener and would *always* have a response to whatever was told him. The response was invariably a cogent observation or a valuable suggestion that would advance the discussion. As a professor at Stanford, Dantzig produced forty-one doctoral students. The subjects of their theses illustrates the range of his interests. The distribution goes about like this: large-scale linear programming (8), stochastic programming (6), combinatorial optimization (4), nonlinear programming (4), continuous linear programming (3), networks and graphs (3), complementarity and computation of economic equilibria (2), dynamic linear and nonlinear programming (2), probability (2), other (7). The skillful supervision Dantzig brought to the dissertation work of these students and the energy they subsequently conveyed to the operations research community shows how effective his earlier plan to have disciples really turned out to be. A nearly accurate version of Dantzig's "academic family tree" can be found at `http://www.genealogy.ams.org/`.

Retirement

George Dantzig became an emeritus professor in 1985, though not willingly. He was "recalled to duty" and continued his teaching and research for another thirteen years. It was during this period that Dantzig's long-standing interest in stochastic optimization got even stronger. In 1989 a young Austrian scholar named Gerd Infanger came to the OR Department as a visitor under the aegis of George Dantzig. Infanger's doctorate at the Vienna Technical University was concerned with energy, economics, and operations research. He expected to continue in that vein working toward his *Habilitation.* Dantzig lured him into the ongoing research program on stochastic optimization which, as he believed, is "where the real problems are." So began a new collaboration. During the 1990s, Dantzig and Infanger co-authored seven papers reporting on powerful methods for solving stochastic programs. Moreover, Infanger obtained his *Habilitation* in 1993.

Dantzig also established another collaboration around this time. Having decided that much progress had been made since the publication of *LP&E* in 1963, he teamed up with Mukund Thapa to write a new book that would bring his *LP&E* more up to date. They completed two volumes [36] and [37] before Dantzig's health went into steep decline (early 2005), leaving two more projected volumes in an incomplete state.

Dantzig's Mathematical Impact

It will not be possible to give a full account of the mathematical impact of Dantzig's work, some of which is still ongoing. Instead, we focus on a few key points.

Before the 1950s, dealing with systems involving linear—and *a fortiori* nonlinear—inequalities was of limited scope. This is not to make light of the pioneering work of Fourier, Monge, Minkowski, the 1930s-Vienna School, and others, but the fact remains that up to the 1950s, this area was in the domain of an exclusive, albeit highly competent, club of mathematicians and associated economists, with little or no impact in the "practical" world. Dantzig's development of the simplex method changed all that. Perhaps his greatest contribution was to demonstrate that one could deal (effectively) with problems involving inequality constraints. His focus and his determination inspired a breakthrough that created a new mathematical paradigm which by now has flourished in a myriad of directions.

Left to right: George Dantzig, Anne Dantzig, President Gerald Ford (National Medal of Honor ceremony, 1971).

Quickly, practitioners ranging from engineers, managers, manufacturers, agricultural economists, ecologists, schedulers, and so on, saw the potential of using linear programming models now that such models came with an efficient solution procedure. In mathematical circles, it acted as a watershed. It encouraged and inspired mathematicians and a few computer scientists to study a so-far-untouched class of problems, not just from a computational but also from a theoretical viewpoint.

On the computational front, it suffices to visit a site like the NEOS Server for Optimization `http://www-neos.mcs.anl.gov/` to get a glimpse at the richness of the methods, now widely available, to solve optimizations problems and related variational problems: equilibrium problems, variational inequalities, cooperative and noncooperative games, complementarity problems, optimal control problems, etc.

On the theoretical front, instead of the classical framework for analysis that essentially restricted functions and mappings to those defined on open sets or differentiable manifolds, a new paradigm emerged. It liberated mathematical objects from this classical framework. A comprehensive theory was developed that could deal with functions that are not necessarily differentiable, not even continuous, and whose domains could be closed sets or manifolds that, at best, had some Lipschitzian properties. This has brought about new notions of (sub)derivatives that can be be used effectively to characterize critical points in situations where classical analysis could not contribute any insight. It created a brand new approximation theory where the classical workhorse of pointwise limits is replaced by that of set limits; they enter the picture because of the intrinsic one-sided (unilateral) nature of the mathematical objects of interest. The study of integral functionals was also given some solid mathematical foundations, and much progress was achieved in solving problems that had puzzled us for a very long time.

Although we could go much further in this direction, we conclude with one shining example. The study of (convex) polyhedral sets was immediately revived once Dantzig's simplex method gained some foothold in the mathematical community. At the outset, in the 1950s, it was led by a Princeton team under the leadership of A. W. Tucker. But quickly, it took on a wider scope spurred by the *Hirsch conjecture*: Given a linear program in standard form, i.e., with the description of the feasible polyhedral set as $S = \mathbb{R}^n_+ \cap M$ where the affine set M is determined by m (not redundant) linear equations, the conjecture was that it is possible to pass from any vertex of S to any other one via a feasible path consisting of no more than $m - 1$ edges, or in simplex method parlance, a path requiring no more than m (feasible) pivot steps. This question went to the core of the efficiency of the simplex method. The conjecture turned out to be incorrect, at least as formulated, when $m > 4$, but it led to intensive and extensive research associated with the names of Victor Klee, David Walkup, Branko Grünbaum, David Gale, Micha Perlis, Peter McMullen, Gil Kalai, Daniel Kleitman, among others, that explored the geometry and the combinatorial properties of polyhedral sets. Notwithstanding its actual performance in the field, it was eventually shown that examples could be created, so that the simplex method would visit every vertex of such a polyhedral set S having a maximal number of vertexes. The efficiency question then turned to research on the "expected" number of steps; partial answers to this question being provided, in the early 1980s, by K. H. Borgwardt [6] and S. Smale [77]. An estimate of the importance attached to this subject area, can be gleaned from Smale's 1998 article [78] in which he states eighteen "mathematical problems for the next century". Problem 9 in this group asks the question: *Is there a polynomial-time algorithm over the real numbers which decides the feasibility of the linear system of inequalities* $Ax \geq b$? (This decision version of the problem rests on duality theory for linear programming.) Problem 9 asks for an algorithm given by a real number machine with time being measured by the number of arithmetic operations.

Epilogue

George Dantzig had a fertile mind and was passionately dedicated to his work throughout his adult life. Although his career began with an interest in mathematical statistics, circumstances guided him to becoming a progenitor of mathematical programming (or optimization). In creating the simplex method, he gave the world what was later to be hailed as one of "the top 10 algorithms" of the twentieth century [44]. A selection of Dantzig's research output appears in the anthology [10].

Dantzig built his life around mathematical programming, but not to the exclusion of colleagues around the world. Through his activities he touched the lives of a vast number of mathematicians, computer scientists, statisticians, operations researchers, engineers, and applied scientists of all sorts. Many of these individuals benefited from their bonds of friendship with him. Dantzig's natural warmth engendered a strong sense of loyalty, and he returned it in full.

Many of Dantzig's major birthdays were celebrated by conferences, banquets, Festschrifts, and the like. Even the simplex method had a fiftieth birthday party in 1997 at the 16^{th} International Symposium on Mathematical Programming (ISMP) held in Lausanne, Switzerland. In the year 2000, George Dantzig was honored as a founder of the field at the 17^{th} ISMP in Atlanta, Georgia. Perhaps the most touching of all the festivities and tributes in Dantzig's honor was the conference and banquet held in November 2004. He had turned ninety just a few days earlier. In attendance were colleagues from the past and present, former students of Dantzig, and current students of the department. To everyone's delight, he was in rare form for a man of his age; moreover, he seemed to relish the entire event. Sadly, though, within two months, his health took a serious turn for the worse, and on the following May 13th (a Friday), he passed away.

George B. Dantzig earned an enduring place in the history of mathematics. He will be warmly remembered for years to come by those who were privileged to know him. By some he will be known only through his work and its impact, far into the future. In part, this will be ensured by the creation of the Dantzig Prize jointly by the Mathematical Programming Society and the Society for Industrial and Applied Mathematics (1982), the Dantzig Dissertation Award by INFORMS (1994), and an endowed Dantzig Operations Research fellowship in the Department of Management Science and Engineering at Stanford University (2006).

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