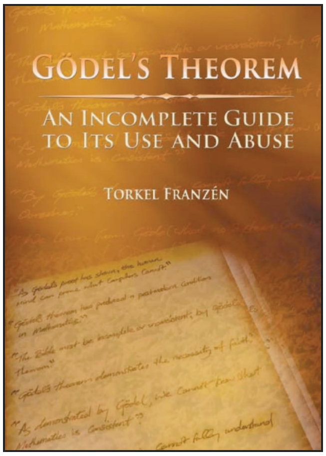
Book Review



Gödel's Theorem: An Incomplete Guide to Its Use and Abuse

Reviewed by Panu Raatikainen

Gödel's Theorem: An Incomplete Guide to Its Use and Abuse
Torkel Franzén
A K Peters, Wellesley, MA
$24.95, paperback, 2005
182 pages, ISBN 1-56881-238-8

Apparently no mathematical theorem has aroused as much interest outside mathematics as Kurt Gödel's celebrated incompleteness result published in 1931. It is invoked not only by mathematicians, logicians, and philosophers but also by physicists, theologians, literary critics, architects, and others. Some eminent physicists have interpreted it as showing that "the theory of everything" demanded by other physicists is impossible to achieve. It is sometimes claimed to prove the existence of God or of free will, the necessary incompleteness of the Bible or of the U.S. Constitution, or the impossibility of genuine knowledge in mathematics—just to mention a few of the many alleged applications (see also [9]).

Gödel is unquestionably among the greatest mathematicians of our times, and he made many important contributions to mathematical logic and other fields. But it is undoubtedly his incompleteness result that made his reputation. In the year 2006 the one-hundredth anniversary of the birth of Gödel was celebrated all over the world with various conferences. The April 2006 issue of the *Notices* was dedicated to Gödel and contains many informative articles on Gödel and on the incompleteness result in particular by some of the leading experts in mathematical logic.

The incompleteness theorem is discussed in countless popular science books, and several are even devoted to Gödel's result. Unfortunately, such books typically show more enthusiasm than competence and tend to be loaded with inadequacies and errors.[1] There is thus still demand for a knowledgeable and reliable exposition of the incompleteness result. Torkel Franzén aims to fulfill this need with his book, and he succeeds outstandingly. He not only provides the reader with adequate understanding of the content of Gödel's theorem and how it is proved but also evaluates critically and thoroughly many applications and misapplications of the theorem and corrects various common misconceptions.

In addition to obvious nonsense, there are among the nonmathematical ideas inspired by Gödel's theorem many that by no means represent postmodernist excesses, but rather come to mind naturally to many people with very different backgrounds when they think about the theorem. It is especially such naturally occurring misunderstandings that Franzén intends to correct.

The book does not pay much attention to Gödel's life and other scientific achievements—only three pages are devoted to them (bits of history are also given along the way). Gödel's strange person and his eventful life are certainly interesting and deserve attention, but fortunately there already exists an excellent biography, *Logical Dilemmas: The Life and*

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[1] *For some apt critiques, see the following reviews in the* Notices*: [1], [3], [5].*

Work of Kurt Gödel by John W. Dawson ([2]; see [1] for a review).

The Incompleteness Theorems

In order to understand Gödel's theorem, one must first explain the key concepts occurring in it: "formal system", "consistency", and "completeness". Very roughly, a *formal system* is a system of axioms equipped with rules of reasoning which allow one to generate new theorems. The set of axioms must be finite or at least decidable; i.e., there must be an algorithm that enables one to mechanically decide whether a given statement is an axiom or not (otherwise, one might stipulate, e.g., taking all true statements of arithmetic as axioms; such a theory is trivially complete but highly abstract and totally useless in practice).

A formal system is *consistent* if there is no statement for which the statement itself and its negation are both derivable in the system. Only consistent systems are interesting in this context, for it is an elementary fact of logic that in an inconsistent formal system every statement is derivable, and consequently such a system is trivially complete. And a formal system is *complete* if for every statement of the language of the system, either the statement or its negation can be derived (i.e., proved) in the system.

Gödel proved two different though related incompleteness theorems, usually called the first incompleteness theorem and the second incompleteness theorem. "Gödel's theorem" is sometimes used to refer to the conjunction of these two and sometimes to either—usually the first—separately. Accommodating an improvement due to J. Barkley Rosser in 1936, the first theorem can be stated as follows:

First incompleteness theorem. Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; there are statements of the language of F which can neither be proved nor disproved in F.

A common misunderstanding is to interpret Gödel's first theorem as showing that there are truths that cannot be proved. This is, as Franzén points out, incorrect, for the incompleteness theorem does not deal with unprovability in any absolute sense, but only unprovability in some particular axiom system. And for any statement S unprovable in a particular formal system F, there are trivially other formal systems in which S is provable. On the other hand, there is the extremely powerful standard axiom system of set theory (the so-called Zermelo-Fraenkel set theory, which is denoted as ZF, or, with the axiom of choice, ZFC), which is more than sufficient for the derivation of all ordinary mathematics. Now there are, by Gödel's theorem, arithmetical truths that are not provable even in ZFC. Proving them would thus require a formal system that incorporates methods going beyond even ZFC. There is thus a sense in which such truths are not provable using today's "ordinary" mathematical methods and axioms or cannot be proved in a way that mathematicians would today regard as unproblematic and conclusive.

Gödel's second theorem concerns the limits of consistency proofs:

Second incompleteness theorem. For any consistent system F within which a certain amount of elementary arithmetic can be carried out, the consistency of F cannot be proved in F itself.

It is important to note that this result, like the first incompleteness theorem, is a theorem about formal provability (which is always relative to some formal system). It does not say anything about whether, for a particular theory T, the statement "T is consistent" can be proved in the sense of being shown to be true by a conclusive argument or by an argument acceptable by mathematicians. For many theories, this is perfectly possible.

Franzén describes in some detail but very informally the ideas of the proofs of the incompleteness theorems. Later, he also explains, again quite informally, the basic notions and results of the theory of computability, essential for proper understanding of the incompleteness results. Franzén also clarifies the relation of the incompleteness theorem to another result of Gödel which is often misleadingly called "the completeness theorem" and to the existence of so-called nonstandard models. The book ends with an appendix that gives a slightly more formal yet still easily understandable explanation of the incompleteness theorems. In all these cases, Franzén has done an admirable job. These sections provide an excellent ground for evaluating various alleged consequences of Gödel's theorem, to which we now turn.

Antimechanism, Faith, and Skepticism

There is a popular view according to which Gödel's theorem shows that the human mind cannot be any sort of computing machine but infinitely surpasses any machine. The alleged justification goes like this: For any formal system, which can be viewed as a computing machine generating theorems, Gödel's proof exhibits an unprovable sentence (often called the Gödel's sentence of the system). We humans can know the truth of this sentence, whereas the formal system or its corresponding machine cannot. There is thus—so the argument goes—something noncomputable about human thinking, perhaps even some irreducibly spiritual, nonmaterial component of the human mind. Such antimechanist conclusions have been drawn from Gödel's theorem, for example, by a philosopher, J. R. Lucas [4], and more recently by a distinguished mathematical physicist, Roger Penrose [6], [7], and these conclusions seem to enjoy some popularity. The idea is apparently quite natural and attractive, for it gets reinvented again and again.

Nevertheless, such conclusions are not justified on the basis of the incompleteness theorem. Franzén explains clearly why this is so: in general, we have no idea whether or not the Gödel sentence of an arbitrary system is true. What we can know is only that the Gödel sentence of a system is true if and only if the system is consistent, and this much is provable in the system itself. But in general we have no way of seeing whether a given system is consistent or not. Later in the book Franzén explores in some detail variants and ramifications of the Gödelian antimechanist argument and shows them all wanting.

Franzén then moves on to discuss various attempts to apply Gödel's theorem outside mathematics. It has been claimed that the incompleteness theorem demonstrates the incompleteness of the Bible, the U.S. Constitution, and Ayn Rand's philosophy of objectivism. He points out that such suggestions ignore the essential condition that the system must be capable of formalizing a certain amount of arithmetic. None of the mentioned "systems" have anything to do with arithmetic. Even worse, they are nothing like a formal system: they do not have an exactly specified formal language, a set of axioms, or rules of inference. Therefore, Gödel's theorem simply is not applicable in such contexts.

More reasonable have been attempts to apply the incompleteness theorem to physics. The hypothetical "theory of everything" (TOE) is sometimes taken to be an ideal of theoretical physics. However, such eminent physicists as Freeman Dyson and Stephen Hawking have invoked Gödel's theorem to suggest that there is no such theory of everything to be had. Now it seems more reasonable to assume that a formalization of theoretical physics would be the subject of the incompleteness theorem by incorporating an arithmetical component. Nevertheless, Franzén adds, Gödel's theorem tells us only that there is an incompleteness in the arithmetical component of the theory. Whether a physical theory is complete when considered as a description of the physical world is not something that the incompleteness theorem tells us anything about.

Franzén also discusses various theological conclusions drawn from Gödel's theorem. Abstracts from the *Bibliography of Christianity and Mathematics* declare, for example, that Gödel's theorem demonstrates that physicists will never be able to formulate a theory of physical reality that is final or that the human mind is more than just a logical machine. Such theological appeals to Gödel's theorem only recycle the above-discussed and deficient Gödelian arguments against the mechanist theory of mind and TOE. But there are some more specifically theological appeals to the incompleteness theorem. Some of these are simply preposterous, and others at best are based on analogies. Sometimes it is suggested that Gödel's theorem shows that the only possible way of avowing an unprovable truth is faith. But, first, Gödel did not exhibit any absolutely unprovable truths, only relative ones; and, second, if we have, on the basis of mathematical reasoning, absolutely no idea whether a given highly complex formal system is consistent or not, it is quite unclear how Christian faith (or anything else) could help.

Gödel's theorem is often thought to support some form of skepticism with regard to mathematics: it is contended that we cannot, strictly speaking, prove anything or that the consistency of our fundamental theories (such as ZFC) is shown to be doubtful. Franzén argues against such claims that nothing in Gödel's theorem in any way contradicts the view that we have absolutely certain knowledge about the truth of the axioms of the system and, consequently, of their consistency. We don't need Gödel's theorem to tell us that we must adopt some basic principles without proof. If we have no doubts about the consistency of, say, ZFC, there is nothing in the second incompleteness theorem to give rise to any such doubts. And if we do have doubts about the consistency of ZFC, we have no reason to believe that a consistency proof of ZFC given in ZFC itself would do anything to remove those doubts.

Franzén also devotes a brief chapter to the variants of incompleteness results arising from the so-called Algorithmic Information Theory, or the theory of Kolmogorov complexity, and especially the various philosophical interpretations of these results by Gregory Chaitin (one of the founders of this theory). For example, Chaitin claims that his results not only explain Gödel's incompleteness theorem but also are the ultimate, or the strongest possible, incompleteness results. Franzén first explains these results and then shows that such claims are in no way justified by mathematical facts (see also [8]).

Concluding Remarks

This is a marvelous book. It is both highly competent and yet enjoyably readable. At some points there are even glimpses of humor, as when Franzén declares in the preface: "For any remaining instances of incompleteness or inconsistency in the book, I consider myself entirely blameless, since after all, Gödel proved that any book on the incompleteness theorem must be incomplete or inconsistent. Well, maybe not" (p. ix). At last there is available a book that one can wholeheartedly recommend for anyone interested in Gödel's incompleteness theorem—one of the most exciting and wide-ranging achievements of scientific thought ever.

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