

The Poincaré Conjecture and Poincaré's Prize

Reviewed by W. B. Raymond Lickorish

The Poincaré Conjecture: In Search of the Shape of the Universe

Donal O'Shea

Walker, March 2007

304 pages, US\$15.95, ISBN 978-0802706545

Poincaré's Prize: The Hundred-Year Quest to Solve One of Math's Greatest Puzzles

George G. Szpiro

Dutton, June 2007

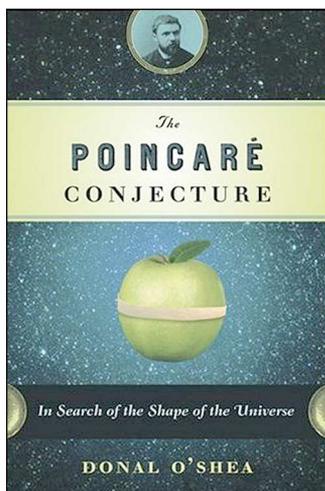
320 pages, US\$24.95, ISBN 978-0525950240

Here are two books that, placing their narratives in abundant historical context and avoiding all symbols and formulae, tell of the triumph of Grigory Perelman. He, by developing ideas of Richard Hamilton concerning curvature, has given an affirmative solution to the famous problem known as the Poincaré Conjecture. This conjecture, posed as a question by Henri Poincaré in 1904, was a fundamental question about three-dimensional topology. It proposed that any closed, simply connected, three-dimensional manifold M be homeomorphic to S^3 , the standard three-dimensional sphere. The conjecture is therefore fairly easy to state and yet, unlike various other conjectures in, say, number theory, just a little mathematical knowledge is needed to understand the concepts involved. The given conditions on M are that it is a Hausdorff topological space, each point having a neighbourhood homeomorphic to \mathbb{R}^3 , it is compact and path connected, and any loop in M

can be shrunk to a point. Armed with a graduate course in topology, and perhaps also aware that any three-manifold can both be triangulated and given a differential structure, an adventurer could set out to find fame and fortune by conquering the Poincaré Conjecture. Gradually the conjecture acquired a draconian reputation: many were its victims. Early attempts at the conjecture produced some interesting incidental results: Poincaré himself, when grappling with the proper statement of the problem, discovered his dodecahedral homology three-sphere that is not S^3 ; J. H. C. Whitehead, some thirty years later, discarded his own erroneous solution on discovering a contractible (all homotopy groups trivial) three-manifold not homeomorphic to R^3 . However, for the last half-century, the conjecture has too often enticed devotees into a fruitless addiction with regrettable consequences.

The allure of the Poincaré Conjecture has certainly provided an underlying motivation for much research in three-manifolds. Nevertheless, three-manifold theory long ago largely bypassed the conjecture. This resulted from a theorem that H. Kneser proved in 1928. It asserts that any compact three-manifold, other than S^3 , can be cut into pieces along a finite collection of two-dimensional spheres, so that, if those spheres are capped off by gluing three-dimensional balls to them, the resulting pieces are prime. A connected three-manifold is *prime* if it is not S^3 and if, whenever a sphere embedded in it separates it into two parts, one of those parts is a ball. Later J. W. Milnor proved an orientable manifold determines its prime pieces, its summands, *uniquely*. Thus it is reasonable to restrict study to prime manifolds. A prime three-manifold might have been a counterexample to

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the Poincaré Conjecture but otherwise, having no proper summands at all, it could not have contained within itself (as a summand) a possible counterexample to the conjecture.

Both of these books have to explain, to non-mathematicians, the meaning of the statement of the Poincaré Conjecture and something about curvature in order to introduce the idea of the Hamilton-Perelman technique. However, Perelman's work also encompasses a proof of the Thurston Geometrisation Conjecture, a much more significant achievement, and explaining the meaning of this is a much more difficult task. A modification of Kneser's argument allows a prime three-manifold to be cut, by a finite collection of (incompressible) tori, into pieces that contain no more such tori (other than parallel copies of tori of the finite set). After a small manoeuvre the collection of tori is unique; that is the Jaco-Shalen-Johannson theorem. The conjecture of W. P. Thurston is that this set of tori cuts the prime manifold into pieces each of which has one of eight possible geometric structures. The Poincaré Conjecture is a special case of this: in a simply connected manifold the set of tori is empty, and so the whole manifold should have a geometric structure which could only be the spherical one. Thurston proved his conjecture for sufficiently large (or Haken) manifolds, a class for which F. Waldhausen had already shown that homotopy equivalence implies homeomorphism. The eight standard geometries are well-known (and well-documented by G. P. Scott). The conjecture is then that each piece of the manifold is the quotient of one of the standard geometries by a discrete group of isometries (acting freely and properly discontinuously), the quotient being complete. Most of the mystery of this has centred around understanding hyperbolic geometry; pieces with the other geometries have Seifert-fibred structures that are understood. In very broad outline, the Hamilton approach, brought to fruition by Perelman, starts by giving a closed three-manifold a smooth structure with arbitrary Riemannian metric (that is easy). Then the metric is allowed to change in time according to a Ricci flow equation that is an analogue of the heat equation of physics. If all goes very well, the metric will converge to a metric of constant curvature (just as a heated body eventually has uniform temperature) and then the manifold has one of the eight geometries. Otherwise it should tend to a metric-with-singularities.

If the singularities can be shown to correspond to the spheres and tori that decompose the manifold, as already described, cuts can be made and the process restarted on the resulting pieces.

The book by Szpiro is a fast-moving account of mathematics in the making. He presents the story of the Poincaré Conjecture, from its formulation to its solution, in terms of the people who, for good or ill, became involved with it. Their lives are painted with all available colour, with details of childhood and education, with accounts of their early mathematical achievements, their hopes, their frustrations, and their disappointments. Mathematicians are shown to have very human passions and fears, they too are caught up with the conventions, the pressures, and the politics of their times, and are involved in the tragedies of war and repression. Details marginally relevant to the Poincaré Conjecture are included if they might illuminate personalities involved. Szpiro has done a great job of biographical research and produced an account that will fill many a gap in a mathematician's historical knowledge. For many it will be an irresistible opportunity to relive contentious times past. The drama of the announcement of Fields Medal winners at the International Congress in Madrid, the refusal by Perelman to accept a medal or participate in the congress, the uncertainty over a possible award of a one-million dollar Millennium Problem prize by the Clay Institute, all these happenings give a breathtaking climax to the century-long saga. It seems doubtful if the reclusive Perelman would want to star in this academic soap opera but a journalist must, it seems, report what he learns. Any educated person with a little curiosity will be able to enjoy this book. He or she will not become bogged down by obtrusive mathematical detail, but will be swept along by this tale of the all-consuming desire to know the truth and the competition to achieve fame thereby, only to find applause eschewed by the eventual winner. Here is an insight into how scientific research really happens and what an emotional affair it can be; mathematics does need books such as this.

The book will be a valuable historical account, a work of reference on the lives of a most impressive number of early topologists. The quantity and spread of the material covered is ambitious and truly impressive. After a brief nod towards Christopher Columbus and the flat earth hypothesis, there follows a 35-page account of the life of Henri Poincaré (1854–1912). His childhood, his distinguished relatives, his schooling and experience of the Franco-Prussian war, his entry to the *École Polytechnique* in Paris and his three years at the *École des Mines* are all described. There is a long account of his short experience in the *Service des Mines* (miner's lamp number 476 caused the pit explosion). Finally he becomes a professor at

the Sorbonne. Next is the story of his winning the prize offered by King Oscar II of Sweden for an essay on the n -body problem, of his dealings on this with G. Mittag-Leffler and of the discovery of an error in his essay. The famous paper on *Analysis situs* and its complements are given due prominence, as is the discovery by P. Heegaard of a mistake in it. There is much here about Heegaard but no mention of the Heegaard splitting of a three-manifold (still much studied in three-manifold theory and the basis for the topical and very productive Heegaard-Floer homology); this is surely the chief thing for which he is remembered today. Homage is paid to the bridges of Königsberg, the Euler characteristic, and a little knot theory, all in a historical setting. The flight from persecution by M. Dehn is graphically described; for Dehn's famous lemma the proof is ascribed to Kneser on page 92 and, correctly, to C. D. Papakyriakopoulos on page 119.

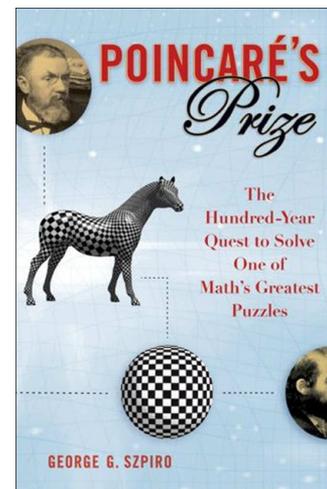
As time and the book progress, history evolves into journalism. There is much discussion of personality clashes, jealousies, failures, and priority disputes, referring to many who are very much still alive. Concerning a bitter employment dispute at Berkeley, which is not at all relevant here, the statement "women's rights may not have been particularly high on Smale's list of priorities" carries unbalanced innuendo. The criticisms eventually provoked by E. C. Zeeman's evangelism for catastrophe theory are superfluous. Placing the famous 3-Manifolds Institute of Georgia, U.S.A., in 1960 rather than in 1961 could affect perceptions of priorities. Attributing to J. Hempel a most puerile observation about the Poincaré Conjecture, whilst overlooking the fact that it is his book that has been the undisputed authority on three-manifolds for thirty years, seems unjust. When discussing a selection of those who tried and failed to solve the Poincaré Conjecture, it is unkind to refer to a respected group theorist, with a goodly list of publications to his credit, as "a rather obscure mathematician". It would have been better to ignore the charming Italian, a retired engineer, who actually turned up at the Berlin congress to present his fallacious proof, rather than give him cavalier treatment.

There is a need for accounts of modern mathematics that are accessible to all. How then does one treat mathematical entities the real understanding of which requires a whole lecture course? Szpiro has some answers to this, and they may be good ones. He tells us that "...manifolds ...can be imagined, for example, as flying carpets floating in the sky". When did the lack of a precise definition affect enjoyment of flying carpets? An attempt to describe the second homotopy group concludes with "part of what is beautiful about topology is the unique way it boggles the mind". A proof by J. R. Stallings is summarised as "stripping a

manifold ...to skeletons, embedding the skeletons in two balls, separating the skin from the inside of one of the balls, then recombining some of the items. And, since a ball is a ball is a ball ...". This is nonsense, and yet it is indeed strangely reminiscent of Stallings' work on the high-dimensional generalisation of the Poincaré Conjecture (a topic, hard for a beginner, which is covered by the book at some length and in some considerable detail).

Geometric structures and the Geometrisation Conjecture are too difficult to describe completely and briefly to the layperson but Szpiro has a try. It is almost 200 pages from the book's start before Ricci flow and the actual Hamilton-Perelman programme get much attention. Plenty of credit is given to those who deserve it, and a brave attempt is made to describe the main idea. Cigar singularities go "plouf" whereas soap bubbles go "pop". But then, neither the author nor this reviewer really understands this theory. Towards the end of the book a detailed account appears of the tensions, somewhat inflamed by an article in the *New Yorker*, between two, or maybe three, teams trying to produce accounts of Perelman's work, more thorough than were his original papers posted on the Internet. Perhaps in time this discord will seem to be of less importance in the whole story.

At the book's conclusion Szpiro acknowledges by name 54 people, mainly mathematicians, who "made suggestions or corrected mathematical or historical errors." It is a pity they did not do a much better job. The Poincaré homology sphere is constructed by identifying opposite faces of a dodecahedron with one tenth of a twist, *not* one fifth (under which identification is impossible). It is *not* the only known homology sphere; there are infinitely many others. Once it was conjectured to be the only one with finite fundamental group. It should have been mentioned that this is now known, a fact following from the very work of Perelman that the book is all about. The descriptive definition given of the fundamental group omits all mention of the homotopy of loops so that the account of the existence of inverses in the group is opaque. Elements of the fundamental group of the figure eight *cannot* be described by three integers *nor* can those of the pretzel be described by four integers. There seems to be an erroneous assumption that fundamental groups are abelian. The torsion of a manifold, the set of homology elements of finite order, is not remotely related to the twistedness of a Möbius strip. The higher-dimensional Poincaré



Conjecture is *not* that a contractible manifold be homeomorphic to a sphere of the appropriate high dimension. S. Smale's decomposition of a high-dimensional manifold into handles does *not* break down in three dimensions; it is his methods of handle *manipulation* that break down. Indeed, attempts to emulate in three dimensions Smale's methods of manipulation have been the basis of many failed attempts at the Poincaré conjecture.

The book by O'Shea tells the same basic story of Perelman's solution to the conjectures of Poincaré and Thurston but in a different way. It uses the total history of mankind's discovery and appreciation of geometry to teach some basic mathematical ideas, which will be needed towards the end of the book to understand the statements of the conjectures. People's lives are described, to give context and some colour to the account, but it is their ideas that get the most attention. We begin, *circa* 500 BCE, with the school of Pythagoras on the Isle of Samos. He is credited with teaching that the earth is spherical, and Eratosthenes (275-195 BCE) is applauded for the accuracy of his measurement of the earth's circumference. This leads to Columbus, mediaeval atlases, and even the possibility that the earth's surface might be a torus. The mathematical idea of an atlas for a manifold thus falls into place, and the classification theorem for closed orientable surfaces is enjoyed. From this it is clear that a two-dimensional version of the Poincaré Conjecture is true. The idea of an atlas is then extended to a consideration of the possibility of three-dimensional charts for the whole universe, and some examples of three-manifolds are given. In a return to ancient Greece, an enthusiastic review of Euclid's *Elements* lists some of Euclid's definitions, his notions, and his postulates. From failed attempts to prove that the parallel postulate follows from the other postulates, the discussion then moves to the nineteenth century discovery, by C. F. Gauss, N. I. Lobachevsky and J. Bolyai, of non-euclidean geometries. We are told that, on learning his son worked on the parallel postulate, Bolyai's father begged, "For God's sake, I beseech you, give it up. ...it too may take all your time, deprive you of your health, peace of mind and happiness in life." A century or more later, many a topologist in thrall to the Poincaré Conjecture could have benefited from similar advice.

The life of B. Riemann was impoverished, sickly, and relatively short. Nevertheless, ideas that he instigated are fundamental to the understanding of modern differential geometry. In O'Shea's book the details of Riemann's *Habilitationsschrift* provide the basis for a careful informal discussion of metrics and curvature. Short biographies of Poincaré and F. Klein follow, detailing the divide, even the animosity, between them engendered by the horrors of the Franco-Prussian war. Returning to mathematics, there is an account of Poincaré's

disc model of the hyperbolic plane together with a good description of how hyperbolic structures can be given to surfaces of genus two or more. Translations of significant passages from Poincaré's great topological papers are given, inspiring a gentle description of Betti numbers, homology, and the fundamental group.

This historical account now moves from pedagogy to narrative. An outline of topology in the first half of the twentieth century, containing a short account of Einstein's use of differential geometry, is followed by an interesting section on the emergence of mathematics in the United States. Just a few pages tell of solutions to the high-dimensional version of the Poincaré Conjecture. Included is a brief idea of the meaning of a differential structure and a mention of Milnor's distinct differential structures on the seven-sphere. M. H. Freedman's topological solution to the conjecture in four dimensions is briefly described (though A. J. Casson also deserves some credit here). An additional comment concerning the lack of knowledge about smooth structures on the four-sphere would have added perspective, for there is mention of the achievements of S. K. Donaldson with four-manifolds. Next come details about Thurston and his Geometrisation Conjecture. The reader has, by now, been carefully prepared and should be able to understand, more or less, the conjecture in the simple way in which it is presented. Likewise the introduction to curvature has prepared the way for the short discussion of Hamilton's programme on Ricci flow that follows. Only a very little space is given to failed attempts to prove the Poincaré Conjecture (but the mention of J. H. Rubinstein's superb recognition algorithm for S^3 , with its interpretation by A. Thompson, would have been better placed in a different section). The final twenty pages of the book pay tribute to Perelman, with a clear qualitative outline of his work, and tell of the congress in Madrid, Fields Medals, and Clay Institute prizes.

As implied by its subtitle, O'Shea's book wonders about "the shape of the universe" and sometimes uses curiosity about the cosmos to motivate the study of three-manifolds. He mentions the speculations of J. R. Weeks on whether the physical universe might be a simply connected closed three-manifold. Most cosmologists, however, do seem to countenance no less than four dimensions for the universe and often prefer many more. This book as a whole is a great success. It sets out to teach: teach it does. It does so without being patronising, flippant, or unkind. It is admirably suited for a serious-minded undergraduate, for a high-school student with mathematical talent, and for their parents and teachers. The few mistakes of detail seem almost intended to amuse: the temple of Hera on Samos is not on

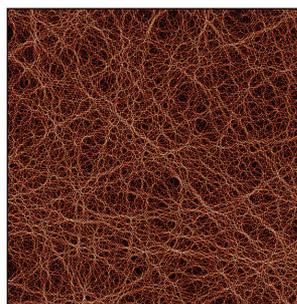
the usual list of the seven wonders of the ancient world; Euclid's Proposition 5 is not usually "known in England as the *pons arsinorum*" (*sic*); R H Bing, famous for his lack of a first name, would turn in his grave on being called "Rudolph"; whatever M. Kervaire's manifold with no possible smooth structure might have been, it could not have been homeomorphic to a nine-dimensional sphere. Appendices to the book are admirably handled. Thirty pages of numbered notes contain detailed references, explanations for experts, and circumstantial details; this permits the main text to proceed smoothly and without too much technicality. A glossary of terms summarises definitions from the text (though that of compactness is spurious). A list of people mentioned, together with their dates and relevance to the plot, is followed by a history timeline spanning some 3,700 years and, of course, a good bibliography and index.

Both these books should certainly find their way into university libraries and onto many a private book shelf. They cater for readers of different temperament and mathematical inclination, but each makes available to anyone the basic story of the solving of one of the greatest problems in mathematics. That this should occur in the twenty-first century, rather than long, long ago, is itself cause for excitement. Most mathematicians must hope that some simplification will be found possible in some of Perelman's work, so that one of the few people who really understands his proof will be able to present it to the general mathematical public. Would-be solvers of the Poincaré Conjecture by combinatorial means would be wise to concede defeat to differential geometry; after all, nobody ever claimed to have a proof of the Poincaré Conjecture that would almost fit into the margin of a page.

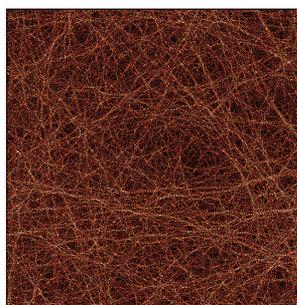
About the Cover

Quantum Chaos

This month's cover accompanies Ze'ev Rudnick's article on quantum chaos. The smaller images illustrate plots of eigenfunctions at progressively higher frequencies for the Dirichlet problem on the curvilinear quadrilateral shown in Rudnick's article. They start from the first and go through the 100,000-th eigenvalue in steps of powers of 10. The background of the cover is a random combination of plane waves, and the point is that in a region giving rise to quantum chaos eigenfunctions of high frequencies are conjectured to be locally similar to such random combinations.



Eigenfunctions of high frequency (cropped from an image on the cover).



Random combination of plane waves. The statistical meaning of the stringy structures in these images does not seem to be well understood (cropped from an image on the cover).

All of the images were produced by Alex Barnett of Dartmouth College, who also provided the data for the figures in Rudnick's article. They were computed using a variant of the Method of Particular Solutions (MPS) called the scaling method, which he has had a hand in developing. It is faster by an order of 10^3 than competing techniques, and is explained in recent papers of his. A good introduction is "Asymptotic rate of quantum ergodicity in chaotic Euclidean billiards", *Comm. Pure Appl. Math.*, 2006.

—Bill Casselman, Graphics Editor
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