Environmental Problems, Uncertainty, and Mathematical Modeling

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In this paper we discuss three rather special characteristics shared by many environmental problems. Namely, that (i) the environmental variables in which we are most interested constitute a stochastic process; (ii) the long-term or limiting behavior and the short-term or transient behavior are often both important; and (iii) the underlying probability distributions are likely to be influenced by the environmental policies or remedies that we choose to impose. This third characteristic implies the need to understand the impact of technologies and controls that influence the dynamics of the system. The control theoretic perspective of environmental engineering problems has, we believe, received less attention than it deserves in the literature. Consequently, after a brief discussion of the exogenous, control-independent case we focus on illustrating some special challenges and opportunities embedded in the control-dependent situations.

Throughout history humans have relied heavily on adaptation to and exploitation of the natural environment. An unintended consequence of civilization and the more recent industrial and technological revolutions has been an ever increasing reliance on industry and technology and a consequent neglect of the natural world as a legitimate planning tool of social and economic development. There are many exemplars. The establishment and growth of water-thirsty, air-conditioned cities such as Dubai and Las Vegas, in hostile desert surroundings, is one such stark reminder of our unbridled desire to dominate the environment and our inability to read the warning signs.

However, the acceleration of a multitude of adverse impacts of human development processes on the environment, including global climate change, the loss of biocapacity and biodiversity, the spread of pollution, and the depletion of natural resources, has, in recent years, served to mobilize public opinion in many countries to tackle environmental problems much more actively. As a result, industries and regulatory agencies in these countries are beginning to show real interest in minimizing undesirable environmental impacts of human activities.

A prerequisite for the design of effective adaptation and mitigation strategies will be to understand the underlying processes and the possible effects of policies and regulatory regimens. Consequently, the forthcoming decades will offer the scientific community unprecedented opportunity to contribute to the development and subsequent refinement of wide-ranging environmental remedies. The majority of these remedies will require evidence to support "proof of concept" before they can be adopted. The latter will often be obtained with

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the help of quantitative—mathematical—models and supporting analysis. Thus there will be ample opportunity for us mathematicians to contribute to these solutions. Indeed, quantitative modeling of environmental problems will be done with or without us; thereby raising the spectre, if we are not engaged, of improper application of the input controls, incorrect interpretation of the output data, and subsequent misguided decision making.

A detailed discussion of minimal requirements that environmental problems should possess in order to make mathematical modeling meaningful is beyond the scope of this short contribution. However, in Filar 2002 [17] certain principles are identified that capture what an applied mathematician might call common sense. Violation of these principles should sound a warning that mathematical modeling may not be appropriate in these situations. The main point made in [17] is that whenever we develop a mathematical model of a phenomenon or a situation that is not itself a mathematical entity a certain, minimal, amount of "domain knowledge" is required. Although the latter may appear obvious, it is clear that in modeling certain environmental phenomena, for instance, the response of the oceans to the doubling or tripling of atmospheric CO_2 concentrations, the issue of what constitutes domain knowledge is a challenging question in its own right.

Nonetheless, in this paper we shall assume that adequate domain knowledge is available and focus instead on three rather special characteristics shared by many environmental problems. We will consider variables (e.g., levels of persistent contamination in a lake)

(1) which constitute a (possibly multi-dimensional) stochastic process $\{X_t\}$, $t \geq 0$;
(2) for which both the short-term or transient behavior and the long-term or asymptotic behavior are equally important; and
(3) where the underlying distributions of the random variables, X_t, are likely to be continually, but only partially, influenced by the policies or remedies we are designing (e.g., regulations or emission filters).

The first of the above characteristics implies that the need to understand and manage risk is usually an essential part of the problem. The second implies that environmental remediation policies and technological remedies that are costly and unpopular in the short term require persuasive advocacy before they will be accepted and adopted. The third characteristic implies the need to understand the impact of the control functions $u(t)$ that influence the dynamics of the stochastic processes. History shows us that, often inadvertently, economic development policies may act as controls that influence the trajectory of key state variables in some important ecosystem. The need for advocacy and the importance of control have, we believe, received less attention than they deserve in the literature. Consequently, after a very brief discussion of the exogenous, control-independent case, we focus on illustrating some special challenges and opportunities embedded in the control-dependent situations.

Extreme and Rare Events in the Exogenous, Uncontrolled, Case

In this section we shall make a simplifying—and increasingly less acceptable—assumption that the majority of our most feared natural disasters such as hurricanes, floods, droughts, crop failures, and bush fires are independent of human activities. In this case, these disasters certainly constitute "extreme events" in the common statistical sense meaning of the phrase.

Consequently, it is prudent to examine what the now classical "extremal value theory" has to offer in our context of modeling the probability of such events occurring. The origins of this theory—that has evolved out of the twin subjects of statistics and stochastic processes—date back to the seminal work of Fisher and Tippett [21] in the first half of the last century. By now, this challenging subject has grown enormously, with researchers following a number of fruitful lines of investigation. For a comprehensive modern text we refer the reader to Embrechts et al. [14].

However, before proceeding, we observe that a substantial portion of the theory of extremal events was motivated by financial considerations such as the "risk of ruin". Thus, to the extent that these techniques and concepts depend on accumulation of losses, they may not correspond very well to the types of problems that are most relevant in our context. For instance—and without in any way advocating the underlying connotations—the phrase "a miss is as good as a mile" captures some of the above distinction. Thus, a severe flood at a level that does not breach existing levees presumably has little or no effect on the probability that future floods will breach these defenses. However, an investor who only just avoided ruin when the market had its last downturn has probably suffered such losses that his or her likelihood of failing to avoid ruin in the next downturn is severely reduced.

Due to the above considerations we will not discuss those aspects of the theory of extremal events that deal with sums of random variables exceeding certain thresholds[1] and will focus instead

[1] *It should be noted that such random sums could still be of interest in our context if, for instance, we were trying to analyze the accumulated degradation of certain natural protective barriers, such as Louisiana's "barrier islands" (e.g., see* [13]*). Thus the issue of partial sums of*

on the aspects that deal with the properties of the so-called "extremal statistics". We shall now introduce some of the notation needed to make the discussion a little more precise.

Consider a sequence $\{X_n\}, n \in \mathbb{N}$, of independent identically distributed random variables (iid rv's, for short), all of which are distributed as a given random variable X that has a cumulative distribution function F. The random variables of interest in the classical extremal value theory are the induced sequence of *sample maxima*

$$M_n := \max(X_1, X_2, \ldots, X_n) \ \text{ for } \ n \geq 2.$$

Note that the analysis of the sample minima, defined analogously, is not any different because of the identity $\min(X_1, X_2, \ldots, X_n) = -\max(-X_1, -X_2, \ldots, -X_n)$ and hence we shall restrict our discussion only to the sequence $\{M_n\}$, $n \in \mathbb{N}$. It is now clear that the distribution function of M_n is simply

$$\begin{aligned} &\mathbb{P}(M_n \leq x) \\ &= \mathbb{P}(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x) = F(x)^n. \end{aligned} \tag{1}$$

Since in this theory we are primarily interested in "failures" corresponding to the sample maxima exceeding certain thresholds, we shall be particularly interested in the "tail" of the distribution of M_n, namely in $\mathbb{P}(M_n > x) = 1 - F(x)^n$. For our underlying distribution $F(x)$ the tail is defined simply by $\bar{F}(x) := 1 - F(x)$.

Of course, for small n, equation (1) provides a means of calculating the tail probabilities for the distribution of M_n, but for large n a direct computation could be very cumbersome. Thus, a major thrust was made to derive asymptotic results that are in the spirit of the celebrated "Central Limit Theorem" of statistics, which states that

$$\frac{\Sigma_n X_n - n\mu}{\sigma\sqrt{n}} \to Z,$$

as $n \to \infty$ where $X_1, \ldots, X_n$ are independent and identically distributed random variables and Z is a standard normal distribution.

This naturally led to the question of whether it is possible to find constants $c_n > 0$ and d_n such that for some nondegenerate probability distribution H

$$\frac{M_n - d_n}{c_n} \to H, \tag{2}$$

in distribution, as n tends to infinity. Clearly, if (2) holds, then the equation

$$\mathbb{P}\left(\frac{M_n - d_n}{c_n} \leq x\right) = \mathbb{P}(M_n \leq u_n) = \int_{-\infty}^{u_n} dH(x), \tag{3}$$

where $u_n = c_n x + d_n$, provides a basis for calculating an approximation of the tail probabilities of interest. The problem contained in equations (2) and (3) is actually more challenging than the analogous problem for random sums. The validity of these equations seems to require delicate conditions on the tail $\bar{F}$ of the distribution F to ensure the existence of a nontrivial limit of $\mathbb{P}(M_n \leq u_n)$ as n tends to infinity. In particular, the tail $\bar{F}(u_n)$ needs to decay at an appropriate rate as u_n tends to infinity with n.

One of the fundamental results is the so-called *Poisson approximation* that states, that for any given nonnegative number τ and a sequence $\{u_n\}$ of real numbers, the following equivalence holds:

$$n\bar{F}(u_n) \to \tau \quad \Longleftrightarrow \quad \mathbb{P}(M_n \leq u_n) \to e^{-\tau} \tag{4}$$

as n tends to infinity. An elegant special case where the above limit can be directly computed is the case when $F(x) = 1 - e^{-x}$, that is, the exponential random variable with parameter 1. In this case a direct calculation shows that

$$\begin{aligned} \mathbb{P}(M_n - \ln\, n \leq x) &= [\mathbb{P}(X \leq x + \ln\, n)]^n \\ &= [1 - n^{-1}e^{-x}]^n \ \to \ \exp\{-e^{-x}\} \end{aligned} \tag{5}$$

as n tends to infinity. Note that $\Lambda(x) := \exp\{-e^{-x}\}$ is the well-known *Gumbel distribution*. Indeed, the remarkable conclusion of the famous Fisher-Tippett theorem is that if H is a nondegenerate distribution such that (2) holds, then H must belong to one of only three families of distributions: Frechet, Gumbel, or Weibull. Thus these three well-known distributions provide a basis for many of the approximations of the probabilities of interest.

Of course, while mathematically very convenient, the independent, identically distributed distribution assumption on the random variables of the process $\{X_t\}$; $t \geq 0$ is not realistic in many environmental applications. After all, for the majority of natural phenomena of interest, such as levels of pollution in the air or water or salinity in the soil, previous values of relevant indicator variables definitely influence current values of these variables. Consequently, perhaps, the mildest and yet still powerful way of relaxing the above assumption is to replace it by the *stationary Markov transition assumption*, which, in discrete time, states that, for every possible pair of values x and x', the probability that $X_{t+1} = x'$ given that $X_t = x$ is independent of time and any previous states and actions. That is, there exist *stationary transition probabilities*:

$$p(x'|x) := \mathbb{P}\{X_{t+1} = x'|X_t = x\} \tag{6}$$

for all $t = 0, 1, 2, \ldots$.

The above—seemingly still quite restrictive—assumption facilitates a lot of very useful modeling, especially when it is reasonable to discretize the range of the random variables X_t by finitely

sequences of random variables exceeding certain thresholds and the amounts by which they exceed these thresholds is very relevant to the assessment of environmental risk. The latter has received considerable attention in the financial risk literature, in which these concepts are known as "Value-at-Risk" and "Conditional-Value-at-Risk". We do not discuss these concepts here but refer the interested reader to Rockafellar and Uryasev [33].

many values $\{x_1, x_2, \ldots, x_N\}$. In such a case, an $N \times N$ *probability transition matrix* P of a Markov chain whose (i, j)th entry is $p(x_j|x_i)$ contains all the required information about the probability distributions of all the random variables X_t for all $t = 0, 1, 2, \ldots$. This simple approximation makes it possible to apply a wide range of computationally effective, matrix analytic methods to understand many important characteristics of the underlying Markov process. The reader is referred to [27] and [30] for both the classical and more modern perspectives on this interesting subject.

We conclude this section with a simplistic but still illustrative, example. Consider a process $\{X_t\};\ \ t \geq 0$, where X_t denotes the stock of a certain species of fish in year t. Assume that, without any harvesting, the natural marine ecosystem cycles ensure that the stock is in one of only three states: abundant (x_1), average (x_2), and low (x_3). Suppose also, for instance, that the corresponding Markov chain is adequately described by the 3×3 probability transition matrix:

$$P = \begin{pmatrix} 0.70 & 0.25 & 0.05 \\ 0.10 & 0.80 & 0.10 \\ 0.00 & 0.30 & 0.70 \end{pmatrix}.$$

The entries in the above matrix are completely fictitious, but they may reflect the anticipated cyclic pattern of the population of that particular species of fish. Furthermore, it is well known that the successive powers of P^n yield probabilities of n-step transitions from state to state. It is easy to verify that, for this particular transition matrix, the limit $P^* := \lim_{n\to\infty} P^n$ exists and has identical rows, each coinciding with the row vector π of so-called *stationary distribution probabilities* satisfying the fixed-point equation $\pi = \pi P$.

Indeed, in this particular instance, the latter is approximately given, given by $\pi = [0.19,\ 0.58,\ 0.23]$. Thus it is possible to conclude that, if the probability transition matrix P continues to describe accurately the stochastic process of interest, then, in the long-run average sense, the population of this particular species of fish will be abundant approximately 19% of the time, average 58% of the time, and low 23% of the time.

Of course, in the above example, all probability distributions were exogenous in that they were assumed to remain unchanged over time and independent of human activities. It is clear, however, that the essence of most environmental problems lies in the violations of such assumptions. For instance, in the fish population example, fishing regulations and market prices for fish are likely to impact the intensity of harvesting and will, therefore, alter these distributions. This naturally leads to the class of models and some of the issues discussed in the next section.

Stochastic Sequential Decision Models

To address some of the issues alluded to in the preceding section, it is natural to move from consideration of Markov chains to the more general framework of Markov decision processes (MDP's, for short). The latter are stochastic, sequential processes in which a "decision maker" has some control over the distributions of a future stream of random benefits or costs frequently referred to as positive or negative "rewards".

More precisely, we shall now consider a process Γ that is observed at discrete time points $t = 0, 1, 2, 3, \ldots$ that will sometimes be called *stages*. At each time point t, the *state* of the process will be denoted by X_t. We shall assume that X_t is a random variable that can take on values from the finite set $\mathbf{X} = \{1, 2, \ldots, N\}$, which from now on will be called the *state space*. The phrase "the process is in state x at time t" will be synonymous with the event $\{X_t = x\}$.

We shall assume that the process is controlled by a *controller* or a *decision maker* who chooses an *action* $a \in \mathbf{A}(x) = \{1, 2, \ldots, m(x)\}$ at time t if the process is in state x at that time. We may regard the action chosen as a realization of a random variable A_t denoting the controller's choice at time t. Furthermore, we shall assume that the choice of $a \in \mathbf{A}(x)$ in state x results in an immediate *reward* or *output* $r(x, a)$ and in a probabilistic transition to a new state $x' \in \mathbf{X}$.

Now the previous stationary transition probabilities assumption of (6) is extended by the assumption that, for every $x, x' \in \mathbf{X}$ and $a \in \mathbf{A}(x)$,

$$(7)\qquad p(x'|x, a) := \mathbb{P}\{X_{t+1} = x'|X_t = x, A_t = a\}$$

for all $t = 0, 1, 2, \ldots$.

Suppose that the decision maker wishes to influence a stream of expected values of these rewards, denoted by

$$\{E_{xf}(R_t)\}_{t=0}^{\infty},$$

where x is the initial "state", f is the control, R_t is the random reward or benefit at stage t, and E denotes the mathematical expectation operator. We assume that specifying x and f uniquely determines the probability distribution of R_t for every time period t in the future.

The decision maker might then wish to choose f so as to maximize either the *discounted performance criterion*

$$v_d(x, f) := \sum_{t=0}^{\infty} \beta^t E_{xf}(R_t),$$

where the parameter $\beta \in [0, 1)$ is called the *discount factor*, or the *long-run average performance criterion*, defined by

$$v_a(x, f) := \liminf_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} E_{xf}(R_t).$$

Note that the discounted criterion has a natural accounting interpretation as the so-called "present value" of the stream of rewards $\{E_{xf}(R_t)\}_{t=0}^{\infty}$. This type of criterion is so widely accepted by economists that, often, the question is not really whether it ought to be a criterion of choice but only of what value the discount factor β should take. However, it can be argued that most dedicated environmentalists would favor the long-run average criterion as the more likely to guarantee sustainability.

Even in this relatively simple setting some conceptual complications quickly arise. What constitutes a "control" in this dynamic, stochastic setting?

The standard approach is to consider a *history* of the process at time t, namely,

$$h_t = (x_0, a_0, x_1, a_1, \ldots, a_{t-1}, x_t)$$

and to define a *decision rule* at time t as the map $f_t : h_t \to f_t(h_t, a) \in [0,1]$. Next, a *control* is defined as a sequence of decision rules, one at each time, denoted by $f := (f_0, f_1, f_2, \ldots, f_t, \ldots)$. Let F_B be the space of *all controls*. If for every t the decision rule f_t depends only on the current state at that stage, then the control f is called *Markov* or *memory-less*. Let F_M be the space of all Markov controls. If $f \in F_M$ and $\forall\, t, x, a$ the probability of choosing any action a, namely, $f_t(x,a)$, is independent of t, then f is called a *stationary* control. Let $f \in F_S$ (the set of stationary controls). Finally, if $\forall\, x, a$ the probability $f(x,a) \in \{0,1\}$, then f is called a *deterministic* control. Let $F_{\mathcal{D}}$ be the set of deterministic controls; then clearly

$$F_{\mathcal{D}} \subset F_S \subset F_M \subset F_B.$$

Next, we consider two "optimal control" problems: (i) find a (simple) control f^0 such that

$$v_d(x) := \max_f v_d(x,f) = v_d(f^0), \tag{8}$$

where $v_d(x)$ will be called the *discounted value* of the corresponding discounted MDP, and (ii) find a (simple) control $f^\star$ such that

$$v_a(x) := \max_f v_a(x,f) = v_a(f^\star), \tag{9}$$

where $v_a(x)$ will be called the *long-run average value* of the corresponding long-run MDP.

It is well known (e.g., see [32]) that both of these problems have simple solutions in deterministic controls. Namely, there exists $f^0 \in F_{\mathcal{D}}$ optimal $\forall x$ in the discounted problem, as well as $f^\star \in F_{\mathcal{D}}$ optimal $\forall x$ in the long-term average problem. Furthermore, there are "good" algorithms for computing f^0, $v_d(x)$, $f^\star$, $v_a(x)$.

In some sense the above means that, separately, with regard to the preferred performance criterion of either the economist or the environmentalist, the corresponding optimal control problem is well posed and well solved. However, it is worth considering what happens if we wish to somehow combine these two performance criteria. Surely the most benign way of attempting to do so would be to choose a *weight parameter* $\lambda \in [0,1]$ and to try to find a control f so as to maximise

$$v_\lambda(x,f) := \lambda(1-\beta)v_d(x,f) + (1-\lambda)v_a(x,f), \tag{10}$$

thereby creating the so-called *weighted reward criterion* and the corresponding *weighted reward MDP*. Clearly it follows that

$$\sup_f v_\lambda(x,f) \le \lambda(1-\beta)v_d(x) + (1-\lambda)v_a(x), \tag{11}$$

where the right-hand side constitutes the *utopian bound* for this new criterion.

Unfortunately, but, perhaps, not surprisingly, in [28] it has been shown that:

(1) The following inequalities hold:

$$\sup_{F_{\mathcal{D}}} v_\lambda(x,f) \le \sup_{F_S} v_\lambda(x,f) \le \sup_{F_{\mathcal{M}}} v_\lambda(x,f),$$

with $<$ possible in both places.

(2) In general, an optimal control for $v_\lambda(x,f)$ need not exist. However, for each λ when the discount factor β is sufficiently near 1, there exists an optimal deterministic control.

(3) Nonetheless, it is reassuring that

$$\sup_{F_{\mathcal{M}}} v_\lambda(x,f) = \sup_{F_{\mathcal{B}}} v_\lambda(x,f).$$

(4) Given any $\varepsilon > 0$ there exists an ε-optimal control f_ε such that

$$v_\lambda(x,f_\varepsilon) \ge \sup_{F_{\mathcal{M}}} v_\lambda(x,f) - \varepsilon. \tag{12}$$

(5) In particular, there exists a "switching time" $\tau(\varepsilon)$:

$$f_\varepsilon = (\underbrace{f_1, f_2, \ldots, f_{\tau(\varepsilon)}}_{\substack{\text{be "greedy"} \\ \text{for a while}}}, \underbrace{f^\star, f^\star, f^\star, \ldots}_{\substack{\text{switch to optimal} \\ \text{long-run average} \\ \text{control}}}),$$

Of course, $\tau(\varepsilon)$ depends critically on the parameter λ and the underlying data.

In a sense, properties 1–5 above capture the essence of the conflict between the "industrialist" and the "environmentalist", a conflict that is captured in—but not reconciled by—the dilemma of the "right choice" of the switching time $\tau(\varepsilon)$ in the structure of nearly optimal controls given in item 5 above. In the context of the previous motivating example of a fishery, this dilemma lies in the dual desires of wishing to profit from harvesting the species and ensuring that its population remains at sustainable levels in perpetuity.

It should be mentioned that the preceding discussion and results are conceptually similar to and consistent with results published in 1996 by Chichilnisky [12]. The latter are presented in a more general axiomatic framework but do not supply the switching structure of nearly optimal controls mentioned in item 5 above.

We conclude this section by pointing out that there is now a substantial literature dealing with weighted reward criteria (including multiple discount factors) in Markov decision processes and stochastic games (e.g., see [19] and [15]). For a survey of that interesting topic the reader is referred to [16].

Environmental Engineering: The Interplay Between Mathematical Modeling, Technology, and Stochastic Control

The main point of this section is to emphasize that environmental engineering is invariably aimed at controlling the evolution of systems that contain inherent uncertainty. There is much that could be said about the mathematical background to stochastic control and many different specialist areas—state space models, Markov decision processes, dynamic programming, control of linear systems, Kalman filtering, system identification, and adaptive control. An excellent introductory reference is the book by Kumar and Varaiya [29]. See also a more modern look at a variety of applications in the edited volume by Abed [1]. Our purpose here is not to survey the existing theory but rather to illustrate the way in which environmental engineering immediately challenges us to come to terms with managing uncertainty. Of all the twentieth-century advances in mathematical control, perhaps the most insightful and elegant is the optimality principle of dynamic programming. However, despite its theoretical elegance, a direct search implementation is still likely to be plagued by the curse of dimensionality—especially in a stochastic situation. We choose an elementary model of a solar-powered desalination plant with which to illustrate the principle.

Illustration: A Model for a Solar-Powered Desalination Unit

We now move to a more specific but in some sense still generic application. Suppose the energy collected by a solar panel can be used immediately to power a desalination unit or stored in a battery for later use. As the level of power supplied to the unit increases, the volume rate of fresh water produced also increases, but the process becomes less efficient. This is a classic case of the "law of diminishing returns". Thus, when energy r is supplied to the desalination unit at constant power for a single day, we assume the volume of fresh water produced is given by a performance function $x : [0, \infty) \mapsto [0, \infty)$, which is increasing and strictly concave with $x(0) = 0$. For convenience suppose $x' : (0, \infty) \mapsto (0, \infty)$ is continuous with $x'(r) \downarrow 0$ as $r \uparrow \infty$ and further that $x''(r) : (0, \infty) \mapsto (-\infty, 0)$ is continuous. The solar energy collected on day t will be modeled as a Markovian random variable $S_t \in S = [0, M]$ with well-defined transition probabilities $\mathbb{P}\{S_t \in [0, v] \mid S_{t-1} = u\}$ for each $(u, v) \in S \times S$. Define $F_u : S \mapsto [0, 1]$ for each $u \in S$ by setting

$$F_u(v) = \mathbb{P}\{S_t \in [0, v] \mid S_{t-1} = u\}$$

for each $v \in S$. For each allowable configuration of the state variables we wish to find an energy usage policy that maximizes the expected volume of fresh water produced by the desalination unit from day $t = n + 1$ to day $t = N$. The state variables are the index t of the day, the amount of energy $b = b_t$ in the battery at the beginning of day t, and the amount of solar energy $u_{t-1} = u$ collected on day $t - 1$. The control variable is the amount of energy $r = r_t$ we decide to use on day t. We use the Bellman principle of dynamic programming [6, 7, 8] to find a stochastic control policy that maximizes the expected total volume of fresh water produced. We show in one special case that a long-term (infinite horizon) optimal strategy uses the same amount of energy each day. Since a long-term strategy must be sustainable, it is intuitively obvious in this case that the energy used each day must be equal to the average solar energy collected.

This model was first formulated to find strategies that maximized the distance traveled by solar-powered racing cars in a given time period. The initial studies [22, 24] treated the problem as a deterministic control problem where the solar radiation was known in advance. These studies evolved from closely related work on optimal train control. For a recent reference see [26]. The solar car problem was later reformulated as a stochastic control problem [25] in which the daily solar radiation evolved according to a known Markov process. The Markovian nature of the process underlying the evolution of solar radiation is well documented. There have been models for discrete space [2, 4, 23] and for continuous space [3, 9]. More recently, Boland [10] has described the similarity of the Markov structure in solar radiation persistence on two time scales, daily and hourly. The cited treatises overcome the seasonality of solar radiation time series in various different ways. In [2], separate Markov transition matrices for each month are constructed to forecast levels of solar radiation on a daily time scale. On the other hand, [4, 10, 31] make extensive use of spectral analysis to identify significant embedded cyclical behavior in the time series and to model that part as a deterministic component using Fourier series. In the present work any seasonal component has been ignored. This is a reasonable assumption if the time period is relatively short. In general we would need to assume that the seasonal component has been identified and removed, in which case the analysis would focus on the remaining stochastic component.

The Mathematical Model

Define value functions $W_N[u] : [0, \infty) \mapsto [0, \infty)$ for each $u \in S$ on day N, the final day of the given period, as the expected volume of fresh water produced on day N given that the energy collected on day $N-1$ was $S_{N-1} = u$ and given that all energy will be used by the end of day N. If b is the energy in the battery at the beginning of the final day, then

$$W_N[u](b) = \int_S x(b + v_N)dF_u(v_N). \tag{13}$$

In general we wish to define value functions $W_t[u] : [0, \infty) \mapsto [0, \infty)$ for each $u \in S$ and each $t = n, n+1, \dots, N$ given that the solar energy collected on day $t-1$ was $u_{t-1} = u$. We begin by defining auxiliary value functions $w_t[u] : [0, \infty) \times [0, \infty) \mapsto [0, \infty)$ for each $u \in S$ on day t given that the solar energy collected on day $t-1$ was $u_{t-1} = u$ and the energy in the battery at the beginning of day t is b. Thus we define

$$w_t[u](b, r) = x(r) + \int_S W_{t+1}[v_t](b + v_t - r)dF_u(v_t). \tag{14}$$

The auxiliary value function determines the expected volume of fresh water produced for every possible level of energy use on day t. The idea now is that the true value function should give the expected volume of fresh water produced given that we make an optimal decision about the level of energy use on day t. Thus the value $W_t[u](b)$ is obtained by maximizing $w_t[u](b, r)$ over all possible values of r. The *optimal control policy* is obtained by solving the following mathematical problem. For each state (t, u, b) with $t < N$ find $r = \varphi_t[u](b)$ such that

$$W_t[u](b) = \max_r w_t[u](b, r).$$

We have used $t = N$ to denote the final time. Thus, in general, we will start at $t = n$ where $n < N$ and where $q = N - n$ is the duration of the operation. In the sequel it is convenient to allow the length of the time interval to increase without bound. Thus we allow $n = N - q$ where q increases without bound.

A Recursive Equation for the Optimal Controls

For each $t < N$ let $r = \varphi_t[u](b)$ denote the energy usage that gives the maximum of the auxiliary value function $w_t[u](b, r)$ over all r. A necessary condition can be found by setting the partial derivative with respect to r equal to zero. The following results are established in [25]. For each $u \in S$ the value $W_{N-1}[u](b)$ is given by the formula

$$\begin{aligned} W_{N-1}[u](b) &= x(\varphi_{N-1}[u](b)) \\ &\quad + \int_S \left[\int_S x([b + v_{N-1} - \varphi_{N-1}[u](b)] + v_N)dF_{v_{N-1}}(v_N) \right] dF_u(v_{N-1}) \end{aligned} \tag{15}$$

and the optimal energy consumption $\varphi_{N-1}[u](b)$ satisfies the equation

$$\begin{aligned} x'(\varphi_{N-1}[u](b)) &= \int_S \left[\int_S x'([b + v_{N-1} - \varphi_{N-1}[u](b)] + v_N)dF_{v_{N-1}}(v_N) \right] dF_u(v_{N-1}). \end{aligned} \tag{16}$$

In general, for each integer $t < N-1$ and each $u \in S$, the value $W_t[u](b)$ is given by the formula

$$\begin{aligned} W_t[u](b) &= x(\varphi_t[u](b)) \\ &\quad + \int_S W_{t+1}[v_t](b + v_t - \varphi_t[u](b))dF_u(v_t) \end{aligned} \tag{17}$$

and the optimal energy consumption $\varphi_t[u](b)$ satisfies the recursive equation

$$\begin{aligned} x'(\varphi_t[u](b)) &= \int_S x'(\varphi_{t+1}[v_t](b + v_t - \varphi_t[u](b)))dF_u(v_t). \end{aligned} \tag{18}$$

A Simple Special Case

Consider the case where the performance function is $x(r) = a[1 - e^{-kr}]$ for some positive constants a and k and the function $F_u : [0, M] \mapsto [0, 1]$ is given by

$$F_u(v) = v/M,$$

where M is the maximum value of the solar irradiance. This is a truly stochastic situation but is simplified by the uniformity of the probability distribution. We need an elementary result before we begin the solution. Let $\theta > 0$ be a constant. The function $f : (0, \infty) \mapsto (0, \infty)$ defined by the formula

$$f(r) = \left[\frac{\sinh r\theta}{r\theta} \right]^{1/r} \tag{19}$$

is strictly increasing. At the first stage equation (16) can be rewritten explicitly in the form

$$\begin{aligned} ak \exp(-k\varphi_{N-1}[u](b)) &= \frac{ak \exp(-kb + k\varphi_{N-1}[u](b))}{M^2} \\ &\quad \times \int_0^M \exp(-kv_1)dv_1 \int_0^M \exp(-kv_0)dv_0 \\ &= \frac{a \exp(-kb + k\varphi_{N-1}[u](b))}{kM^2} \\ &\quad \times [1 - \exp(-kM)]^2. \end{aligned}$$

If we define $\theta = kM$ and if f is the function defined in equation (19), then some elementary algebra gives

$$\varphi_{N-1}[u](b) = b/2 + M/2 - c_1,$$

where $c_1 = (1/(2k)) \ln f(1/2)$. If $c_q = (q/(q+1))[c_{q-1} + (1/(2qk)) \ln f(1/(2q))]$ for each $q > 1$ and we make the inductive assumption that

$$\varphi_{N-q+1}[u](b) = b/q + M/2 - c_{q-1},$$

then equation (18) and some elementary algebra can be used to show that

$$\varphi_{N-q}[u](b) = b/(q+1) + M/2 - c_q.$$

Thus the formula is true for all $q > 1$. From the formula it follows that $c_q < c_{q-1}$ if and only if $c_{q-1} > (1/(2k)) \ln f(1/2q)$. Once again a simple inductive argument shows that this is true. Since $\{c_q\}$ is positive and strictly decreasing $c_q \downarrow c \geq 0$ as $q \uparrow \infty$. By applying the recursive formula one can show by induction that

$$c_q = (2/(q+1))c_1 + (1/2k(q+1)) \times [\ln f(1/4) + \cdots + \ln f(1/2q)].$$

For each $\epsilon > 0$ choose $Q = Q(\epsilon)$ such that $(1/2k) \ln f(1/2q) < \epsilon$ for all $q > Q$. It follows that

$$\begin{aligned} c_q &= (2/(q+1))c_1 + (1/2k(q+1)) \\ &\quad \times [\ln f(1/4) + \cdots + \ln f(1/2Q)] \\ &\quad + (1/2k(q+1)) \\ &\quad \times [\ln f(1/2(Q+1)) + \cdots + \ln f(1/2q)] \\ &\leq (2/(q+1))c_1 + (1/2k(q+1)) \\ &\quad \times [\ln f(1/4) + \cdots + \ln f(1/2Q)] \\ &\quad + [q - (Q+1)]\epsilon/(q+1). \end{aligned}$$

By taking the limit as $q \uparrow \infty$ we see that $c \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $c = 0$ and that $\varphi_{N-q}[u](b) \to M/2 = \bar{s}$ as $q \uparrow \infty$ for all $u \in [0, M]$.

An Open Question

The elementary example suggests that long-term strategies may exist and that such strategies may be independent of the present state. For an ergodic system we believe that the limit $\varphi[u](b) = \lim_{q\to\infty} \varphi_{N-q}[u](b)$ exists and is well defined. Furthermore we conjecture that in such cases $\varphi[u](b) = \bar{s}$ for all (u, b). This formula is certainly true for the example considered above. Is it true in general?

Conclusions

In such a short, expository article, it is impossible to do justice to the fast-exploding research field aimed at developing mathematical models and techniques that adequately deal with the problems of uncertainty encountered in environmental decision making. There are many fast-developing branches of mathematics that contain concepts, techniques, theorems, and algorithms that have much to offer in this area. The latter include the theories of signal processing, stochastic and robust programming, large deviation theory, and the theory of singular perturbations of operators. However, a special mention is made here of viability theory, pioneered by Aubin in [5], as its central concept of a "viability kernel" offers a very general perspective of managing the natural environment without pushing it to potentially unacceptable regions of an appropriate state space. We also refer interested readers to a recent collection of papers on this topic, contained in [18].

In these concluding remarks we recall that Rachel Carson's *Silent Spring* [11] is widely credited with helping launch the environmental movement. This book inspired widespread public concern with pesticides and pollution of the environment. *Silent Spring* facilitated the ban of the pesticide DDT in 1972 in the United States. The book documented detrimental effects of pesticides on the environment, particularly on birds. Carson said that DDT had been found to cause thinner egg shells and to result in reproductive problems and death. She also accused the chemical industry of spreading disinformation and public officials of accepting industry claims uncritically. Most recently, *Silent Spring* was named one of the twenty-five greatest science books of all time by the editors of *Discover* magazine.

Most environmentalists believe that, early in our development, the human race lived within the environment and adapted to it, but as the industrial revolution turned into a technological revolution we outgrew our environment and began to change it. On June 23, 1886, the *New York Times* reported that the Reverend J. P. Newman delivered an address before the literary societies of St. John College. The address was titled "The March of Civilization" and was reported thus: "All civilization," he said, "has been abnormal in that some one element has tyrannized over the others. A perfect civilization is that wherein all the elements essential to individual development and social progress blend harmoniously." But he warned that "the master thought of our civilization is the power of wealth; to the prosperity of our commerce we subordinate education, morality, Government, and religion". One cannot know precisely what was in the reverend gentleman's mind at the time, but 123 years later his warning has passed into history, and the march has become a stampede that threatens everything in its path.

On a more positive note, we should observe that at the dawn of the twenty-first century, modern societies are beginning to focus on slowing and, ultimately, stopping at least those aspects of the above-mentioned stampede that threaten our life support systems. As with any great endeavor of civilization, mathematics and mathematicians have an important part to play in this global effort.

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References

[1] E. H. Abed, *Advances in Control, Communication Networks, and Transportation Systems*, in honor of Pravin Varaiya, Birkhäuser, 2005.

[2] R. J. Aguiar, M. Collares-Pereira, and J. P. Conde, Simple procedure for generating sequences of daily radiation values using a library of Markov transition matrices, *Solar Energy* **40** (1988), 269–279.

[3] R. Aguiar and M. Collares-Pereira, Gaussian model for generating synthetic hourly radiation, *Solar Energy* **49** (1992), 167–174.

[4] U. Amato, A. Andretta, B. Bartoli, B. Coluzzi, V. Cuomo, F. Fontana, and C. Serio, Markov processes and Fourier analysis as a tool to describe and simulate daily solar irradiance, *Solar Energy* **37** (1986), 179–194.

[5] J.-P. Aubin, *Dynamical Economic Theory—A Viability Approach*, Springer-Verlag, Boston, 1997.

[6] Richard Bellman, On the theory of dynamic programming, *Proceedings of the National Academy of Sciences* **38** (1952), 716–719.

[7] ______, Some applications of the theory of dynamic programming, *Journal of the Operations Research Society of America* **2** (1954), 275–288.

[8] ______, The theory of dynamic programming (Invited address at Laramie, Wyoming), *Bulletin of the American Mathematical Society* **60** (1954), 503–515.

[9] J. Boland, Time-series analysis of climatic variables, *Solar Energy* **55**(5) (1995), 377–388.

[10] ______, Time series and statistical spectre of solar radiation. In: V. Badescu (ed.), *Recent Advances in Solar Radiation modeling*, Springer-Verlag, 2008, pp. 283–312.

[11] R. L. Carson, *Silent Spring*, Houghton Mifflin Company, New York, 1962.

[12] G. Chichilnisky, An axiomatic approach to sustainable development, *Social Choice and Welfare* **13** (1996), 231–257.

[13] I. Curiel, J. A. Filar, and R. Zapert, Relative contribution of the enhanced greenhouse effect on the coastal changes in Louisiana. In: *Modeling Environmental Policy*, (W. E. Martin and L. A. McDonald, eds.), Kluwer, New York, 1997, pp. 161–184.

[14] P. Embrechts, C. Klüppelberg, and T. Mikosch, *modeling Extremal Events, Applications of Mathematics: Stochastic modeling and Applied Probability*, Springer, Berlin, 1997.

[15] E. A. Feinberg and A. Schwartz, Constrained dynamic programming with two discount factors: Applications and an algorithm. *IEEE Transactions on Automatic Control* **44** (1999), 628–631.

[16] ______, Mixed criteria. In: *Handbook of Markov Decision Processes: Methods and Applications*, Kluwer, 2002, pp. 209–230.

[17] J. A. Filar, Mathematical models. In: *Knowledge for Sustainable Systems. An Insight into the Encyclopedia of Life Support Systems*, Vol. 1., pp. 339–354, UNESCO Publishing-Eolss Publishers, Paris, France, Oxford, UK, 2002.

[18] J. A. Filar and A. Haurie, *Uncertainty and Environmental Decision Making: A Handbook of Research and Best Practice Series*, International Series in Operations Research and Management Science, Vol. 138, Springer, New York, 2009.

[19] J. A. Filar and O. J. Vrieze, Weighted reward criteria in competitive Markov decision processes, *Zeitschrift fr O.R.* **36** (1992), 343–358.

[20] ______, *Competitive Markov Decision Processes*, Springer-Verlag, New York, 1997.

[21] R. A. Fisher and T. H. C. Tippett, Limiting forms of the frequency distribution of the largest or smallest member of a sample, *Proc. Cambridge Philos. Soc.* **24** (1928), 180–190.

[22] D. J. Gates and M. R. Westcott, Solar cars and variational problems equivalent to shortest paths, *SIAM Journal on Control and Optimization* **34**(2) (1996), 428–436.

[23] J. M. Gordon and T. A. Reddy, Time series analysis of daily horizontal solar radiation, *Solar Energy* **44** (1988), 215–226.

[24] P. G. Howlett and P. J. Pudney, An optimal driving strategy for a solar powered car on an undulating road, *Dynamics of Continuous, Discrete and Impulsive Systems* **4** (1998), 553–567.

[25] P. G. Howlett, A Markov model for the stochastic optimal control of a solar-powered car. In: *Optimization Methods and Applications*, (X. Yang, K. L. Teo, and L. Caccetta, eds.), Kluwer, 2001, pp. 331–341.

[26] Phil Howlett, Peter Pudney, and Xuan Vu, Local energy minimization in optimal train control, *Automatica* **45**(11) (2009), 2692–2698.

[27] J. G. Kemeny and K. L. Snell, *Finite Markov Chains*, Van Nostrand, Princeton, New Jersey, 1960.

[28] D. Krass, J. A. Filar, and S. Sinha, A weighted Markov decision process, *Operations Research* **40** (1992), 1180–1187.

[29] P. R. Kumar and Pravin Varaiya, *Stochastic Systems: Estimation, Identification and Adaptive Control*, Information and System Sciences Series, (Thomas Kailath, ed.), Prentice Hall, 1986.

[30] G. Latouche and V. Ramaswami, *Introduction to Matrix Geometric Methods in Stochastic Modeling*, ASA-SIAM Series on Statistics and Applied Probability, SIAM, Philadelphia, 1999.

[31] W. F. Phillips, Harmonic analysis of climatic data, *Solar Energy* **32** (1984), 319–328.

[32] M. L. Puterman, *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, Wiley, New York, 1994.

[33] R. T. Rockafellar and S. Uryasev, Conditional value-at-risk for general loss distributions, *J. Banking and Finance* **26** (2002), 1443–1471.