

# Notices

of the American Mathematical Society

March 2011

Volume 58, Number 3

Special Issue  
on Education

Introduction  
page 367

Memories of  
Martin Gardner  
page 418

About the Cover:  
Martin Gardner  
1914-2010  
(see page 475)

MARTIN GARDNER

10 EUCLID AVENUE  
HASTINGS-ON-HUDSON  
NEW YORK 10706

Dear Don:

8 Jan 76

That line from C. S. Lewis is perfect to use as a quote at the top of a column on everything, and I really appreciate your passing it along. I will check the source, to make sure I have it right, and see if Lewis has anything else to say on the topic. I have not yet written the column, so it will have to run after the April issue. What's held me up is that ~~now~~ I'm trying to <sup>find</sup> an evaluation, by a logician, of G. Spencer Brown's book, Laws of Form. This has become very popular with counter-culture types who are into occultism, and has influenced (I'm told) John ~~Wheeler~~ Wheeler. In his big book, Gravity, Wheeler puts our space into what he calls superspace, and speculates on the most basic physical laws which operate on superspace. He comes to the (to me) surprising conclusion that the rock-bottom laws are the laws of the propositional calculus! (Boolean set theory would have been a better term for what he means.) Anyway, I'm told this idea comes from his reading of Brown's book. I found the book trivial (he simply starts with a distinction between a symbol and nothing, and slowly builds up Boolean algebra, using his private notation, but accomplishing nothing that you can't do with binary notation. The notion that you can describe the entire phenomenal world by positing the relation of likeness and difference is ~~maintained~~ the central idea of Carnap's Aufbau. But Brown's book won high praise from B. Russell, a rave review in British J. of Philos. of Science, so I want to be cautious before I attack it. I've written to Alonzo Church about it, but he hasn't replied. (He may be ill. I assume he's still living).

But enough of that. I have page proofs of Conway's forthcoming On Numbers and Games (supposedly on sale in April). If you would like to see the first section (he calls it the Zeroeth Section) on his number construction system, let me know and I'll send it to you. You may have already seen it.

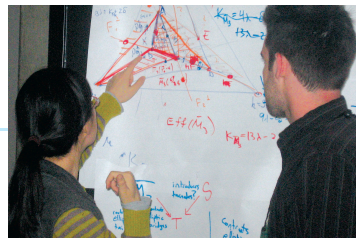
I have been wasting many hours on a combinatorial problem that apparently is enormously difficult, although seemingly simple. Can 42 "canonical bricks" ( $1 \times 2 \times 4$ ) be put into a  $7 \times 7 \times 7$  box? My column on brick packing runs in Feb. Here's a quick background on this particular problem. It

\* If you know of any reviews of Laws of Form, aside from the one I mention, I'd like the references. Bantam has a paperback ed. of the book.

all best,

Martin

# AMS-Simons Travel Grants



The AMS is launching a new program, the AMS-Simons Travel Grants, with support provided by the Simons Foundation. Each grant provides an early career mathematician with \$2,000 per year for two years to reimburse travel expenses related to research. Sixty new awards will be made in each of the next three years (2011, 2012, and 2013). Individuals who are not more than four years past the completion of the PhD are eligible. The department of the awardee will also receive a small amount of funding to help enhance its research atmosphere.

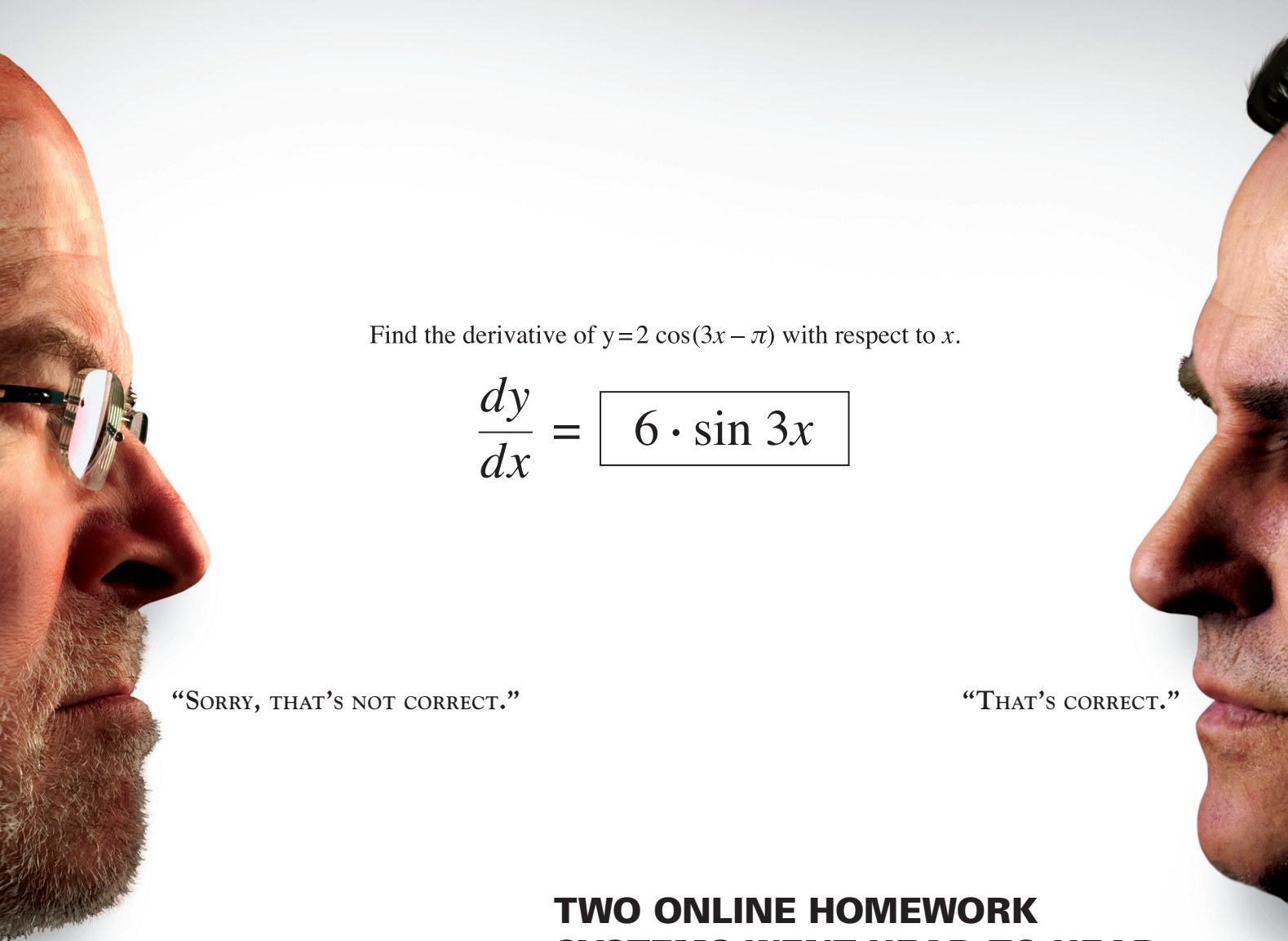
**The deadline for 2011 applications is March 31, 2011.**

Applicants must be located in the United States or be U.S. citizens. For complete details of eligibility and application instructions, visit:

[www.ams.org/programs/travel-grants/AMS-SimonsTG](http://www.ams.org/programs/travel-grants/AMS-SimonsTG)







Find the derivative of  $y = 2 \cos(3x - \pi)$  with respect to  $x$ .

$$\frac{dy}{dx} = \boxed{6 \cdot \sin 3x}$$

“SORRY, THAT’S NOT CORRECT.”

“THAT’S CORRECT.”

## TWO ONLINE HOMEWORK SYSTEMS WENT HEAD TO HEAD. ONLY ONE MADE THE GRADE.

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So, for those of you who thought that other system was the right answer for math, we respectfully say, “Sorry, that’s not correct.”

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*Christian Science Monitor*

**"Journeys to the Distant Fields of Prime"**  
*The New York Times*

**"He's Too Good at Math"**  
*Slate Magazine*

**"Sensor Sensibility"**  
*Science News*

**"Into the Fold"**  
*Smithsonian*

**"Fast Routing in Road Networks with Transit Nodes"**  
*Science*

**"Puzzle Me This"**  
*Chronicle of Higher Education*

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*Nature*

**"Students Learn to Rhythmic Beat of Rap"**  
*NCTimes.com*

**"The Science of Steadying a Wobbly Table"**  
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*The Washington Post*

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# Notices

of the American Mathematical Society

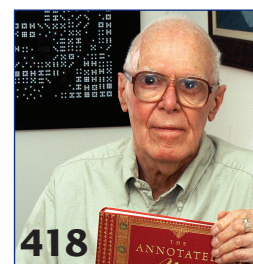
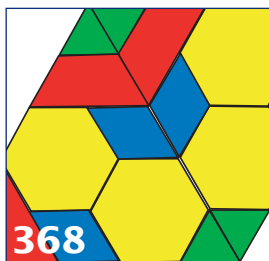
March 2011

## Communications

- 418** Memories of Martin Gardner  
*Steven G. Krantz,*  
*Coordinating Editor*
- 423** WHAT IS...Equivariant Cohomology?  
*Loring W. Tu*
- 432** *Doceamus*: Using Mathematics to Improve Fluid Intelligence  
*Vali Siadat*
- 434** Nefarious Numbers  
*Douglas N. Arnold and*  
*Kristine K. Fowler*
- 438** 2010–2011 Faculty Salaries Report  
*Richard Cleary, James W. Maxwell, and Colleen Rose*
- 444** Interview with Abel Laureate John Tate  
*Martin Raussen and*  
*Christian Skau*
- 453** The Mathematical Work of the 2010 Fields Medalists  
*Thomas C. Hales, Benjamin Weiss, Wendelin Werner, and Luigi Ambrosio*
- 469** On the Work of Louis Nirenberg  
*Simon Donaldson*
- 473** Presidential Views: Interview with Eric Friedlander  
*Allyn Jackson*

## Commentary

- 365** Opinion: Mathematical Community  
*John Swallow*
- 366** Letters to the Editor
- 427** The Black Swan: The Impact of the Highly Improbable—A Book Review  
*Reviewed by David Aldous*



## Features

The theme this month is mathematics education. This is a complex subject with many dimensions. The articles herein explore different means by which research mathematicians can get involved in the education process. The articles are written by seasoned veterans who can speak in detail of first-hand experiences. Associate Editor Mark Saul has assembled a valuable testimony to ongoing efforts to improve the flow of knowledge.

—Steven G. Krantz, *Editor*

- 367** Introduction  
*Mark Saul*
- 368** The Community of Math Teachers, from Elementary School to Graduate School  
*Sybilla Beckmann*
- 372** The Mis-Education of Mathematics Teachers  
*H. Wu*
- 385** A Mathematician Writes for High Schools  
*Dan Fendel*
- 389** Strengthening the Mathematical Content Knowledge of Middle and Secondary Mathematics Teachers  
*Ira J. Papick*
- 394** A Mathematician–Mathematics Educator Partnership to Teach Teachers  
*Ruth M. Heaton and W. James Lewis*
- 401** Philosophy, Math Research, Math Ed Research, K–16 Education, and the Civil Rights Movement: A Synthesis  
*Ed Dubinsky and Robert P. Moses*
- 410** More Than a System: What We Can Learn from the International Mathematical Olympiad  
*Mark Saul*

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of the American Mathematical Society

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## Departments

About the Cover . . . . .	475
Mathematics People . . . . .	476
<i>Lewis Awarded CRM-Fields-PIMS Prize, 2010 ICTP Prize Awarded, Gualtieri and Tang Awarded Lichnerowicz Prize, Coates Receives Leverhulme Prize, Montalban Awarded Packard Fellowship, Dillon Awarded Marshall Sherfield Fellowship, 2010 Siemens Competition.</i>	
Mathematics Opportunities . . . . .	479
<i>AMS-Simons Travel Grants for Early-Career Mathematicians, Call for Nominations for Prizes of the Academy of Sciences for the Developing World, News from the CRM.</i>	
Inside the AMS . . . . .	480
<i>From the AMS Public Awareness Office, Deaths of AMS Members.</i>	
Reference and Book List . . . . .	481
Mathematics Calendar . . . . .	488
New Publications Offered by the AMS . . . . .	498
Classified Advertisements . . . . .	507
Meetings and Conferences of the AMS . . . . .	510
Meetings and Conferences Table of Contents . . . . .	519

## From the AMS Secretary

**Call for Nominations for the Position of AMS Secretary . . . . 485**

**Call for Nominations for AMS Award for Mathematics Programs That Make a Difference . . . . . 486**

I thank Randi D. Ruden for her splendid editorial work, and for helping to assemble this issue. She is essential to everything that I do.

—Steven G. Krantz  
Editor



# Mathematical Community

## Only we can tell the story—and it's time we did

The common image of a mathematician is of someone isolated and working alone: someone without a community. Some six generations ago, such perception was reality. What would become the AMS was in 1889 an organization of only sixteen people. Not surprisingly, books about mathematicians have long emphasized individuals, their work, and sometimes their wonderful idiosyncrasies.

That small community is now long past. The AMS alone counts over 30,000 members. Mathematicians meet often, through conferences and visits. We see our colleagues more regularly and know them better. More and more of us collaborate on projects and get together for walks or hikes at institutes.

In the United States, Project NExT has played a signal role in amplifying this wave, helping a new generation of mathematicians create their own community from the very beginning of their careers. And the growth of research experiences for undergraduates, as well as the AMS's own Mathematics Research Communities program, offer the promise of creating a community larger still.

It's time that books and articles about mathematicians reflect mathematical society. The stories of our community—the peculiar predispositions we share, what distinguishes us from other academics, or other scientists—haven't been told.

The writers among us have continued to focus on individual mathematicians: esteemed researchers, or authors of definitive textbooks. Their stories are certainly valuable, and I wouldn't want to lose them. But they don't tell us about our community, about what it's like, generally speaking, to be a contemporary mathematician in contemporary mathematical society.

We need stories about our community for several reasons.

First, there's more public interest than ever in what it means to be a mathematician. Playwrights and screenwriters have sensed this for some time; David Auburn's *Proof*, Tanya Barfield's *Blue Door*, Tom Stoppard's *Arcadia*, and of course Nicolas Falacci and Cheryl Heuton's *NUMB3RS* confirm that we're long past having Ted Kaczynski define for the public what it means to be a mathematician.

Yet these stories are not ours, and they serve purposes other than accurate representation. Alice Silverberg drives this point home in her 2006 *MAA FOCUS* article "Alice in NUMB3Rland" [26 (2006), no. 8, pp. 12–13]. Our own stories, authentic and insightful, will better meet the public's interest—and likely create more empathy for mathematicians.

Second, our stories would benefit the profession, helping us recruit those for whom life as a mathematician would be desirable if only they knew what it would be like. Graduate school in mathematics is certainly not for tourists—it's just too hard—but we can do more to tell potential graduate students what their future, beyond teaching classes and doing research, is likely to be. Taking students to undergraduate conferences, introducing them to visiting speakers, and advising them one-on-one: all these are good. But offering extended portraits of mathematical communities, written by mathematicians, would be even better.

Finally, as mathematicians, we share—and continue to create—our own mathematical culture, and we should communicate that culture as a means of consciously shaping it. We all know anecdotes about mathematicians, and we can use these as starting points for insights into who we are and where we're going. By finding patterns and disseminating them, we'll begin to involve the community at large in exploring, developing, and even celebrating mathematical culture.

What do we need, then? We need mathematicians willing to pen a few words about what they observe when they sit down with other mathematicians, and to compare us to other groups, of faculty or of researchers.

These observations don't need to be scientific. It's not as if we're considering the theoretical underpinnings of a sociology of mathematicians. And we should let go of any notion that our observations will all agree, as proof of some essential consistency in mathematical society. But short observational pieces, whether humorous or serious, will inspire us to think more deeply about ourselves.

Potential topics abound. Does mathematics attract lovers of the outdoors? Why the emphasis, after all, on places to walk or hike at mathematical institutes? Or, do mathematicians approach travel to other countries differently from the way other academics do—with more familiarity, or predispositions? Are we, as a group, truly more eccentric than others on campus? Are mathematicians at the forefront of collaboration, with the advent of the Polymath Projects and Math Overflow?

I wonder: are others as interested in this project as I am? I hope so, and I'd be interested in hearing from them. Essays on mathematical life and society will be fascinating and meaningful, both to us and to others outside mathematics—and they'll help us create an even richer mathematical culture.

—John Swallow  
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joswallow@davidson.edu

### Affordable Higher Education

Current debates on affordability of higher education focus almost entirely on financial matters. However, there is an important academic component that we cannot ignore or disregard: peculiar pedagogical methods employed by instructors and administrators in their efforts to make undergraduate education affordable for all who want to have a diploma. It is an open secret in academe that these methods include the following:

1. Sample or practice tests and review sessions.

On Monday students receive a “sample” set of questions and problems. On Tuesday during the review session the lecturer or teaching assistant gives a complete analysis, including solutions, of the problems. On Wednesday the same problems with minor changes—perhaps the number 120 is changed to 150—are given as test problems. Students love this; “Sample exams are how we roll”.

2. Take-home midterm and final tests.

The chairman could request that the instructor make the final exam a take-home final, “in order to give students a thorough opportunity to demonstrate their knowledge of the material and what they’ve learned in the class”.

3. Curving.

This is a simple way to push grades up; D+ becomes B–, etc. When the instructor does not cave in and does not curve, students might lecture the instructor: “Curving the test occurs after the test, you calculate the class average and adjust the grades based on that. When students are still doing so poorly on exams even with twenty possible bonus points, usually a teacher curves. All of my math and science teachers have curved. In fact, in past math classes here, they have adjusted the grading scale previous to any scores being submitted, as well as curved all the test scores. Otherwise the majority of the class doesn’t do well.”

4. Self-evaluation by students.

Who knows better than the student himself how good or bad his

performance is? So would it not be quite natural to ask him which grade he deserves and for the instructor to simply record it in the grade roster?

5. Grade roster adjustment or rosters of nonexistent classes.

Maybe manipulations of grade rosters are rare and extreme; they violate rules and regulations. However, as a top administrator of a major midwestern university explained once: we should be flexible and commend faculty who assign phony grades if it helps to guarantee high quality and integrity of undergraduate education.

This list is not exhaustive. Educators and educrats have many sophisticated and innovative ways to level the playing field and guarantee that any student—independently of his/her competence or skills—can sail smoothly from admission to graduation and get a diploma that presumably certifies excellent knowledge and high professional qualifications. Does it?

—Boris Mityagin

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(Received November 29, 2010)

### Topical Bias and Journal Backlog

I found the article “Topical bias in generalist mathematics journals” quite enlightening. Although the author does point out the self-perpetuating nature of the bias and the influence of editorial boards and topic culture, he neglects to mention what is perhaps the most obvious observation: that the most negatively biased subjects are those which have a more “applied” orientation and have significant overlap with other fields (e.g., computer science, physics, biology). Many of these topics are relatively new (e.g., computer science, information theory, game theory), and most are very fast moving. Consequently, the long lag between submission and publication in most generalist mathematics journals (e.g., seventeen months for *Proceedings* of the AMS, twenty-eight months for *Transactions* of the AMS (source: *AMS Notices*, Oct. 2009, p. 1316)) is

not acceptable, and authors prefer journals with more rapid publication cycles—largely specialized/electronic journals. It seems to me that the trend is getting worse, i.e., longer lag times rather than shorter, and this deficiency must be addressed if journals wish to broaden the scope of their publications to include more papers in these areas.

—Kevin Ford

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(Received December 1, 2010)

### Educational Failures

In “Commentary on Education Legislation: A Mathematical Perspective” (January 2011), Matthew Pascal and Mary Gray mischaracterized me as a “back to basics” advocate” and wrote that I agreed with Cathy Seeley “that NCLB provided little incentive for engaging students in learning more.” This was offered as counterpoint to Pat Connell Ross’s assertion that “if teachers are not teaching better, that’s not NCLB’s fault.”

I do not disagree with Ross’s statement and, while recognizing the essential role of technical fluency in sound mathematics education, I do not advocate “back to basics”. The NCLB legislation was a coercive, blunt instrument whose main thrust was to demand that more students achieve “proficiency” as measured by state tests. The details, including the definition of “proficiency”, the quality of the assessments, and the standards on which they were based, were left to the states.

No agency had greater influence on all of these than the National Council of Teachers of Mathematics which bears far greater responsibility than NCLB for poor results.

—David Klein

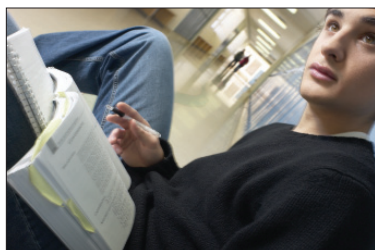
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(Received December 15, 2010)



Welcome to this special theme issue of the *AMS Notices*, which highlights the many possible roles of mathematicians in precollege education.



Sybilla Beckman's piece offers a rare and valuable frog's-eye view of a mathematician who has spent significant time and effort working directly in an elementary school classroom. Dan Fendel discusses his work, slightly removed from the classroom, developing a high school curriculum for the Interactive Mathematics Program. Ed Dubinsky and Robert Moses write about curriculum development and also about its implementation, making important connections between mathematics teaching and the civil rights movement. Jim Lewis and Ruth Heaton, as well as Ira Papick, write about teacher preparation, and particularly about professors of education and mathematics collaborating to develop exemplary practices. Hung-Hsi Wu's piece also addresses teacher preparation. It offers valuable insights into the role that research mathematicians can play in certifying teacher content knowledge and gives an historical overview of the development of teacher education policy. My own contribution on the International Mathematical Olympiad describes a venue for mathematicians that is less widely known in the United States than in other countries. It describes work with those precollege students who are most likely to become our next generation of mathematicians.

So what is missing? Well, there is a dark side to the work of mathematicians in education. The landscape includes instances of squabbling, on intellectual, political, and even financial levels, over who knows best. Such dissension is not so much a role as a rather regrettable mode of communication, one that we have avoided in choosing articles for this issue.

Likewise absent from these essays is the role of the mathematician as corrector of errors in textbooks. That's too easy. There are errors in virtually any textbook, on any level. Of course errors are bad, and of course mathematicians can help by making sure that the mathematics in textbooks is correct. But the mathematics must also be appropriate—the *right* material, not just the correct material. So this role of "refining" the mathematics is one that belongs to the entire mathematics community, and not solely to the mathematician.

This last point is perhaps the most important one to be made here. Our community has a wide span. It includes not only researchers but also mathematics educators, policy setters, and teacher trainers. It includes classroom teachers of mathematics, some of who teach mathematics exclusively and some who teach mathematics within a context of wider responsibilities. And it includes consumers of mathematics: scientists, engineers, medical personnel, and lately also librarians and bankers. The community expands as our understanding of how to use mathematics in our lives expands.

Each of these smaller communities has a contribution—and each thinks its contribution is central. But in fact the task of education is so difficult and subtle that the expertise will have to remain distributed. The mathematics community must nevertheless find ways to synthesize the various contributions. The authors of these articles have begun this work.

—Mark Saul  
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# The Community of Math Teachers, from Elementary School to Graduate School

Sybilla Beckmann

**W**hy should mathematicians be interested and involved in pre-K-12 mathematics education? What are the benefits of mathematicians working with school teachers and mathematics educators?<sup>1</sup> I will answer these questions from my perspective of research mathematician who became interested in mathematics education, wrote a book for prospective elementary teachers, and taught sixth-grade math a few years ago. I think my answers may surprise you because they would have surprised me not long ago.

## It's Interesting!

If you had told me twenty-five years ago, when I was in graduate school studying arithmetic geometry, that my work would shift toward improving pre-K-12 mathematics education, I would have told you that you were crazy. Sure, I would have said, that is

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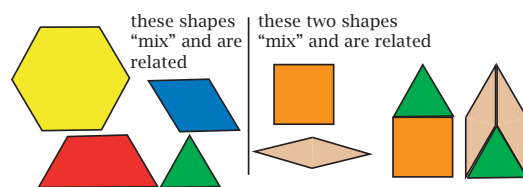
*Sybilla Beckmann is professor of mathematics at the University of Georgia. Her email address is sybilla@math.uga.edu.*

<sup>1</sup>A note on terminology: By "mathematician" I mean individuals in mathematics departments at colleges and universities who teach mathematics courses and who have done research in math. By "teacher" I mean individuals who teach within pre-kindergarten through grade 12. By "mathematics educator" I mean individuals who teach mathematics methods courses, professional development seminars, or workshops or who supervise or coordinate math teaching or curricula in schools and who have done research in mathematics education. I acknowledge that these categories are neither exhaustive among mathematics professionals nor mutually exclusive, that the descriptions of these categories should be viewed as somewhat fuzzy and approximate, and that the names of these categories are not fully descriptive.

important work, it's probably hard, and somebody needs to do it, but it doesn't sound very interesting. Much to my surprise, this is the work I am now fully engaged in. It's hard, and I believe what I'm doing is useful to improving education, but most surprising of all is how interesting the work is.

Yes, I find it interesting to work on improving pre-K-12 math! And in retrospect, it's easy to see how it could be interesting. Math at every level is beautiful and has a wonderful mixture of intricacy, big truths, and surprising connections. Even preschool math is no exception.

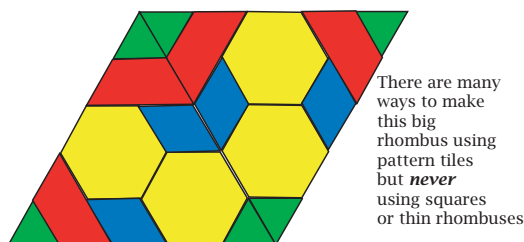
Consider this connection between preschool math and number theory. Young children play with pattern tile sets that consist of the shapes shown in Figure 1. Playing with these shapes,



**Figure 1.** Pattern tiles that young children play with.

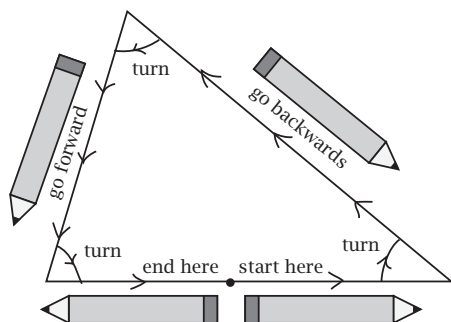
children discover that some of them can be put together to make others (e.g., three triangles fit together to make the trapezoid) but that the squares and thin rhombuses are different. In fact, shapes that are made without the squares and thin rhombuses, such as the shape in Figure 2, can never be made in a different way using the squares or thin rhombuses. Why not? Because the square root of three is irrational! The square and thin rhombus have rational area (in terms of square inches), but the other shapes' areas are rational multiples of the square root of three.





**Figure 2. A shape made from pattern tiles without using the square or thin rhombus.**

Of course it is interesting to find connections between elementary math and more advanced math (such as my example with the pattern tiles, which delighted me to discover). We can discover these connections without ever interacting with children, their teachers, or with mathematics educators. But what I have learned from mathematics educators is how interesting it is to find out how students—our own students in college classes as well as younger students in school—think about mathematical ideas. I've always enjoyed teaching but before I interacted with mathematics educators, I didn't realize it would be both a useful teaching tool and also *interesting* to find out how my students were thinking about the math I was trying to teach them. In retrospect, this lack of awareness is surprising. Most (all?) mathematicians enjoy talking to each other about math to find out how others are approaching problems and thinking about ideas. But all humans are capable of mathematical thought. Why not delight in it at every level? From the four-year-old who realizes that  $8 + 9$  is 17 because she knows  $8 + 8$  is 16 and so  $8 + 9$  must be one more, to the prospective middle-grades teachers in my geometry class this semester who devised the argument for why the sum of the angles in a triangle is  $180^\circ$  that is sketched in Figure 3, students can come up with ways to solve problems that we might not have thought of ourselves.



Going all the way around the triangle, the pencil turned a half-turn, which was the sum of the angles in the triangle.

**Figure 3. An explanation for why the sum of the angles in a triangle is  $180^\circ$ .**

One surprise in listening to how students think about math is to find that insightful ideas can

come even from students who have big gaps in their mathematical knowledge. I have found this not only with college students, but also with schoolchildren. A few years ago I taught sixth-grade math, every morning for a whole year, to a group of students who were acknowledged by other teachers at the school to be functioning below grade level in math. Many of the students were still struggling with basic arithmetic facts. Near the beginning of the year, when I asked my students to write a word problem for whole number division, most of the students couldn't write any problems at all. But, despite the deficits, students still came up with valuable comments and insights throughout the year, and their interest in abstract mathematical ideas surprised me at times. When we discussed the circumference and area of a circle, I showed the students a printout of several thousand digits of pi. I told them that the digits go on forever without stopping and without a repeating pattern. Their eyes grew big. "For real!?" they said. When I asked the students where pi would be on a number line, Santiago described how he thought about the location of pi, explaining that we'd have to keep zooming in forever on the number line to see exactly where pi is located.

In all my teaching, whether sixth grade or at the college and graduate levels, I've found that gaps and difficulties can coexist with insightful thoughts and interest in mathematical ideas and with enthusiasm for math. It's easy to get frustrated with our students' knowledge gaps and misconceptions, but by recognizing that all of our students have mathematical potential and by seeking out our students' ideas, we can make our teaching more satisfying and more interesting.

### What Can We Contribute to Pre-K-12 Education and What Can We Learn?

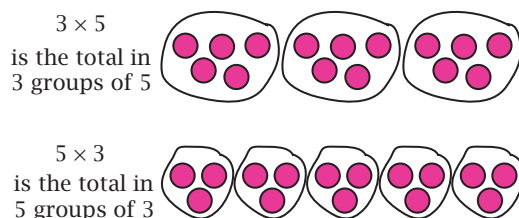
It's not surprising when I say that mathematicians have much to offer teachers and mathematics educators because of their broader, deeper view of mathematics. Mathematicians can help teachers and mathematics educators learn more math and learn connections between school math and more advanced math. But, perhaps surprisingly, there is plenty of *mathematics* that teachers and mathematics educators know but that mathematicians may not know explicitly or may not know in a way that applies to school mathematics.

For example, imagine that you are teaching third graders about division and that you want them to solve a variety of division word problems. What kinds of problems will you give them for  $15 \div 3$ ? You will surely have the students solve problems about dividing 15 objects equally among 3 groups, such as dividing 15 cookies equally among 3 people, or dividing 15 blocks equally among 3 containers. But you might not think to have students solve problems that involve dividing

15 objects into groups of 3 each, such as dividing 15 cookies into packages of 3 each, or dividing 15 blocks into containers that each hold 3 blocks. These two different perspectives on what division means correspond to two different equations, which are related by commutativity.

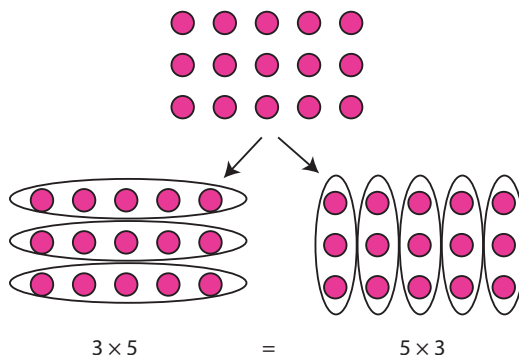
$$\begin{aligned} 3 \times ? &= 15 \\ ? \times 3 &= 15 \end{aligned}$$

We know not to take commutativity for granted because of the existence of nonabelian groups and noncommutative rings. The commutative property of multiplication is important to third graders too because it helps them lighten the load of learning the single-digit multiplication facts. But will third graders understand that whole number multiplication is commutative? Is it obvious? In fact, no; even from a third-grade perspective, the commutativity of multiplication of whole numbers is not obvious, as shown in Figure 4.



**Figure 4. A third-grade perspective on why commutativity of multiplication is not obvious.**

After seeing many examples, third graders may come to expect that multiplication really is commutative, but what is a third-grade way to see why whole number multiplication is commutative? (Note: no Peano axioms!) The existence of two-dimensional arrays, which can be decomposed either into equal rows or into equal columns, as in Figure 5, shows why whole number multiplication is commutative. As simple as arrays are, the existence of these structures now strikes me as saying something much deeper and more surprising about two-dimensional Euclidean space than I had previously appreciated.



**Figure 5. A third-grade perspective on the commutativity of multiplication.**

My examples so far have concerned only whole number multiplication and division. But examples of surprisingly intricate details that are involved in understanding elementary math are everywhere. Did you know, for example, that there are many ways to explain why it makes sense to divide fractions according to the “invert and multiply” rule, including ways that involve analyzing word problems and drawing simple pictures? Who knew!

Even if you aren’t interested in learning cool ways of explaining why “invert and multiply” is valid, what can mathematicians learn from the work of mathematics educators and teachers? I can summarize the most important thing I have learned: to improve teaching and learning in mathematics, we must take into account not only the mathematics itself—how to organize it, how to explain lines of reasoning clearly and logically, how the mathematical ideas are connected to other ideas both in and outside of math—but also *what students think*—what paths they tend to take as they develop understanding of mathematical ideas, where the difficulties lie, what errors and misconceptions tend to occur, what captures students’ interest. We must attend to where our students are in their understanding of the material we are trying to teach them, not just by marking their answers right or wrong (which of course is important), but also by looking into the source of our students’ errors. What ideas have our students not yet grasped and how can we help them learn those ideas? What misconceptions do they have and how can we help them see why these are misconceptions? What gets students excited about math and interested in learning it?

We might think that studying student thinking is only the job of mathematics education researchers and that the rest of us who teach math could safely dispense with it. Top-notch teaching might seem to be just a matter of having a well-structured course and a good book and then presenting the material clearly and enthusiastically in class, assigning good homework, and holding students accountable by giving tests. All these things are components of good teaching and can contribute to student learning, but they are not enough for excellent teaching. Most of us who teach have had the experience of delivering some beautifully polished lessons and carefully designed homework sets only to find out from students’ performance on the test that they didn’t actually grasp the ideas. What was missing? Most likely, our lectures didn’t connect with students’ existing knowledge and didn’t help students engage with the material at a level where they could make sense of it. In our enthusiasm to share exciting mathematical ideas, we might have failed to see that our students weren’t ready to appreciate the ideas. We probably gave answers to questions before students even grasped what the questions were and why the questions were significant. We

showed students mathematical tools for solving problems before the students saw the need for those tools. We didn't learn how our students were thinking and therefore we weren't able to help them build the ideas up in their own minds.

So I have learned from mathematics educators that there is no "royal road" to mathematics teaching and learning. It will never be just a matter of getting students who are adequately prepared when they enter our classes, and it will never be just a matter of delivering polished material. Teaching is a deeply human activity because, like conversation, it requires a give and take between the teacher and the students. Good teaching will always be hard work because it requires a teacher to know the mathematics *and* to take his or her students' thinking into account when making instructional decisions. Good teaching requires knowing the mathematical ideas and how to connect and scaffold them to make them accessible to students, *and* it requires finding out how students are thinking and then using this information in lectures, problems, and activities. Good learning will always be hard work for students because it requires them to engage actively with the material, to think about what they do and don't understand, and to persevere in making sense of the ideas.

Even if you are not interested in learning more about pre-K-12 math or in learning about the work of mathematics educators and about results from mathematics education research, why should mathematicians, mathematics educators, and teachers work together?

### **We Are All in This Together: Collective Responsibility for Improving Pre-K-College Math Education**

If we care about the pipeline of students going into math and about the strength of our profession in the future, then we simply must take the whole system of mathematics education into account. Students arrive at college with a long history of learning math, and that history affects their initial choices of math in college and their attitudes toward math as they enter their initial college math classes. These initial classes, together with a student's mathematical background, affect a student's decision to take further math classes or not, and they affect whether the student decides to become a math teacher. This means that all of us who teach math, pre-K teachers, elementary school teachers, middle school teachers, high school teachers, college teachers—all of us—must think collectively and systemically about improving our system.

Think about this: if you teach a calculus course, some of your students may go on to become teachers who will teach high school, middle school, or elementary school students. These students' experiences in your math class inform them about what math is and how it's done. Do your students

view explaining ideas and making sense of lines of reasoning as an important part of math? Or do they see math as plowing through a large volume of stuff that doesn't make sense? Students' experiences and views—not just your intentions—will inform their future teaching if they become teachers. So whether you want to be involved in pre-K-12 mathematics education or not, if you teach math to college students, you *are* involved in pre-K-12 mathematics education because some of your students might someday become teachers.

If we—mathematicians, mathematics educators, and teachers—are the community that is responsible for improving the mathematics education of all students, then we all bear *collective* as well as *individual* responsibility for improvement of the mathematics education system as a whole. Individually, we are responsible for constantly seeking to improve our own teaching. Collectively, enough of us must work together to cause the community as a whole to move along a path of constant improvement.

But here is something puzzling: why is it that our system of doing research promotes vigorous activity and striving for excellence, whereas at no level of teaching, from pre-K through the graduate level, do we have such a system? In research, we have a system of publication, presentation, and peer review in which we build on each other's ideas and constantly strive to move the field forward. The acts of publishing and presenting research findings are public activities, and because these activities are filtered by a peer review system, they allow us to compete for each other's admiration, and thus they provide us with an incentive to think hard about our work and to keep trying to improve it.

Wouldn't it be wonderful if teaching were a public activity, in the way that research is, in which we build on other people's good ideas and compete for each other's admiration? Wouldn't it be great if all of us who teach math were to take pride in the things we know well and yet at the same time be humble, expect to learn more, and recognize that in each one of us, knowledge, skill, and insightfulness coexist with gaps and areas that need improvement? I think it would be truly exciting to have a vibrant community of math teachers at all levels—the community of math teachers from pre-kindergarten through graduate school—thinking together about mathematics teaching and spurring each other on to do better and better work for the sake of all of our students.

### **Acknowledgments**

I would like to thank Michael Ching, Pete Clark, and Mark Saul for commenting on earlier drafts.



# The Mis-Education of Mathematics Teachers

*H. Wu*

If we want to produce good French teachers in schools, should we require them to learn Latin in college but not French? After all, Latin is the mother language of French and is linguistically more complex than French; by mastering a more complex language teachers could enhance their understanding of the French they already know from their school days. To correlate their knowledge of French with their students' achievements, we could look at their grades in Latin!

As ridiculous as this scenario is, its exact analogue in mathematics education turns out to be central to an understanding of the field as of 2011. A natural question is why the mathematics research community should be bothered with a problem in education. The answer is that the freshmen in our calculus classes year after year, and ultimately our math graduate students, are products of this educational philosophy. The purpose of this article is to alert the mathematics community to the urgent need of active participation in the education enterprise. It is a call for action. We will begin by reviewing the state of the mathematical education of teachers in the past four decades, and then give an indication of what needs to be done to improve teachers' content knowledge and why knowledgeable mathematicians' input is essential.

## The Early Work of Begle

No one doubts that improvement in school mathematics education depends critically on having effective mathematics teachers in the classroom. The common notion that "you cannot teach what you don't know" underscores our need to produce teachers with a solid knowledge of mathematics. Yet the mathematics education establishment has

not maintained a sharp focus on the professional development of both preservice and inservice teachers, in part because what-you-need-to-know turns out to be a contentious issue. It appears that educators<sup>1</sup> are content to let the mathematics community decide what secondary teachers should know and to deal only with the professional development of elementary teachers. In the case of the former, there is too much of the Latin-French syndrome. Mathematicians feed secondary teachers the kind of advanced mathematics that future math researchers should learn and expect the Intellectual Trickle-Down Theory to work overtime to give these teachers the mathematical content knowledge they need in the school classroom. In the case of elementary teachers, too often the quality of the mathematics they are taught leaves much to be desired: the negative evaluation, mostly by mathematicians, of the commonly used textbooks for elementary teachers ([NCTQ]<sup>2</sup>) paints a dismal picture of how poorly elementary teachers are served.

A related issue, of course, is whether any correlation exists between mathematics teachers' content knowledge and student achievement. Among the early researchers who tried to establish this correlation was E. G. Begle, the director of SMSG (School Mathematics Study Group), the group that was most identified with the "New Math" of the period 1955–1975. In a 1972 study of 308 teachers of first-year high school algebra ([Begle 1972]), he gave both teachers and students multiple-choice tests to measure teachers' knowledge and student achievement gains.<sup>3</sup> Broadly speaking, he found "little empirical evidence to substantiate any claim that, for example, training in mathematics for

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<sup>1</sup>I will use "educator" in this article to refer to the mathematics education faculty in universities.

<sup>2</sup>See pp. 34–37 and 76–81.

<sup>3</sup>Students were given a pretest and a posttest.

mathematics teachers will have payoff in increased mathematics achievement for their students". Subsequently, he surveyed the empirical literature in mathematics education research and again confirmed that the available evidence did not support the belief that "the more one knows about one's subject, the more effective one can be as a teacher" (p. 51, [Begle1979]).

The 1972 work of Begle is best known for casting doubt on the relevance of mathematical content knowledge to the effectiveness of teaching, but a close examination of this report is extremely instructive. Begle administered two tests to teachers, one on the algebra of the real number system and the other on the level of the abstract algebra of groups, rings, and fields. Analysis of the results indicated to Begle that

...teacher understanding of modern algebra (groups, rings, and fields) has no significant correlation with student achievement in algebraic computation or in the understanding of ninth grade algebra. ...However, teacher understanding of the algebra of the real number system does have significant positive correlation with student achievement in the understanding of ninth grade algebra. (Page 8 of original text in [Begle 1972].)

From these findings, Begle arrived at the following two remarkable recommendations:

The nonsignificant relationship between the teacher modern algebra scores and student achievement would suggest the recommendation that courses not directly relevant to the courses they will teach not be imposed on teachers. The small, but positive, correlation between teacher understanding of the real number system and student achievement in ninth grade algebra would lead to the recommendation that teachers should be provided with a solid understanding of the courses they are expected to teach... (ibid.).

It is to be regretted that Begle did not follow through with his own recommendations. Had that been done, there would have been no need for the present article to be written. Let us put this statement in context. Begle was dealing with high school teachers who are traditionally required to complete the equivalent of a major in mathematics. However, the requirements for math majors are designed mainly to enable them to succeed as mathematics graduate students and, for this very reason, are full of "courses not directly relevant

to the courses [teachers] will teach" in the school classroom. Implicitly, Begle recognized back in 1972 a critical flaw in the preservice professional development of high school teachers, namely, they are fed information that doesn't directly help them with their work. In other words, we teach Latin to French teachers and hope that they will become proficient in teaching French. Begle's second recommendation hinted at his awareness of the complementary fact, namely that high school teachers *do* need courses that provide them with a solid understanding of what they teach.

### Basic Criteria of Professional Development

Begle's work was carried on by others in the intervening years, notably by [Goldhaber-Brewer] and [Monk]. But the work that is most relevant to the present article is that of Deborah Ball, who some twenty years after Begle considered what teachers need to know about the mathematics of elementary school ([Ball]). Her survey of both elementary and secondary teachers showed that even teachers with a major in mathematics could not explain something as basic as the division of fractions (a basic topic in grades 5 and 6) in a way that is mathematically and pedagogically adequate. Her conclusion is that "the subject matter preparation of teachers is rarely the focus of any phase of teacher education" (p. 465, [Ball]).

A few years later, as a result of my work with the California Mathematics Project (cf. [Wu1999c]), I became alarmed by the deficiency of mathematics teachers' content knowledge and argued on theoretical grounds that improvement must be sought in the way universities teach prospective mathematics teachers ([Wu1999a], [Wu1999b]).<sup>4</sup> The conclusions I arrived at are entirely consistent with those of Begle and Ball, and a slightly sharpened version may be stated as follows. To help teachers teach effectively, we must provide them with a body of mathematical knowledge that satisfies both of the following conditions:

- (A) It is **relevant to teaching**, i.e., does not stray far from the material they teach in school.
- (B) It is consistent with the **fundamental principles of mathematics**.

The rest of this article will amplify on these two statements.

### Three Examples

The almost contradictory demands of these two considerations on professional development is illustrated nowhere better than in the teaching of fractions in school mathematics. Although

<sup>4</sup>I wish I could say I was aware of the work of Begle and Ball at the time that those articles were written, but I can't.

fractions are sometimes taught as early as second grade nowadays, the most substantial instruction occurs mainly in grades 5–7, and students’ difficulty with learning fractions in these grades is part of American folklore.<sup>5</sup> We will henceforth concentrate on fractions in grades 5–7.

Mathematicians who have a dim memory of their K–12 days may of course wonder why the teachers of these grades must be provided with a knowledge of fractions that is *relevant* to the school classroom. What is so hard about equivalence classes of ordered pairs of integers? Let us recall how fractions are taught in university mathematics courses. As usual, let  $\mathbb{Z}$  be the integers, and let  $S$  be the subset of ordered pairs of integers  $\mathbb{Z} \times \mathbb{Z}$  consisting of all the elements  $(x, y)$  so that  $y \neq 0$ . Introduce an equivalence relation  $\sim$  in  $S$  by defining  $(x, y) \sim (z, w)$  if  $xw = yz$ . Denoting the equivalence class of  $(x, y)$  in  $S$  by  $\frac{x}{y}$ , we call the set of all such  $\frac{x}{y}$  the rational numbers  $\mathbb{Q}$ . Identify  $\mathbb{Z}$  with the set of all elements of the form  $\frac{x}{1}$ , and we have  $\mathbb{Z} \subset \mathbb{Q}$ . Finally, we convert  $\mathbb{Q}$  into a ring by *defining* addition and multiplication in  $\mathbb{Q}$  as

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + zy}{yw}, \quad \text{and} \quad \frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}.$$

Of course we routinely check the compatibility of these definitions with the equivalence relation. This is what we normally teach our math majors in two to three lectures; it is without a doubt consistent with the fundamental principles of mathematics. The question is: what could a teacher do with this information in grades 5–7? Probably nothing.

Let us analyze this definition a bit: it requires an understanding of the partition of  $S$  into equivalence classes and the ability to consider each equivalence class as one element. Acquiring such an understanding is a major step in the education of beginning math majors. In addition, understanding the identification of  $\mathbb{Z}$  with  $\{\frac{x}{1} : x \in \mathbb{Z}\}$ , or as we say, the injective homomorphism of  $\mathbb{Z}$  into  $\mathbb{Q}$ , requires another level of sophistication.

Surely very little of the preceding discussion is comprehensible to students of ages 10–12, but even more problematic are the definitions of addition and multiplication of rational numbers. For example, consider multiplication once addition has been defined. The definition  $\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$  makes sense to us because we *want* to introduce a ring structure in  $\mathbb{Q}$  and this is the most obvious way to make it work. But can we explain to an average pre-teenager that rings are important and that therefore this definition of multiplication is the *right* definition? If so, what is wrong with defining  $\frac{x}{y} + \frac{z}{w}$  as  $\frac{x+z}{y+w}$  in accordance with every school student’s dream?

Because schools were in existence before the introduction of fractions in the 1930s as equivalence classes of ordered pairs of integers, and because fractions have been taught in schools from the beginning, it is a foregone conclusion that some version of fractions has been taught to teachers for a long time. But this version of mathematics makes no pretense of teaching *mathematics*. At least in this case, the relevance to the school classroom has been achieved at an unconscionable cost, namely at the expense of the fundamental principles of mathematics.

Mathematics depends on precise and literal *definitions*, but the way fractions are taught to elementary teachers has almost *no* definitions. The following is a typical example. A fraction is presented as three things all at once: it is a **part of a whole**, it is a **ratio**, and it is a **division**. Thus  $\frac{3}{4}$  is 3 parts when the whole is divided into 4 equal parts. Because it is not clear what a “whole” is, the education literature generally resorts to metaphors. Thus a prototypical “whole” is *like* a pizza. Now do we divide a pizza into 4 equal parts according to shape? Weight? Or is it area? The education literature doesn’t say. And how to multiply or divide two pieces of pizza? (See [Hart].) As to a fraction being a ratio,  $\frac{3}{4}$  can represent a “ratio situation”, as 3 boys for every 4 girls. What is the logical connection of boys and girls to pizzas? The education literature is again silent on this point, except to make it clear that every fifth grader had better acquire such a conceptual understanding of a fraction, namely that it can be two things simultaneously. Finally, the fraction  $\frac{3}{4}$  is also “3 divided by 4”. Now there are *many* things wrong with this statement, foremost being the fact that when students approach fractions, they are either in the process of learning about division of whole numbers or just coming out of it. In the latter situation, they understand  $m \div n$  (for whole numbers  $m$  and  $n$ ,  $n \neq 0$ ) to be a partition into equal groups or as a measurement *only when  $m$  is a multiple of  $n$* . If  $m$  is not a multiple of  $n$ , then students learn about division-with-remainder, in which case  $m \div n$  yields *two* numbers, namely, the quotient and the remainder. The concept of a *single* number  $3 \div 4$  is therefore entirely new to a student trying to learn fractions, and to define  $\frac{3}{4}$  in terms of  $3 \div 4$  is thus a shocking travesty of mathematics. What is true is that, when “part of a whole” is suitably defined and when  $m \div n$  is also suitably defined for arbitrary whole numbers  $m$  and  $n$  ( $n \neq 0$ ), it is a provable **theorem** that, indeed,  $\frac{m}{n} = m \div n$ . Yet, there is no mention of this fact in the education literature, and such *absence of reasoning* pervades almost all such presentations of fractions.

As a result of this kind of professional development, a typical elementary teacher asks her

<sup>5</sup>If in doubt, look up *Peanuts* and *FoxTrot* comic strips.



students to believe that there is a mysterious quantity called *fraction* that possesses three totally unrelated properties and then also asks them to compute with this mysterious quantity in equally mysterious ways. To add two fractions, take their *least common denominator* and then do some unusual things with the numerators to get the sum.<sup>6</sup> Of why, and of how, this concept of addition is related to the concept of adding whole numbers as “combining things”, there is no explanation at all.

Recall that we are here discussing the *mathematical* education of elementary teachers. We have to teach them mathematics so that, with proper pedagogical modifications, they can teach it to primary students, and so that, with essentially no modification, they can teach it to students in the upper elementary grades. So how can a teacher teach the addition of fractions in grades 5–7? If a fraction  $\frac{a}{b}$  is defined to be a point on the number line, then the sum of two fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  is, by definition, the total length of the two intervals  $[0, \frac{a}{b}]$  and  $[0, \frac{c}{d}]$  joined end-to-end—just as is the case of the sum of two whole numbers. In this way, adding fractions is “combining things” again. A simple reasoning then gives  $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ . See, for example, pp. 46–49 of [Wu2002].

Next, consider division. The rote teaching of the division of fractions is a good example of the total neglect of the fundamental principles of mathematics, and it has inspired the jingle, “Ours is not to reason why, just invert and multiply.” One recent response to such rote teaching is to imitate division between whole numbers by teaching the division of fractions as *repeated subtraction*. Unfortunately, the concept of division in a field cannot be equated with the division algorithm in a Euclidean domain, and the reaction against a defective mathematical practice has resulted in the introduction of another defective mathematical practice. Such a turn of events seems to be typical of the state of school mathematics education in recent times.

In any intellectual endeavor, a crisis of this nature naturally calls for research and the infusion of new ideas for a resolution. What is at present missing is the kind of education research that addresses students’ cognitive development without sacrificing precise definitions, reasoning, and mathematical coherence in the teaching of fractions (see pp. 33–38 in [Wu2008a] for a brief discussion of the research literature).

To improve on fraction instruction in schools, we first need to produce school textbooks that present a mathematically coherent way of approaching the subject, one that proceeds by reason rather than by decree. Several experiments along this

line were tried in the past two decades, but let us just say that, from the present perspective, they were not successes. An easier task would be to produce professional development materials for elementary teachers that are sufficiently elementary for students in grades 5–7. This would require a presentation of the mathematics of fractions different from the mathematically incoherent one described above. One way that has been thoroughly worked out is to define a fraction, in an explicit manner, as a point on the number line ([Jensen] and [Wu2002]). It does not matter whether teachers are taught this or possibly other approaches to fractions for school students; the important thing is that teachers *are* taught some version that is valid in the sense of conditions (A) and (B) above so that they can teach it in the school classroom. *It is simply not realistic to expect teachers to develop by themselves the kind of knowledge that satisfies (A) and (B).*

Two additional comments on fractions will further illuminate why we need to specifically address the special knowledge for teaching. At present, a major stumbling block in the learning path of school students is the fact that fractions are taught as *different numbers* from whole numbers. For example, it is believed that “Children must adopt new rules for fractions that often conflict with well-established ideas about whole numbers” ([Bezuk-Cramer], p. 156). The rules here presumably refer to the rules of arithmetic; if so, we can say categorically that there is a complete parallel between these two sets of rules for whole numbers and fractions; the similarity in question is a main point of emphasis in [Wu2002]. If mathematicians who take for granted that  $\mathbb{Z}$  is a subring of  $\mathbb{Q}$  are surprised by this misconception about fractions and whole numbers, they would do well to ask at which point of teachers’ education in K–16 (or, for that matter, a teacher’s education, period) they would get an *explicit* understanding of this basic algebraic fact. The unfortunate answer is probably “nowhere”, because until the last two years in college, mathematics courses are traditionally more about techniques than ideas, and even for those junior- and senior-level courses, our usual mode of instruction often allows the ideas to be overwhelmed by procedures and formalism (cf. [Wu1999a]). It should be an achievable goal for all teachers to acquire an understanding of the structural similarity between  $\mathbb{Z}$  and  $\mathbb{Q}$  so that they can teach fractions by emphasizing the similarity rather than the difference between whole numbers and fractions.

A second comment is that school mathematics is built on  $\mathbb{Q}$  (the rationals) and not on  $\mathbb{R}$  (the reals).  $\mathbb{Q}$  is everything in K–12, while  $\mathbb{R}$  appears only as a

<sup>6</sup>This is tantamount to saying that addition cannot be defined in the quotient field of a domain unless the latter is something like a UFD.

pale shadow.<sup>7</sup> It is this fact that accounts for the need to teach fractions well. We hope all teachers are aware of the dominance of  $\mathbb{Q}$  in their day-to-day work,<sup>8</sup> but few are, for the simple reason that we have never brought it to their attention.

In terms of the nitty-gritty of classroom instruction, real numbers are handled in K-12 by what is called the **Fundamental Assumption of School Mathematics** (FASM; see p. 101 of [Wu2002] and p. 62 of [Wu2008b]). It states that any formula or weak inequality that is valid for all rational numbers is also valid for all real numbers.<sup>9</sup> For example, in the seventh grade, let us say, the formula for the addition of fractions,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

where  $a, b, c, d$  are whole numbers, can be (and should be) proved to be valid when  $a, b, c, d$  are rational numbers. By FASM, the formula is also valid for all real numbers  $a, b, c, d$ . Thus high school students can write, without blinking an eye, that

$$\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} = \frac{\sqrt{3} + 2\sqrt{2}}{\sqrt{2}\sqrt{3}},$$

even if they know nothing about what  $1/\sqrt{2}$  or  $\sqrt{2}\sqrt{3}$  means. If this seems a little cut-and-dried and irrelevant, consider the useful identity

$$\frac{1}{1-x} + \frac{1}{1+x} = \frac{2}{1-x^2} \quad \text{for all real numbers } x.$$

If  $x$  is rational, this identity is easily verified (see preceding addition formula). But the identity implies also

$$\frac{1}{1-\pi} + \frac{1}{1+\pi} = \frac{2}{1-\pi^2}.$$

Without FASM, there is no way to confirm this equality in K-12, so its validity is entirely an article of faith in school mathematics.

As a final example, let  $a$  be any positive number  $\neq 1$ . Then for all rational numbers  $\frac{m}{n}$  and  $\frac{p}{q}$ , the following law of exponents for rational exponents can be verified (even if the proof is tedious):

$$a^{m/n} \cdot a^{p/q} = a^{m/n + p/q}.$$

Now, FASM implies that we may assume that the following identity holds for all real numbers  $s$  and  $t$ :

$$a^s \cdot a^t = a^{s+t}.$$

Of course, school mathematics cannot make sense of any of the numbers  $a^s$ ,  $a^t$ , and  $a^{s+t}$  when  $s$  and  $t$  are irrational, much less explain why this equality is valid. Nevertheless, this equality is of more

than purely academic interest because it is needed to describe a basic property of the exponential function  $a^x : \mathbb{R} \rightarrow (0, \infty)$ .

The preceding discussion brings out the fact that any discussion in high school mathematics is bound to be full of holes, and FASM is needed to fill in those holes. We would like to believe that FASM is a basic part of the professional development of mathematics teachers. Yet, to our knowledge, FASM has never been part of such professional development, with the result that schoolteachers are forced to fake their way through the awkward transition from fractions to real numbers in middle school. It is difficult to believe that, when teachers make a habit of blurring the distinction between what is known and what is not, their teaching can be wholly beneficial to the students. There is definitely room for improvement in our education of mathematics teachers.

Another illustration of the difference between the teaching of mathematics to the average university student and to prospective teachers is the concept of *constant speed*. Consider the following staple problem in fifth or sixth grade:

If Ina can walk  $3\frac{2}{5}$  miles in 90 minutes, how long would it take her to walk half a mile?

A common solution is to *set up a proportion*: Suppose it takes Ina  $x$  minutes to walk half a mile; then proportional reasoning shows that “the distances are to each other as the times”. Therefore  $3\frac{2}{5}$  is to  $\frac{1}{2}$  as 90 is to  $x$ . So

$$\frac{3\frac{2}{5}}{\frac{1}{2}} = \frac{90}{x}.$$

By the cross-multiplication algorithm:

$$3\frac{2}{5} \cdot x = \frac{1}{2} \cdot 90, \quad \text{so that } x = 13\frac{4}{17} \text{ minutes.}$$

The answer is undoubtedly correct, but what is the reasoning behind the setting up of a proportion? This rote procedure cannot be explained because *the assumption that makes possible the explanation has been suppressed*, the fact that Ina walks at a constant speed. As we know, if there is no assumption, then there is no deduction either. It therefore comes to pass that problem solving in this case is reduced to the rote procedure of setting up a proportion.

How did school mathematics get to the point that “constant speed” is not even mentioned or, if mentioned, is not explained in the school classroom? It comes back to the issue of how we educate our teachers. The only time university mathematics deals with constant speed is in calculus, where a motion along a line  $f(t)$  describing the distance from a fixed point as a function of time  $t$  is said to have *constant speed* if its derivative  $f'(t)$  is a constant. There are teachers who don’t take calculus, of course, but even those who do

<sup>7</sup>This fundamental fact seems to have escaped Begle, as evidenced by his tests for teachers ([Begle1972]).

<sup>8</sup>Better yet, one hopes that all state and national standards reflect an awareness of this fact as well, but that is just a forlorn hope.

<sup>9</sup>A trivial consequence of continuity and the density of  $\mathbb{Q}$  in  $\mathbb{R}$ .

will see constant speed as a calculus concept and nothing else. Because we do not see fit to help prospective teachers relate university mathematics to school mathematics, such a misconception about constant speed will remain with them. In the school classroom, they realize that there is no place for the derivative and therefore conclude that it is impossible to discuss constant speed. Once this realization sets in, they fall back on what they learned as students in K-12, which is not to talk about constant speed at all. So the tradition continues, not just in the classroom instruction but also in textbooks.

Having taken calculus is usually considered a badge of honor among middle and elementary school teachers, and some professional development programs go out of their way to include calculus exactly for this reason. The example of constant speed is but one of the innumerable reasons why having taken a standard calculus course does not ensure a teacher's effectiveness in the school classroom.<sup>10</sup>

Professional development of teachers ideally should include the instruction that in the school curriculum the concept of *speed* is too subtle to be made precise, but that one should use instead the concept of **average speed in a time interval**  $[t, t']$ , which is the quotient

$$\frac{(\text{the distance traveled from time } t \text{ to time } t')}{(t' - t)}.$$

A motion is said to have **constant speed**  $K$  if, for *every* time interval  $[t, t']$ , the average speed is always equal to  $K$ , i.e.,

$$\frac{(\text{the distance traveled from time } t \text{ to time } t')}{(t' - t)} = K.$$

Once this concept is introduced, the setting up of a proportion in the preceding example can be explained *provided Ina is assumed to walk at a constant speed*. For then her average speeds in the two time intervals  $[0, 90]$  and  $[0, x]$  are the same, and therefore

$$\frac{3\frac{2}{5}}{90} = \frac{1}{x},$$

and this equality is equivalent to the proportion above.

Of course school students would find it difficult to grasp the idea that the average speed in *every* time interval is a fixed number, and education researchers should consider how to lighten the attendant cognitive load. But that is a different story. Our concern here is whether prospective teachers are taught what they need to know in order to carry out their duties, and once again we see the gulf that separates what is mathematically correct in a university setting from what is pedagogically feasible in a school

classroom. What is needed to bridge this gulf is the concept of *customizing abstract mathematics for use in the school classroom*. This is the essence of mathematics education (see [Wu2006] for a full discussion). In this case, it is a matter of taking apart the concept of the constancy of the derivative of a function and reconstructing it so that it makes sense to school students.

As a final example to illustrate the chasm between what we teach teachers and what they need to know, consider the fundamental concepts of congruence and similarity in geometry. The gaps in our teachers' knowledge of these two concepts are reflected in the existing school geometry curricula. For example:

(i) In middle school, two figures (not necessarily polygons) are defined to be *congruent* if they have the same size and same shape and to be *similar* if they have the same shape but not necessarily the same size. In high school, congruence and similarity are defined in terms of angles and sides, but only for polygons. There is no attempt to reconcile the more precise definitions in high school with the general ones in middle school.

(ii) In middle school, the purpose of learning about congruence is to perceive the inherent symmetries in nature as well as in artistic designs such as Escher's prints, tessellations, and mosaic art. Likewise, the purpose of learning about similarity is to engage in fun activities about enlarging pictures. In high school, students prove theorems about congruent and similar triangles in a geometry course but otherwise never again encounter these concepts in another course in school mathematics.

(iii) Because similarity is more general than congruence and because two figures are more likely to be similar than congruent, some curricula ask teachers to teach similarity before congruence in middle school.<sup>11</sup>

As a result of the neglect by universities, our teachers' conception of congruence and similarity is largely as fragmented and incoherent as the practices described in (i)–(iii) above. Not every

<sup>10</sup>Calculus is by definition, as well as by design, a technique-oriented subject.

<sup>11</sup>It is possible to define similarity as a bijection of the plane that changes distance of any two points by a fixed scale factor  $k$  and to define a congruence as the case of  $k = 1$ . This approach is, however, basically impossible to bring off in a school classroom.



school geometry curriculum is guilty of all three, but most are guilty of the first two. So long as university mathematics courses do not address issues arising from school mathematics, teachers will not be sufficiently well informed to reject such mathematical illiteracy, and publishers will continue to get away with the promotion of this kind of illiteracy. We must create a university mathematics curriculum for prospective teachers to help them *look back* at such school concerns as the meaning of congruence and similarity and why these concepts are important in mathematics. By contrast, preservice teachers are given at least some access to such topics as the curvature of curves, Gaussian curvature of surfaces, finite geometry, projective geometry, non-Euclidean geometries, and the foundations of geometry. They are *not*, however, taught plane Euclidean geometry. This last is exactly what teachers need because it is usually taught poorly in schools. They desperately need solid information about school geometry in order to better teach their own geometry classes.

Thus we see in this case the same scenario that we saw with fractions played all over again: mangled definitions, critical gaps in mathematical reasoning, and insufficient attention to mathematical coherence; above all, students are given no purpose for learning these concepts except for fun, for art appreciation, or for the learning of boring geometric proofs.

However, we should not accept these results of years of neglect as immutable, because there are ways to make mathematical sense of school geometry and, in particular, of congruence and similarity. We can begin with the instructions on the *basic rigid motions* of the plane (translations, rotations, and reflections) more or less informally by the use of hands-on activities; after all, one has to accept the fact that the concept of a transformation is difficult for students, and it won't do to insist on too much formalism at the outset. We can do the same with the concept of a *dilation* from a point (i.e., central projection of a fixed scale factor from that point). Then we can define **congruence** as a finite composition of basic rigid motions and **similarity** as the composition of a dilation and a congruence. But, as in all things mathematical, precision is not pursued for its own sake. In the present situation, students can now make direct use of translations, rotations, and reflections to prove the congruence of segments and angles; such proofs are far more intuitive than those using the traditional criteria of ASA, SAS, and SSS. In addition, it is a rather simple exercise to *assume* the abundant existence of basic rigid motions in the plane in order to prove all the usual theorems in Euclidean geometry, including those on similar triangles (cf. [CCSS] and Chapter 11 in Volume II of [Wu2011b]). The requirement of "invariance

under congruence" in such a mathematical development further highlights the fundamental role of congruence in the definitions of length, area, and volume (cf. Chapter 7 of [Wu2010] and Chapter 18 in Volume III of [Wu2011b]). This is one way to make teachers aware of what congruence and similarity are and why they are part of the basic fabric of mathematics itself.

In advanced mathematics, the basic rigid motions of Euclidean  $n$ -space  $\mathbb{R}^n$  are defined in terms of orthogonal transformations and coordinates, and a dilation is also defined in terms of coordinates. Here we use rigid motions and dilations instead as the basic building blocks of geometry in order to define coordinate systems in the plane in a way that is usable in middle and high schools. This is another example of the *customization* of abstract mathematics for use in schools.

Such content knowledge for mathematics teachers is not yet standard fare in preservice professional development, but it should be.

### The Role of Mathematicians

The three mathematical examples above indicate what needs to be done to customize abstract mathematics for use in the K–12 classroom, but they are only the tip of the iceberg. Almost the entire K–12 curriculum needs careful revamping in order to meet the *minimum* standards of mathematics, and this kind of work calls for input by mathematicians. The mathematical defects of the present curriculum are, in my opinion, too pronounced to be undone by people outside of mathematics. Research mathematicians have their work cut out for them: consult with education colleagues, help design new mathematics courses for teachers, teach those courses, and offer constructive criticisms in every phase of this reorientation in preservice professional development. My own systematic attempt to address the problem is given in [Wu2011a] (for elementary school teachers) and in [Wu2011b] (for high school teachers); a third volume for middle school teachers will include [Wu2010]).<sup>12</sup>

For those who don't care about the details, an outline of what is possible for the K–8 curriculum can be found in [Wu2008b]. Such an outline also appears in [MET], which was written in 2001 to give guidance on the mathematical education of math teachers to university math departments. Its main point was to bring research mathematicians into the discussion of mathematics education. Although others may disagree with me, my own opinion is that its language is not one that speaks persuasively to mathematicians and that the mathematics therein

<sup>12</sup>[Wu2011b] is the text for the sequence of three-semester courses, *Mathematics of the Secondary School Curriculum*, which is required of all math majors at UC Berkeley with a teaching concentration.

fails to respect the fundamental principles of mathematics more often than it should.

To (research) mathematicians, the mathematics of K-12 obviously holds no mystery. If they have to develop the whole body of knowledge *ab initio*, strictly as mathematics, they can do it with ease. But if they hope to make the exposition speak to the teachers, then they will have to spend time to learn about the school classroom. If one mathematician's experience is to be trusted, the pedagogical pitfalls of such an undertaking can be avoided only if mathematicians *can get substantive input about the K-12 classroom*. For starters, one can go to the local school district office to look at the textbooks being used; reading them should be an eye-opener. One should also try to talk to inservice teachers about their experiences and their students' learning difficulties; make an effort to visit a school classroom if possible. But the ultimate test is, of course, to get to teach (inservice or preservice) teachers *the mathematics of K-12* and solicit honest feedback about their reactions. If the mathematics department and the school of education on campus are on good terms, then the whole process of getting in touch with teachers can be expedited with the help of one's education colleagues.

There is another crucial contribution that research mathematicians can make, one that seems to be insufficiently emphasized in education discussions up to this point. In their routine grappling with new ideas, mathematicians need to know, for survival if nothing else, the intuitive meaning of a concept perhaps not yet precisely formulated and the motivation behind the creation of a particular skill and to have a vague understanding of the direction they have to pursue. These needs completely parallel those of students in their initial attempt to learn something new. This part of a research mathematician's knowledge would surely shed light on students' learning processes. Here, then, is another important resource that should not go to waste in our attempt to help teachers and educators better understand teaching.

## The Fundamental Principles of Mathematics

Having invoked the "fundamental principles of mathematics" several times throughout this article, I will now summarize and make explicit what they are and why they are important. I believe there are at least five of them. They are interrelated and, to the extent that they are routinely violated in school textbooks and in the school education literature (to be explained below), teachers have to be aware of them if they hope to teach well.

(1) *Every concept is precisely defined, and definitions furnish the basis for logical deductions.* At the moment, the neglect of definitions in school

mathematics has reached the point at which many teachers no longer know the difference between a definition and a theorem. The general perception among teachers is that a definition is "one more thing to memorize". We have already pointed out that the concepts of a fraction, constant speed, congruence, and similarity are in general not defined in the school mathematics education literature. It is sobering to point out that many more bread-and-butter concepts of K-12 mathematics are also not correctly defined or, if defined, are not put to use as an integral part of reasoning. These include: number, rational number (in middle school), decimal (as a fraction in upper elementary school), ordering of fractions, length-area-volume (for different grade levels), slope of a line, half-plane of a line, equation, graph of an equation, inequality between functions, rational exponents of a positive number, and polynomial.

(2) *Mathematical statements are precise. At any moment, it is clear what is known and what is not known.* Yet there are too many places in school mathematics in which textbooks and education materials fudge the boundary between what is true and what is not. Often a heuristic argument is conflated with correct logical reasoning. For example, the identity  $\sqrt{a}\sqrt{b} = \sqrt{ab}$  for positive numbers  $a$  and  $b$  is often explained by assigning a few specific values to  $a$  and  $b$  and then checking for these values by a calculator. (For other examples, see pp. 3-5 of [Wu1998].) Sometimes the lack of precision comes from an abuse of notation or terminology, such as using  $25 \div 6 = 4 \text{ R } 1$  to express "25 divided by 6 has quotient equal to 4 and remainder 1" (this is an equality of neither two whole numbers nor two fractions). At other times an implicit assumption is made but is not brought to the fore; perhaps the absence of any explicit statement about FASM is the most obvious example of this kind of transgression.

(3) *Every assertion can be backed by logical reasoning.* Reasoning is the lifeblood of mathematics and the platform that launches problem solving. Given the too frequent absence of reasoning in school mathematics (cf. the discussion of fractions and constant speed above), how can we ask students to solve problems if teachers do not have the ability to engage students in logical reasoning on a consistent basis?

(4) *Mathematics is coherent; it is a tapestry in which all the concepts and skills are logically interwoven to form a single piece.* The professional development of math teachers usually emphasizes either procedures (in days of yore) or intuition (in modern times) but not the coherence (structure) of mathematics. The last may be the one aspect of mathematics that most teachers (and dare I say also educators) find most elusive. The lack of awareness

of the coherence of the number systems in K-12<sup>13</sup> may account for teaching fractions as “different from” whole numbers (so that the learning of whole numbers becomes almost divorced from the learning of fractions). We mentioned earlier an example of curricular incoherence when similarity is discussed before congruence. A more common example is the almost universal “proof” of the theorem on *equivalent fractions*, which states: *For all fractions  $\frac{m}{n}$  and for any nonzero whole number  $c$ ,*

$$\frac{m}{n} = \frac{cm}{cn}.$$

The “proof” in question goes as follows:

$$\frac{m}{n} = \frac{m}{n} \times 1 = \frac{m}{n} \times \frac{c}{c} = \frac{cm}{cn}.$$

The problem with this argument is that this theorem must be proved essentially as soon as a fraction is defined, but multiplication of fractions, the most sophisticated of the four arithmetic operations on fractions,<sup>14</sup> comes much later in the usual development of fractions.

The coherence of mathematics includes (but of course is not limited to) the sequential development of concepts and theorems; the progression from the logically simple to the logically complex cannot be subverted at will. However, for people who have not been immersed in mathematics systematically and for a long time, it is almost impossible to resist the temptation to subvert this sequential development. The two preceding examples testify eloquently to this fact.

(5) *Mathematics is goal-oriented, and every concept or skill in the standard curriculum is there for a purpose.* Teachers who recognize the purposefulness of mathematics gain an extra tool to make their lessons more compelling. When congruence and similarity are taught with no mathematical purpose except to do “fun activities”, students lose sight of the mathematics and wonder why they were made to learn it.<sup>15</sup> When students see the technique of completing the square merely as a trick to get the quadratic formula rather than as the central idea underlying the study of quadratic functions, their understanding of the technique is

superficial. But perhaps the most telling example of teaching mathematics without a purpose is teaching students by rote to round off whole numbers, to the nearest hundreds or to the nearest thousands, *without* telling them why it is useful (cf. section 10.3 of [Wu2011a]). Most elementary students consider rounding a completely useless skill that is needed only for exams. If teachers can put rounding off in the context of the how and the why of estimations, they are likely to achieve better results.

## The Mathematics Teachers Need to Know

I hope that this discussion of the fundamental principles of mathematics convinces the reader that there is substantive mathematics about the K-12 curriculum that a teacher must learn. This body of knowledge may be elementary, but it is by no means trivial, in the same sense that the theory behind the laptop computer may be elementary (just nineteenth-century electromagnetic theory as of 2000) but decidedly not trivial. This discussion in fact strongly bears on the central question of the moment in mathematics education: *exactly* what kind of content knowledge for teachers would lead to improved student achievement? (Cf. Begle’s work, mentioned at the beginning of this article.) Although research evidence on this issue is lacking, it is not needed as a first step toward a better mathematics education for teachers. For whatever this knowledge may be, it must *include* the mathematics of the school curriculum presented in a way that is consistent with the fundamental principles of mathematics. Let me be as explicit as I can: I am not making any extravagant claims about the advanced mathematics teachers need to know or even whether they need to know advanced mathematics, only that *they must know the content of what they teach to their students*. Here I am using the word “know” in the unambiguous sense that mathematicians understand this term:<sup>16</sup> *knowing* a concept means knowing its precise definition, its intuitive content, why it is needed, and in what contexts it plays a role, and *knowing* a technique<sup>17</sup> means knowing its precise statement, when it is appropriate to apply it, how to prove that it is correct, the motivation for its creation, and, of course, the ability to use it correctly in diverse situations. In this unambiguous sense, teachers cannot claim to know the mathematics of a particular grade without also knowing a substantial amount of the mathematics of three or four grades *before* and *after* the grade in question (see Recommendation 19 of [NMP1]). This necessity that math teachers actually *know* the mathematics

<sup>13</sup>Whole numbers, integers, fractions, rational numbers, real numbers, and complex numbers.

<sup>14</sup>The sophistication comes from the fact that at least three things must be explained about  $m/n \times k/\ell$  before it can be effectively used by students: (1) it is the area of a rectangle of sides  $m/n$  and  $k/\ell$ , (2) it is the number that is the total of  $m$  parts when  $k/\ell$  is partitioned into  $n$  equal parts, and (3) it is equal to  $(mk)/(n\ell)$ . Either (1) or (2) can be used as the definition of  $m/n \times k/\ell$  and the other will have to be proved, and then the seductive formula (3) must also be proved. Too often, the deceptive simplicity of (3) is the siren song that causes many shipwrecks in the teaching of fraction multiplication.

<sup>15</sup>At least according to math majors I have taught at Berkeley.

<sup>16</sup>Educators usually use the word “know” in its literal sense: being able to memorize a fact, a definition, or a procedure.

<sup>17</sup>Usually referred to as “skill” in the education literature.



they teach sheds light, in particular, on why we want all high school teachers to know some abstract algebra: this knowledge allows them to really understand why there are only two arithmetic operations (+ and  $\times$ ) instead of four, in what way the rational functions are similar to rational numbers, and that the axiomatic system they encounter in geometry<sup>18</sup> is part of a universal practice in mathematics. The necessity that teachers know the mathematics they teach also explains why we want all teachers of high school calculus to know some analysis rather than just lower-division calculus.

At the moment, most of our teachers do not know the materials of the three grades above and below what they teach, because our education system has not seen to it that they do. We have the obligation to correct this oversight.

### Content Knowledge and Pedagogical Knowledge

The title of this article is about the education of mathematics teachers, but we have talked thus far only about learning mathematics, not about the methodology of teaching it. While knowing mathematics is undoubtedly necessary for a teacher to be effective, it is clearly not sufficient. For example, while we want all teachers to know precise definitions and their role in the development of mathematical skills and ideas, we do not wish to suggest that they teach school mathematics in the definition-theorem-proof style of graduate mathematics courses. The fact remains, however, that the more teachers know about a definition (the historical need it fulfills, why a particular formulation is favored, what ramifications it has, etc.), the more likely it is that they can make it accessible to their students. The same comment applies to every one of the fundamental principles of mathematics.

This then brings up the tension that exists at present between some mathematicians' perception of the most urgent task in a mathematics teacher's education and some educators' perception of the same. Mathematicians tend to believe that, because the most difficult step in mathematics teachers' education is to learn the necessary mathematics, giving them this knowledge is the number one priority in professional development. Quite understandably, some educators believe that the really hard work lies in the pedagogical part of the education that channels the teacher's content knowledge into the school classroom. As this theory goes, teachers learn the mathematics better if it is taught hand in hand with pedagogy. The main point of these conflicting perceptions—whether learning the pedagogy or learning the mathematics is more difficult to achieve—can at

<sup>18</sup>This is not to be interpreted as an advocacy of teaching high school geometry by the use of axioms.

some point be resolved by a large-scale study to see whether it is a lack of genuine understanding of content knowledge or weak pedagogical skills that contribute more to student nonlearning in the classroom.<sup>19</sup> In the meantime, some small-scale studies, e.g., [Ball] and [Ma], indicate that teachers' lack of content knowledge is the more severe problem. The available anecdotal evidence points in the same direction.

My personal experience, from having taught elementary and middle school teachers for eleven summers in four states (sometimes more than once in a given year) and having taught prospective high school teachers for four years at Berkeley, is that, in an overwhelming majority of cases, their mathematical preparation leaves a lot to be desired.<sup>20</sup> It is also the case that even when I inject pedagogical issues into my teaching from time to time, the teachers are usually so preoccupied with learning the mathematics that the pedagogical discussion hardly ever takes place. Some standard statistics, such as those in *A Nation at Risk* (see "Findings Regarding Teaching" in [NAR]), are consistent with this overall picture. It is for this reason that I have focused exclusively in this article on teachers' content knowledge.

This discussion of content knowledge should be put in the context of Lee Shulman's 1985 address ([Shulman]) on pedagogical content knowledge, i.e., the kind of pedagogical knowledge *specific to the teaching of mathematics* that a math teacher needs in order to be effective. There are two things that need clarification in such a discussion: what this *mathematical content knowledge* is and what the *associated pedagogical knowledge* is. Deborah Ball and her colleagues have recently begun to codify both kinds of knowledge in their attempt to reform math teachers' education (cf. [Ball-TP2008]). What must not be left unsaid is the obvious fact that, without a solid mathematical knowledge base, it is futile to talk about pedagogical content knowledge.

### The Need for Inservice Professional Development

At the beginning of this article, I mentioned the disheartening results of Deborah Ball's survey of teachers on their understanding of fraction division ([Ball]). I would venture a guess that, had her teachers been taught *the mathematics of K-12* in a way that respects the five fundamental principles of mathematics, the results of the survey

<sup>19</sup>For teachers in first and third grades, the large-scale study of [Hill-RB] found positive correlation between teachers' content knowledge and student achievement. So content knowledge is likely a major factor even at such an early stage of student learning.

<sup>20</sup>Again, see [Hill-RB].

would have been far more satisfactory.<sup>21</sup> Until we improve on how we teach mathematics to teachers in the universities, defective mathematics will continue to be the rule of the day in our schools. It is time for us to break out of the vicious cycle by exposing teachers to a mathematically principled version of the mathematics taught in K-12.

Unfortunately, such short-term exposure in the university may not be enough to undo thirteen years of mis-education of prospective teachers in K-12. Uniform achievement in the content knowledge of all math teachers will thus require heavy investments by the state and federal governments in sustained inservice professional development. To this end we need inservice professional development that directly addresses content knowledge. Funding for such professional development, however, may be hard to get, for content knowledge does not seem to be a high-priority consideration among government agencies. For example, in a recent survey by Loveless, Henriques, and Kelly of winning proposals among the state-administered Mathematics Science Partnership (MSP) grants from forty-one states ([Loveless-HK]), it was found that: "Some of the MSPs appear to be offering sound professional development. Many, however, are vague in describing what teachers will learn." Typically, these "MSPs' professional development activities tip decisively towards pedagogy". For example, although the professional workshops described in [TAMS] were not part of the review in [Loveless-HK], they nevertheless fit the description of this review. The [TAMS] document begins with the promising statement that the "TAMS-style teacher training increases teachers' content knowledge". But other than mentioning "teacher workshops focused on data analysis and measurement... Early grade teachers also studied length, area, and volume", the rest of the discussion of mathematics professional development focuses on persuading teachers to adopt "constructivist, inquiry-based instruction". The lack of awareness in [TAMS] about what content knowledge elementary teachers need in their classrooms is far from uncommon. It is time to face the fact that the need for change in the funding of inservice professional development is every bit as urgent as the need for more focus on content knowledge in the preservice arena.

### Concluding Remarks

To conclude, let me add two observations. The mathematics taught in K-12 is the main source of the mathematical information of not only our schoolteachers but also of the mathematics

education faculties and school administrators.<sup>22</sup> Mathematics education cannot improve so long as educators and administrators remain mathematically ill-informed as a result of the negligence of the mathematics community. It is doubtful, for example, that the research literature on fractions would slight logical reasoning (cf. pp. 33-38 in [Wu2008a]) had the researchers been exposed to a presentation of K-12 mathematics consistent with the five principles above. Many mathematics educators have likewise been denied this exposure and, as a result, have developed a distorted view of what mathematics is about. As this article tries to show, the cumulative gap between what (research) mathematicians take for granted as mathematics and what teachers and educators *perceive* to be mathematics has caused enormous damage in mathematics education. It is imperative that we minimize this damage by straightening out at least the *mathematics of K-12*, and we cannot possibly do that without first creating a corps of mathematically informed teachers. The latter has to be the mathematics community's immediate goal.

To lend some perspective on the communication gap between mathematicians and educators, it must be said that such miscommunication is by no means unusual in any interdisciplinary undertaking. In his celebrated account of the discovery of the double-helix model of DNA ([Watson]), James Watson recalled that at one point of his and Francis Crick's model building,<sup>23</sup> they followed the standard reference on organic chemistry<sup>24</sup> to pair the bases like-with-like. By luck, the American crystallographer Jerry Donohue happened to be visiting and was sharing an office with them, and Donohue told Watson not only that his (Watson's) scheme of pairing was wrong but also that such information given in most textbooks of chemistry was incorrect (p. 190, *ibid.*). In Watson's own words:

If he [Donohue] had not been with us in Cambridge, I might still have been pumping for a like-with-like structure. (p. 209)

In other words, but for the fortuitous presence of someone truly knowledgeable about physical chemistry, Crick and Watson might not have been able to guess the double helix model, or at least the discovery would have been much delayed.

The moral one can draw from this story is that, if such misinformation could exist in high-level science, one should expect the same in mathematics education, which is much more freewheeling. This suggests that real progress in teacher education will

<sup>21</sup>Note that the work of Hill, Rowan, and Ball ([Hill-RB]), while not directly verifying this hypothesis, is nevertheless fully consistent with it.

<sup>22</sup>If anyone wonders where administrators come in, let me say that the number of horrendous decisions in school districts on mathematics textbooks and professional development would easily fill a volume.

<sup>23</sup>In Cambridge, England.

<sup>24</sup>The Biochemistry of Nucleic Acids by J. N. Davidson.

require both the education and the mathematics communities to collaborate very closely and to be vigilant in separating the wheat from the chaff. In particular, given the long years during which incorrect information about mathematics has been accumulating in the education literature and school textbooks, there should be strong incentive for educators to seek information about the K-12 mathematics curriculum anew and to begin some critical rethinking.

Last but not least, all through this article I have put great emphasis on getting teachers and (consequently) educators to *know* the mathematics of K-12. This should in no way be interpreted as saying that the mathematics of K-12 is all a teacher needs to know. Contrary to Begle's belief, there is no such thing as knowing (in the sense described above) *too much* mathematics in mathematics education. Every bit of mathematical knowledge will help in the long run. However, faced with the almost intractable problem of improving the education of *all* math teachers, it is only proper that we focus on a modest and doable first step: make sure that mathematics teachers all know the mathematics of K-12. Let us get this done.

## Acknowledgments

I am extremely grateful for the two referees' extensive and constructive comments, and to Ralph Raimi for many factual and linguistic corrections. They have made this article substantially better. I am also indebted to Richard Askey, Deborah Ball, Raven McCrory, Xiaoxia Newton, and Rebecca Poon for their invaluable contributions in the preparation of this article.

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# A Mathematician Writes for High Schools

Dan Fendel

## A Brief Personal History

I started out my graduate school education fully intending to be a research mathematician. As a Harvard undergraduate (A.B. 1966), I had been inspired by Richard Brauer's courses in finite group theory and, at his recommendation, went to Yale to work with Walter Feit. I received my Ph.D. in 1970, and my dissertation ("A characterization of Conway's group .3") appeared in *Journal of Algebra*, January 1973.

But, in early 1969, while still in graduate school, I decided that my skills could be put to greater public service if I focused my career on helping to improve public school education. During and right after graduate school, I worked in public school systems in New Haven, Connecticut, and in Compton and Oakland, California. In these settings, I was a mathematics specialist working with full classes of elementary or middle school students. Although I enjoyed this work, I grew to see the need for regular teachers to have a deeper understanding of basic mathematics, and I decided that my impact would be greater if I were working directly in the preparation of teachers. In 1973 I joined the mathematics department at San Francisco State University, where I worked for more than thirty years with preservice teachers and current classroom teachers at all levels, meanwhile continuing to teach service courses such as calculus, upper division courses for mathematics majors, and graduate courses in our master's program.

In 1989 I got the opportunity of a lifetime. I was invited to join a project, then in its very early stages, whose goal was to create a problem-based curriculum for high schools that would embody the ideas and recommendations of a recent series of reports on the need for reform.

I accepted that invitation, and, along with Diane Resek (Ph.D., 1975, U.C. Berkeley), became one of the principal authors of the *Interactive Mathematics*

*Program* (IMP).<sup>1</sup> Over the next decade or so, I worked with a wonderful team of mathematics educators and teachers, in an intensive process of writing, testing, and revising, revising, revising, to produce a four-year program. In 1999 the IMP curriculum was one of a handful of mathematics programs designated by the U.S. Department of Education as "exemplary" (the highest rating). The curriculum has been used in over 1,000 schools throughout the United States, as well as in many schools outside the United States.

## Writing Curriculum

One of the great challenges of mathematics curriculum development—at any level—is to make complex ideas meaningful and comprehensible to students while maintaining the integrity of the ideas. Anyone who has seen eyes glaze over as mathematics is presented with great rigor and elegance will agree that mathematical knowledge and correctness do not, by themselves, make for good teaching. Curriculum development requires both a deep understanding of mathematics and a realistic view of how students think. And so, no matter how mathematically elegant or aesthetically satisfying an approach may be, a curriculum writer must be willing to discard it if it doesn't work with students. He or she must then struggle to find something else that is more effective.

I offer here two examples of our curriculum development process. I hope that the description of how I was able to contribute may aid other mathematicians to develop similar curricula.

## Expected Value

We want our citizens to be able to make intelligent decisions on issues that involve chance and data,

<sup>1</sup>Interactive Mathematics Program (*Years 1 to 4*), by Dan Fendel and Diane Resek, with Lynne Alper and Sherry Fraser; published by Key Curriculum Press, Emeryville, CA (first edition © 1997 through 2000; second edition © 2009 through 2012 [*Years 1 through 3* currently available; *Year 4* available spring 2011]).

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yet most students even today leave high school with only the vaguest ideas about probability and statistics. Therefore our project leaders decided that we would give these topics much greater emphasis than they'd had in the traditional high school curriculum. In particular, we identified *expected value* as a key concept in probability.<sup>2</sup>

One important step was to limit consideration to situations with finite sample spaces. We recognized that, in this context, it's easy to give a formal definition of expected value as a sum of products of values and probabilities. For example, the expected value for the roll of a single (balanced) die is the sum  $1 \cdot (1/6) + 2 \cdot (1/6) + 3 \cdot (1/6) + 4 \cdot (1/6) + 5 \cdot (1/6) + 6 \cdot (1/6)$ , which comes to  $21/6$ , or  $3.5$ . And, for other situations, even where probabilities are not all the same, the definition is similar. A textbook could define expected value simply as  $\sum_{i=1}^n x_i \cdot \mathbf{p}(x_i)$ , where  $\{x_1, x_2, \dots, x_n\}$  is the sample space of possible outcomes, and  $\mathbf{p}(x_i)$  is the probability of outcome  $x_i$ .

But, though it might have been aesthetically pleasing to me as a mathematician to use that definition, doing so would have doomed our work to failure with students (and here I mean the vast majority of high school students—not the rare abstract thinker who might become a mathematics Ph.D.).

As a mathematician, I know that many definitions can be equivalent to one another. As a person with experience in high school classrooms, I know that the phrase “sample space” and the use of summation notation, subscripts, and other formal symbolism will lead to “glaze-over” among students.

So, instead of the formal definition described earlier, we chose an equivalent definition based on the idea of “average in the long run”. Before using the formal phrase “expected value”, the IMP curriculum thus gives students concrete experiences, such as asking them to imagine rolling a die many, many times and to compute what they might expect for the average of those rolls.

Students understand, based on experiments and intuition, that if the number of rolls is “large enough”, then the fraction of rolls giving each result will be “pretty close” to the value given by the probability. (Indeed, students using IMP come to see that as, essentially, the meaning of probability.) For example, if they use 600 rolls, they can expect about 100 rolls for each possible outcome of the die. This leads IMP students to a computation such as  $1 \cdot 100 + 2 \cdot 100 + 3 \cdot 100 + 4 \cdot 100 + 5 \cdot 100 + 6 \cdot 100 (= 2100)$  for the total value of all the rolls, giving  $2100/600 = 3.5$  for the average.

In this context, students' intuition about probability serves them well. They see that if the number of rolls for each outcome were off a bit from the “perfect  $1/6$ ”, this would change the total value of

the rolls, but it would not change the average value much, because the total value is being divided by “a big number”. After working with several such computations, students with sufficient understanding of the distributive property can see that the result of such a computation is independent of the actual number of rolls.

### Gravitational Fall

One of my favorite units in the IMP curriculum involves the following circus scenario:

A performer is on a Ferris wheel that is turning at a constant rate. A cart with a tub of water is moving along a straight track at a constant speed. The track passes under the Ferris wheel, and the performer is to be dropped from the moving Ferris wheel so that he lands in the tub.

Based on specific parameters provided (such as the rates of motion, the dimensions of the Ferris wheel, and starting positions of the cart and the performer), when should the performer be dropped?<sup>3</sup>

The problem involves many mathematical considerations. It serves as the IMP curriculum's vehicle for generalization from right-triangle trigonometry to circular functions. It also involves the idea of vector decomposition, as students take into account the initial “airborne” velocity of the performer—that is, the performer's velocity at the moment of release, due to the motion of the Ferris wheel itself.

Here I want to focus on the simplified version of the problem that students do first, in which they disregard the performer's initial velocity. (If the Ferris wheel is moving slowly enough, this initial velocity has only a small effect on the performer's fall.) As part of the analysis, students must determine how long it will take for the performer to fall a given distance.

Although some high school students are familiar with the formula  $s = \frac{1}{2}gt^2$  from their science courses, few have any understanding of where this formula comes from. In particular, even if they know that gravitational fall involves a constant rate of acceleration, they don't understand how this is connected to the formula.

We decided to take the principle of constant acceleration as a given—as an *axiom*, to put it in mathematical terms. As a mathematician and curriculum developer, I was faced with the challenge of finding a way to get from that principle to the formula, and I needed to do so within the restrictions of what would be meaningful to high

<sup>2</sup>The introduction of this concept appears in the Year 1 IMP unit The Game of Pig.

<sup>3</sup>This scenario was the central problem in the Year 4 unit High Dive in IMP's first edition. In the second edition, the discussion of this scenario is in two separate units, one at the end of Year 3 and one at the start of Year 4.

school students, based on assumptions of what they knew from earlier elements of the IMP curriculum.

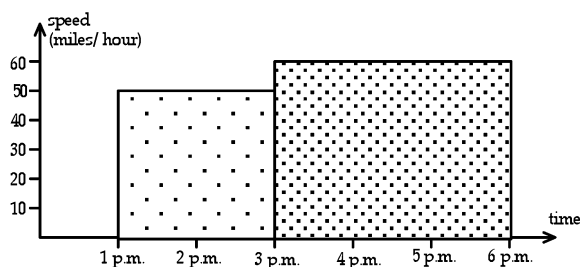
We tried several approaches, with both high school teachers and students, before hitting on one that worked. On a sophisticated mathematical level, the definition of velocity involves the derivative. But this also means that one can find distance traveled by finding the integral of the velocity function.

This last insight suggested to me a way of using students' understanding of area to develop an expression for position in terms of time. The first step was to establish an intuitive connection between area under the graph of the velocity function and total distance traveled. As part of this development, we opted to have IMP students work on an activity with the following as its first part:

1. Curt drove from 1 p.m. to 3 p.m. at an average speed of 50 miles per hour, and then drove from 3 p.m. until 6 p.m. at an average speed of 60 mph.

- Draw a graph showing Curt's speed as a function of time for the entire period from 1 p.m. to 6 p.m., treating his speed as constant for each of the two time periods—from 1 p.m. to 3 p.m. and from 3 p.m. until 6 p.m.
- Describe how to use areas in this graph to represent the total distance he traveled.

The graphs students create look like the one below, and they see that, in using the familiar "rate  $\cdot$  time = distance" idea to find the distance traveled, they are doing the same computation that they would use to find the areas of the two rectangles.



Through their discussion of Question 1, IMP students generally see intuitively that this connection between area and distance traveled should remain valid if the velocity is not constant. (In bringing out this insight, their teachers are laying a foundation for students who may later study the idea of defining area via approximating rectangles.)

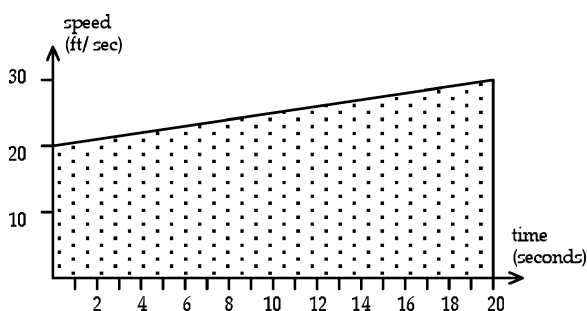
Building from that insight, students move on to the second part of the activity:

2. Consider a runner who is going at a steady twenty feet per second. At exactly noon, he starts to increase

his speed. His speed increases at a constant rate so that twenty seconds later, he is going thirty feet per second.

- Graph the runner's speed as a function of time for this twenty-second time interval.
- What is his average speed for this twenty-second interval?
- Explain how to use area to find the total distance he runs during this twenty-second interval.

On Question 2b, IMP students generally take a purely intuitive approach, saying that the speed increases at a constant rate from 20 feet per second to 30 feet per second, so the average speed is simply 25 feet per second. But they also recognize that Question 2c involves a variation on the earlier Question 1b—here the area is a trapezoid, as shown below, instead of the combination of rectangles in Question 1.



Working from the idea that the total distance traveled is again the area, they can confirm their insight that the average speed is the "midpoint" between the initial speed and the final speed. Moreover, they can see the role of the "constant acceleration" assumption. Putting these ideas together leads to this conclusion:

**If an object is traveling with constant acceleration, then its average speed over any time interval is the average of its beginning speed and its final speed during that time interval.**

Just a few small steps lead from this conclusion to the formula for gravitational fall:

- If velocity starts at 0 and increases at a rate  $g$ , then after  $t$  seconds, the velocity is  $gt$ .
- Therefore, the average velocity over  $t$  seconds is  $\frac{0+gt}{2}$ .
- Therefore, the total distance traveled over  $t$  seconds is  $\frac{gt}{2} \cdot t$ .

By first going through these steps for some specific examples, IMP students are able to develop the general formula.

## Some Lessons

The above examples illustrate some important points that may be of use to future curriculum developers:

- A flexible and deep understanding of a mathematical concept can provide insights into how to present that concept to students. Such an understanding profoundly informed our development of the IMP curriculum.
  - In the case of expected value, it was important to recognize that there was another definition of expected value that is mathematically equivalent to the standard definition.
  - In the case of gravitational fall, the key was recognizing how distance traveled could be represented via area.
- Creating curricula for high school students requires a clear picture of what they know, what they don't know, and the depth of their understanding of what they do know, as well as a clear picture of what their intuitions are likely to tell them.
  - For expected value, IMP students knew how to find totals and averages. They also appreciated intuitively that if the denominator of a fraction is “big”, then a small change in the numerator won't affect the fraction very much. But, since most high school students are not comfortable with symbolic formalism, an appeal to the distributive property to prove that the size of the sample doesn't matter would have had little meaning for them.
  - For gravitational fall, it was important to know that students were comfortable with the “rate · time = distance” idea and that they would also know how to find the relevant areas. (Note: If this material had not yet been a part of students' background, the curriculum writer would have needed to think about how to introduce it prior to its use here.)
- If we want students to apply a definition or formula with understanding, we need to build gradually, using concrete situations.
  - With expected value, the definition was preceded by concrete work with dice and examples involving averages. A deep understanding of the meaning of probability was crucial for building the concept of expected value.
  - For gravitational fall, we started with students' intuition about situations involving constant speed and their ideas about area. We combined this with the use of specific examples to strengthen

and expand those intuitions and to build the necessary connections.

- The true test of a curriculum element is its effectiveness in classrooms.
  - For expected value, some teachers wanted to define expected value using “the fraction method”. [This was their term for a definition along the lines of the expression  $\sum_{i=1}^n x_i \cdot p(x_i)$ . Generally, they wanted to use this approach because it was the definition they had learned in their own college courses.] Teachers who tried “the fraction method” reported considerable confusion among students, so they switched to the “in the long run” approach and got better student understanding.
  - For gravitational fall, we had tried some other approaches before coming up with the one described here. Teachers reported that students accepted the principle of “averaging the endpoints” but had no intuitive understanding of why that was legitimate. They indicated that the approach described here was successful because it both appealed to students' intuition and made meaningful use of their prior knowledge.

Carrying out these guidelines made different demands on me. As a mathematician, I already had a “flexible and deep understanding” of most of the mathematical concepts, but I soon realized that I needed to learn much more about statistics, which is an important part of the curriculum. Determining what the students knew and what their intuitions were telling them involved many hours spent in high school classrooms and talking with students (and that was fun). The principle of building ideas gradually and concretely had been driven home to me through many years of experience as a college-level teacher but needed constant attention in this work, especially because I was working at a different level of mathematics learning.

As to whether something really worked in the classroom, that dimension involved a more personal challenge. I needed to be willing to tear up something I'd spent months developing and start over with a new approach. I had to do that more often than I liked, but the final rewards made it worthwhile.

Overall, the work was as challenging and satisfying to me as a mathematician as any theorem or proof I ever developed, and certainly gave me a greater sense of contributing to society than I ever hoped to do through mathematical research.



# Strengthening the Mathematical Content Knowledge of Middle and Secondary Mathematics Teachers

Ira J. Papick

As a research mathematician and a teacher of mathematics, I have continually enjoyed the thrill of discovering (or re-discovering) mathematics and the excitement of discussing the beauty and utility of mathematics with colleagues and students. Stimulating mathematical conversations have made each day interesting and unique. Although I have taught several different mathematics undergraduate and graduate courses involving a variety of majors, much of my career has been spent working with prospective/practicing middle and secondary mathematics teachers. This teaching trajectory was not accidental, since my original occupational plan was to become a high school mathematics teacher. A desire to continue my studies of advanced mathematics altered this plan, and although my enthusiasm and passion for the subject has not directly impacted middle and secondary students, the mathematical preparation of their teachers has been central to my collegiate life.

Since my primary research area is in commutative algebra, I have taught and developed courses in linear and abstract algebra for prospective/practicing middle and high school teachers. These courses were rich in mathematical ideas, but the connections to important concepts in school mathematics were not always explicitly detailed. Several factors contributed to this shortcoming, but a most significant one was the lack of excellent curricular materials (both at the school and

university levels) to help illustrate and demonstrate the critical links. Without such materials, it is challenging and time-consuming for mathematicians, who primarily teach content courses for prospective teachers and who are typically unfamiliar with school mathematics curricula, to make these critical connections. A natural consequence of this predicament was frustrated students (What does this have to do with teaching mathematics in middle or secondary school?) and my own bewilderment (Why can't these students appreciate the beautiful theorems we are proving?).

To help address the need for specialized courses and materials for mathematics teachers, the Conference Board of Mathematical Sciences, in concert with the Mathematical Association of America (with funding provided by the United States Department of Education), developed the *Mathematical Education of Teachers Report (MET Report)*, 2001. This document carefully articulates a framework for mathematics content courses for prospective teachers that is built upon the premise that "the mathematical knowledge needed for teaching is quite different from that required by college students pursuing other mathematics-related professions."

Mathematics teachers should deeply understand the mathematical ideas (concepts, procedures, reasoning skills) that are central to the grade levels they will be teaching and be able to communicate these ideas in a developmentally appropriate manner. They should know how to represent and connect mathematical ideas so that students may comprehend them and appreciate the power, utility, and diversity of these ideas, and they should be able to

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understand student thinking (questions, solution strategies, misconceptions, etc.) and address it in a manner that supports student learning.

To further clarify the notion of “mathematical knowledge for teaching”, consider some authentic mathematics students’ questions that school algebra teachers regularly encounter in their teaching practice and should be prepared to address in a mathematically meaningful way.

1. My teacher from last year told me that whatever I do to one side of an equation, I must do the same thing to the other side to keep the equality true. I can’t figure out what I’m doing wrong by adding 1 to the numerator of both fractions in the equality  $\frac{1}{2} = \frac{2}{4}$  and getting  $\frac{2}{2} = \frac{3}{4}$ .
2. Why does the book say that a polynomial  $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$  if and only if each  $a_i = 0$ , and then later says that  $2x^2 + 5x + 3 = 0$ ?
3. You always ask us to explain our thinking. I know that two fractions can be equal, but their numerators and denominators don’t have to be equal. What about if  $\frac{a}{b} = \frac{c}{d}$ , and they are both reduced to simplest form. Does  $a = c$  and  $b = d$ , and how should we explain this?
4. I don’t understand why  $(-3) \times (-5) = 15$ . Can you please explain it to me?
5. The homework assignment asked us to find the next term in the list of numbers 3, 5, 7, ...? John said the answer is 9 (he was thinking of odd numbers), I said the answer is 11 (I was thinking odd prime numbers), and Mary said the answer is 3 (she was thinking of a periodic pattern). Who is right?
6. We know how to find  $2^2$ , but how do we find  $2^{2.5}$  or  $2^{\sqrt{2}}$ ?
7. My algebra teacher said  $\frac{x^2+x-6}{x-2} = \frac{[(x+3)(x-2)]}{(x-2)} = x + 3$ , but my sister’s boyfriend (who is in college) says that they are not equal, because the original expression is not defined at 2, but the other expression equals 5 when evaluated at 2.
8. My father was helping me with my homework last night and he said the book is wrong. He said that  $\sqrt{4} = 2$  and  $\sqrt{4} = -2$ , because  $2^2 = 4$  and  $(-2)^2 = 4$ , but the book says that  $\sqrt{4} \neq -2$ . He wants to know why we are using a book that has mistakes.
9. Why should we learn the quadratic formula when our calculators can find the roots to 8 decimal places?
10. The carpenter who is remodeling our kitchen told me that geometry is important. He said he uses his tape measure and the Pythagorean theorem to tell if a corner is square. He marks off 3 inches on one edge of the corner, 4 inches on the other

edge, and then connects the marks. If the line connecting them is 5 inches long, he knows by the Pythagorean theorem that the corner is square. This seems different from the way we learned the Pythagorean theorem.

*Remark.* The Situations Project, a collaborative project of the Mid-Atlantic Center for Mathematics Teaching and Learning and the Center for Proficiency in Teaching Mathematics, is developing a practice-based framework for mathematical knowledge for teaching at the secondary school level. The framework creates a structure for identifying and describing important mathematics that underlies authentic classroom questions (“situations”) arising in teaching practice (e.g., identifying and describing various mathematical ideas connected to questions such as those previously listed).

In addition to knowing and communicating mathematics, teachers of mathematics must be prepared to:

- Assess student learning through a variety of methods.
- Make mathematical curricular decisions (choosing and implementing curriculum), understand the mathematical content of state standards and grade-level expectations, communicate mathematics learning goals to parents, principals, etc.

This kind of mathematical knowledge is beyond what most teachers experience in standard mathematics courses in the United States (*Principles and Standards for School Mathematics*, NCTM, 2000), but there are a growing number of institutions, mathematicians, and mathematics educators who are determined to improve their teacher education programs along the lines recommended in the *MET Report*.

## Two Collaborative Projects: Mathematicians and Mathematics Educators Working Together to Improve Mathematics Teacher Education

Collaborative efforts between mathematicians, mathematics teacher educators, classroom teachers, statisticians, and cognitive scientists have yielded (and continue to yield) innovative foundational mathematics and mathematics education courses and materials for prospective and practicing teachers that fundamentally address the need to improve the mathematical and pedagogical content knowledge of teachers. These collaborations have provided a greater understanding of the varying perspectives on important issues regarding the teaching and learning of mathematics and have significantly contributed (and continue to do so) to the improvement of mathematics teacher education in the United States. What follows are two examples of such fruitful collaboration.

## I. Connecting Middle School and College Mathematics

Using the *MET Report* as a basic framework, a group of research mathematicians and mathematics educators at the University of Missouri-Columbia, in combination with a group of classroom teachers from Missouri, jointly developed four foundational college-level mathematics courses for prospective and practicing middle-grade teachers and accompanying textbooks as part of the NSF-funded project, *Connecting Middle School and College Mathematics* [(CM)<sup>2</sup>], ESI 0101822, 2001–2006. These courses and materials were designed to provide middle-grade mathematics teachers with a strong mathematical foundation and to connect the mathematics they are learning with the mathematics they will be teaching. The four mathematics courses focus on algebra and number theory, geometric structures, data analysis and probability, and mathematics of change and serve as the core of the twenty-nine-credit-hour mathematics content area of the College of Education's middle-school mathematics certificate program at the University of Missouri-Columbia. For those practicing elementary or middle-grade teachers seeking graduate mathematics experiences to improve their mathematics content knowledge, cross listed, extended versions of the four core courses developed under the (CM)<sup>2</sup> Project are offered for graduate credit.

In an effort to help students explore and learn mathematics in greater depth, the four companion textbooks that were developed as part of the (CM)<sup>2</sup> Project (*Algebra Connections*, *Geometry Connections*, *Calculus Connections*, *Data and Probability Connections*, Prentice Hall, 2005, 2006), employ a unique design feature that utilizes current middle-grade mathematics curricular materials in the following multiple ways:

- As a springboard to college-level mathematics.
- To expose future (or present) teachers to current middle-grade curricular materials.
- To provide strong motivation to learn more and deeper mathematics.
- To support curriculum dissection—critically analyzing middle-school curriculum content—developing improved middle-grade lessons through lesson study approach.
- To use college content to gain new perspectives on middle-grade content and vice versa.
- To apply middle-grade instructional strategies and multiple forms of assessment to the college classroom.

Throughout each book, the reader finds a number of classroom connections, classroom discussions, and classroom problems. These instructional components are designed to deepen the connections between the mathematics that students are studying and the mathematics that they will be teaching.

The classroom connections are middle-grade investigations that serve as launching pads to the college-level classroom discussions, classroom problems, and other related collegiate mathematics. The classroom discussions are intended to be detailed mathematical conversations between college teachers and preservice middle-grade teachers and are used to introduce and explore a variety of important concepts during class periods. The classroom problems are a collection of problems with complete or partially complete solutions and are meant to illustrate and engage preservice teachers in various problem-solving techniques and strategies. The continual process of connecting what they are learning in the college classroom to what they will be teaching in their own classrooms provides teachers with real motivation to strengthen their mathematical content knowledge.

## II. Nebraska Algebra

(Part of the NSF project, NebraskaMATH, DUE-0831835, 2009–2014). For several decades, school algebra has occupied a unique position in the middle and secondary curricula and even more so in recent times with the expectations of “algebra for all” (Kilpatrick et al., 2001). Not only is algebra a critical prerequisite for higher-level mathematics and science courses, but also it is essential for success in the work force (ACT, 2005). Most recently, several national reports have called for an intensified focus on the learning and teaching of school algebra (National Mathematics Advisory Panel; NCTM Focal Points; MET; MAA report, *Algebra: Gateway to a Technological Future*). Although the specific recommendations of these reports have some differences, all of them agree that “strategies for improving the algebra achievement of middle and high school students depend in fundamental ways on improving the content and pedagogical knowledge of their teachers” (Katz, 2007).

Employing the teacher-education recommendations of the aforementioned reports, with the ultimate goal of extending success in algebra to all students in Nebraska, a collaborative group of mathematicians, mathematics educators, classroom teachers, statisticians, and cognitive psychologists recently developed an integrated nine-graduate-credit-hour sequence designed to help practicing Nebraska Algebra I teachers to become master Algebra I teachers with special strengths in algebraic thinking and knowledge for teaching algebra to middle and high school students. Two of the courses in the program (Algebra for Algebra Teachers and Seminar in Educational Psychology: Cognition, Motivation, and Instruction for Algebra Teachers) are taught in a two-week summer institute (the first two cohorts completed these courses in the summers of 2009 and 2010) and, during the academic year following their participation in the Nebraska algebra summer institute, teachers return to the classroom and work with

an instructional coach or teaching mentor as they strive to transfer knowledge gained in the summer institute into improved classroom practice. In addition, teachers take a three-graduate-credit-hour yearlong pedagogy class focused on enhancing their ability to teach algebra to all students and to become reflective practitioners.

The Algebra for Algebra Teachers course was designed to help teachers better understand the conceptual underpinnings of school algebra and how to leverage that understanding into improved classroom practice. Course content and pedagogy development was strongly influenced by national reports and research findings, as well as by the collaborative expertise of mathematicians, mathematics educators, and classroom teachers.

The course content begins with a review of key facts about the integers, including the Euclidean algorithm and the fundamental theorem of arithmetic. The integers modulo  $n$  are studied as a tool to broaden and deepen students' knowledge of the integers, since questions concerning integers can often be settled by translating and analyzing them within the framework of this allied system. From this foundation, the course of study involves polynomials, roots, polynomial functions, polynomial interpolation, and polynomial rings  $k[x]$ , where  $k$  is the field of rationals, reals, or complex numbers. Special attention is paid to linear and quadratic polynomials/functions in connection to their importance in school algebra. Fundamental theorems established in the context of the ring of integers are studied in the context of  $k[x]$ , e.g., the division algorithm, Euclidean algorithm, irreducibility, and unique factorization. Additionally, applications (such as the remainder theorem, factor theorem, etc.) are considered, and other results in polynomial algebra (such as the rational root test, multiple roots and formal derivatives, Newton's method, etc.) are studied in depth.

The course pedagogy combines collaborative learning with direct instruction and was designed to provide teachers with dynamic learning and teaching models that can be employed in the school classroom. Course assessments include individual and collective presentations, written assignments, historical assignments, mathematical analyses of school curricula, extended mathematics projects, and a final course assessment.

## Conclusion

A fundamental tenet of our courses and material development is that mathematics teachers should not only learn important mathematics, but they should also explicitly see the fundamental connections between what they are learning and what they teach (or will teach) in their own classrooms. Moreover, while learning this mathematics, they should directly experience exemplary classroom practice, creative applications to a wide variety of state-of-the-art technology, and multiple forms of

authentic assessment. The work we have accomplished, and that which we hope to accomplish, has occurred through the collaboration of mathematicians, mathematics educators, classroom teachers, statisticians, and cognitive psychologists. The combined expertise and perspective of these professionals have significantly strengthened our efforts, and we are hopeful that these and future collaborations will help contribute to the improvement of mathematics teacher education in the United States.

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# Search for the Executive Director of the Mathematical Association of America

## Position

The Executive Committee of the Mathematical Association of America seeks candidates for the position of Executive Director to succeed Dr. Tina Straley, who will retire in December 2011 after twelve years of outstanding service. This position offers the appropriate candidate the opportunity to have a strong influence on all activities of the Association, as well as the responsibility of overseeing a large, complex, and diverse operation. The desired starting date is January 1, 2012.

## Duties and terms of appointment

The Mathematical Association of America is the largest professional society that focuses on mathematics accessible at the undergraduate level. The approximately 20,000 members include university, college, and high school teachers; graduate and undergraduate students; pure and applied mathematicians; computer scientists; statisticians; and many others in academia, government, business, and industry. Through its active program of publications, meetings, and conferences, the Association provides expository mathematics, professional development programs for faculty, and resources for teaching and learning. Its programs include the American Mathematics Competitions (AMC), the Putnam Examination, and Project NExT. The Association has its headquarters in Washington, DC. The AMC office is located in Lincoln, Nebraska.

The economic condition of the Association is healthy with an annual operating budget of approximately \$8 million. There is a staff of just over 40 people in the two offices.

The Executive Director is a full-time employee of the Association with administrative responsibility for the Association, is in charge of the facilities and staff of the Association, carries out such other duties as may be assigned by the Board, and is empowered to employ persons to discharge these duties. The directors of the various divisions report directly to the Executive Director. Besides these management duties, the Executive Director, together with the officers, provides leadership to the Association in furthering its mission to advance the mathematical sciences, especially at the collegiate level. The Executive Director, together with the President, represents the Association to outside groups and individuals.

The Executive Director serves at the pleasure of the Board. The terms of appointment, salary, and benefits will be consistent with the nature and responsibilities of the position and will be determined by mutual agreement between the Executive Committee and the prospective appointee.

## Qualifications

A candidate for the office of Executive Director should be a mathematician with significant administrative experience. The position calls for interaction with the staff, membership, and patrons of the Association as well as leaders of other scientific societies. Leadership, communication skills, and diplomacy are prime requisites.

## Applications

A search committee chaired by Ron Graham <graham@ucsd.edu> has been formed to seek and review applications. All communication with the committee will be held in confidence. Suggestions of suitable candidates are most welcome. Applicants should submit a CV, letter of interest, and an explanation of how their qualifications and experience will contribute to support the mission and build the future of the MAA. For full consideration, these should be sent by April 1, 2011, to:

Executive Director Search Committee  
c/o Julie Forster, Office of the MAA Secretary  
Box 15, Patterson 301  
Westminster College  
New Wilmington, PA 16172-0001  
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# MAA

MATHEMATICAL ASSOCIATION OF AMERICA



# A Mathematician– Mathematics Educator Partnership to Teach Teachers

*Ruth M. Heaton and W. James Lewis*

**T**he mathematical preparation of teachers is both a core problem in and a central solution to improving K–12 mathematics education nationwide ([1], [2], [3], [4]). Currently, there is broad agreement that most teachers, and especially elementary teachers, lack the depth of mathematical and pedagogical knowledge needed to teach mathematics (e.g., [5]). The mathematics knowledge that future teachers gain from their own K–12 education, including competency with basic skills and modest knowledge of algebra and geometry, is insufficient for the work of teaching elementary mathematics. Unfortunately, higher education is not seen as doing its part to “fix” this problem. For example, *Educating Teachers* [6] argues:

The preparation of beginning teachers by many colleges and universities... does not meet the needs of the modern classroom.... Professional development for continuing teachers... may do little to enhance teachers' content knowledge or the techniques and skills they need to teach science and mathematics effectively. [6, p. 31]

In this article, we emphasize how those with advanced mathematical knowledge can help to resolve the problems of mathematics teacher education. We address two questions:

- What knowledge, especially mathematical knowledge, do teachers need to have to teach mathematics effectively?

- How can teachers best learn what they need to know?

## **A Partnership of Expertise**

Simply requiring teachers to take more mathematics courses is an inefficient, impractical, and, almost certainly, inadequate response to the problem ([7], [8]). Most courses that teachers might take are not designed to prepare teachers, so the content is far removed from the work of teaching. Furthermore, teachers' collegiate mathematics education has historically been disconnected from their pedagogical preparation. What connections there are remain invisible to students and are not discussed among mathematics and pedagogy instructors. In fact, often there is deep-rooted distrust between the mathematicians and mathematics educators teaching these courses.

At the University of Nebraska-Lincoln (UNL), Lewis, a mathematician, and Heaton, a mathematics educator, have developed a ten-year partnership designed to address problems of elementary mathematics teacher preparation. Following the recommendations for forming interdisciplinary partnerships by the Conference Board of the Mathematical Sciences (CBMS) [9] and the National Research Council (NRC) [6], Lewis<sup>1</sup> and Heaton have integrated the intellectual content of school mathematics and the special blend of mathematical and pedagogical knowledge needed for teaching [10].

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<sup>1</sup>Lewis was chair of the steering committee for [9] and co-chair of the NRC committee that produced [6].

CBMS [9] proposes what teachers need to know and how best to learn it:

Prospective teachers need a solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power. Mathematicians are particularly qualified to teach mathematics in the connected, sense-making way that teachers need. For maximum effectiveness, the design of this instruction requires collaboration between mathematicians and mathematics educators and close connections with classroom practice. [9, p. xi]

To our partnership, Lewis brings his expertise as a mathematician and Heaton brings her understanding of learning to teach and research on teaching, coupled with ten years of experience as an elementary classroom teacher. We work closely with teachers in a local public school to connect their courses to real classrooms [6] in an effort to build prospective teachers' deep understanding of mathematics, children, and teaching.

A number of writers have described the intertwined nature of mathematical and pedagogical knowledge that is central to the goals of our program. Ma's [5] description of the profound understanding of fundamental mathematics needed for teaching gives one clear picture. Ball, Thames, and Phelps [10] describe the nature and structure of mathematical knowledge needed for successful teaching as:

[t]he mathematical knowledge "entailed by teaching"—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students. To avoid a strictly reductionist and utilitarian perspective, however, we seek a generous conception of "need" that allows for the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline. [10, p. 399]

In our teaching, Lewis focuses on helping prospective teachers acquire a deep understanding of the content of school mathematics and the attributes of mathematicians seriously engaged in doing mathematics. CBMS refers to these as the "habits of mind of a mathematical thinker" [9, p. 8]. Simultaneously, Heaton works with teachers to use their understanding of mathematics to find the mathematics in the many tasks of teaching mathematics [10]. This helps prospective teachers to develop productive habits of pedagogy [11] and to understand mathematics from the child's point of view. Our public school teacher partners help prospective teachers see the relevance of their

coursework in managing the realities of mathematics teaching and learning.

## Teaching Mathematical Content for Elementary Teaching

Most mathematicians, including many who take the work of educating teachers seriously, work in isolation from those more directly involved in teacher education. Our view is that this approach is less successful than having mathematicians and educators work in partnership.

Helping future elementary teachers learn the mathematics they need to know is hard work. Our students choose to become elementary teachers because they love children, not because they love mathematics. Many are weak mathematically. Past experiences have led them to believe they cannot be good in mathematics. They may believe that school textbooks and a teacher's guide are all they need to teach effectively. They may not fully appreciate the "intellectual substance in school mathematics" [9, p. xi]. Few understand the need or expect to be challenged by the need to understand thoroughly the mathematics of the elementary curriculum and can react negatively if their mathematics class proves to be harder than expected. Still, many are quite bright, are driven by a genuine passion to help children learn, and are quite willing to work hard. Thus mathematicians charged with the task of educating future elementary teachers often face a tough audience of learners. Teaching future elementary teachers can be a positive experience for both the students and the mathematician. A partnership between the latter and a mathematics educator and classroom teachers who support and communicate the importance of understanding mathematics helps ease the students' resistance to the mathematician's expectations.

We believe that mathematicians should hold high expectations for what they ask future elementary teachers to learn. Simultaneously, they need to support teachers as learners as they struggle to learn mathematics. Teachers should leave their mathematics courses believing in their ability to do mathematics and to reason about mathematical situations. They need to understand that mathematics is something that can and should make sense.

As Roger Howe wrote to Lewis, "For most future elementary school teachers the level of need is so basic, that what a mathematician might envision as an appropriate course can be hopelessly over the heads of most of the students" [12]. Most mathematicians need mathematics educators to help them to define the core mathematical knowledge of the elementary curriculum. Courses should focus on a thorough development of basic mathematical ideas, and teachers should be encouraged to develop flexibility in their ability to think mathematically, to develop careful reasoning

skills, and to acquire mathematical “common sense” [9].

Across the math courses Lewis teaches, his goal is to help teachers become productive mathematical thinkers with a toolbox of skills and knowledge to use to experiment, conjecture, reason, and ultimately solve problems. Developing “mathematical habits of mind” (e.g., [13], [14]) means helping learners to acquire understanding of and experience in using these tools. Although a complete mathematical toolbox includes algorithms, a person with well-developed habits of mind knows why algorithms work and under what circumstances an algorithm will be most effective. Mathematical habits of mind are marked by ease of calculation and estimation as well as persistence in pursuing solutions to problems. A person with well-developed habits of mind will want to analyze all situations, will believe that he or she can make progress toward a solution, and will engage in metacognition: monitoring and reflecting on the processes of reasoning, conjecturing, proving, and problem solving. In the pedagogy courses Heaton teaches, her goal is to help teachers develop pedagogical knowledge and skills that support the development of mathematical habits for elementary students.

### The Context of Teacher Education

UNL elementary education majors take twelve hours of mathematics. The first course is typically a general education course, Contemporary Mathematics, which introduces students to many ways in which mathematics is important to our daily lives. It covers topics such as Euler circuits and fair division. Next, they take a number and number sense course with the goal of developing a deep understanding of the arithmetic that is taught in the K–6 curriculum (place value, basic operations, fractions, primes). This is followed by a descriptive geometry course that focuses on understanding the measurement and geometry topics taught in the K–6 curriculum. Lastly, students choose a fourth course from a list that includes several courses designed for future mathematics teachers.

In 2000 we received a Course, Curriculum, and Laboratory Improvement (CCLI) grant from the National Science Foundation (NSF) to rethink elementary mathematics teacher preparation at UNL. The grant helped “purchase” cooperation within Heaton’s teacher education department. Lewis was chair of the mathematics department at the time, so cooperation from mathematics was assured. As funding ended, UNL adopted The Mathematics Semester [15], a four-course (ten hours), one-semester integrated immersion program for acquiring and learning how to apply mathematical knowledge to elementary teaching.

In addition to the arithmetic course, The Mathematics Semester includes two pedagogy courses

and a two-credit-hour field experience. One goal of the pedagogy courses is to help students understand mathematics from children’s perspectives. Another goal is to help them learn how to teach specific mathematical ideas to children. A third is to teach them how to establish and sustain a classroom culture that supports all children in studying mathematics, as well as other subjects. Prior to the development of The Mathematics Semester, the second pedagogy course and the field experience did not include any mathematics. Now they emphasize mathematics, effectively adding two mathematics education courses to the elementary education program. Students participate in The Mathematics Semester as a cohort, taking all four courses with the same group of students. Most of our students end the semester with a positive attitude toward learning and teaching mathematics and a significant increase in their mathematical knowledge for teaching.

### The Mathematical Content

Although future elementary teachers need to understand the mathematics they will teach, it is important that a college-level math class for teachers not completely mirror the elementary mathematics curriculum. Mathematicians will be pulled by their students to teach useful pedagogical strategies, such as providing math activities that could be done with fourth graders or a prescribed method for teaching fractions. A mathematician working with a mathematics educator can—and should—resist this pull. In the context of a partnership, the former can concentrate on mathematics while the latter attends to pedagogy.

Lewis, for example, teaches core mathematical content to teachers [16, 17], emphasizing problem solving, communication, and reasoning and proof [18]. In doing so, he models for future teachers a way of moving away from thinking about mathematics only as mastery of basic skills and computational fluency toward a definition of mathematics that recognizes and values mathematical proficiency as defined in *Adding It Up* [19]—including strategic competence, adaptive reasoning, and productive disposition.

Procedural fluency and conceptual understanding are important to both teachers and their students. It is a mistake to dismiss one in favor of the other [20]. Either the NCTM process standards [18] or the components of mathematical proficiency outlined in *Adding It Up* [19] can be used as organizational structures for mathematicians’ pedagogy. The mathematics educator can then, with students in the pedagogy course, analyze and reflect on the mathematician’s teaching, considering specific pedagogical strategies used by the mathematician and the varied outcomes.

## A Mathematical Example from Practice

In developing the habits of mathematical thinkers, instructors must identify interesting problems that are accessible but for which solutions or means of finding solutions are not immediately obvious. Such problems make an important contribution to the mathematical education of teachers, even though they may not connect directly to the particular content being studied in class. The problems should be challenging enough that students will want to seek out other members of the class with whom to work.

Lewis assigns these problems as weekly homework and expects the work to be accomplished outside of class time. The problems and their solutions are rarely discussed in class. If the problems are particularly challenging, Lewis might hand out a solution he himself has created after students have worked on them and turned in their own solutions. The solution then serves as a model for how they might communicate their solutions. By analyzing someone else's mathematical arguments and explanations, students can learn how to construct their own arguments.

Students just learning the careful reasoning necessary in mathematics have trouble if their first experience is with a mathematical proof. Even if they are asked for a straightforward proof (such as the proof that an odd number plus an odd number is even), they are not likely to engage. Students have much more success if they are given a problem to be solved, an answer to be found, or a solution to be justified. Such contexts offer students opportunities to ask and answer questions to help them move through the construction of a mathematical argument.

As a semester begins and students are adjusting to new courses and a new schedule, Lewis's first goal is to have students understand that there is important mathematics they need to understand but do not yet understand well. He believes that it is important that the first homework assignment establish his expectation that students will put time into their mathematics course. One such problem was based on a Puzzler used on the National Public Radio show *Car Talk* [21].

The Chicken Nugget Conundrum involved both intentional and unintentional complexity. The mathematical complexity was planned. Not planned, and a surprise to Lewis, was the unintentional linguistic complexity in the problem. As a result, some students found themselves off track, unable to solve the problem because they did not understand the question being asked. For example, some students interpreted the sentence, "You can only buy them in a box of six, a box of nine, or a box of twenty" to mean they could only consider multiples of six, nine, or twenty, not different combinations, despite the example that clarifies this point. Others interpreted "Explain why it is

## The Chicken Nugget Conundrum

*There's a famous fast-food restaurant you can go to where you can order chicken nuggets. They come in boxes of various sizes. You can only buy them in a box of six, a box of nine, or a box of twenty. Using these order sizes, you can order, for example, thirty-two pieces of chicken if you want. You'd order a box of twenty and two boxes of six. Here's the question: What is the largest number of chicken pieces that you cannot order? For example, if you wanted, say, thirty-one of them, could you get thirty-one? No. Is there a larger number of chicken nuggets that you cannot get? And if there is, what number is it? How do you know your answer is correct?*

*A complete answer will:*

- i) Choose a whole number "N" that is your answer to the question.*
- ii) Explain why it is not possible to have a combination of "boxes of six" and "boxes of nine" and "boxes of twenty" chicken nuggets that add to exactly N pieces of chicken.*
- iii) Explain why it is possible to have a combination that equals any number larger than N.*

possible to have a combination that equals any number larger than  $N$ " to mean that they needed to show that some number larger than, say, forty-three was possible. These examples point out not only linguistic but also cultural differences. Whereas precision in language is highly valued by mathematicians, teachers (and many others) tend to be linguistically imprecise. Teachers need to acquire the habit of mind that precise language is important in mathematics.

Below are several solutions Lewis received: Parts i and ii:

*1) I've found that the largest number of chicken pieces that I cannot order is forty-three. I know that forty-three is correct because it is not divisible by any number except one and forty-three. This makes a prime number. I also know that forty-three is the correct number because it cannot be broken down into any combination of the numbers twenty, nine, and six. With the number forty-three, it is not possible to have a combination of multiple "boxes of six", "boxes of nine", or "boxes of twenty" because you cannot use these numbers to reach forty-three.*

*2) You cannot have any combination that adds to forty-three because it can't evenly divide by six, nine, or twenty. It is not a multiple of fifteen and it can't be evenly divided in half.*



3) *There is no combination for forty-three, but there is for some of the numbers below, and all the numbers above. I had noticed in the early numbers that it was skipping by ten. Five did not work, fifteen did, twenty-five did not. Then once it got higher they doubled or added.*

4) *It is not possible because the numbers don't have common divisors. Although forty-three is a prime number. if [sic] we had a three-pack we could fulfill the order with one three-pack and two orders of twenty.*

Part iii:

5) *It's possible to have a combination greater then forty-three. This is because you can buy all the multiples of the numbers. For example, if you buy eighteen, you can buy thirty-six and seventy. Or if you by [sic] twenty you can buy forty, sixty, eighty, one hundred, etc.*

6) *After forty-three, I went up to sixty-five and everything between forty-three and sixty-five could be done. Beyond forty-three, each number would be able to work because they are multiples of six, nine, or twenty. This could go on forever, but I figured after forty-three they will somehow work!*

7) *Every number after forty-three is possible whether you add six, nine or twenty multiple combinations will give you any number will give you more than forty-three.[sic]*

These answers include poor grammar, imprecise language, information irrelevant to the solution, "trust me" arguments, and evidence that some of the students did not understand the question they were trying to answer. For example, in argument #3, it is not useful to note that one cannot order five nuggets. Nor is it useful to talk about what would be possible under different conditions from those given in the problem (see argument #4). Argument #6 indicates the student believes a proof is found in the existence of a large number of examples. We leave it to readers to complete an assessment of these answers.

Lewis was surprised by the level of difficulty the problem posed for students and the many weaknesses in the justifications of their solutions. Lewis assigned low marks to our students' work but then took time in class to discuss the weaknesses in students' solutions by sharing some of the false explanations and asking students to discuss them.

Lewis changed what he first feared was a disaster into a lesson on communication, using the false explanations to help students learn about the nature of mathematical argument, the importance of using precise language, and different reasonable methods of justification. After the discussion, students were offered the opportunity to redo their explanations. Most took advantage of the opportunity, meeting with Lewis and putting considerable energy into communicating their understanding of a solution. It was common to receive two-page typed solutions for what was only a ten-point assignment. Susan's solution, here considerably shortened, was typical:

*It is easy to see that no more than two boxes of twenty could be used. If you subtracted the two boxes of twenty from forty-three, only three would remain... (and) there are no boxes that offer only three chicken nuggets.... If you were to use only one box of twenty, you would subtract twenty from forty-three resulting in twenty-three. These twenty-three would have to come from a combination of boxes of nine or six. Both nine and six are multiples of three, so any combination resulting from boxes of nine, boxes of six, or both would have to be divisible by three also. Twenty-three is not divisible by three so any combination... would not work. ...the only other option is to use no boxes of twenty.... Just like twenty-three, forty-three is not divisible by three either. ...With all the possible options eliminated it is clear that forty-three cannot be created using a combination of twenty, nine, or six.*

Despite the stress of this process (for everyone concerned), our students made significant progress quickly, responding to the challenge to reason about mathematics, to use careful language, and to communicate their understanding. By the third homework assignment, most students were meeting regularly in groups to work on the homework and to produce careful explanations. They regularly offered a two- to three-page solution to a ten-point homework problem. The high quality of their work made the homework easier to grade. The homework problems thus established a culture of mathematical explanation that carried over to class and our study of the mathematics taught in the elementary classroom.

### Translating Mathematical Knowledge into Classroom Practice

Over time, we have integrated the goals and practices of our teaching mathematics and pedagogy. We began simply by scheduling the mathematics and pedagogy classes back-to-back in the same



university classroom and by requiring students to take all four classes in the same semester. We used separate syllabi but experimented with assignments that were given in multiple courses. Currently, we have one integrated syllabus for the entire Mathematics Semester, including several assignments that require students to apply what they learn from one course setting to another. For these assignments, we evaluate students together, and they receive single grades that count in more than one course.

These integrated assignments give students practice in some of the “mathematical tasks of teaching” [10]. Assignments include such things as modifying tasks to be easier or harder. We also ask students to plan, teach, and reflect on math lessons, in elementary school settings that are one-on-one, in small groups, or with an entire class of students. Students are also asked to appraise mathematical topics within reform curricula and to identify within these topics intellectually rich problems for children. The students are required as well to recognize the mathematical knowledge that teachers need to teach the topics well.

One major assignment is the Learning and Teaching Project. Students use a challenging homework problem from Lewis’s class to plan a lesson in which they work with a K–5 child in the field. The goal is to adapt the problem so as to offer the child a successful learning experience. Students need to consider vocabulary, instructional representational tools, a sequence of tasks, and possible questions to move the child along in his or her thinking about the problem.

One such assignment begins with Crossing the River [22], a problem that Lewis first saw at a conference that Ira Papick organized in Missouri. An edited version of the problem is shown in the box.

When the problem is assigned in the second week of the semester, students often struggle to explain their reasoning and to state and solve the problem for  $a$  adults and  $s$  students. When they receive the Learning and Teaching assignment later in the semester, they may think it is an unreasonable problem for a second-grade or even fifth-grade student and look to their cooperating teacher at the elementary school to validate their belief. Fortunately, the teachers always support us because Heaton has developed a strong partnership with them.

Our students videotape themselves teaching and write a paper about their experience. The assignment thus proves valuable in many ways. Students come to realize that they cannot teach mathematics successfully unless they understand it themselves. Students also must use their pedagogical knowledge to prepare appropriate manipulatives, to plan how they will present information and ask questions, and to anticipate the difficulties the children will have. Often, in their papers,

### **Crossing the River**

*A group of adults go on a camping trip with a group of fourth-grade students. They come to a river that is too deep to wade across. They find a boat, but it isn’t very big. The adults are rather big, and only one adult can fit in the boat at one time, but the boat can hold any two fourth-grade students. The students have experience boating, and each can safely row across the river by themselves.*

*If there are four adults and two students on the trip, is it possible to get all of them across the river? If yes, how many one-way trips across the river will it take? What if there were five adults and only one student? What if there were five adults and two students or four adults and six students? How can the problem be generalized? Solve the general problem or at least several more cases.*

students write about their surprise that young children can be creative and successful with challenging mathematics assignments. This integrated learning experience would not be possible without our mathematician/educator/teacher partnership.

### **Expanding the Partnership**

The key to improving K–12 mathematics education is to build teachers’ mathematical and pedagogical knowledge, and the need is not limited to the context of preparing future elementary teachers. Many current K–12 teachers have similar needs. The separate expertise of a mathematician and a mathematics educator, joined in a successful partnership, is the right foundation to support this kind of work. At Nebraska, our partnership has resulted in two large NSF grants for Math Science Partnerships (MSP), Math in the Middle Institute Partnership and NebraskaMATH. Information about these grants is available on our website (<http://scimath.unl.edu/>).

### **Conclusion**

As we look back on ten years of working together, we are convinced that our partnership has been the key to our success. Heaton’s courses are more mathematical than they were a decade ago. Lewis’s courses have a much stronger connection to the work of teaching elementary mathematics. By supporting each other, we are able to hold our students to high standards and to help them learn both mathematics and how to teach mathematics so that they end the semester with a positive attitude toward mathematics.

The partnership that began with a CCLI grant has given us the opportunity to work with hundreds of mathematics teachers who are eager to learn more mathematics in a context that enables them to be more successful teachers. Many of our

colleagues are now involved in teaching teachers, either as part of The Mathematics Semester or through one of our MSP grants. Over the past five years, over thirty mathematics graduate students have benefited from working in our projects as part of instructional teams, thus enhancing their ability to teach and their knowledge of teachers. This experience has proven quite valuable as they earn Ph.D.s and apply for jobs. The partnership is also supporting a substantial research program in mathematics education.

We encourage others to join us in working across department lines to benefit both their departments and the future teachers they educate.

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# Philosophy, Math Research, Math Ed Research, K-16 Education, and the Civil Rights Movement: A Synthesis

*Ed Dubinsky and Robert P. Moses*

**R**obert Moses studied philosophy of mathematics under W. V. O. Quine at Harvard. He has taught high school in New York City; Jackson, Mississippi; and Miami, Florida. A major figure in the civil rights movement of the 1960s, he is currently developing an algebra curriculum for middle and high school mathematics that reaches out effectively to students from underrepresented groups.

Ed Dubinsky spent twenty-five years doing research in functional analysis (including a solution to Grothendieck's bounded approximation problem) and another twenty-five years doing research in undergraduate mathematics education. He has developed a theory of learning topics in undergraduate mathematics and has designed and disseminated innovative curricula in several undergraduate courses. He, too, was active in the civil rights movement of the 1960s.

It is probably our common experiences in struggles for human rights and our commitment to understanding how the mind might work when a student is trying to learn mathematics that has allowed us, in spite of disparate backgrounds and life experiences, to communicate about high school algebra. In any case, the mathematician has been able to contribute to the philosopher-educator's

Algebra Project, which has grown and which we both hope will continue to grow in coming years.

It is the purpose of this article to discuss the thinking that has gone into this work and to describe some examples of what has come out of it. First we will give our separate points of view about the epistemology of learning mathematics, then discuss a synthesis of the two approaches, and then describe our high school algebra curriculum as it relates to modular arithmetic. Finally, we will describe how the Algebra Project, founded by Moses, relates to the civil rights movement.

## **Ed's Story**

Throughout the twenty-five years I spent doing research in functional analysis and teaching undergraduate mathematics at six universities in five countries and on three continents, I was always interested in effective teaching. Unfortunately, in spite of trying a myriad of popular methods (modified Socratic, self-paced instruction, mastery learning, etc.), what I produced, more often than not, was *ineffective* teaching. I was a good lecturer, enthusiastic about teaching, serious in my attempt to do it well, and I cared about my students. They liked me and my courses, but from everything I could see, they were not learning much more than students of other teachers, and that was woefully inadequate—as many national reports of the 1970s and 1980s concluded.

At one point, I decided in my frustration that if I were to significantly improve my students' learning, I was going to have to figure out something about the process of learning mathematics. That is, I would need to study what might be going on

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in a student's mind when he or she is trying to understand a mathematical concept. What mental activities need to take place in order for a student to be successful in such learning? I thought that, as I came to know more about the learning process in mathematics, I would be able to figure out pedagogical strategies that would help students engage in appropriate mental activities so as to be more successful. So I began to read. I read a lot over the first two years of my new career (new because, shortly after I started, all of my interest in functional analysis shriveled up). Some of the education literature I read was good; most was not very helpful. It was not until I came across the work of Piaget that I thought I had found an author who understood the mental processes of learning mathematics. I remembered that, as a young student of functional analysis, I had considerable difficulty with the idea of the dual of a locally convex space. I was fine with the notion of a linear functional that acted on elements of a locally convex space to produce numbers—linearly. But the idea of applying actions to these transformations, putting them together in a set, equipping the set with arithmetic and even topologies, was really tough for me. These linear functionals were doing things to elements of a vector space, so how could things be done to them? It was terribly confusing. I struggled for a long time and eventually mastered the mathematics. But I can't say I understood what had gone on in my mind.

It was when I read Piaget's discussions of transformations, the content which they transformed, the fact that these dynamic transformations could be stabilized in one's mind and thereby become contents for higher level transformations, and that this latter step was very *difficult* both historically and for individual students, that I knew I had come home. I began to see that it might be possible to identify mental constructions required to understand a mathematical concept. Working with the ideas of Piaget, I began to express them in an explicit theory called APOS theory. APOS is an acronym for Actions, Processes, Objects, and Schemas. It was developed by a team of mathematicians and mathematics education researchers led by me (see Asiala et al., 1996)

### APOS Theory

APOS theory is based on Piaget's principle that an individual learns (e.g., mathematics) by applying certain mental mechanisms to build specific mental structures and uses these structures to deal with mathematical problem situations. According to this principle, for each mathematical concept, there are mental structures one can develop that are appropriate for this concept and that can be used to learn it, understand it, and use it (Asiala et al., 1996). If one has built appropriate structures, very elementary concepts can be grasped easily and early through normal life experiences,

trial and error, and discussions with peers. Later, with such structures, more advanced concepts can be learned without undue difficulty via any pedagogical method that relates the concept to the structures. If, however, one does not possess structures appropriate for a concept, it is nearly impossible to learn it.

This aspect of Piaget's theory can explain a phenomenon that seems to be almost universal with respect to learning mathematics. Just about everyone learns the most elementary mathematics: counting, sequential ordering, forming sets, the concept of number. Even as the mathematics becomes less elementary, an individual may feel for a while that the mathematical ideas are almost obvious. One need only have a concept mentioned, perhaps explained, and then it is understood almost immediately and automatically. This period of "automaticity" can last for very different periods of time depending on the individual (from months to decades), but, for everyone, the time comes when the ideas become more difficult. Intervention of others (teachers, colleagues, books) becomes necessary, and learning can be delayed, eventually even stopped. What is happening, according to Piaget's principle—what needs intervention and takes time—is that the individual is building new mental structures to deal with the more complex concepts. At first, with the elementary concepts, the mental structures are built more or less automatically through normal day-to-day experiences. Later, as the mathematics becomes more sophisticated and the requisite structures more complex, intervention, or at least reflection over a period of time, is necessary and, for even the most powerful research mathematician, there are, eventually, mathematical concepts he or she cannot fully understand. The stopping point comes at different places for different people, and one measure of mathematical talent can be the extent of mental structures one is able to build with minimal intervention.

This principle has important consequences for education. Simply put, it says that teaching should consist of helping students use the mental structures they have to develop an understanding of as much mathematics as those available structures can handle. For students to move further, teaching should help them build new, more powerful structures to handle more and more advanced mathematics.

These ideas raise certain questions. Given a mathematical concept, what are the mental structures that can be used to learn it, and, knowing that, how can we help students build them? It is these questions that APOS theory and a pedagogical strategy based on it try to answer.

According to APOS theory, the mental structures are what we call *actions*, *processes*, *objects*, and *schemas*. The mental mechanisms used to build these mental structures are called



*interiorization* and *encapsulation*. An *action* is a transformation of a physical or mental object that requires specific instruction and must be performed explicitly, one step at a time. A mathematical concept begins to be formed when an action transforms objects to obtain other objects. As an individual repeats and reflects on an action, it may be *interiorized* into a mental process. A *process* is a mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly. Given a process structure, one can reverse it to obtain a new process or even coordinate two or more processes to form a new process via composition. If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has *encapsulated* the process into a mental *object*. In some situations, when working with a mental object, it is necessary to *de-encapsulate* the object back to the process from which it came. While these structures describe how an individual constructs a single transformation, a mathematical topic often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, which is called a *schema*. The mental structures of action, process, object, and schema constitute the acronym APOS.

Determining the specific actions, processes, objects, and schemas for a given concept requires research and a specific methodology that I will not discuss in this article. It may be helpful, however, to consider an example from elementary mathematics that will also allow a proposed explanation for a difficulty in arithmetic that is widespread among students and even some teachers. I am talking about the concept of division by a fraction. One understanding of division by a number requires that the number be understood as an object, and the division question is: How many of this object can be found in the dividend? Now think about the notion of a fraction, say  $2/3$ . Initially, one can take a specific object (e.g., a pie or a rectangle), divide it into 3 equal pieces, and pick two of them. If an individual can think of  $2/3$  only in terms of such an activity, then he or she has an action conception of  $2/3$ . After repeating such an action and reflecting on it, the individual may construct an internal process that allows her or him to imagine dividing an unspecified object into 3 parts and taking 2 of them. This is a process conception of  $2/3$ , and most people, as the result of normal human activity, will come to this point without too much difficulty. It is the next step, necessary for understanding division by  $2/3$ , that is difficult. In order to divide, say, 5 by  $2/3$ , that is, to ask: "How many  $2/3$ s are there in 5?" one must

understand that this question requires thinking of  $2/3$  as an object. Without such an understanding, one can't begin to think about an answer to the division problem. Thus, one must encapsulate the process conception of dividing into 3 parts and selecting 2 into an object which becomes a somewhat abstract entity in the mind of the individual. Most people need help with this process, and it is not immediately obvious how to help students to use the mechanism of encapsulation to come to see the  $2/3$  process as an object, also called  $2/3$ . In the next section we discuss methods to help students do this.

The above is a very brief description of an analysis that requires considerable research and that must be done for every mathematical concept one wishes one's students to learn. After reading about these ideas applied to very elementary mathematics, we developed APOS theory as a formulation of Piaget's theories that could be applied to more advanced mathematical concepts.

### **APOS-Based Pedagogy: Writing Computer Code and Programs**

I began to look for pedagogical approaches to fit with this theory. I wanted to find ways to induce students to make the mental constructions called for by the theoretical analyses of concepts. I found that one could go a long way in this direction by having the students write certain computer programs or just code. That is, for each mental construction that comes out of an APOS analysis, one can find a computer task of writing a program or code such that, if a student engages in that task, he or she is fairly likely to make the mental construction that leads to learning the mathematics. I am not saying that the computer task is the mental structure but rather that performing the task is an experience that leads to one or more mental constructions.

Here is an example. Consider the concept of function. As with fractions, an APOS analysis says that development of understanding the function concept begins with an action understanding. That is, a function is understood to be an algebraic/trigonometric expression with numbers and a symbol, usually  $x$ . The action consists in replacing  $x$  with a number, making the calculation specified by the expression, and getting a number as the answer. It is externally directed in the sense that it follows a formula that is external to the individual performing the action. With repetition and reflection, the learner can interiorize this action, which means that he or she builds a mental structure that does the same thing internally that the action does externally. This mental structure is called a process, and it allows an individual to imagine the action as being performed without actually having to perform it. It is then possible to think of the function in terms of "something comes in, something is done to it, something



comes out". With a process conception one can coordinate two or more processes to obtain a new process and reverse a process, first in one's mind and then, if needed, with pencil and paper. Finally, if an individual wishes to perform an action on this mental process, he or she first has to see it as a totality and encapsulate it mentally into an object. Then the individual can act on it. (For more details, see Asiala et al., 1996.) Now what kind of pedagogy can be based on such a theoretical analysis?

First, the teacher needs to have an idea of where the students are relative to the construction of requisite mental structures. Is the student restricted to thinking about functions as actions, or is he or she able to understand a function as a process but is still unable to encapsulate these processes as objects? The teacher needs to know this mental activity in order to navigate through the course material. The students may also need to know this in order to have a good idea of their progress. The research provides indicators that can help make reasonable conjectures about where students are relative to an APOS analysis. For example, if a student insists (as many do) that unless there is an explicit formula, there is no function, then such a student is probably at the action level for functions. On the other hand, if he or she is comfortable with forming sets of functions or realizes that the derivative can be interpreted as an operation that transforms a function into another function, then the student may be thinking at the object level for functions.

Working together with several colleagues, we found that a host of mathematical concepts could be analyzed in terms of these actions, processes, and objects. Such analyses could explain student difficulties in terms of mental constructions not made. On the other hand, we found that if we asked students to perform a mathematical action and write a computer program expressing that action, then, in performing this task, the student tended to interiorize the action into a process. Even more exciting was that if the student then wrote another program that accepted the first program as an input, transformed it in some way, and returned a new program, then this student was very likely to encapsulate the process and see it as an object. Going back and forth between object and process conceptualizations of a mathematical idea, so necessary in doing mathematics, resulted from this pedagogy almost effortlessly (Weller et al., 2003).

Based on these ideas, we devised a structured pedagogical approach. It works by a division of the course material into small units, each to last about one week. Each week is a cycle of three kinds of work. First, the students work (usually in cooperative groups) in a computer lab to write programs and code designed to foster mental constructions that can help them build an understanding of the

concepts in that unit. They complete this work outside of class. Second, meeting in a classroom, the students work (again in groups) on tasks designed to help them convert the mental structures they have built into understandings of mathematical concepts. Third, based on the assumption that most of the students have at least begun to build understandings that fit with the mathematical ideas held by mathematicians, they are given exercises designed for practice, reinforcing the knowledge they are building, and extending that knowledge (Asiala et al., 1996).

We have designed and implemented undergraduate courses that follow this approach. Textbooks have been written that in their structure and content reflect the three-part cycle. We have conducted empirical studies using both qualitative and quantitative research methodologies of student performance and attitudes. Our results suggest that this approach can be highly effective in helping students learn various advanced mathematical concepts that appear in subjects such as precalculus, calculus, discrete mathematics, abstract algebra, and linear algebra (Weller et al., 2003). It must be acknowledged, however, that this pedagogical strategy requires teachers not only to significantly alter their thinking about learning and teaching but also to exert considerable effort to learn the method. We believe that these requirements are among the things that have limited the widespread adoption of such a strategy in undergraduate mathematics teaching.

### Bob's Story

In the 1987–1988 school year, I was a parent volunteer teaching algebra to eighth graders in the open program at the Martin Luther King Jr. school in Cambridge, Massachusetts. My son, Omo, was in the class and wanted very much for some of his friends to be part of the class. He said he felt lonely when he was doing algebra. One of his friends wanted to be part of the group but didn't know his multiplication tables. I agreed to take him in the group and we worked side by side, one on one, every day. When we came to questions about the number line, adding integers on the number line, he always got the same kind of answers. That is, he consistently answered a question different from the one the book was asking. He had only one question about numbers in his mind, namely the "how many" question. My problem was to figure out another question about numbers that he needed to get into his mind.

I finally settled on a "which way" question. This question was a part of his daily routines and vocabulary. He knew how to ask: "Which way to the mall?" or "Which way to a friend's house?" But he didn't have his "how many" questions together with his "which way" questions as part of his concept of number. My problem became how to get his "which way" questions into his number

concept on an equal footing with his “how many” questions.

One day, while traveling from Cambridge to Boston, I entered the T-stop on the Red Line at Central Square and noticed that all passengers are called upon to decide whether they are going inbound or outbound—two answers to a “which way” question. At this point, I recalled Quine’s ideas about the process of generating elementary mathematics along with the concepts of experiential learning that had been a part of pedagogy at the open program in the Martin Luther King Jr. school. I, along with other teachers, then organized students to take trips on the T and asked them to write, talk, and draw pictures about their trips. We thought of these representations as their commonsense representations, what Quine calls “ordinary discourse”. We then asked them to identify important aspects, called *features*, of these representations and discussed with them obvious features that they may not have paid attention to, such as the start and finish of the trip, as well as features that were not so obvious, such as locations and relative positions of stops.

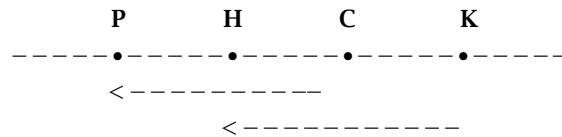
This process, which Quine identifies as a process for mathematizing events, involves moving from ordinary discourse to *regimented language*, that is, the language used in mathematics. Adapting his theories to the classroom, we called the commonsense representations *people-talk* and the regimented or strait-jacketed representations *feature-talk*. We engaged the students in the process of constructing iconic symbols, that is, symbols that are also pictorial representations, as well as abstract symbols for the features that we intended to mathematize, and we developed iconic, as well as abstract, representations for various mathematical features of these trips.

Over time, it became clear that students mathematizing these trips acquire powerful metaphors and concepts for addition and subtraction very different from their arithmetic metaphors for those operations, including the concept of displacement as a mathematical object representing answers to both the “how many” and the “which way” questions. For example, consider the following two questions: “Where is Porter Square in relation to Central Square on the Red Line in Cambridge?” and “Where is Harvard Square in relation to Kendall Square?”

Underlying both questions is the concept of the relative position of two stops on the Red Line. The answer to both questions is the same: two stops outbound, an answer to both “how many” and “which way”.

The geometrical representation of this answer is a displacement two units outbound. Students thought of the movement from Central Square to Porter Square as starting at Central Square and moving two units outbound, and of the movement from Kendall to Harvard as starting at Kendall and

moving two units outbound. Thus we have two movements which have the same number of stops and are in the same direction. That is, these two movements represent the same displacement.



We call this diagram an *iconic* representation of the trips. The people-talk representations are the statements:

Porter Square is two stops outbound from Central Square.

Harvard is two stops outbound from Kendall.

Feature-talk involves explicit reference to location and relative positions of stops. This gives us addition as movement from the location of one stop to the location of another in one of two directions, and subtraction as the comparison of the location of the ending to the location of the starting stop. In other words,

starting at the location of Kendall  
and moving two stops outbound  
one arrives at the location of  
Harvard

is feature-talk leading to addition, and

the location of Harvard compared  
to the location of Kendall is 2 stops  
outbound

is feature-talk leading to subtraction.

To obtain this mathematization, we select some stop as the benchmark. We then discuss with the students assigning symbols such as 0 for the benchmark,  $x_1$  for the location of Kendall,  $x_2$  for the location of Harvard, and  $\Delta x$  for the displacement. Then the first feature-talk sentence becomes

$$x_1 + \Delta x = x_2,$$

and the second becomes

$$x_2 - x_1 = \Delta x.$$

We can summarize the mathematization of this type of sentence in the following eight steps:

1. Identify the observation sentence.  
Harvard is two stops outbound from Kendall.
2. Identify the name(s) in the sentences.  
Harvard, Kendall.
3. Identify the predicate of the sentences.  
The predicate in this case is the relation of equality (“is”) between a name (“Harvard”) and the object resulting from applying an operation (“two stops outbound”) to a name (“Kendall”).
4. Construct an icon for the name(s).  
The students will do this.

5. Construct an icon for the predicate.  
The students will do this.
6. Construct an iconic representation of this sentence.  
This is the Trip Line diagram shown above.  
The students will do this.
7. Translate the observation sentence into a sentence using regimented language.

In this case there are two ways of doing so:

- a. Starting at the location of Kendall and moving two stops outbound one arrives at the location of Harvard.
  - b. The location of Harvard compared to the location of Kendall is 2 stops outbound.
8. Identify the conventional symbols that are needed to translate the regimented language into conventional mathematical symbols and make that translation.

We might take  $L(H)$ ,  $L(K)$  for the locations of Harvard and Kendall, respectively, and we take  $+$  for “move” and  $-$  for “compared to”. This leads to the following abstract symbolic representation of the two sentences:

- a.  $L(K) +^{-} 2 = L(H)$
- b.  $L(H) - L(K) =^{-} 2$

This recipe for converting an experience into a mathematical expression can be applied in a wide variety of situations and, together with students actually experiencing the situation, represents our main contribution to the pedagogy referred to as experiential learning.

## A Synthesis

The synthesis of the above sets of ideas in our curriculum materials uses the structure described in Bob’s story as the basic navigational framework of the material while paying attention to possible actions, processes, and objects that students might be constructing in their minds, as described in Ed’s story. Thus writing computer programs has been replaced by playing certain games, discussing them, and writing about them. On the other hand, many of the specifics of the games are driven by the need to make certain mental constructions suggested by APOS theory.

We can make other uses of a synthesis of the two “stories”. Consider, for example, the relation that appears in every Algebra 1 high school textbook:

$$a - b = a +^{-} b.$$

Here,  $a, b$  are any two integers. As we saw in the discussion of trips in Bob’s story, an integer can be interpreted as a movement of a certain number of steps in a certain direction or as a location on a line. So is an integer a movement or a location? The APOS theory in Ed’s story resolves this seeming ambiguity. If an integer is interpreted as a movement, then this is a process in the sense of APOS theory. The encapsulation of that process is

an object that, in the case of an integer, is a location on the number line. With the mechanism of encapsulation and its opposite, de-encapsulation, we may go back and forth between interpreting an integer as either a movement or a location.

Now, suppose we start at a location  $b$  and make the movement

$$a +^{-} b.$$

This movement is constructed by moving from the benchmark to the location  $a$ , making the movement  $^{-}b$  to arrive at the location  $a +^{-} b$ , which is then de-encapsulated to a movement that we also call  $a +^{-} b$ .

Now we can start at the location  $b$  and make the movement  $a +^{-} b$ , which, by our interpretation of addition, brings us to the location

$$b + (a +^{-} b),$$

which, using standard properties of integers,<sup>1</sup> is equal to the location  $a$ . To summarize, we have said that if we start at  $b$  and make the movement  $(a +^{-} b)$ , then we arrive at  $a$ . According to our interpretation of subtraction, this movement is just  $a - b$ . So we have:

$$a - b = a +^{-} b.$$

Now this relation may seem too obvious to mention to experienced mathematicians, but it appears explicitly in almost every high school algebra text and is one of the more difficult parts of beginning algebra.

To develop this material for the classroom, we divide the content into segments. Each segment begins with an experience, such as a game. The students play the game and record salient information. Each student then writes a description of what happened in the plays of the game. They are encouraged to write in complete sentences, organized in paragraphs (people-talk). Then, in a classroom discussion, the teacher helps them identify the features of the game (feature-talk), the operations that were performed with these features, and the predicate that describes the goal of the game (process of mathematization). The students are then asked to work in teams to answer certain questions designed to move them further toward mathematization of the situation. This is completed with the teacher describing the mathematics in language and symbols that are used by mathematicians.

We can also use this approach to interpret two equations that are so important in the mathematics that comes after algebra:

$$x_2 - x_1 = \Delta x,$$

$$x_1 + \Delta x = x_2.$$

The first relation says, according to our interpretation of subtraction, that the comparison of  $x_2$  with  $x_1$  is  $\Delta x$ . That is, it is the movement that takes

<sup>1</sup>These properties are developed in our curriculum before the treatment being described.

us from  $x_1$  to  $x_2$ . In other words, if we begin at  $x_1$  and make the movement  $\Delta x$ , we arrive at  $x_2$ , which, according to our interpretation of addition, is precisely the second equation.

Of course these two equations involve no more than very simple arithmetic, but, in order to do that arithmetic with any kind of understanding, students need to have useful interpretations—metaphors if you like—for the equations. We believe that the metaphors we have presented for addition and subtraction of integers can provide the necessary interpretations.

## An Example

As a final example, here is a brief outline of curriculum material based on certain games for the topic of modular arithmetic. In discussing these games and what happens in the classroom, we will explain how this pedagogy relates to the ideas in Ed's story and in Bob's.

The first goal of this unit is for students to understand the mathematical operation of division-with-remainder of a positive integer  $a$  by a positive integer  $b$  in terms of the classic equation,

$$(1) \quad a = qb + r, \quad r = 0, 1, 2, \dots, b - 1.$$

The curriculum begins with a game called Winding Around Positions. There are twelve stations that could represent hours on a clock or the Chinese years zodiac. A reference station is selected (in general, selections are made by the class with some input from the teacher), and one student sits at that station throughout the game. The class selects an integer, and a second student goes to the starting position and then moves through the stations, counting until the selected number is reached. While the student is moving, note is taken of the number of times the second student passes by the first and of the final position reached by the second student.

The features of this game are: the starting position, number of positions to be moved, number of winds, and final position. The operation is to count the positions, and the predicate asks how many winds there are. The purpose of this game is for the students to construct a mental process of moving through the stations and winding around the circle. We do this by first getting the students to perform the action of multiplying explicit numbers  $b$  by numbers  $q$  and adding quantities  $r$  that are less than  $b$  and second by interiorizing this action into a process that does the same. The reason for doing this is that an APOS analysis expresses the mental process underlying (1) as the reversal of the process of multiplying  $b$  by  $q$  and adding  $r$ .

The next game is played with the same setup but, instead of beginning with a single number, the students select a number of winds and an increment (which must be between 0 and 11). Here the features are essentially the same, but the operation is to multiply the number of winds by 12 and add

the increment to respond to the predicate, which is: how many positions have been traversed? The mathematization to which the students are led is the basic division-with-remainder formula (1). This expresses a mental process in which a single traversal of all twelve stations has been encapsulated into a "wind".

The next game is designed to help the students reverse the mental process of multiplying the number of winds by 12 and adding the increment. It is also played with twelve stations representing the hours on a clock. A number of hours is given to represent time elapsed. Working in teams, the students begin at 12 and count around the clock to determine the number of winds and the increment that gives the final time on the clock. In this game, the features are: the time elapsed, the number of winds, and the remainder or end time. The operation consists of dividing the time elapsed by 12 to find the number of winds (quotient) and the end time (remainder). The mathematization of this game is division-with-remainder. It is symbolized by the same formula (1), which now is seen as expressing the reversal of a process. That process consists in multiplying a number of winds by 12 and adding an increment to obtain a total. The reversal consists in starting with the total, determining the number of winds, and determining the remainder.

All of the games are now repeated, with the twelve hours on a clock replaced by the seven days of the week. Then there is a summary discussion in which the ideas are mathematized to obtain the notion of an integer mod  $n$  where  $n$  is 12, 7, or any positive integer. This permits a discussion of *equivalence mod  $n$* , partitions of a set of integers, and the relationship between equivalence and partition.

One can then return to the clock and days-of-the-week games to do arithmetic, using the same epistemological perspective and the same pedagogy. For addition, one simply plays the winding game with two numbers. With the first number, one begins at the starting point (12 o'clock or Sunday) and then, with the second number, one begins at the ending point reached by the first number. A deep mathematical idea that can be represented in the game (and hence is likely to be accessible to the students) is that one can add two numbers  $a$  and  $b \bmod n$  by either adding first and then finding the equivalent mod  $n$  or finding the equivalents first and then adding mod  $n$ . Of course the standard group properties of  $\mathbb{Z}_n$  with addition mod  $n$  can be discussed entirely in terms of trips around the clock or in the calendar.

For multiplication, we play the addition game several times using the same number. This leads to multiplication as repeated addition through the use of all of the same pedagogy, including people-talk, feature-talk, mathematization through operations on the features and evaluating a predicate,



and assigning symbols. The result is the concept of multiplication mod  $n$ . Since the bases 12 and 7 are used, the students can experience directly the mathematical phenomena of the axioms for a field being satisfied in the mod 7 system but not in the mod 12 system. Some of the brighter students may even be interested in thinking about the properties of 12 and 7 that lead to this difference.

### The Algebra Project and the Civil Rights Movement

The United States, to lay down the economic foundations for the caste system established after the Civil War, built a steel industry on the backs of the indentured slavery of young black men in Alabama (Blackmon, *Slavery by Another Name*) and established its textile industry on the pittance doled out to sharecroppers picking cotton in Mississippi (Barry, *Rising Tide*; Lemann, *Redemption and The Promised Land*). The civil rights movement dismantled the manifestations of the caste system in public accommodations, voting, and the National Democratic Party; however, the clearest manifestation of this caste system remains in its public schools (*U.S. v. State of Mississippi*, Civil Action 3312). The Algebra Project, a direct descendent of the 1961 to 1965 Mississippi Theater of the civil rights movement, tackles head-on this dimension of the nation's unfinished work (Moses, testimony to the U.S. Senate Judiciary Committee).

It is our contention that, with the ascendance of information technology and the increasing complexity of our society, mathematics joins reading and writing as a literacy needed for full citizenship. Like it or not, history has thrust mathematicians and specialists in mathematics education into the middle of a central American dilemma: the reconciliation of the ideals in the Declaration of Independence and the United States Constitution with the structures of race and caste and the legacies of slavery and Jim Crow.

Briefly, in 1875, Congress refused to consider President Grant's appeal for a constitutional amendment to guarantee at the level of the federal government the right to an education for all children, including those of the freed slaves. It did pass a civil rights bill, but the Supreme Court of 1883 declared that Congress had no right to do this, thus setting the stage for eighty-one years of rigid race and class divisions (Civil Rights Cases, 1883; see also Justice Harlan's dissent).

The Court decided that, for the purpose of access to public accommodations, the nation's constitutional people were decisively citizens of states rather than citizens of the nation, a constitutional status applicable to the vote and membership in the national political party structures as well as to public school education.

The Supreme Court's landmark 1954 decision did not challenge, with respect to their education, this constitutional status of the nation's children.

Rather it affirmed the "equal protection" clause of the Fourteenth Amendment: states, rather than the federal government, have a constitutional obligation to provide their citizens equal access to public school education. As James Bryant Conant reminded us in 1961, the nation's caste system thus found its clearest manifestation in its education system (Conant, 1961).

Such inequality was confirmed in 1968, when four hundred Mexican American high school students left school to march on their school board to demand better physical facilities and better teachers. Their mothers sued, and their case, "San Antonio Independent School District v. Rodriguez" was decided March 21, 1973:

*Justice Lewis Powell's majority opinion in Rodriguez held that education was not a fundamental right, since it was guaranteed neither explicitly nor implicitly in the Constitution.*

Powell's decision, in effect, guaranteed that public school education remained the clearest manifestation of the nation's caste system, which now extended over class as well as race. This situation still holds today.

When, in 1960, Kennedy stepped into the presidency, black students at historically black universities and colleges stepped into history: "On February 1, 1960, four African American college students sat down at a lunch counter at Woolworth's in Greensboro, North Carolina, and politely asked for service. Their request was refused. When asked to leave, they remained in their seats. Their passive resistance and peaceful sit-down demand helped ignite a youth-led movement to challenge racial inequality throughout the South" (C. Vann Woodward, 2001).

The sit-in students demanded, in effect, a change in their constitutional status: for purposes of access to public accommodations, they demanded status as citizens of the nation rather than citizens of a state. This demand was made crystal clear a year later, with the Freedom Rides.

Thanks largely to Ella Baker, the sit-in movement was transformed into a network of sit-in leaders called the Student Nonviolent Coordinating Committee, or SNCC. Then, thanks largely to Amzie Moore, SNCC transported the sit-in energy into Mississippi to focus on the constitutional status of sharecroppers in the Mississippi Delta, especially with respect to the right to vote. SNCC organized sharecroppers not only to demand constitutional status as citizens of the nation with respect to voting rights but also to demand an equivalent status with respect to participation in the National Democratic Party structure, making it possible for a Democratic Party Convention to consider an African American as its presidential nominee.

Robert (Bob) Moses, coauthor of this article and president and founder of the Algebra Project, was the director of SNCC's Mississippi operations. He



left Mississippi in 1965, left the country in 1966, and made his way to Tanzania with his wife Janet, where they started their family. They returned to the United States in 1976 with their four children: Maisha, Omowale (Omo), Tabasuri (Taba), and Malika. Bob's job in the family was to make sure the kids did their math, a job he enlarged as a parent volunteer in the Open Program of the Martin Luther King School in Cambridge, Massachusetts, to teach Maisha and three of her classmates algebra when she hit the eighth grade in 1982. Bob got a MacArthur fellowship in 1982 and settled into the issue of algebra for all the eighth graders in the Open Program, thereby launching the "Algebra Project", which inevitably found its way into Mississippi and the issue left hanging from the Mississippi civil rights movement of 1961-1965: the constitutional status of children in the nation with respect to their public school education. It seems clear that, unless children become decisively citizens of the nation for the purposes of their public school education, public school education will remain the clearest manifestation of the nation's caste system.

## Conclusion

Today, the Algebra Project, working together with sister organizations such as the Young People's Project, with support from the National Science Foundation and other public as well as private agencies, is a national movement that is trying to transform the educational experiences of children from the underserved lowest quartile of our population. It is a prime example of how people from the academic fields of philosophy, mathematics research, mathematics education research as well as teachers and administrators from the field of K-16 education and also those of us who struggle for social and economic justice in the United States can find common ground, work together, and contribute to solving some of the major problems facing our country in the twenty-first century.

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# More Than a System: What We Can Learn from the International Mathematical Olympiad

Mark Saul

## Brigadoon

The northern European sky is often ambiguous. Patches of intense blue alternate with lowering grays. Fog conceals the landscape, lifting to reveal sky and water, then descending like a huge curtain. An enormous glass wall, one side of a hotel dining room, highlights the drama of the sky over the North Sea in the German city of Bremerhaven.

The jury of the 2009 International Mathematical Olympiad (the IMO), representatives of 104 countries, trickles in. The conversation reveals a community coming together, a community that, like the fabled Scottish village Brigadoon, comes to life once a year, for just ten days. Old friends greet each other, fill each other in on personal news, on prospects for their team, on the uncertainties of international travel. They sort themselves by language: English, French, Spanish, Russian, Chinese, and many smaller communities. It is a peculiarity of today's political geography that official languages are typically shared by two or more countries.

The full jury meets the next morning to decide on the problems to be set for the students. The discussion ranges from mathematics to pedagogy to the art of problem solving. Sequestered (by tradition) from the students, who are housed twenty miles away in Bremen, they look for problems that will cover a range of levels of difficulty and a variety of mathematical topics.<sup>1</sup> The problems must not favor routine methods studied in the school systems of participating countries. They must be true problems, not exercises.

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<sup>1</sup>See <http://www.imo-official.org/problems.aspx>.

The discussion is in earnest. Not only must each team be treated fairly, but also each student must get something out of participation. So the problems must have a certain difficulty—and a certain significance. The choices are not easy ones to make.

## The Beauty Contest

A system, not quite an algorithm, emerges from the chaos of opinion. Representatives rank the problems not just by difficulty but also by “beauty”, a bit of ironic language that acknowledges a measure of arbitrariness in the judgments. Problems can be OK, pretty, gorgeous, and so forth. Delegates have observed that even these terms are “politically correct”. “OK” is more like homely, “pretty” is just “plain”, and so on. Diplomacy reigns throughout.

Indeed, for an international body, there is little contention. The lion and the lamb are equally powerful, mathematically. Historically, disagreement arises only on mathematical issues or on levels of difficulty.

And, occasionally, even on honesty. Over the years there have been just a few incidents of dishonesty and also of incorrect accusations of dishonesty. We hear that members of this team all gave the same unlikely solution for problem A. Members of that team visited the restroom too often. A journal in one country had a problem similar to problem B or one that gave a hint for problem C. Unfortunate, and requiring a most delicate sort of diplomacy. Enjoyment of the competition and fulfillment of its goals depends on people working together to achieve these goals. In making your team as excellent as possible, you are working toward a common goal. In making it merely look better than another team, you are working just for your own.

The search for problems is thus a process of

achieving consensus.<sup>2</sup> The debate touches on numerous issues in the field:

In problem D, you can write down all the coordinates, and then it's an unpleasant but routine matter to compute the lengths of the two segments in question. I'm too old to do this, but for a student who sees it, the solution becomes tedious and routine. On the other hand, the official solution, classic and synthetic, is very nice. The student who chooses this method will not suffer.

Motion that we start by choosing the easy problems. Good easy problems are the hardest to find.

Query: Our delegation would like to hear comments on the content of problem E. What is the significant mathematics in this problem?

Some of these problems are harder than they look. Problem F involves geometry, which is always difficult. Weaker students will fatuously chase angles. Motion to label Problem F "medium" rather than "easy".

Problem G falls to a technique which is standard in our curriculum. I'm not sure this is best, either for our students or for others. Motion to strike problem G.

The jury eventually reaches consensus. The significance of the entire event hinges on this consensus, a different significance for each audience. The participants themselves mostly find it fun and challenging and enjoy being with peers having similar interests, working on tasks involving just those interests.

## Ripples

But, like a stone dropped in a pond, the ripples of the event have wider, albeit less intense, impact beyond that on the students gathered in Bremen. John Webb is the secretary of the IMO Advisory Board and has worked on mathematical competitions in South Africa for more than thirty years: "The IMO had a profound effect on our national

mathematics scene. Our Olympiad used to be severely elitist. Entering a team in the IMO stimulated the creation of a broad-based talent search, which in turn increased the number of students taking part in the national Olympiad. This change was influenced by the need to find students to represent the country in the IMO."<sup>3</sup>

József Pelikán from Hungary is a veteran of many IMOs and has been chair of the IMO Advisory Board for the last eight years: "Perhaps numbers alone will not tell the story. In some countries even the very idea that math can be a topic for a competition and not just an endless source of drill is such a novel one that it drives change. This phenomenon would not be easy to summarize in a research report."

Here, at the epicenter of IMO activity, I found evidence of some of the ways that Pelikán's observations played out. The clearest picture, in an event like this, is of the peaks of achievement: the students who are most successful. An examination of their success can yield a broader picture of how we can build and maintain systems that discover and nurture this sort of talent.

## Omer Cerrahoglu

One of these is Omer Cerrahoglu. Born in Istanbul but educated in Romania, fourteen-year-old Omer distinguished himself in a chain of local and regional contests and was among the youngest students at the 2009 IMO, where he received its highest honor, a gold medal.

I first heard of him from a Romanian mathematician friend in the United States, then again from Radu Gologan, the leader of the Romanian team. Their comments carry weight. Romania is among the leading countries in the IMO. The very first IMO was organized by Romania, and they have consistently done well—better than the size of their population or their economy would predict—on these events. So the Romanians have seen it all, and if they say that this student is outstanding, I pay attention.

Gologan gave a firsthand description of Omer's thinking style: "Omer is intuitive. Like Ramanujan, he looks and sees and writes it down—then later he proves it. He has gotten many classical results by himself. I've seen him solve a problem about the altitudes of a triangle without knowing what an orthocenter was. 'The altitudes must intersect, and the intersection has this and this property', he reasoned, and eventually solved the problem."

But at what cost? Can a fourteen-year-old personality support a mind like this? Gologan

<sup>2</sup>A more detailed account of the process of selecting IMO problems can be found at <http://www.win.tue.nl/~wstomv/publications/imo2002report.pdf> or at [http://www.maths.otago.ac.nz/home/schools/gifted\\_children/olympiad.pdf](http://www.maths.otago.ac.nz/home/schools/gifted_children/olympiad.pdf).

<sup>3</sup>For more information about the South African system, see, for example, Mark Saul, "A distant mirror", *AMS Notices*, April 2001, at <http://www.ams.org/notices/200104/comm-saul.pdf>.

answered the question before I asked it, starting with an oxymoron: “He is an extremely normal kid. His colleagues love him. In the last few days of preparation, we have a psychologist interview the students about handling the competition emotionally, how to remain calm, to allow their minds to function under pressure, and so on. This psychologist had not seen a child with Omer’s range of intellectual and emotional maturity. Her comment? ‘If Omer didn’t exist we would have to invent him’.”

The important point is that Romania’s system of talent development was able to support Omer. He lives in the north of Romania, not in the capital. Because he went to school in Romania, he entered an Olympiad. The system thus “discovered” him, then provided him with peers and mentors who would encourage and reward his achievement. This is a mature system, typical of those in eastern Europe, which has been described often before.<sup>4</sup>

### Raul Sarmiento

And even he was not the youngest in history. Terry Tao was slightly younger in 1986 when he wrote his paper for a bronze medal. He was ten, then had a birthday that week, so he was barely eleven years old when he received the bronze medal.

Omer was also not the youngest student at the 2009 IMO. That distinction belonged to eleven-year-old Raul Arturo Chavez Sarmiento from Peru, who won a bronze medal. Whereas Omer is the product of a mature and robust system, Raul has been served by a successful nascent system. Like Omer, Raul lived in a city far from the capital. His gifts showed up on the national Olympiad, so one of the schools in Lima gave him a scholarship, and the rest is history.

Maria de Losada, the Colombian team leader, is also one of the prime movers of the IMO. She started the program in her own country and leveraged her success there to bring virtually the entire Latin American region into the IMO. Losada comments: “The Peruvians have been working very hard. Their Olympiad is supported by their ministry of education and reaches virtually all the schools in the country. Peru has a population of about twenty-five million, with about three million students in the national Olympiad. The Peruvians get a larger number of students, and a higher percentage of students, to participate in their national Olympiad, than many countries with a similar population.

<sup>4</sup>See Mark Saul, “Mathematics in a small place: Notes on the mathematics of Romania and Bulgaria”, in *AMS Notices*, May 2003 (at <http://www.ams.org/notices/200104/comm-saul.pdf>) or Saul, “Love among the ruins: The education of high-ability mathematics students in the USSR”, *Focus*, Vol. 12, No. 1, February, 1992.

“In Latin America, we work with each other and learn from each other. More than most other regions, we are unified by language, history, and culture. In 1985 UNESCO helped us initiate the Ibero-American competition, which continues to this day. The IMO in Argentina (1997) brought us still closer together. The organizers of these events were able to attract coordinators [question graders] from all over Latin America, who learned from each other about the mathematics and logistics of the Olympiad. Our Colombian team has trained with teams from Venezuela, Ecuador, Costa Rica, Panama, and Peru.

“One of the keys is an early start. We have found the most success when students and teachers learn early about competitions in mathematics. Another is the forging of a community which includes both the mathematicians and the teachers. For example, in Brazil, one of the national Olympiads is funded by the government, which pays professors at the public universities to develop and grade the contests.

“Difficulties? One difficulty we find, particularly from the smaller countries in our region, is that students don’t see themselves as potential winners in the IMO. They’ve triumphed in the national contests, but when they meet students from larger countries, or from countries with older and deeper traditions, they are intimidated. They tend to lose confidence and frequently don’t do their best. We see this time after time and are not sure how we can help.”

### Systems Make the Difference

As I talked with IMO students and coaches, the importance of systems of support emerged. A mind like Omer’s or Raul’s is a great gift. Where do such talents come from? How do we find them? The answer seems to lie in large-scale systems of support. Had Omer been born in Senegal, had Raul been born in Peru forty years ago, their gifts would very likely have gone unnoticed. The systems serving them made possible the emergence of their gifts.

The Romanian system is old and robust. The Peruvian system is younger but sturdy. In other regions of the world, the system of talent development in mathematics is much more fragile.

African countries, for example, are not well represented at the IMO. Just five African countries (out of about fifty) had a team enter prior to 2005, and the number is growing only slowly. In 2009 two “new” countries sent teams, and two more sent observers (a prerequisite to being invited in the next year). Similar conditions hold in the Middle East.

### Building Capacity

How does a country build the capacity to field an IMO team? And how can it use the opportunity to build such a team to stimulate wider and deeper interest in mathematics? One model effort has



been launched by Saudi Arabia. The Saudis have fielded an IMO team for several years but have not been satisfied with its achievement. Rather than sending a team in 2009, they sent observers, who have been looking at best practices for team development throughout the world.

One of these observers was Dr. Abdulaziz Salem Al-Harthi, the IMO Project Manager for Saudi Arabia: “We are concerned about the standard of mathematics in our schools on the precollege level. We see IMO participation as a motivation to stimulate learning and appreciation of the subject in our country. Our mission here is supported by the Mawhiba Foundation, headed by the king.”

Al-Harthi described his program. They use an SAT-style test, taken widely throughout the country, to retain a broad base yet identify the most promising students. These students are urged to take a national Olympiad test. The winners are then trained and a team identified from among them.

Al-Harthi said: “Selection and training are the key. We have visited a number of countries and invited experts in training Olympiad students to our country from around the world. Not only will they train our team, but they will give us ideas about how we can do this ourselves in the future.

“Good mathematics students in our country tend to go into medicine or engineering, fields in which their contributions, and their monetary rewards, are much more visible to the public. We are hoping that the IMO will help us change this public view of mathematics.”

Even well-established teams give evidence that their systems work. The Faroe Islands, a part of Denmark, are in the North Atlantic, more than one hundred watery miles from the next center of population. Yet twice in the history of the IMO, students from the Faroe Islands were chosen to represent Denmark in the IMO. It was a system of talent identification that allowed this: a local teacher got the students to enter the national contest.

## The Loftiest Peaks

So systems catch talent. But do they support it? Nurture it? Bring it to fruition? Or is success in mathematics solely the result of determined individuals?

The 2009 IMO was special: it was the fiftieth event of its kind.<sup>5</sup> The first IMO was initiated by Romania in 1959, and early events were attended only by countries of the Eastern bloc at the time. The first Western country (Finland) participated in 1965, and the United States joined in 1974.<sup>6</sup>

With the support of the John Templeton Foundation, the German organizers of the 2010 IMO brought six Fields Medal winners, all IMO alumni,

to speak with the students. They spoke both formally and informally, and their words give a picture of a community, by now international, that creates systems, by now global, that produce mathematics.

## Bollobás

Béla Bollobás was at the first three IMOs, 1959–1961: “It was completely different then. In those days, Hungarians couldn’t travel, and even a trip to neighboring Romania was a treat. I was very excited to meet so many students from other countries. It was great to know that being among the top in Hungary also meant being among the elite young mathematicians in the world—or at least the Eastern bloc.

“My mathematical talent showed pretty early. Until I was nine years old, my father, a doctor, wanted me also to be a doctor. I have no idea how, but father managed to persuade a medical instrument factory to produce tiny toy medical instruments for me to play with, so that I’d long to use the real instruments later. But when I was nine my father had a chat with my teachers in school, who told him that I was rather good at maths. Soon I was some years ahead of my peers, and later this lead stretched to about four years. When I went to the university, I unofficially knew the mathematics of the first three or four years. I took all the courses, and I loved them, even though I knew most of the material already.”

Bollobás here was putting a positive spin on a comment I had heard from many much younger students at the IMO: school mathematics was boring. These minds need more of a challenge.

Bollobás continued. “In school I did not have to pay any attention to the maths lessons. My mathematical inspiration came from private lessons from an excellent lecturer, István Reimann, at the University, and from *KöMaL*, our Hungarian student journal. Reimann guided me in various areas of mathematics. We always had a ball talking about maths. In *KöMaL* I could read about what my peers were doing mathematically, even if I hadn’t met them. And this journal gave the students plenty of motivation, because they published the best solutions. That seems to be an ideal system. In some ways, this is better than the IMO, since at the IMO you get no credit for giving two different solutions, or for a generalization. Working for *KöMaL* was much closer to research.”

## Gowers

Timothy Gowers is tall and thin, with a shock of white hair. He speaks slowly and deliberately—and one wants to hang on to his every word. He is a great communicator. I enjoy his writing. Brisk and terse, it is as effective as, but completely different from, his speech.

One of Gowers’s remarkable contributions to our intellectual life does not concern mathematics itself but “doing” mathematics. He has proposed a

<sup>5</sup>See, for example, <http://imo.wolfram.com/facts.html> and <http://imo.wolfram.com/morefacts.html>.

<sup>6</sup>See <http://www.imo-official.org/results.aspx>.



## Some Achievements of Mathematicians Mentioned in this Article

### *Béla Bollobás*

IMO 1959 Bronze Medal  
IMO 1960 Gold Medal  
IMO 1961 Gold Medal

### *Timothy Gowers*

IMO 1981 Gold Medal  
Fields Medal, 1998

### *László Lovász*

IMO 1963 Silver Medal  
IMO 1964 Gold Medal  
IMO 1965 Gold Medal  
IMO 1966 Gold Medal  
Wolf Prize, 1999

### *József Pelikán*

IMO 1963 Silver Medal  
IMO 1964 Gold Medal  
IMO 1965 Gold Medal  
IMO 1966 Gold Medal

### *Stanislav Smirnov*

IMO 1986 Gold Medal  
IMO 1987 Gold Medal  
Fields Medal, 2010

### *Terence Tao*

IMO 1986 Bronze Medal  
IMO 1987 Silver Medal  
IMO 1988 Gold Medal  
Fields Medal, 2006  
MacArthur Fellowship, 2006

### *Jean-Christophe Yoccoz*

IMO 1973 Silver Medal  
IMO 1974 Gold Medal  
Fields Medal, 1994

### *Günter M. Ziegler*

IMO 1981 Gold Medal

### *Radu Gologan*

IMO 1970 Silver Medal  
IMO 1971 Silver Medal

Sources:

<http://www.imo-official.org/hall.aspx>  
[http://en.wikipedia.org/wiki/Wolf\\_Prize\\_in\\_Mathematics](http://en.wikipedia.org/wiki/Wolf_Prize_in_Mathematics)  
<http://www.infoplease.com/ipa/A0192505.html#axzz0zEgA62dW>  
[http://en.wikipedia.org/wiki/MacArthur\\_Fellows\\_Program](http://en.wikipedia.org/wiki/MacArthur_Fellows_Program)

form, or forum, called “polymath”, in which anyone who cares to can contribute to the research of a mathematical problem. The structure of the form (it does have a structure!) can be viewed<sup>7</sup> at <http://gowers.wordpress.com/2009/01/27/is-massively-collaborative-mathematics-possible/>.

But Gowers did not talk with me about his polymath idea. He talked more directly about the IMO: “I was not a prodigy. In school, I knew that I was good at maths. But I didn’t know what I was capable of until I came to the IMO.

“The IMO was a positive experience, a life-changing experience. For the first time I realized that I might actually be good enough at mathematics to be a mathematician. Had I not gone to the IMO, I would still have gone up to Cambridge to read maths. Having got there, I would have found other students who were clearly better than me. Without the confidence I gained at the IMO, I might

have been demoralized. But as it was, the IMO had given me a faith in my ability that could survive the times when I did less well relative to my peers. So I persisted.

“By the way: I also know some people for whom it was not a good influence in their lives, people who were unable to make the transition to research-level mathematics.

“The IMO problems are different from mathematical research. In doing research, one can actually change the problem one is working on, shape it, make it more tractable and one’s efforts more fruitful. IMO problems are fixed.”

József Pelikán had a different metaphor for this point: “IMO problems are like animals in a zoo. Mathematical research is like studying animals in the wild.”

Is it the competition at the IMO that is a driving force in mathematical creativity? Is it rivalry, or even just communication, with one’s peers that makes people want to explore mathematics?

“That’s part of it,” explained Gowers. “But not everything. Let’s put it this way. Suppose I went to prison for some crime. Suppose I were allowed access to a library, to the tools of mathematical

<sup>7</sup>For another of Gowers’s contributions to the problem-solving community, see <http://www.tricki.org/>. See also <http://numberwarrior.wordpress.com/2009/03/25/a-gentle-introduction-to-the-polymath-project/>.

research, but not to my colleagues. In that situation, I do not know whether I would still do mathematical research. The subject is intrinsically interesting, and there are certain problems I would love to solve, but solving problems requires a great deal of dedication: the reactions of other mathematicians are one of the main reasons I continue with it.”

#### Yoccoz

Jean-Christophe Yoccoz addressed this subject in a slightly different way: “Is mathematics useful? Usually, but I don’t care. Still, as a Platonist, I have the feeling that when you discover something in mathematics, you are discovering something for real, something that exists in the real world. And that gives me satisfaction.”

Yoccoz, like Tao and Bollobás, knew early that he would be a mathematician. “I came from an academic family. My father was a physicist, my mother a translator, so I had a good idea that one could make a career as an academic. I discovered my love for mathematics very early, and it still motivates my work.”

#### Tao

Terence Tao’s particular brand of intellectual enthusiasm showed through in one of his first remarks: “The IMO was one of the best times of my life. A week’s vacation, and all you have to do is answer six simple questions.”

Did the IMO influence his decision to become a mathematician? “My course was set on mathematics quite early. The IMO experience meant something else to me. First of all, it was fun being with other students who enjoyed solving hard math problems. I don’t remember any particular bit of mathematics I learned at the IMO that proved significant later on. It’s not like that. But the habits of problem solving—taking special cases, forming a subproblem or subgoal, proving something more general, and so on—these became useful skills later on. It’s worth teaching students those skills through Olympiad work.

“But serious mathematical research involves other skills: acquiring an overview of a body of knowledge, getting a feeling for what sorts of techniques will work for a certain problem, putting in long and sustained effort to accomplish something. These are skills I acquired later, through other means. I would not want students to think that IMO-type problems are all there is to mathematics. I mean, the twin primes question [the existence of infinitely many twin primes] seems like an Olympiad problem if you look at it shallowly. Even getting one pair of large primes is an achievement, an achievement which can take more than three hours.”

As it happens, Tao has used Gowers’s polymath concept to give us an idea of how IMO problems can grow into more serious mathematical research. At <http://terrytao.wordpress.com/2009/07/20/imo-2009-q6-as-a-mini-polymath-project/>, he has made a polymath project out of problem 6 on the 2009 IMO, the “grasshopper problem”. This turned out to be one of the most difficult problems ever posed at an IMO, fully solved by only three students.

com/2009/07/20/imo-2009-q6-as-a-mini-polymath-project/, he has made a polymath project out of problem 6 on the 2009 IMO, the “grasshopper problem”. This turned out to be one of the most difficult problems ever posed at an IMO, fully solved by only three students.

#### Smirnov

So Tao is using Gowers’s idea to show us how the IMO experience can grow into a more serious mathematical endeavor. Stanislav Smirnov addressed this relationship as well, in his remarks to the students:

“Mathematics research has become a truly collaborative effort, in that it is different from the actual IMO competition. It is much more interesting to work on problems together, and sharing ideas is always a rewarding experience. And in one aspect mathematical research is much like the IMO—both are truly international.”

Smirnov concluded with a personal welcome into the mathematical community: “I hope that many IMO participants will go on to become mathematicians, and that we will meet again.”

#### Lovász

Günter M. Ziegler, a much-honored Berlin mathematician, won an IMO gold medal in 1981 and acted as the personable and articulate host of the awards ceremonies. On stage, he asked László Lovász how much he earns from being president of the International Mathematical Union.

Lovász replied, “The amount is actually negative, because I forget to submit travel bills.”

“And do you get bribes?” asked Ziegler, joking.

Lovász grinned: “Well, not exactly bribes. But if you want to have a lower Erdős number, maybe you can advertise that you will reward someone with a lower number to write an article with you....”

This idea is not likely to be very influential. So Lovász then gave a more serious talk about how ideas spread through the mathematical community and how mathematics itself gives us tools to study that spread.

His talk was about graphs—huge graphs, such as the Internet. “Just as a crystal is a huge network of atoms, so the human brain is a network of neurons. And the same is true of human society as a whole.

“As one gets older one sits more with other scientists from different areas. And more scientists are using networks to describe what they do. Historians sometime call this the ‘network of human interactions’. And history itself—not just of mathematics—may depend on how religion, ideas, disease, news, and so on spread through these networks. So it is important to look at their structure. Understanding large graphs is a very important task for mathematics.

“A lot depends on asking the right questions. Does the Internet have an odd or even number of nodes? This is probably a meaningless question.



### **Worldwide Search for Talent**

City University of Hong Kong is a dynamic, fast developing university distinguished by scholarship in research and professional education. As a publicly funded institution, the University is committed to nurturing and developing students' talent and creating applicable knowledge to support social and economic advancement. Currently, the University has six Colleges/Schools. Within the next few years, the University aims to recruit **200 more scholars** in various disciplines from all over the world, including **science, engineering, business, social sciences, humanities, law, creative media, energy, environment**, and other strategic growth areas.

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#### **Salary and Conditions of Service**

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Further information on the posts and the University is available at <http://www.cityu.edu.hk>, or from the Human Resources Office, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong [Fax : (852) 2788 1154 or (852) 3442 0311/email : [hrojob@cityu.edu.hk](mailto:hrojob@cityu.edu.hk)].

Please send the application with a current curriculum vitae to Human Resources Office. **Applications will be considered until positions are filled.** Please quote the reference of the post in the application and on the envelope. The University reserves the right to consider late applications and nominations, and not to fill the positions. Personal data provided by applicants will be used for recruitment and other employment-related purposes. The University is an equal opportunity employer.

City University of Hong Kong is an equal opportunity employer and we are committed to the principle of diversity. We encourage applications from all qualified candidates, especially applicants who will enhance the diversity of our staff.

We don't know the answer, and the Internet itself is not well defined. But if we ask: How dense is the graph? Or: What is the average degree of a node? This is a very useful piece of information to know about the Internet.

"Is the Internet connected? This is a tricky question. The answer is probably 'no'. Somewhere there is a bad router and an unhappy group of people who don't have a connection except among themselves.

"But we can shape this question to have still more importance. Suppose there is an event, say an earthquake, which severs connections between the old and new worlds. Will the Internet still be connected? Or, in the bad old days, were there 'socialist' and 'capitalist' Internets, with no connection between them? These questions are meaningful, but interesting only if asked in the right way. What we really want to know is if the Internet decomposes into big parts.

"When I was young it looked like mathematics was going to decompose into just such big parts. Now there are many more connections between these parts. So my advice to young mathematicians is to be prepared to go and learn some area of mathematics which you thought you were not interested in. They might impose themselves on you, and you should be happy about this. It might lead to interesting developments."

#### **Pangea**

I understand Lovász to be saying that the mathematical network, the mathematical community, is somehow strongly connected, with more connections appearing all the time, like the continents drifting together to form the complex geologists call Pangea. It is this community that has appeared, almost magically, on the north coast of Germany. It is these close connections that allow the IMO community to exist, coming together only once a year. It is these connections, too, that create the systems—local, national, and regional—that discover and support new talent, young people who rejuvenate and extend the system that supported them.

So it is more than a group of individuals that creates our mathematics: it is a system. And it is more than a system that keeps itself going: it is a community that forges the system.

# William Benter Prize in Applied Mathematics

## Call for **NOMINATIONS**

The Liu Bie Ju Centre for Mathematical Sciences of City University of Hong Kong is inviting nominations of candidates for the William Benter Prize in Applied Mathematics, an international award.

### The Prize

The Prize recognizes outstanding mathematical contributions that have had a direct and fundamental impact on scientific, business, financial, and engineering applications.

It will be awarded to a single person for a single contribution or for a body of related contributions of his/her research or for his/her lifetime achievement.

The Prize is presented every two years and the amount of the award is US\$100,000.

### Nominations

Nomination is open to everyone. Nominations should not be disclosed to the nominees and self-nominations will not be accepted.

A nomination should include a covering letter with justifications, the CV of the nominee, and two supporting letters. Nominations should be submitted to:

#### Selection Committee

c/o Liu Bie Ju Centre for Mathematical Sciences  
City University of Hong Kong  
Tat Chee Avenue  
Kowloon  
Hong Kong

Or by email to: [mclbj@cityu.edu.hk](mailto:mclbj@cityu.edu.hk)

**Deadline for nominations: 30 September 2011**

### Presentation of Prize

The recipient of the Prize will be announced at the **International Conference on Applied Mathematics 2012: Modeling, Analysis, and Computation** from 28 May to 1 June 2012. The Prize Laureate is expected to attend the award ceremony and to present a lecture at the conference.

The Prize was set up in 2008 in honor of Mr William Benter for his dedication and generous support to the enhancement of the University's strength in mathematics. The first Prize was presented in 2010 to George Papanicolaou, Robert Grimmett Professor of Mathematics at Stanford University.

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The Liu Bie Ju Centre for Mathematical Sciences was established in 1995 with the aim of supporting world-class research in applied mathematics and in computational mathematics. As a leading research centre in the Asia-Pacific region, its basic objective is to strive for excellence in applied mathematical sciences. For more information, visit <http://www.cityu.edu.hk/lbj/>





# Memories of Martin Gardner

## Steven G. Krantz

Martin Gardner (1914–2010) took no mathematics courses after high school. He attempted to learn calculus in college but failed. He graduated from the University of Chicago with a bachelor's degree in philosophy. He did a year of graduate study, but earned no advanced degree.

Gardner was the ultimate polymath. His passion for mathematics stayed with him his entire life. He wrote seventy books on mathematics and related topics. His column in *Scientific American*, which ran for more than twenty-five years, was read worldwide and had an enormous influence over popular interest in mathematical topics.

Perhaps Gardner's most successful book was one of his first. *The Annotated Alice* was greatly popular, and is still in print today. He got his start in publishing as the editor of *Humpty Dumpty* magazine, a children's periodical. The paper-folding puzzles that Gardner designed for *Humpty Dumpty* led to his first contact with *Scientific American*. Gardner began his *Mathematical Games* column in the latter magazine in 1956 and continued it until 1981.

Gardner is remembered for introducing his reading public to

- Flexagons
- John Horton Conway's Game of Life
- Polyominoes
- Paradoxes such as the unexpected hanging
- Fractals
- The work of M. C. Escher
- Penrose tiling
- Piet Hein's superellipse
- Random walks
- Graceful graphs
- Worm paths
- Minimula sculpture
- Newcomb's paradox

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*Steven G. Krantz is professor of mathematics at Washington University in St. Louis and current editor of the Notices. His email address is sk@math.wustl.edu.*

- Nontransitive dice
- The board game *Hex*
- Public key cryptography
- The Kakeya needle problem

And there are dozens more.

Gardner had many interests. He was an expert magician. He was a noted skeptic and took great interest in debunking pseudoscience and fraudulent psychic phenomena. He had a great interest in religion. His friends and professional acquaintances ranged from John Nash to Douglas Hofstadter to John Milnor to magician James Randi to Ron Graham and Donald Knuth.

Martin Gardner thought he had an advantage as a mathematical writer not to have any background in mathematics. He said that, if he could not understand an idea, then his readers would not understand it either. Gardner prepared each of his columns in a painstaking and scholarly fashion and conducted copious correspondence to be sure that he got all the ideas straight. He was humble and straightforward and was at ease approaching even great minds with his questions.

Every few years there is a gathering to celebrate Martin Gardner and his contributions to our intellectual culture. The last such meeting was attended by 1,400 people. Clearly Gardner will be remembered for many years to come.

## Persi W. Diaconis

A blurb on the dust jacket of Martin Gardner's recent *The Colossal Book of Mathematics* says:

*Warning:* Martin Gardner has turned dozens of innocent youngsters into math professors and thousands of math professors into innocent youngsters.

And it's true.

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*Persi W. Diaconis is professor of mathematics and statistics at Stanford University. His email address is diaconis@math.stanford.edu.*



I met Martin when I was thirteen. He helped get me into mathematics. His books and columns made mathematical ideas accessible and elevated mathematics. More directly, he sometimes helped me do my homework and wrote letters of recommendation for my graduate school admissions. I'm a grown-up mathematician now, and paging through the book mentioned above constantly opens my eyes to lovely things.

Martin was a great explainer and debunker of various fads and fallacies. He left us with a mystery: How did he do it? How does a man with an undergraduate degree in philosophy touch youngsters and professionals? By clarity. By content: the ratio of examples and theorems to filler is high. By harnessing the best contributions of millions of readers. By hard work: Martin told me that he spent about twenty-five days a month on his *Scientific American* column. By his enthusiasm for what he explained.

Yet there is something more. Martin's work stands up to multiple readings. Go take a look.

## Ronald L. Graham

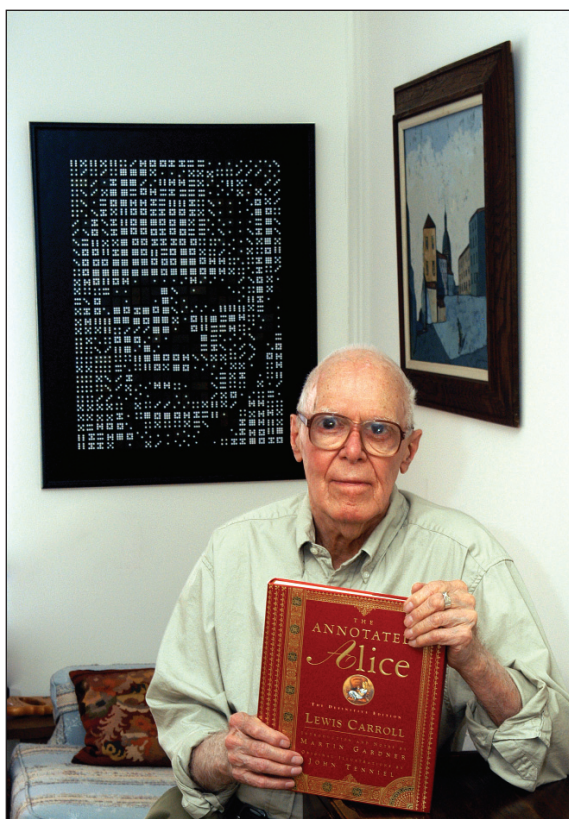
Martin Gardner was a gem. There is absolutely no question that he, more than anyone else in the world, was responsible for turning people of all ages on to the pleasures of mathematical recreations. His infectious enthusiasm, brilliant topic selection, and seductive prose in this activity are unrivaled. Many have tried to emulate him—no body has succeeded. What is more remarkable is how little formal mathematical training Martin actually had. In fact, he felt that this was to his great advantage, since if something wasn't clear to him, then it would probably also be unclear to many of his readers.

It is extraordinary how little Martin seemed to change over the forty-five years that I knew him. He was inevitably curious and excited about some new mathematical teaser, a neat card trick, or a subtle logical puzzle. Of course Martin's interests spanned much more than mathematical recreations and included magic, philosophy, and debunking pseudo-science, among others. He was modest, self-effacing, and always careful to give full credit to any reader who made a contribution to what he was writing about. Thousands of them did over the twenty-plus-year period he wrote his celebrated column in *Scientific American*.

I personally owe Martin a lot. But I think that this is true for many of us as well.

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Ronald L. Graham is professor of mathematics at the University of California, San Diego. His email address is [graham@ucsd.edu](mailto:graham@ucsd.edu).



Photograph by Gilbert Jain Photography.

**Martin Gardner at home in Norman, OK, 2005, holding his book, *The Annotated Alice*.**

## Donald E. Knuth

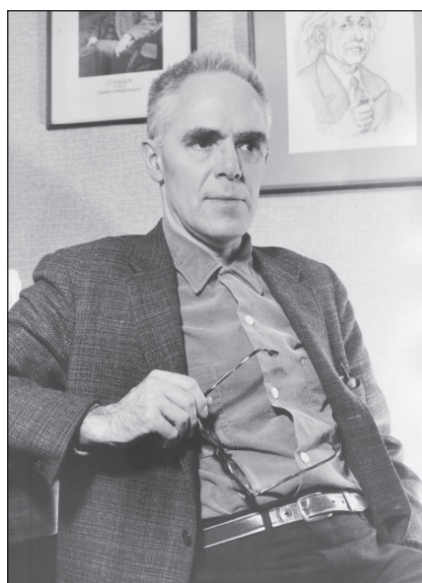
Most Americans over sixty remember the moment that they first learned that President Kennedy had been shot. I shall always remember the moment that I first learned of Martin Gardner's death.

I was staying for two weeks with one of my cousins in Ohio, using spare time to put the finishing touches on parts of a book that I was dedicating to Martin. At dinner one night I had explained to my hosts how I was preparing a special part of the preface in his honor, and why I was thankful for his ongoing inspiration. Then, at dinner two nights later, my cousin said that she'd just heard an obituary notice for him, while listening to NPR on her way home. Alas! Martin had told me how much he was looking forward to seeing this book, and I had been writing much of it especially for his personal pleasure.

But I believe in celebrating the joyous experiences of life, rather than mourning what might have been. Martin brought me and countless others a steady stream of intellectual stimulation and delight, over a period of many decades. A piece of writing from him often caused me to drop everything else for several days so that I could work on

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Donald E. Knuth is professor of computer science at Stanford University.



**Gardner in the early 1960s, Dobbs Ferry, NY.**

a fascinating puzzle. His fifteen precious volumes, in which twenty-five years' worth of monthly columns for *Scientific American* have been collected and amplified, sit prominently on a shelf next to the chair in which I read and write every day. For me, those volumes are the Canon.

Indeed, more people have probably learned more good mathematical ideas from Martin Gardner than from any other person in the history of the world, in spite of (or perhaps because of) the fact that he claimed not to be a

mathematician himself. He was the consummate master of the art of teaching by storytelling. Yet he didn't stick to the easy aspects of the subjects that he treated; he dug deeply into the origins of every idea that he was explaining, with superb scholarship. (On dozens of occasions when it turned out that he and I had independently researched the history of some topic, he had invariably located some aspects of the story that had escaped my notice.) Most amazingly, he did all this while faced with relentless monthly deadlines—spending two weeks per month on *Scientific American* while devoting the remaining two weeks to a wide variety of other pursuits.

I first had the opportunity to meet him in person at his home on Euclid Avenue, Hastings-on-Hudson, in December 1968. I was especially impressed by his efficient filing system using tiny cards, and by the fact that he did all of his writing while *standing up*, at a typewriter on a raised pedestal. I eventually followed his lead by getting my own stand-up computer desk.

In 1994, after many years of continued friendship, he invited me to spend two unforgettable weeks at the condominium in Hendersonville, North Carolina, where all of the notes and correspondence from his days of writing for *Scientific American* were currently stored. I systematically went through about fifty large boxes of material, barely able to sleep at night because of all the exciting things I was finding among those letters. He had carried on incredibly interesting exchanges with hundreds of mathematicians, as well as with artists and polymaths such as Maurits Escher and Piet Hein, all recorded in these files, mixed in of course with a fair amount of forgettable trivia. Already when he began his monthly series in 1956

and 1957, he was corresponding with the likes of Claude Shannon, John Nash, John Milnor, and David Gale. Later he would receive mail from budding mathematicians John Conway, Persi Diaconis, Jeffrey Shallit, Ron Rivest, et al. These files of correspondence now have a permanent home at Stanford University Archives, where I continue to consult them frequently.

While writing the present note, I took the opportunity to reread dozens of the letters that Martin had sent to me over the years, most recently a month or so before his death. In one of those letters he remarked that he regularly devoted one full day each week to answering mail. Thus I know that thousands of people like me have been able to benefit in a direct and personal way from his wisdom and generosity, in addition to the millions who have been edified by his publications. Countless more will surely benefit from his classic works, because those beautifully written volumes continue to remain in print, and someday they will be online.

## James Randi

I knew Martin Gardner for some sixty-plus years, I'm proud to say, and at our just-held annual conference of the James Randi Educational Foundation in Las Vegas, we held a celebration of his marvelous career, with his son Jim and his grandson Martin present. I say "celebration", you'll note, not "memorial". We all agreed that Martin would have been quite embarrassed to know that almost 1,400 of our members joined in the celebration. I'd wanted balloons and dancing girls, as well, but I was out-voted on that point.

This exceedingly modest man could never quite understand why so much fuss was made over him. I had no problem understanding this, and as I traveled around the world and occasionally mentioned that I knew the genius, I was immediately pestered with inquiries about him. He seemed almost a mythical character, this man who never took a course in mathematics after leaving high school, yet remains an icon to mathematicians all over this planet who quote him and flaunt their collection of his insightful books. I'm proud to say that my own copy of *Fads & Fallacies in the Name of Science* bears the inscription "To Randi—The Amazing Non-Gulliblist, from Martin." How good can life get...?

No, I don't mourn Martin's passing. I celebrate the fact that he was with us for .9559 of a century. He lived a rich, full life and enjoyed every discovery that he made about the world that he so improved with his wit and perception. His massive

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*James Randi is a magician and an investigator of paranormal and pseudoscientific claims. His email address is randi@randi.org.*



files featured a section that simply listed numbers from 0000 all the way up into the “alephs” that so fascinated him, and when I needed to know everything that he knew about the number 370, he told me that it was one of only four possible numbers that is the sum of the cubes of each of its digits. He then asked me what one of the others was (0 and 1 being ineligible), and I was stymied. When he told me, I experienced an “Aha!”<sup>1</sup>—which Martin designated to describe a very obvious fact that should be sobering to anyone who missed it. I was quite sobered....

Martin Gardner was number three on my automatic phone dialer. He’s not available that way now, but more than two feet of my library shelves bear his books. No, it’s not quite enough, but it will have to do.

## Peter Renz

I worked with Martin Gardner as his editor. We met in 1974, when I joined W. H. Freeman and Company. Freeman was a subsidiary of *Scientific American*, and Gerard Piel, the magazine’s publisher, sent me off to meet Martin. We worked together on Freeman projects for ten years and on projects at the MAA and elsewhere until his death.

**The View from *Scientific American*.** Dennis Flanagan, editor of the magazine, told me that columns like Martin’s freed him for other work. Reviewing Martin’s *Colossal Book of Mathematics* in *American Scientist* (2002), Dennis wrote that the column “was a big hit with the readers and contributed substantially to the magazine’s success.” Dennis Flanagan and Gerry Piel protected Martin’s interests. When Morris Kline put together his reader *Mathematics in the Modern World* (1968), he wanted to draw on Martin’s columns. Gerry Piel ruled this out, saying Martin controlled the rights. In 1976 Morris was working on a second reader, *Mathematics: Introduction to its Spirit and Use*. He wanted Martin’s coverage and exposition and chafed at Gerry’s prohibition. Knowing Martin to be generous about permissions, I asked him. He said, “Yes,” and fourteen of the articles that Kline used were Martin’s.

**How Did He Do It?** What were the keys to Martin’s success? A powerful mind, superb memory, writing skill, and great energy. His *Scientific American* audience devoured his columns and showered him with ideas. Many of you contributed.

How did Martin work? Partly as a reporter, starting from a primary source and working outward: John Conway on the Game of Life, Benoit Mandelbrot on fractals, Ron Rivest on public-key cryptography. Sometimes he drew a column

from a book, for example, his April 1961 column on H. S. M. Coxeter’s *Invitation to Geometry*. Some columns he drew from many sources: for example, his February 1963 column, “Curves of Constant Width”, draws on *The Enjoyment of Mathematics* by Rademacher and Toeplitz and on papers by Michael Goldberg on “rotors” from the *Monthly*.

This “Curves” column winds up with the Kakeya problem and Besicovitch’s result that there is no minimal-area solution. Martin uses an asteroidal shape from Ogilvy’s *Through the Mathescope* to suggest how a needle can be turned in smaller and smaller areas using overlapping turns.

**Lasting Impact, Long Tail.** Recreational problems often tie in to deeper mathematics, as the Kakeya example shows. Looking at Martin’s columns, I am struck by their lasting interest.

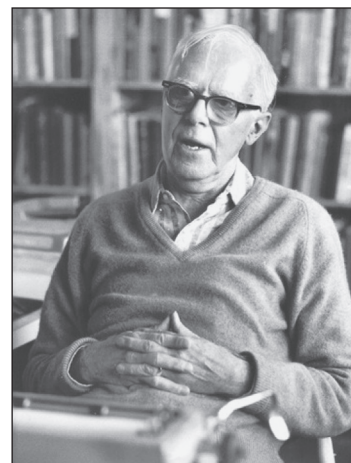
Flexagons, the Game of Googol (Secretary Problem), and the Unexpected Hanging launched small industries. We will be chewing on new forms of his puzzles for decades. His trapdoor cipher column jolted cryptography. His Game of Life columns energized cellular automata. His *Gödel, Escher, Bach* and *Planiverse* columns popularized the work of Doug Hofstadter and Kee Dewdney—both of whom became *Scientific American* columnists.

**Many, Diverse, and Continuing Contributions.** Martin could not rest from writing. After his wife died in 2000 he mentioned he probably wouldn’t write any more books. What is his record? From 2001 on he published twenty-two books and seventy-eight articles, reviews, or magic tricks.

Martin’s columns became books and the books became a CD—*Martin Gardner’s Mathematical Games*. In 2006 he began working on second editions. Three of these Games books have appeared; the rest should follow, based on Martin’s files and pending resolution of issues with *Scientific American*. The Gatherings 4 Gardner will carry forward Martin’s tradition. See the downloadable proceedings of G4G1—*The Mathematician and the Pied Puzzler*.

Many a book carries a preface or blurb of Martin’s. He defended reason and attacked folly. He had to expose fraud or injustice. See his “False Memory Wars” in *The Skeptical Inquirer*. He was my source for the latest on wild ideas and hypocrisy.

As a hard-nosed Platonist, Martin wrote critical reviews of *The Mathematical Experience* and New New Math textbooks in *The New York Review of Books*. We disagreed about Platonism, but his



Gardner, probably around 1994.

Photograph courtesy of Jim Gardner.

<sup>1</sup>The “obvious” number is 371. Aha!

Peter Renz was Martin Gardner’s book editor at W. H. Freeman. His email address is peter.renz@verizon.net.



**Gathering organized by Neil Patterson in 1983, to work on development of educational material. Front row: left to right: Neil Patterson, Martin Gardner, Herbert Kohl.**

barbs were aimed at my positions, not me. Martin had no animosity against those whose positions he attacked.

His delight in intellectual play, his regard for reason, his interest in and sympathy for human foibles, and his skill and productivity enriched us greatly. It was a pleasure to have worked with him.

## *Raymond M. Smullyan*

I first knew Martin when we were students at the University of Chicago. He has been a most wonderful friend, and to him I owe a good deal of my success as a puzzle writer. At the expense of appearing immodest (which unfortunately I am) I must tell you that he once paid me the supreme compliment of telling me: "Your puzzles have charm."

Unexpected as they were, I found the religious writings of Martin Gardner to be of extreme interest. Some have criticized them as being too mystical, but I don't believe they are mystical in the least! Martin was indeed devoutly religious, but that is something very different. His religious novel *The Flight of Peter Fromm* is a superb gem and shows profound psychological insight. Less impressive, in my opinion, are the religious chapters of his book *The Whys of a Philosophical Scrivener*. I had several objections to parts of it, all of which I wrote to Martin. He graciously wrote me back that he could not imagine a more fair review. Among my objections, Martin tended to identify belief in God with belief in an afterlife, which I believe to be a complete mistake, since I know so many people who believe in God but firmly disbelieve in an afterlife. Secondly, Martin wrote: "Is there any

significant difference between not believing in God and believing there is no God, or not believing in an afterlife and believing there is no afterlife?" I pointed out that there is an enormous difference, because, for one thing, one who does not believe in an afterlife but also doesn't disbelieve can at least have hopes that there may be one, whereas one who disbelieves can have no such hope. As the famous agnostic Robert Ingersoll said: "We agnostics also have our creed, 'help for the living; hope for the dead'." After much thought, Martin wrote me that he subsequently realized that there was much merit in what I said.

Two cute incidents: Martin was great on making April fool jokes, but I once pulled one off on him, which he fell for for a while. On the phone I said: "What do you think of that fantastic article in today's *New York Times* about Leonardo da Vinci? There is now incontrovertible evidence that da Vinci was really a woman. Isn't that remarkable?" At first, Martin believed there was really such an article, until he suddenly realized it was April 1. On another occasion I phoned him about his book *Confessions of a Psychic*, written under the pseudonym "Uriah Fuller", in which he so cleverly exposed psychic fraudulence, and in a terribly threatening voice said: "LOOK, THIS IS URIAH FULLER AND I WANT YOU TO KEEP OUT OF MY TERRITORY, SEE!!" In his sweet gentle voice he said, "Oh hi, Raymond."

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WHAT IS . . .

# Equivariant Cohomology?

Loring W. Tu

Many invariants in geometry and topology can be computed as integrals. For example, in classical differential geometry the Gauss-Bonnet theorem states that if  $M$  is a compact, oriented surface in Euclidean 3-space with Gaussian curvature  $K$  and volume form  $\text{vol}$ , then its Euler characteristic is

$$\chi(M) = \frac{1}{2\pi} \int_M K \text{vol}.$$

On the other hand, if there is a continuous vector field  $X$  with isolated zeros on a compact, oriented manifold, the Hopf index theorem in topology computes the Euler characteristic of the manifold as the sum of the indices at the zeros of the vector field  $X$ . Putting the two theorems together, one obtains

$$(1) \quad \frac{1}{2\pi} \int_M K \text{vol} = \sum_{p \in \text{Zero}(X)} i_X(p),$$

where  $i_X(p)$  is the index of the vector field  $X$  at the zero  $p$ . This is an example of a localization formula, for it computes a global integral in terms of local information at a finite set of points. More generally, one might ask what kind of integrals can be computed as finite sums. A natural context for studying this problem is the situation in which there is a group acting on the manifold with isolated fixed points. In this case, one can try to relate an integral over the manifold to a sum over the fixed point set.

Rotating the unit sphere  $S^2$  in  $\mathbb{R}^3$  about the  $z$ -axis is an example of an action of the circle  $S^1$  on the sphere. It has two fixed points, the north pole and the south pole. This circle action generates a continuous vector field on the sphere, and the

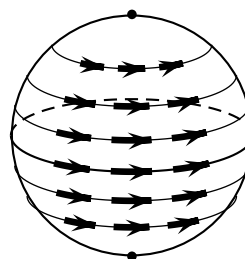


Figure 1. A circle action on a sphere.

zeros of the vector field are precisely the fixed points of the action (see Figure 1).

Recall that the familiar theory of singular cohomology gives a functor from the category of topological spaces and continuous maps to the category of graded rings and their homomorphisms. When the topological space has a group action, one would like a functor that reflects both the topology of the space and the action of the group. Equivariant cohomology is one such functor.

The origin of equivariant cohomology is somewhat convoluted. In 1959 Borel defined equivariant singular cohomology in the topological category using a construction now called the Borel construction. Nine years earlier, in 1950, in two influential articles on the cohomology of a manifold  $M$  acted on by a compact, connected Lie group  $G$ , Cartan constructed a differential complex  $(\Omega_G^*(M), d_G)$  out of the differential forms on  $M$  and the Lie algebra of  $G$ . Although the term “equivariant cohomology” never occurs in Cartan’s papers,

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Cartan's complex turns out to compute the real equivariant singular cohomology of a  $G$ -manifold (a manifold with an action of a Lie group  $G$ ), in much the same way that the de Rham complex of smooth differential forms computes the real singular cohomology of a manifold. Without explicitly stating it, Cartan provided the key step in a proof of the equivariant de Rham theorem, before equivariant cohomology was even defined! In fact, a special case of the Borel construction was already present in Cartan's earlier article (Colloque de Topologie, C.B.R.M., Bruxelles, 1950, p. 62). Elements of Cartan's complex are called *equivariant differential forms* or *equivariant forms*. Let  $S(\mathfrak{g}^*)$  be the polynomial algebra on the Lie algebra  $\mathfrak{g}$  of  $G$ ; it is the algebra of all polynomials in linear forms on  $\mathfrak{g}$ . An equivariant form on a  $G$ -manifold  $M$  is a differential form  $\omega$  on  $M$  with values in the polynomial algebra  $S(\mathfrak{g}^*)$  satisfying the equivariance condition:

$$\ell_g^* \omega = (\text{Ad } g^{-1}) \circ \omega \quad \text{for all } g \in G,$$

where  $\ell_g^*$  is the pullback by left multiplication by  $g$  and  $\text{Ad}$  is the adjoint representation. An equivariant form  $\omega$  is said to be *closed* if it satisfies  $d_G \omega = 0$ .

What makes equivariant cohomology particularly useful in the computation of integrals is the equivariant integration formula of Atiyah-Bott (1984) and Berline-Vergne (1982). In case a torus acts on a compact, oriented manifold with isolated fixed points, this formula computes the integral of a closed equivariant form as a finite sum over the fixed point set. Although stated in terms of equivariant cohomology, the equivariant integration formula, also called the equivariant localization formula in the literature, can often be used to compute the integrals of ordinary differential forms. It opens up the possibility of machine computation of integrals on a manifold.

## Equivariant Cohomology

Suppose a topological group  $G$  acts continuously on a topological space  $M$ . A first candidate for equivariant cohomology might be the singular cohomology of the orbit space  $M/G$ . The example above of a circle  $G = S^1$  acting on  $M = S^2$  by rotation shows that this is not a good candidate, since the orbit space  $M/G$  is a closed interval, a contractible space, so that its cohomology is trivial. In this example, we lose all information about the group action by passing to the quotient  $M/G$ .

A more serious deficiency of this example is that it is the quotient of a nonfree action. In general, a group action is said to be *free* if the stabilizer of every point is the trivial subgroup. It is well known that the orbit space of a nonfree action is often "not nice"—not smooth or not Hausdorff. However, the topologist has a way of

turning every action into a free action without changing the homotopy type of the space. The idea is to find a contractible space  $EG$  on which the group  $G$  acts freely. Then  $EG \times M$  will have the same homotopy type as  $M$ , and no matter how  $G$  acts on  $M$ , the diagonal action of  $G$  on  $EG \times M$  will always be free. The *homotopy quotient*  $M_G$  of  $M$  by  $G$ , also called the Borel construction, is defined to be the quotient of  $EG \times M$  by the diagonal action of  $G$ , and the *equivariant cohomology*  $H_G^*(M)$  of  $M$  is defined to be the cohomology  $H^*(M_G)$  of the homotopy quotient  $M_G$ . Here  $H^*(\ )$  denotes singular cohomology with any coefficient ring.

A contractible space on which a topological group  $G$  acts freely is familiar from homotopy theory as the total space of a *universal principal  $G$ -bundle*  $\pi: EG \rightarrow BG$ , of which every principal  $G$ -bundle is a pullback. More precisely, if  $P \rightarrow M$  is any principal  $G$ -bundle, then there is a map  $f: M \rightarrow BG$ , unique up to homotopy and called a *classifying map* of  $P \rightarrow M$ , such that the bundle  $P$  is isomorphic to the pullback bundle  $f^*(EG)$ . The base space  $BG$  of a universal bundle, uniquely defined up to homotopy equivalence, is called the *classifying space* of the group  $G$ . The classifying space  $BG$  plays a key role in equivariant cohomology, because it is the homotopy quotient of a point:

$$\text{pt}_G = (EG \times \text{pt})/G = EG/G = BG,$$

so that the equivariant cohomology  $H_G^*(\text{pt})$  of a point is the ordinary cohomology  $H^*(BG)$  of the classifying space  $BG$ .

It is instructive to see a universal bundle for the circle group. Let  $S^{2n+1}$  be the unit sphere in  $\mathbb{C}^{n+1}$ . The circle  $S^1$  acts on  $\mathbb{C}^{n+1}$  by scalar multiplication. This action induces a free action of  $S^1$  on  $S^{2n+1}$ , and the quotient space is by definition the complex projective space  $\mathbb{C}P^n$ . Let  $S^\infty$  be the union  $\bigcup_{n=0}^\infty S^{2n+1}$ , and let  $\mathbb{C}P^\infty$  be the union  $\bigcup_{n=0}^\infty \mathbb{C}P^n$ . Since the actions of the circle on the spheres are compatible with the inclusion of one sphere inside the next, there is an induced action of  $S^1$  on  $S^\infty$ . This action is free with quotient space  $\mathbb{C}P^\infty$ . It is easy to see that all homotopy groups of  $S^\infty$  vanish, for if a sphere  $S^k$  maps into the infinite sphere  $S^\infty$ , then by compactness its image lies in a finite-dimensional sphere  $S^{2n+1}$ . If  $n$  is large enough, any map from  $S^k$  to  $S^{2n+1}$  will be null-homotopic. Since  $S^\infty$  is a CW complex, the vanishing of all homotopy groups implies that it is contractible. Thus the projection  $S^\infty \rightarrow \mathbb{C}P^\infty$  is a universal  $S^1$ -bundle and, up to homotopy equivalence,  $\mathbb{C}P^\infty$  is the classifying space  $BS^1$  of the circle.

If  $H^*(\ )$  is a cohomology functor, the constant map  $M \rightarrow \text{pt}$  from any space  $M$  to a point induces a ring homomorphism  $H^*(\text{pt}) \rightarrow H^*(M)$ , which gives  $H^*(M)$  the structure of a module over the ring  $H^*(\text{pt})$ . Thus the cohomology of a point serves

as the coefficient ring in any cohomology theory. For the equivariant real singular cohomology of a circle action, the coefficient ring is

$$\begin{aligned} H_{S^1}^*(\text{pt}; \mathbb{R}) &= H^*(\text{pt}_{S^1}; \mathbb{R}) = H^*(BS^1; \mathbb{R}) \\ &= H^*(\mathbb{C}P^\infty; \mathbb{R}) \simeq \mathbb{R}[u], \end{aligned}$$

the polynomial ring generated by an element  $u$  of degree 2. For the action of a torus  $T = S^1 \times \cdots \times S^1 = (S^1)^\ell$ , the coefficient ring is the polynomial ring  $H_T^*(\text{pt}; \mathbb{R}) = \mathbb{R}[u_1, \dots, u_\ell]$ , where each  $u_i$  has degree 2.

## Equivariant Integration

Let  $G$  be a compact, connected Lie group. Over a compact, oriented  $G$ -manifold, equivariant forms can be integrated, but the values are in the coefficient ring  $H_G^*(\text{pt}; \mathbb{R})$ , which is generally a ring of polynomials. According to Cartan, in the case of a circle action on a compact, oriented manifold, an equivariant form of degree  $2n$  is a sum

$$(2) \quad \omega = \omega_{2n} + \omega_{2n-2} u + \omega_{2n-4} u^2 + \cdots + \omega_0 u^n,$$

where  $\omega_{2j} \in \Omega^{2j}(M)^{S^1}$  is an  $S^1$ -invariant  $2j$ -form on  $M$ . If  $\omega$  is closed under the Cartan differential, then it is called an *equivariantly closed extension* of the ordinary differential form  $\omega_{2n}$ . The equivariant integral  $\int_M \omega$  is obtained by integrating each  $\omega_{2j}$  over  $M$ . If  $M$  has dimension  $2n$ , then the integral  $\int_M \omega_{2j}$  vanishes except when  $j = n$ , and one has

$$\begin{aligned} \int_M \omega &= \int_M \omega_{2n} + \left( \int_M \omega_{2n-2} \right) u \\ &\quad + \cdots + \left( \int_M \omega_0 \right) u^n \\ &= \int_M \omega_{2n} + 0 + \cdots + 0 = \int_M \omega_{2n}. \end{aligned}$$

One peculiarity of equivariant integration is the possibility of obtaining a nonzero answer while integrating a form over a manifold whose dimension is not equal to the degree of the form. For example, if  $M$  has dimension  $2n - 2$  instead of  $2n$ , then the integral over  $M$  of the equivariant  $2n$ -form  $\omega$  above is

$$\int_M \omega = \left( \int_M \omega_{2n-2} \right) u,$$

since for dimensional reasons all other terms are zero. From this, one sees that an equivariant integral for a circle action is in general not a real number, but a polynomial in  $u$ .

## Localization

What kind of information can be mined from the fixed points of an action? If a Lie group  $G$  acts smoothly on a manifold, then for each  $g \in G$ , the action induces a diffeomorphism  $\ell_g: M \rightarrow M$ . At a fixed point  $p \in M$ , the differential  $\ell_{g*}: T_p M \rightarrow T_p M$  is a linear automorphism of the tangent

space, giving rise to a representation of the group  $G$  on the tangent space  $T_p M$ . Invariants of the representation are then invariants of the action at the fixed point. For a circle action, at an isolated fixed point  $p$ , the tangent space  $T_p M$  decomposes into a direct sum  $L^{m_1} \oplus \cdots \oplus L^{m_n}$ , where  $L$  is the standard representation of the circle on the complex plane  $\mathbb{C}$  and  $m_1, \dots, m_n$  are nonzero integers. The integers  $m_1, \dots, m_n$  are called the *exponents* of the circle action at the fixed point  $p$ . They are defined only up to sign, but if  $M$  is oriented, the sign of the product  $m_1 \cdots m_n$  is well defined by the orientation of  $M$ .

When a torus  $T = (S^1)^\ell$  acts on a compact, oriented manifold  $M$  with isolated fixed point set  $F$ , for any closed  $T$ -equivariant form  $\omega$  on  $M$ , the equivariant integration formula states that

$$(3) \quad \int_M \omega = \sum_{p \in F} \frac{i_p^* \omega}{e^T(v_p)},$$

where  $i_p^* \omega$  is the restriction of the equivariantly closed form  $\omega$  to a fixed point  $p$  and  $e^T(v_p)$  is the equivariant Euler class of the normal bundle  $v_p$  to  $p$  in  $M$ . Of course, the normal bundle to a point  $p$  in a manifold  $M$  is simply the tangent space  $T_p M$ , but formula (3) is stated in a way to allow easy generalization: when  $F$  has positive-dimensional components, the sum over the fixed points is replaced by an integral over the components  $C$  of the fixed point set and  $v_p$  is replaced by  $v_C$ , the normal bundle to the component  $C$ . In formula (3), the degree of the form  $\omega$  is not assumed to be equal to the dimension of the manifold  $M$ , and so the left-hand side is a polynomial in  $u_1, \dots, u_\ell$ , while the right-hand side is a sum of rational expressions in  $u_1, \dots, u_\ell$ , and it is part of the theorem that the equivariant Euler classes  $e^T(v_p)$  are nonzero and that there will be cancellation on the right-hand side so that the sum becomes a polynomial.

Return now to a circle action with isolated fixed points on a compact, oriented manifold  $M$  of dimension  $2n$ . Let  $\omega$  be a closed equivariant form of degree  $2n$  on  $M$ . Since the restriction of a form of positive degree to a point is zero, on the right-hand side of (3) all terms in  $\omega$  except  $\omega_0 u^n$  restrict to zero at a fixed point  $p \in M$ :

$$\begin{aligned} i_p^* \omega &= \sum_{j=0}^n (i_p^* \omega_{2n-2j}) u^j \\ &= (i_p^* \omega_0) u^n = \omega_0(p) u^n. \end{aligned}$$

The equivariant Euler class  $e^{S^1}(v_p)$  turns out to be  $m_1 \cdots m_n u^n$ , where  $m_1, \dots, m_n$  are the exponents of the circle action at the fixed point  $p$ . Therefore, the equivariant integration formula for a circle action assumes the form

$$\int_M \omega_{2n} = \int_M \omega = \sum_{p \in F} \frac{\omega_0}{m_1 \cdots m_n}(p).$$

In this formula,  $\omega_{2n}$  is an ordinary differential form of degree  $2n$  on  $M$ ,  $\omega$  is an equivariantly closed extension of  $\omega_{2n}$ , and  $\omega_0$  is the coefficient of the  $u^n$  term in  $\omega$  as in (2).

## Applications

In general, an integral of an ordinary differential form on a compact, oriented manifold can be computed as a finite sum using the equivariant integration formula if the manifold has a torus action with isolated fixed points and the form has an equivariantly closed extension. These conditions are not as restrictive as they seem, since many problems come naturally with the action of a compact Lie group, and one can always restrict the action to that of a maximal torus. It makes sense to restrict to a maximal torus, instead of any torus in the group, because the larger the torus, the smaller the fixed point set, and hence the easier the computation.

As for the question of whether a form has an equivariantly closed extension, in fact a large collection of forms automatically do. These include characteristic classes of vector bundles on a manifold. If a vector bundle has a group action compatible with the group action on the manifold, then the equivariant characteristic classes of the vector bundle will be equivariantly closed extensions of its ordinary characteristic classes.

A manifold on which every closed form has an equivariantly closed extension is said to be *equivariantly formal*. Equivariantly formal manifolds include all manifolds whose cohomology vanishes in odd degrees. In particular, a homogeneous space  $G/H$ , where  $G$  is a compact Lie group and  $H$  is a closed subgroup of maximal rank, is equivariantly formal.

The equivariant integration formula is a powerful tool for computing integrals on a manifold. If a geometric problem with an underlying torus action can be formulated in terms of integrals, then there is a good chance that the formula applies. For example, it has been applied to show that the stationary phase approximation formula is exact for a symplectic action (Atiyah-Bott 1984), to calculate the number of rational curves in a quintic threefold (Kontsevich 1995, Ellingsrud-Strømme 1996), to calculate the characteristic numbers of a compact homogeneous space (Tu 2010), and to derive the Gysin formula for a fiber bundle with homogeneous space fibers (Tu preprint 2011). In the special case in which the vector field  $X$  is generated by a circle action, the Gauss-Bonnet-Hopf formula (1) is a consequence of the equivariant integration formula. Equivariant cohomology has also helped to elucidate the work of Witten on supersymmetry, Morse theory, and Hamiltonian actions (Atiyah-Bott 1984, Jeffrey-Kirwan 1995).

The formalism of equivariant cohomology carries over from singular cohomology to other cohomology theories such as  $K$ -theory, Chow rings, and quantum cohomology. There are similar localization formulas that compare the equivariant functor of a  $G$ -space to that of the fixed point set of  $G$  or of some subgroup of  $G$  (for example, Segal 1968 and Atiyah-Segal 1968). In the fifty years since its inception, equivariant cohomology has found applications in topology, differential geometry, symplectic geometry, algebraic geometry,  $K$ -theory, representation theory, and combinatorics, among other fields, and is currently a vibrant area of research.

## Acknowledgments

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# The Black Swan: The Impact of the Highly Improbable

*Reviewed by David Aldous*

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**The Black Swan: The Impact of the Highly Improbable**

Nassim Nicholas Taleb

Random House, 2007

US\$28.00, 400 pages

ISBN: 978-1-4000-6351-2

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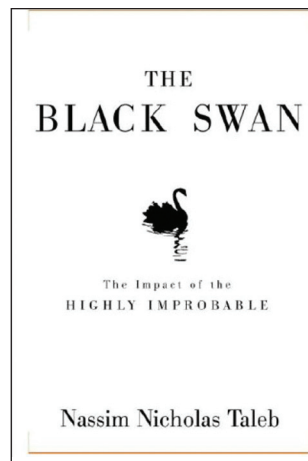
Taleb has made his living (and a small fortune, perhaps transformed into a large fortune by the 2008 market) in an unusual way—by financial speculation in contexts in which he spots a small chance of making a very large gain. As with others who have had unusual careers (say, Neil Armstrong or Marcel Marceau), it is interesting to hear his experiences, but when such a person declares *I am a philosopher of ideas*, one is wise to be cautious (italics denote quotes from Taleb, boldface denotes my own emphasis).

The phrase “Black Swan” (arising earlier in the different context of Popperian falsification) is here defined as an event characterized [p. xviii] by *rarity, extreme impact, and retrospective (though not prospective) predictability*, and Taleb’s thesis is that such events have much greater effect, in financial markets and the broader world of human affairs, than we usually suppose. The book is challenging to review because it requires considerable effort to separate the content from the style. The style is rambling and pugnacious—well described by one reviewer as “with few exceptions, the writers and professionals Taleb describes are knaves or fools, mostly fools. His writing is full of irrelevances,

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*This is a slight revision of an article posted January 2009 on the reviewer’s website <http://www.stat.berkeley.edu/~aldous/>, which contains further argumentative essays.*



asides and colloquialisms, reading like the conversation of a raconteur rather than a tightly argued thesis”. And clearly this is perfectly deliberate. Such a book invites a review that reflects the reviewer’s opinions more than is customary in the *Notices*. My own overall reaction is that Taleb is sensible (going on prescient) in his discussion of financial markets and

in some of his general philosophical thought but tends toward irrelevance or ridiculous exaggeration otherwise. Let me run through some discussion topics, first six on which I broadly agree with Taleb, then six on which I broadly disagree, then five final thoughts.

(1) [p. 286] *The sterilized randomness of games does not resemble randomness in real life*; thinking it does constitutes the *Ludic Fallacy* (his neologism). This is exactly right, and mathematicians should pay attention. In my own list of one hundred instances of chance in the real world, exactly one item is “Explicit games of chance based on artifacts with physical symmetry—exemplified by dice, roulette, lotteries, playing cards, etc.”

(2) Taleb is dismissive of prediction and models (explicitly in finance and econometrics, and implicitly almost everywhere). For instance [p. 138], *Why on earth do we predict so much? Worse, even, and more interesting: why don’t we talk about our record in predicting? Why don’t we see how we (almost) always miss the big events? I call this the scandal of prediction.* And [p. 267] *In the absence*

of a feedback mechanism [not making decisions on the basis of data] you look at models and think they confirm reality. He's right; people want forecasts in economics, and so economists give forecasts, even knowing they're not particularly accurate. The culture of academic research in numerous disciplines encourages theoretical modeling which is never seriously compared with data.

(3) Taleb is scathing about stock prediction models based on Brownian motion (Black-Scholes and variants) and of the whole idea of measuring risk by standard deviation [p. 232]: *You cannot use one single measure for randomness called standard deviation (and call it "risk"); you cannot expect a simple answer to characterize uncertainty. And [p. 278] if you read a mutual fund prospectus, or a description of a hedge fund's exposure, odds are that it will supply you...with some quantitative summary claiming to measure "risk". That measure will be based on one of the above buzzwords [sigma, variance, standard deviation, correlation, R square, Sharpe ratio] derived from the bell curve and its kin.... If there is a problem, they can claim that they relied on standard scientific method.*

(4) Ask someone what happened in a movie they've just watched; their answer will not be just a list (this happened, then this happened, then this happened...) but will also give reasons (he left town **because** he thought she didn't love him...). We habitually think about the past in this way, as events linked by causal explanations. As Taleb writes [p. 73]: *narrativity causes us to see past events as more predictable, more expected, and less random than they actually were...* and he calls this the *Narrative Fallacy*.

(5) Chapter 3 introduces neologisms *Mediocristan* and *Extremistan* for settings in which outcomes do [do not] have finite variance. His writing is lively and memorable, and his examples are apposite, so that it would make a useful reading accompaniment to a technical statistics course, though as indicated below I disagree with his interpretation of the relative significance of the two categories.

(6) Given that Taleb's thesis is already well expressed by the bumper sticker "Expect the unexpected", what more is there to say? Well, actually he makes several memorable points, such as his summary [p. 50] of themes related to Black Swans:

(a) *We focus on preselected segments of the seen and generalize from it to the unseen: the error of confirmation.*

(b) *We fool ourselves with stories that cater to our Platonic thirst for distinct patterns: the narrative fallacy.*

(c) *We behave as if the Black Swan does not exist; human nature is not programmed for Black Swans.*

(d) *What we see is not necessarily all that is there. History hides Black Swans from us [if they didn't happen] and gives a mistaken idea about the odds of these events: this is the distortion of silent evidence.*

(e) *We "tunnel": that is, we focus on a few well-defined sources of uncertainty, on too specific a list of Black Swans (at the expense of others that do not come so readily to mind).*

And here is his investment strategy [pp. 295–296]: *Half the time I am hyperconservative in the conduct of my own [financial] affairs; the other half I am hyperaggressive. This may not seem exceptional, except that my conservatism applies to what others call risk-taking, and my aggressiveness to areas where others recommend caution. I worry less about small failures, more about large, potentially terminal ones. I worry far more about the "promising" stock market, particularly the "safe" blue chip stocks, than I do about speculative ventures—the former present invisible risks, the latter offer no surprises since you know how volatile they are and can limit your downside by investing smaller amounts.... In the end this is a trivial decision making rule: I am very aggressive when I can gain exposure to positive Black Swans—when a failure would be of small moment—and very conservative when I am under threat from a negative Black Swan. I am very aggressive when an error in a model can benefit me, and paranoid when an error can hurt. This may not be too interesting except that it is exactly what other people do not do. In finance, for instance, people use flimsy theories to manage their risks and put wild ideas under "rational" scrutiny.*

Maybe not easy for you or me to emulate, but surely conceptually useful for us to keep in mind.

## Criticisms

(7) Taleb dismisses *Mediocristan* as uninteresting and basically attributes Life, The Universe, and Everything to *Extremistan* [p. xix]: *it is easy to see that life is the cumulative effect of a handful of significant shocks*. Now power laws (in the present context, distributions with power law tails, roughly what *Extremistan* is; pedantically, I am now talking about Gray Swans) have received much attention in popular science and popular economics over the last twenty years, and they really do arise in various aspects of the natural world, and (for different reasons) in various aspects of the human economic world. But my view is that

(a) the apparent prevalence of *Extremistan* is exaggerated by several cognitive biases;

(b) outside rather narrow economic contexts, each example of *Extremistan* in the human world is surrounded by numerous equally significant examples of *Mediocristan*—it's just a small part of a big picture.

In other words Taleb's assertion quoted above, like much of the popular literature, wildly overstates the significance of *Extremistan*. A building might be damaged in a few seconds by an earthquake, in a few minutes by a fire, in a few hours by a flood, or in a few decades by termites. The



first three are visually dramatic and may affect a large and unpredictable number of buildings at once (Extremistan); not so the fourth (Mediocristan); the first three appear in the news as “natural disasters” but the fourth doesn’t. But none of this is relevant to the quantitative impact of such events, which is an empirical matter (termites win). Similarly, “number of deaths in different wars” is in Extremistan; childhood deaths from poor sanitation and consequent disease is in Mediocristan. Guess which caused more deaths worldwide in the twentieth century. That’s an empirical matter (poor sanitation wins). So:

**Extremistan is sometimes dramatic; Mediocristan is never dramatic. But this has no necessary connection with quantitative impact.**

Setting aside drama aspects, the simple fact is that our minds focus on the variable aspects of life because we don’t **need** to focus on the nonvariable aspects. If I ask you what you did yesterday, you don’t tell me the usual things (commuting to work, brushing teeth, breathing), you tell me what was different about yesterday. If I ask you to describe your dog, you don’t say “four legs, one tail, vocalizes by barking”, you tell me how your dog differs from the average dog. So:

**Our minds focus on variability. Extremistan is, by definition, more variable than Mediocristan, so it attracts relatively more of our attention. But this has no necessary connection with quantitative impact.**

Turning to (b), take any example, even a standard “economic” one such as financial success of different movies. Most movies lose money; a few make enormous profits. So this aspect of the movie sector of the economy is indeed in Extremistan. But how much, and to whom, does this matter? The size of the sector (number of employed actors and technicians, number of cinemas) isn’t affected in any obvious way by this variability, just by our taste for watching movies as opposed to other entertainment. Of the movies you and I enjoy, some were commercial successes and some were flops—how would our experience be different if the successes and failures were less extreme? Even an investor diversified across the movie business isn’t much affected. It’s hard to think of any very substantial consequences—for instance, logic suggests that in Extremistan one should “take risks” by making unconventional movies, but Hollywood is generally criticized for exactly the opposite, for making formulaic movies.

(8) In other words the whole *Extremistan* metaphor, suggesting a country in which everything is ruled by power laws, is misleading. A better metaphor is an **agora**, a marketplace, which is a useful component of a city but is surrounded by other components with different roles. This provides a segue to a quotable proclamation of my own.

**Financial markets differ from casinos in many ways, but they are almost equally unrepresentative of the operation of chance in other aspects of the real world. Thinking otherwise is the *Agoran fallacy*.**

Here are three facets of this fallacy.

(a) Money is “simply additive”—your career investment profit is the sum of your profits and losses each day. The rest of life doesn’t work that way—your happiness today isn’t a sum of incremental happiness and unhappiness of previous days.

(b) In financial speculation one doesn’t care about actual outcomes, merely about the competitive issue of being able to guess outcomes better than others can, like “betting against the spread” on football. But in most important decisions under uncertainty (choosing a spouse, choosing a cancer treatment), one seeks desirable outcomes rather than to beat others.

(c) Imagine you have woken from a twenty-five-year sleep and want to catch up on what’s happened. Taleb and I agree that looking at the roughly nine thousand daily headlines you missed would not be helpful—these are “just noise” from a long-term perspective. Taleb views Black Swans as the only alternative. But he ignores the cumulative effect of slow trends (because they are uninteresting to a speculator?). One can think of an endless list of slow changes in the United States over the last generation (increase in childhood obesity, increased consumption of espresso, increased proportion of occupations requiring a college education, increased visibility of pornography), as well as the more prominent ones (acceptability of a black president, increase in health care sector to around 16% of GDP). Consider a fifty-five-year-old thinking about changes in the United States over the last thirty years—how is the experience of being twenty-five in 2011 different from the experience of being twenty-five in 1981? Perhaps most obvious is the Internet (more precisely, the things we now do using the Internet) and the prevalence of laptop computers. This is a change that our fifty-five-year-old experienced as an individual—we remember the first time we used a browser or a search engine. We have a natural cognitive bias toward changes such as the Internet that we experienced as individuals rather than those such as “increase in childhood obesity” that we didn’t. One can hardly quantify such matters, but contrary to Taleb I would assert

**Most of the differences in life experience from one generation to the next are the cumulative results of slow changes that do not have much impact on a typical individual and therefore that we don’t pay much attention to. Of course in the long term the nature, time of origination, and duration of slow trends is unpredictable—but it**

is this, not Black Swans, that actually constitute long-term unpredictability.

(9) The word **prediction** has a range of meaning. Stating “Microsoft stock will rise about 20% next year” is a deterministic prediction, whereas stating your opinion about the stock’s performance as a probability distribution is a statistical prediction. Any attempt by a reader to make more precise sense of Taleb’s rhetoric about prediction requires the reader to keep firmly in mind which meaning is under discussion, since Taleb isn’t careful to do so. For instance, Taleb discusses [p. 150] data showing that security analysts’ predictions are useless, as if this were a novel insight. But in this setting he is talking about deterministic prediction, and he is just repeating a central tenet of thirty years of academic theory (the **efficient market hypothesis** and all that), not to mention the classic best-seller [1]. On the other hand, the standard mathematical theory of finance starts with some statistical assumption—that prices will move like Brownian motion or some variant. Taleb’s criticisms of this theory—that it ignores Black Swans, and that future probabilities are intrinsically impossible to assess well—have considerable validity, but he doesn’t make sufficiently clear the distinction between this and traditional stockbroker advice.

(10) A book on (say) the impact of empires on human history might be expected to contain an explicit list of entities the author considered as empires; that way, a reader could analyze any asserted generality about empires by pondering whether it applied to at least most empires on the list. Similarly, one might expect this book to contain some explicit list of past events the author considered Black Swans (here I am thinking of unique Black Swans, not Gray Swans). But it doesn’t; various instances are certainly mentioned, but mostly via asides and anecdotes. If you read the book and extracted the mentioned instances, and then read it again to see how much of the material was directly relevant to most of the listed Black Swans, then it would be a very small proportion. In other words, the summary (6) of Taleb’s views is interesting, but instead of expanding the summary to more concrete and detailed analysis, the book rambles around scattered philosophical thoughts.

(11) The style of Taleb’s philosophizing can be seen in the table [p. 284] “Skeptical Empiricism vs Platonism”, in which he writes a column of ideas that he explicitly identifies with and contrasts this with another column that no one would explicitly identify with. This is straw man rhetoric. Indeed, much of the book is rhetoric about empiricism, with a remarkable lack of actual empiricism, i.e., rational argument from data.

(12) This love of rhetoric causes Taleb to largely ignore what I would consider interesting philosophical questions related to Black Swans. Here are two such. There are a gazillion things we might

think about during a day, but (unlike a computer rebooting) we don’t wake up, run through the gazillion, and consciously choose which to actually think about. For obvious reasons, in everyday life this question—What comes to one’s conscious attention as matters one might want to think about?—is no big deal. But it’s a central issue with Black Swans: if we believe there may be many low-probability high-impact future events that we can’t imagine this moment, how much effort should we put into trying to imagine them, and how do we go about doing so, anyway? Taleb’s comments [p. 207]—*For your exposure to the positive Black Swans, you do not need to have any precise understanding of the structure of uncertainty* [here Taleb is assuming power-law payoffs] and [p. 210] *the probabilities of very rare events are not computable; the effect of an event on us is considerably easier to ascertain*—are partially true, but don’t tell us how and where to look for potential Black Swans.

Second, it is easy to cite, say [p. xviii], *the precipitous demise of the Soviet bloc* as having been unpredictable, but what does this mean? If you had asked an expert in 1985 what might happen to the USSR over the next ten years—“give me a range of possibilities and a probability for each”—then they would surely have included something like “peaceful breakup into constituent republics” and assigned it some small probability. What does it mean to say such a prediction is right or wrong? In 2008, the day before John McCain was scheduled to announce his VP choice, the Intrade prediction market gave Sarah Palin a 4% chance. Was this right or wrong? Unlikely events will sometimes happen just by chance. Taleb’s whole thesis is that experts and markets do not assess small probabilities correctly, but he supports it with anecdote and rhetoric, not with data and analysis.

### Five Final Thoughts

(13) If you haven’t read *The Black Swan*, Taleb’s online essay [3] is a shorter and more cohesive account of some of his ideas.

(14) Taleb often seems to imagine that the views he disagrees with come from some hypothetical Financial Math 101 course, though in one case it was an actual course [p. 278]: *It seemed better to teach [MBA students at Wharton] a theory based on the Gaussian than to teach them no theory at all*. It is easy to criticize introductory courses in any subject as concentrating on some oversimplified but easy-to-explain theory that is not so relevant to reality (e.g., many introductory statistics courses exaggerate the relevance and scope of tests of significance; physics courses say more about gravity than about friction). It is much harder to rewrite such a course to make it more realistic without degenerating into vague qualitative assertions or scattered facts.

(15) I am always puzzled that writers on financial mathematics (Taleb included) tend to ignore what strikes me as the most important insight that mathematics provides. Common sense and standard advice correctly emphasize a trade-off between short-term risk and long-term reward, implicitly suggesting that this spectrum goes on forever. But it doesn't. At least, if one could predict **probabilities** accurately, there is a "Kelly strategy" that optimizes long-term return. This strategy, the subject of the popular book [2], carries a very specific level of short-term risk, given by the remarkable formula

with chance  $p\%$  your portfolio value  
will sometimes drop below  $p\%$  of its  
initial value.

Now actual stock markets are less volatile, and consequently one of the best (fixed, simple) investment strategies for a U.S. investor over the last fifty years has been to invest about 140% of their net financial assets in stocks (by borrowing money). It is easy to say that [p. 61] *The sources of Black Swans today have multiplied beyond measurability* and imply that this is a source of increased market volatility, but it is equally plausible or implausible to conjecture that mathematically based speculative activity is pushing the stock market toward the "Kelly" level of volatility.

(16) My own investment philosophy, as someone who devotes three hours a year to his investments, is:

**As a default, assume the future will be statistically similar to the past. Not because this is true in any Platonic sense, but because anyone who says different is trying to sell you something.**

(17) *The Black Swan* illustrates a general phenomenon that authors who deal with chance in specific contexts (finance, the logic of scientific inference, physics, luck in everyday life, philosophy, risks to the world economy, evolution, algorithmic complexity,...) can be very perceptive within these contexts, yet, by not keeping in mind the full extent of real-world occurrences of chance, assert generalizations about chance that are silly outside their particular context. An amusing antidote to such generalizations is to examine the contexts in which "ordinary people" perceive chance. For some data on this, derived from 100,000 queries to a search engine, see [http://www.stat.berkeley.edu/~aldous/Real-World/bing\\_chance.html](http://www.stat.berkeley.edu/~aldous/Real-World/bing_chance.html).

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# Using Mathematics to Improve Fluid Intelligence

*Vali Siadat*

In the past several decades the mathematical community has witnessed a fervent debate about the relationship between the development of basic mathematical skills and higher-order thinking [9]. This debate addresses both the learning processes and facilitating the retention of what is learned.

One important educational tool in promoting the learning process is testing. My experience suggests that, to be most effective, testing should be both cumulative and time restricted. By cumulative, I mean individual tests, including quizzes, that include items from material covered earlier in the term. When students realize that testing will be cumulative, there is strong motivation to understand, practice, and review all the material taught from the beginning of the course. As opposed to tests on chapters or modules, cumulative tests center on the essence of education, recognizing that the integration of knowledge is the very heart of learning. By the very hierarchical nature of mathematics, understanding of new material depends upon what has been learned before, and so the learning of the new topic becomes intimately tied to the knowledge of the previous topics. The effectiveness of cumulative vs. narrowly focused testing is supported by studies we have conducted in mathematics (using control groups) and by research of other educators [1, 5, 7]. Related work by other respected scholars on test-enhanced learning has appeared in [4, 6] as well.

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Time-restricted tests provide additional benefits to the educational process. Time limits build students' concentration skills; the student must fully focus attention on the task. This requirement addresses the contemporary habit of living with constant disruptions, which is reflected in students' thought processes that meander, lacking the ability to focus. Frequent, time-restricted testing in mathematics trains students to fully concentrate on a task, targeting the development of their ability to learn and retain knowledge. Recent collaborative research that I have conducted has shown that using frequent and time-restricted tests in mathematics improves students' outcomes not only in mathematics, but also in "unrelated" subjects such as reading comprehension [7, 8]. One of my colleagues has personally reported similar results in the social sciences. These examples seem very likely attributable to students' improved concentration skills.

Beyond this, we have found that timed tests, much more than paper and pencil routines, enhance students' ability to do mental mathematics by training them to instantly build images of multi-step problems in their minds and solve them rapidly. For example, to solve the equation  $x/2 - 7 = -3$ , they create an image of the problem, perform the additive and multiplicative properties mentally and arrive at  $x = 8$  as the solution. In trigonometry, to simplify the identity  $1 + \tan^2 x$ , they can rapidly create an image  $1 + \tan^2 x$  as  $1 + \sin^2 x / \cos^2 x$ , equal to  $\cos^2 x / \cos^2 x + \sin^2 x / \cos^2 x$  which results in  $\cos^2 x + \sin^2 x / \cos^2 x$  leading to  $1 / \cos^2 x$  which equals  $\sec^2 x$ .



In calculus also, to calculate the derivative  $(e^{\sin 2x})'$ , they form a mental image of the chain rule on composition of functions, and form the product of  $e^{\sin 2x} \cdot (\sin 2x)'$ , which is  $e^{\sin 2x} \cdot (\cos 2x) \cdot 2$ . This is what mathematicians do when they confront problems of such characteristics. They do problems in their minds. I have seen very good students begin to reach this point.

Finally, timed tests build students' automaticity of basic skills in mathematics. This may sound like nothing more than inducing stress during tests, but many cognitive scientists have determined that stress within the context of a learning experience induces focused attention and improves memory of relevant information [3]. This enables the mind to perform at that level of conceptual thought devoted to higher-order thinking and problem-solving activities. The comfort that one experiences by achieving fluency in basic skills far transcends any initial stress as such fluency removes educational barriers which exist on the way to performing at higher domains of thought.

Recently there has been exciting seminal research suggesting that fluid intelligence can be improved with training on working memory [2]. Fluid intelligence is considered to be one of the most important factors in learning and comprises those sets of abilities associated with abstract reasoning and higher-order thinking. Intelligence has always been thought to have a strong hereditary component, immutable even with training, but the new research shows that training in continuous performance tasks (dual  $n$ -back tasks) stimulates brain activity leading to improved results as reflected through intelligence tests. Psychologists have experimented with dual  $n$ -back tasks to provide simultaneous auditory and visual stimuli on subjects in time-restricted intervals. Such tasks rely heavily on attentional control, which is required in the performance of demanding working-memory tasks. The new research on fluid intelligence is also important in that it shows that the training effect occurs across all ability levels, i.e., people with low IQ, as well as those at the higher end of the spectrum. The results of the new research are important contributions in learning sciences, as they show that cognitive training improves fluid intelligence. Beyond that, these findings also have important implications in the mathematical sciences.

In mathematics we have a natural paradigm for training the brain to deal in a focused manner with demanding tasks. This is what I refer to as concentration, automaticity, and mental mathematics. Our earlier research suggests that work in mathematics using frequent, cumulative, and time-restricted testing can improve the working memory. Training students to perform multi-level problems mentally in timed intervals has a close resemblance to dual  $n$ -back tasks, in the related

working-memory research in psychology: both demand full concentration, speed, and accuracy in the processing of stimuli. If this correlation is valid, and training in working memory can correlate with gains in fluid intelligence, then disciplined training in mathematics utilizing cumulative, time-restricted testing can improve fluid intelligence and students' ability to reason and solve problems in any field and in all disciplines.

While more research is required—including the relative value of this protocol at various levels of mathematical study—the possible implications for mathematics education are dramatic. As we continue to explore this premise, one cannot help but reflect once again upon Plato's penetrating insight on the richness and value of training in mathematics.

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# Nefarious Numbers

*Douglas N. Arnold and Kristine K. Fowler*

The impact factor has been widely adopted as a proxy for journal quality. It is used by libraries to guide purchase and renewal decisions, by researchers deciding where to publish and what to read, by tenure and promotion committees laboring under the assumption that publication in a higher-impact-factor journal represents better work, and by editors and publishers as a means to evaluate and promote their journals. The impact factor for a journal in a given year is calculated by ISI (Thomson Reuters) as the average number of citations in that year to the articles the journal published in the preceding two years. It has been widely criticized on a variety of grounds:<sup>1,2,3,4</sup>

- A journal's distribution of citations does not determine its quality.
- The impact factor is a crude statistic, reporting only one particular item of information from the citation distribution.
- It is a flawed statistic. For one thing, the distribution of citations among papers is highly skewed, so the mean for the journal tends to be misleading. For another, the impact factor only refers to citations within the first two years after publication (a particularly serious deficiency for mathematics, in which around 90% of citations occur after two years).
- The underlying database is flawed, containing errors and including a biased selection of journals.

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<sup>1</sup>P. O. Seglen, *Why the impact factor of journals should not be used for evaluating research*, *BMJ* **314** (1997), 498–502.

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- Many confounding factors are ignored, for example, article type (editorials, reviews, and letters versus original research articles), multiple authorship, self-citation, language of publication, etc.

Despite these difficulties, the allure of the impact factor as a single, readily available number—not requiring complex judgments or expert input, but purporting to represent journal quality—has proven irresistible to many. Writing in 2000 in a newsletter for journal editors, Amin and Mabe<sup>5</sup> noted that the “impact factor has moved in recent years from an obscure bibliometric indicator to become the chief quantitative measure of the quality of a journal, its research papers, the researchers who wrote those papers and even the institution they work in.” It has become commonplace for journals to issue absurd announcements touting their impact factors, such as this one, which was mailed around the world by World Scientific, the publisher of the *International Journal of Algebra and Computation*: “IJAC’s Impact Factor has improved from 0.414 in 2007 to 0.421 in 2008! Congratulations to the Editorial Board and contributors of IJAC.” In this case, the 1.7% increase in the impact factor represents a single additional citation to one of the 145 articles published by the journal in the preceding two years.

Because of the (misplaced) emphasis on impact factors, this measure has become a target at which journal editors and publishers aim. This has in turn led to another major source of problems with the factor. Goodhart’s law warns us that “when a measure becomes a target, it ceases to be a good measure.”<sup>6</sup> This is precisely the case with impact factors. Their limited utility has been further compromised by impact factor manipulation, the engineering of this supposed measure of journal quality, in ways that increase the measure but do not add to—indeed, subtract from—journal quality.

Impact factor manipulation can take numerous forms. In a 2007 essay on the deleterious effects of

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<sup>5</sup>M. Amin and M. Mabe, *Impact factors: Use and abuse*, *Perspectives in Publishing* **1** (2000), 1–6.

<sup>6</sup>This succinct formulation is from M. Strathern, “Improving ratings”: Audit in the British University system, *European Review* **5** (1997), 305–321.

impact factor manipulation, Macdonald and Kam<sup>7</sup> noted wryly that “the canny editor cultivates a cadre of regulars who can be relied upon to boost the measured quality of the journal by citing themselves and each other shamelessly.” There have also been widespread complaints by authors of manuscripts under review, who were asked or required by editors to cite other papers from the journal. Given the dependence of the author on the editor’s decision for publication, this practice borders on extortion, even when posed as a suggestion. In most cases, one can only guess about the presence of such pressures, but overt instances were reported already in 2005 by Monastersky<sup>8</sup> in the *Chronicle of Higher Education* and Begley<sup>9</sup> in the *Wall Street Journal*. A third well-established technique by which editors raise their journals’ impact factors is by publishing review items with large numbers of citations to the journal. For example, the editor-in-chief of the *Journal of Gerontology A* made a practice of authoring and publishing a review article every January focusing on the preceding two years; in 2004, 195 of the 277 references were to the *Journal of Gerontology A*. Though the distortions these unscientific practices wreak upon the scientific literature have raised occasional alarms, many suppose that they either have minimal effect or are so easily detectable that they can be disregarded. A counterexample should confirm the need for alarm.

### The Case of IJNSNS

The field of applied mathematics provides an illuminating case in which we can study such impact-factor distortion. For the last several years, the *International Journal of Nonlinear Sciences and Numerical Simulation* (IJNSNS) has dominated the impact-factor charts in the “Mathematics, Applied” category. It took first place in each year 2006, 2007, 2008, and 2009, generally by a wide margin, and came in second in 2005. However, as we shall see, a more careful look indicates that IJNSNS is nowhere near the top of its field. Thus we set out to understand the origin of its large impact factor.

In 2008, the year we shall consider in most detail, IJNSNS had an impact factor of 8.91, easily the highest among the 175 journals in the applied math category in ISI’s *Journal Citation Reports* (JCR). As controls, we will also look at the two journals in the category with the second and third highest impact factors, *Communications on Pure and Applied Mathematics* (CPAM) and *SIAM Review* (SIREV), with 2008 impact factors of 3.69 and 2.80,

respectively. CPAM is closely associated with the Courant Institute of Mathematical Sciences, and SIREV is the flagship journal of the Society for Industrial and Applied Mathematics (SIAM).<sup>10</sup> Both journals have a reputation for excellence.

Evaluation based on expert judgment is the best alternative to citation-based measures for journals. Though not without potential problems of its own, a careful rating by experts is likely to provide a much more accurate and holistic guide to journal quality than impact factor or similar metrics. In mathematics, as in many fields, researchers are widely in agreement about which are the best journals in their specialties. The Australian Research Council recently released such an evaluation, listing quality ratings for over 20,000 peer-reviewed journals across disciplines. The list was developed through an extensive review process involving learned academies (such as the Australian Academy of Science), disciplinary bodies (such as the Australian Mathematical Society), and many researchers and expert reviewers.<sup>11</sup> This rating is being used in 2010 for the Excellence in Research Australia assessment initiative and is referred to as the ERA 2010 Journal List. The assigned quality rating, which is intended to represent “the overall quality of the journal,” is one of four values:

- A\*: one of the best in its field or subfield
- A: very high quality
- B: solid, though not outstanding, reputation
- C: does not meet the criteria of the higher tiers.

The ERA list included all but five of the 175 journals assigned a 2008 impact factor by JCR in the category “Mathematics, Applied”. Figure 1 shows the impact factors for journals in each of the four rating tiers. We see that, as a proxy for expert opinion, the impact factor does rather poorly. There are many examples of journals with a higher impact factor than other journals that are one, two, and even three rating tiers higher. The red line is drawn so that 20% of the A\* journals are below it; it is notable that 51% of the A journals have an impact factor above that level, as do 23% of the B journals and even 17% of those in the C category.

The most extreme outlier is IJNSNS, which, despite its relatively astronomical impact factor, is not in the first or second but, rather, third tier. The ERA rating assigned its highest score, A\*, to 25 journals. Most of the journals with the highest impact factors are here, including CPAM and SIREV, but, of the top 10 journals by impact factor, two were assigned an A, and only IJNSNS was assigned a B. There were 53 A-rated journals and 69 B-rated journals altogether. If IJNSNS were assumed to be the best of the B journals, there would be 78

<sup>7</sup>S. Macdonald and J. Kam, Aardvark et al.: *Quality journals and gamesmanship in management studies*, *Journal of Information Science* 33 (2007), 702–717.

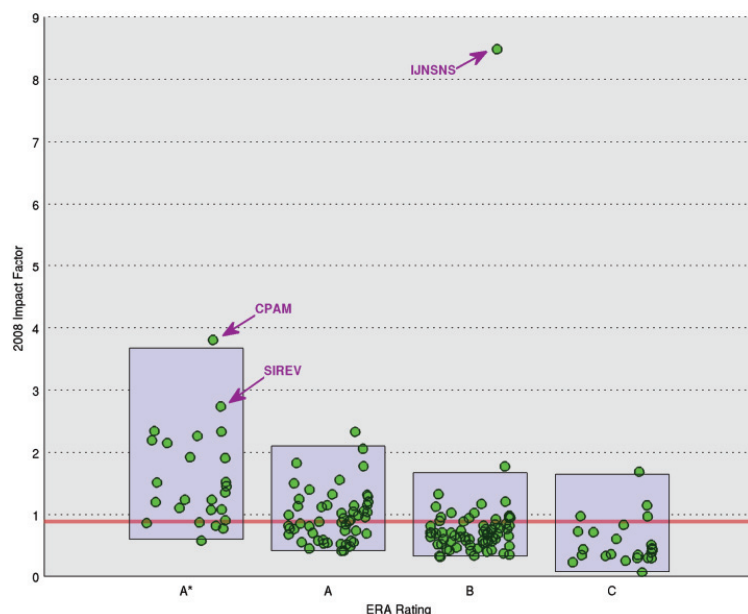
<sup>8</sup>R. Monastersky, *The number that’s devouring science*, *Chronicle of Higher Education* 52 (2005).

<sup>9</sup>S. Begley, *Science journals artfully try to boost their rankings*, *Wall Street Journal*, 5 June 2006, B1.

<sup>10</sup>The first author is the immediate past president of SIAM.

<sup>11</sup>Australian Research Council, *Ranked Journal List Development*, [http://www.arc.gov.au/era/journal\\_list\\_dev.htm](http://www.arc.gov.au/era/journal_list_dev.htm).





**Figure 1.** 2008 impact factors of 170 applied math journals grouped according to their 2010 ERA rating tier. In each tier, the band runs from the 2.5th to the 97.5th percentile, outlining the middle 95%. Horizontal position of the data points within tiers is assigned randomly to improve visibility. The red line is at the 20th percentile of the A\* tier.

journals with higher ERA ratings, whereas if it were the worst, its ranking would fall to 147. In short, the ERA ratings suggest that IJNSNS is not only not the top applied math journal but also that its rank should be somewhere in the range 75–150. This remarkable mismatch between reputation and impact factor needs an explanation.

### Makings of a High Impact Factor

A first step to understanding IJNSNS's high impact factor is to look at how many authors contributed substantially to the counted citations and who they were. The top-citing author to IJNSNS in 2008 was the journal's editor-in-chief, Ji-Huan He, who cited the journal (within the two-year window) 243 times. The second top citer, D. D. Ganji, with 114 cites, is also a member of the editorial board, as is the third, regional editor Mohamed El Naschie, with 58 cites. Together these three account for 29% of the citations counted toward the impact factor. For comparison, the top three citers to SIREV contributed only 7, 4, and 4 citations, respectively, accounting for less than 12% of the counted citations, and none of these authors is involved in editing the journal. For CPAM the top three citers (9, 8, and 8) contributed about 7% of the citations and, again, were not on the editorial board.

Another significant phenomenon is the extent to which citations to IJNSNS are concentrated within the two-year window used in the impact-factor calculation. Our analysis of 2008 citations to articles published since 2000 shows that 16% of the citations to CPAM fell within that two-year

window and only 8% of those to SIREV did; in contrast, 71.5% of the 2008 citations to IJNSNS fell within the two-year window. In Table 1, we show the 2008 impact factors for the three journals, as well as a modified impact factor, which gives the average number of citations in 2008 to articles the journals published not in 2006 and 2007 but in the preceding six years. Since the cited half-life (the time it takes to generate half of all the eventual citations to an article) for applied mathematics is nearly 10 years,<sup>12</sup> this measure is at least as reasonable as the impact factor. It is also independent, unlike JCR's 5-Year Impact Factor, as its time period does not overlap with that targeted by the

Journal	2008 impact factor with normal 2006–7 window	Modified 2008 “impact factor” with 2000–5 window
IJNSNS	8.91	1.27
CPAM	3.69	3.46
SIREV	2.8	10.4

**Table 1.** 2008 impact factors computed with the usual two-preceding-years window, and with a window going back eight years but neglecting the two immediately preceding.

impact factor. Note that the impact factor of IJNSNS drops precipitously, by a factor of seven, when we consider a different citation window. By contrast, the impact factor of CPAM stays about the same, and that of SIREV increases markedly. One may simply note that, in distinction to the controls, the citations made to IJNSNS in 2008 greatly favor articles published in precisely the two years that are used to calculate the impact factor.

Further striking insights arise when we examine the high-citing journals rather than high-citing authors. The counting of journal self-citations in the impact factor is frequently criticized, and indeed it does come into play in this case. In 2008 IJNSNS supplied 102, or 7%, of its own impact factor citations. The corresponding numbers are 1 citation (0.8%) for SIREV and 8 citations (2.4%) for CPAM. The disparity in other recent years is similarly large or larger.

However, it was *Journal of Physics: Conference Series* that provided the greatest number of IJNSNS citations. A single issue of that journal provided 294 citations to IJNSNS in the impact-factor window, accounting for more than 20% of its impact factor. What was this issue? It was the proceedings of a conference organized by IJNSNS editor-in-chief He at his home university. He was responsible for the peer review of the issue. The second top-citing journal for IJNSNS was *Topological Methods in Nonlinear Analysis*, which contributed 206 citations (14%), again with all citations coming from a single issue. This was a special

<sup>12</sup>In 2010, Journal Citation Reports assigned the category “Mathematics, Applied” an aggregate cited half-life of 9.5 years.



issue with Ji-Huan He as the guest editor; his co-editor, Lan Xu, is also on the IJNSNS editorial board. J.-H. He himself contributed a brief article to the special issue, consisting of three pages of text and thirty references. Of these, twenty were citations to IJNSNS within the impact-factor window. The remaining ten consisted of eight citations to He and two to Xu.

Continuing down the list of IJNSNS high-citing journals, another similar circumstance comes to light: 50 citations from a single issue of the *Journal of Polymer Engineering* (which, like IJNSNS, is published by Freund), guest edited by the same pair, Ji-Huan He and Lan Xu. However, third place is held by the journal *Chaos, Solitons and Fractals*, with 154 citations spread over numerous issues. These are again citations that may be viewed as subject to editorial influence or control. In 2008 Ji-Huan He served on the editorial board of CS&F, and its editor-in-chief was Mohamed El Naschie, who was also a coeditor of IJNSNS. In a highly publicized case, the entire editorial board of CS&F was recently replaced, but El Naschie remained coeditor of IJNSNS.

Many other citations to IJNSNS came from papers published in journals for which He served as editor, such as *Zeitschrift für Naturforschung A*, which provided forty citations; there are too many others to list here, since He serves in an editorial capacity on more than twenty journals (and has just been named editor-in-chief of four more journals from the newly formed Asian Academic Publishers). Yet another source of citations came from papers authored by IJNSNS editors other than He, which accounted for many more. All told, the aggregation of such editor-connected citations, which are time-consuming to detect, account for more than 70% of all the citations contributing to the IJNSNS impact factor.

### Bibliometrics for Individuals

Bibliometrics are also used to evaluate individuals, articles, institutions, and even nations. Essential Science Indicators, which is produced by Thomson Reuters, is promoted as a tool for ranking “top countries, journals, scientists, papers, and institutions by field of research”. However, these metrics are primarily based on the same citation data used for journal impact factors, and thus they can be manipulated just as easily, indeed simultaneously. The special issue of *Journal of Physics: Conference Series* that He edited and that garnered 243 citations for his journal also garnered 353 citations to He himself. He claims a total citation count of over 6,800.<sup>13</sup> Even half that is considered highly noteworthy, as evidenced by this announcement in

ScienceWatch.com:<sup>14</sup> “According to a recent analysis of Essential Science Indicators from Thomson Scientific, Professor Ji-Huan He has been named a Rising Star in the field of Computer Science... His citation record in the Web of Science includes 137 papers cited a total of 3,193 times to date.” Together with only a dozen other scientists in all fields of science, He was cited by ESI for the “Hottest Research of 2007–8” and again for the “Hottest Research of 2009”.

The h-index is another popular citation-based metric for researchers, intended to measure productivity as well as impact. An individual’s h-index is the largest number such that that many of his or her papers have been cited at least that many times. It too is not immune from Goodhart’s law. J.-H. He claims an h-index of 39, while Hirsch estimated the median for Nobel prize winners in physics to be 35.<sup>15</sup> Whether for judgment of individuals or journals, citation-based designations are no substitute for an informed judgment of quality.

### Closing Thoughts

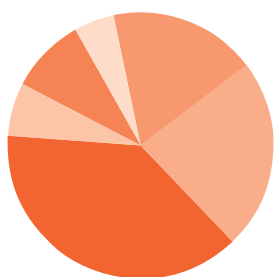
Despite numerous flaws, the impact factor has been widely used as a measure of quality for journals and even for papers and authors. This creates an incentive to manipulate it. Moreover, it is possible to vastly increase impact factor without increasing journal quality at all. The actions of a few interested individuals can make a huge difference, yet it requires considerable digging to reveal them. We primarily discussed one extreme example, but there is little reason to doubt that such techniques are being used to a lesser—and therefore less easily detected—degree by many journals. The cumulative result of the design flaws and manipulation is that impact factor gives a very inaccurate view of journal quality. More generally, the citations that form the basis of the impact factor and various other bibliometrics are inherently untrustworthy.

The consequences of this unfortunate situation are great. Rewards are wrongly distributed, the scientific literature and enterprise are distorted, and cynicism about them grows. What is to be done? Just as for scientific research itself, the temptation to embrace simplicity when it seriously compromises accuracy must be resisted. Scientists who give in to the temptation to suppress data or fiddle with statistics to draw a clearer point are censured. We must bring a similar level of integrity to the evaluation of research products. Administrators, funding agencies, librarians, and others needing such evaluations should just say no to simplistic solutions and approach important decisions with thoughtfulness, wisdom, and expertise.

<sup>14</sup>ScienceWatch.com, April 2008, <http://sciencewatch.com/inter/aut/2008/08-apr/08aprHe/>.

<sup>15</sup>J. Hirsch, *An index to quantify an individual’s scientific research output*. PNAS 102 (2005), 16569–16572.

<sup>13</sup>This claim, and that of an h-index of 39, are made in the biographical notes of one of his recent papers (Nonl. Sci. Letters 1 (2010), page 1).



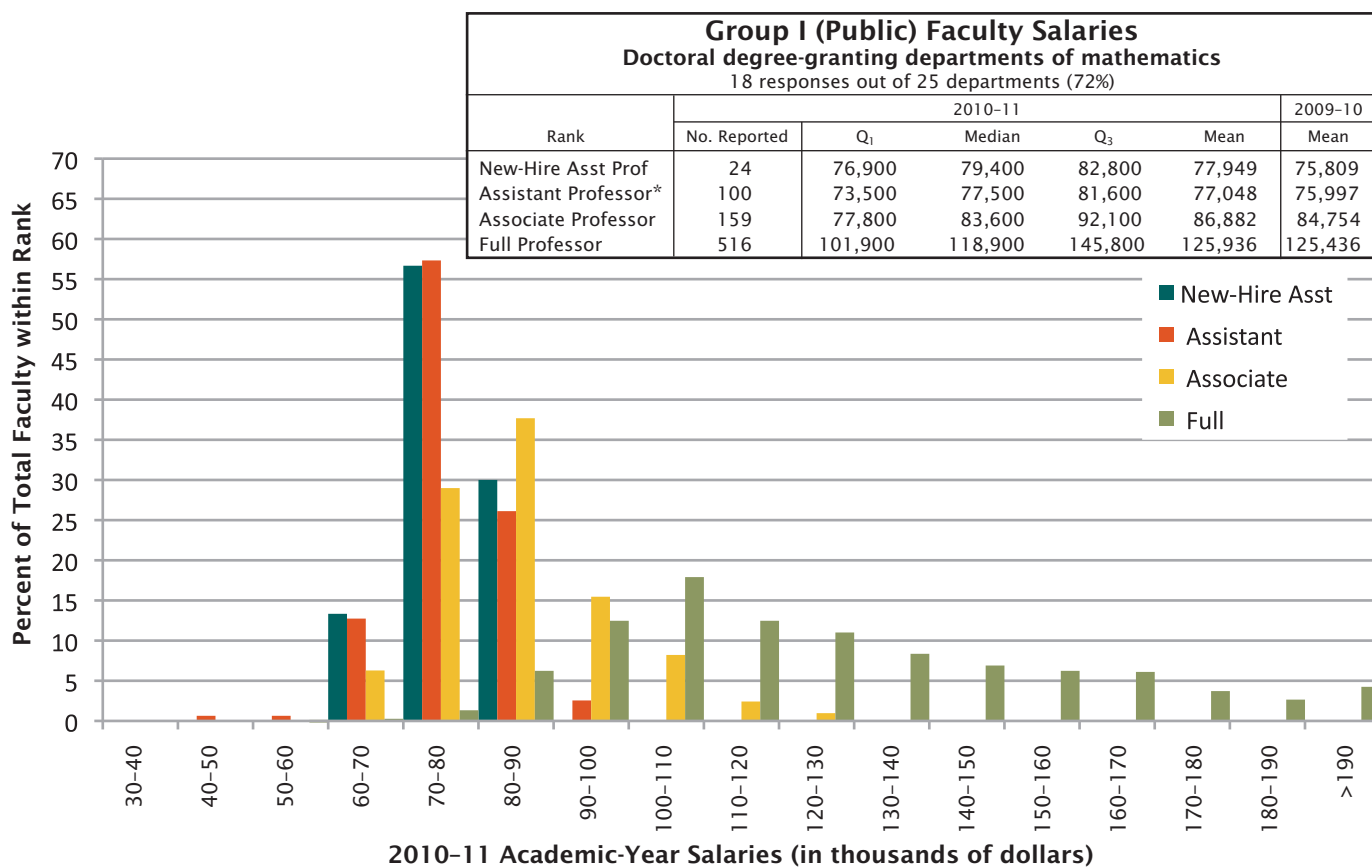
# 2010-2011 Faculty Salaries Report

*Richard Cleary, James W. Maxwell, and Colleen Rose*

This report provides information on the distribution of 2010–2011 academic-year salaries for tenured and tenure-track faculty at four-year mathematical sciences departments in the U.S. by the departmental groupings used in the Annual Survey. (See page 443 for the definitions of the various departmental groupings.) Salaries are described separately by rank. Salaries are reported in current dollars (at time of data collection). Results reported here are based on the departments which responded to the survey with no adjustment for non-response.

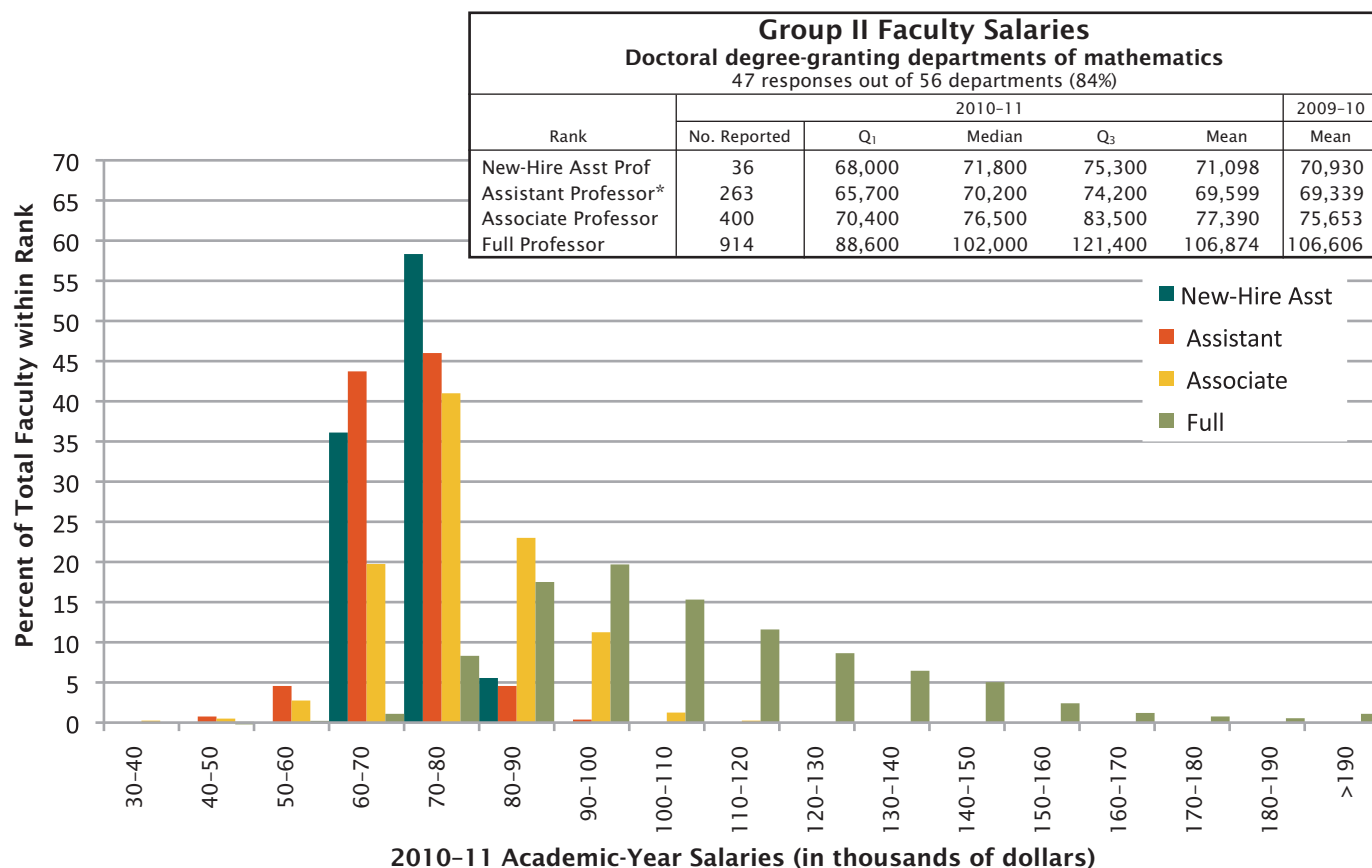
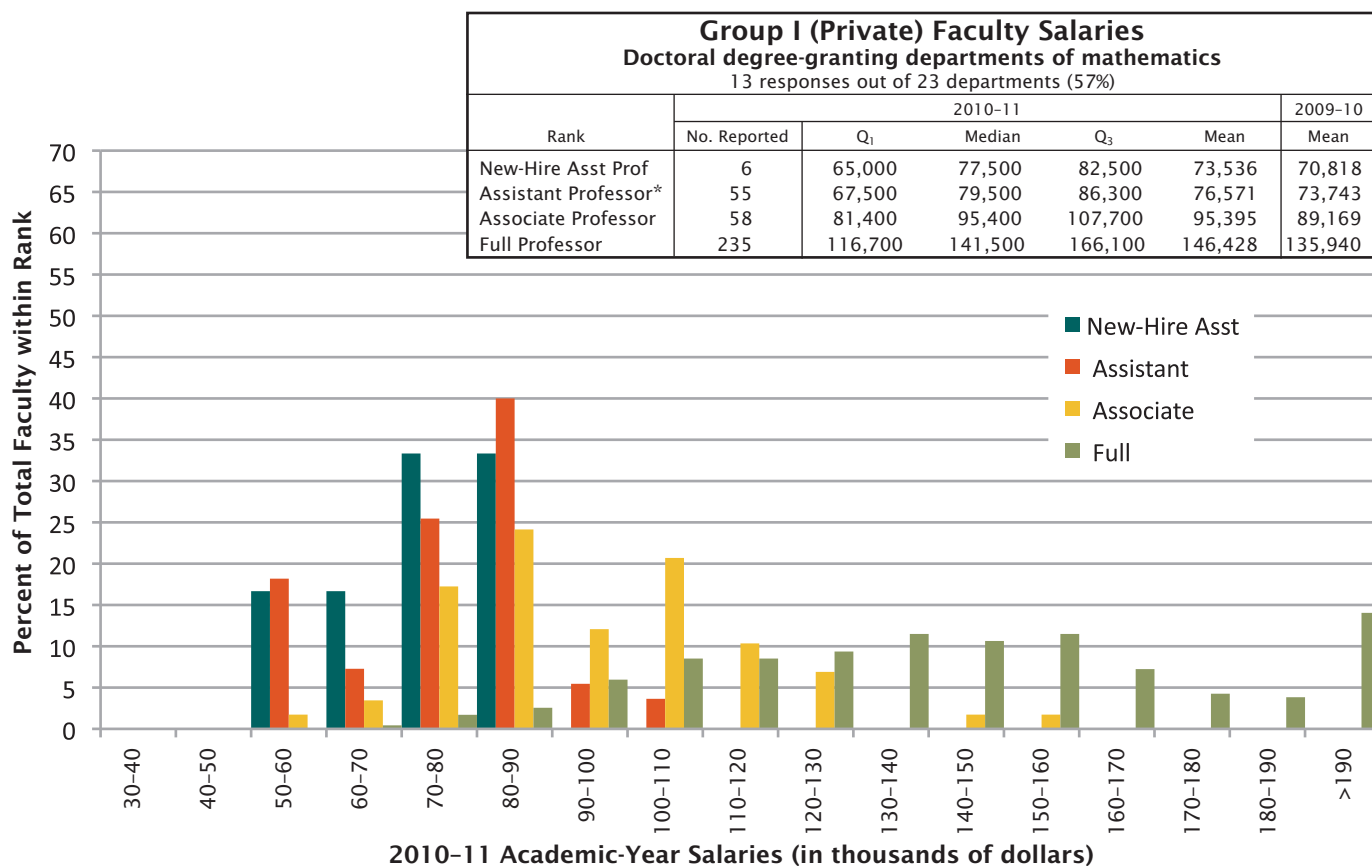
Departments were asked to report for each rank the number of tenured and tenure-track faculty whose 2010–11 academic-year salaries fell within given salary intervals. Reporting salary data in this fashion eliminates some of the concerns about confidentiality but does not permit determination of actual quartiles. Although the actual quartiles cannot be determined from the data gathered, these quartiles have been estimated assuming that the density over each interval is uniform.

When comparing current and prior year figures, one should keep in mind that differences in the set of responding departments may be one of the most important factors in the change in the reported mean salaries.



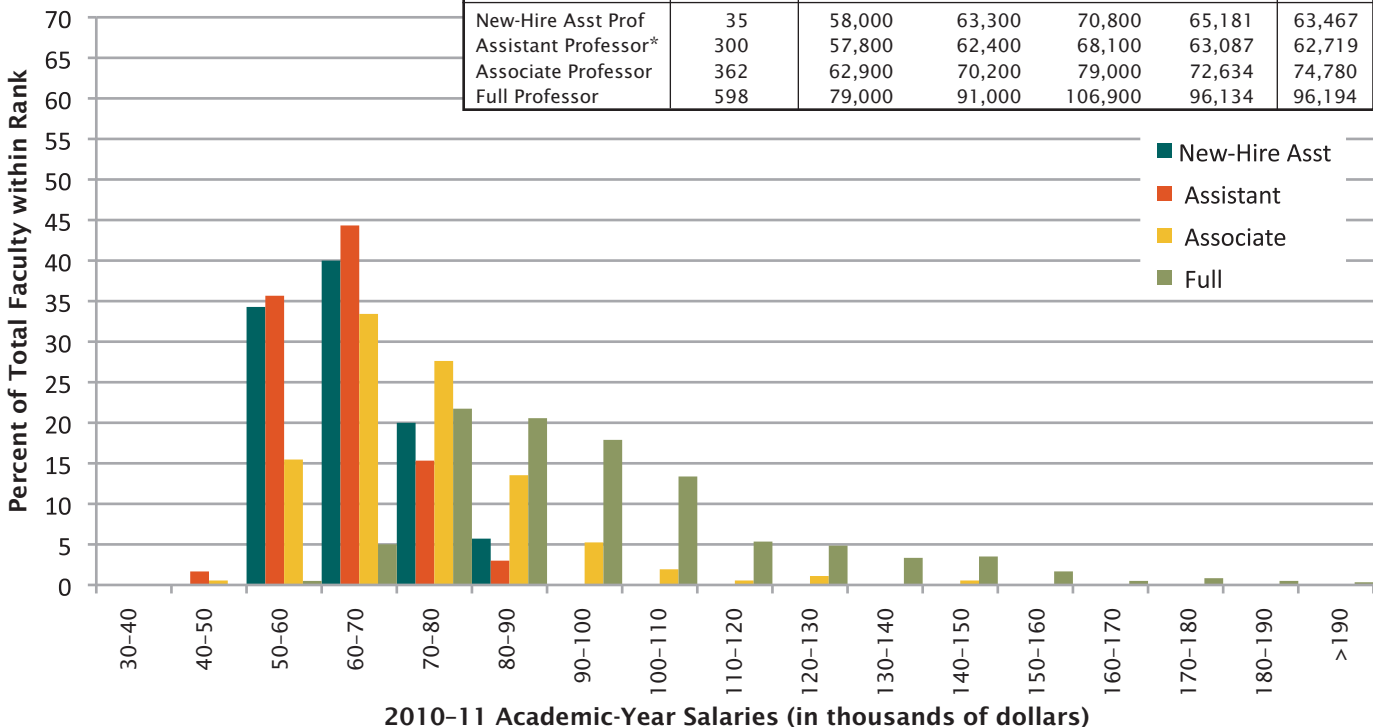
\*Includes new hires.

*Richard Cleary is professor and chair of the Department of Mathematical Sciences at Bentley University. James W. Maxwell is AMS associate executive director for special projects. Colleen A. Rose is AMS survey analyst.*

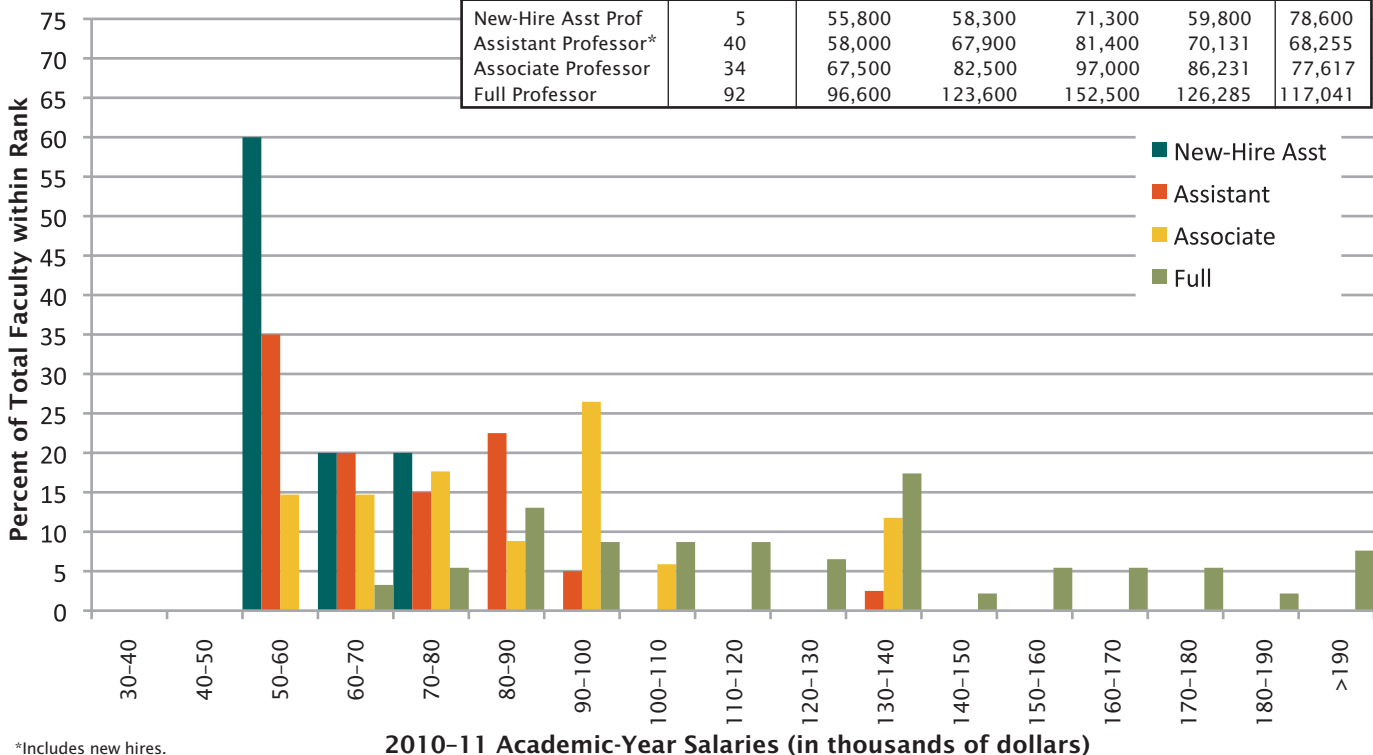


\*Includes new hires.

<b>Group III Faculty Salaries</b> <b>Doctoral degree-granting departments of mathematics</b> 57 responses out of 80 departments (71%)						
Rank	2010-11					2009-10
	No. Reported	Q <sub>1</sub>	Median	Q <sub>3</sub>	Mean	Mean
New-Hire Asst Prof	35	58,000	63,300	70,800	65,181	63,467
Assistant Professor*	300	57,800	62,400	68,100	63,087	62,719
Associate Professor	362	62,900	70,200	79,000	72,634	74,780
Full Professor	598	79,000	91,000	106,900	96,134	96,194

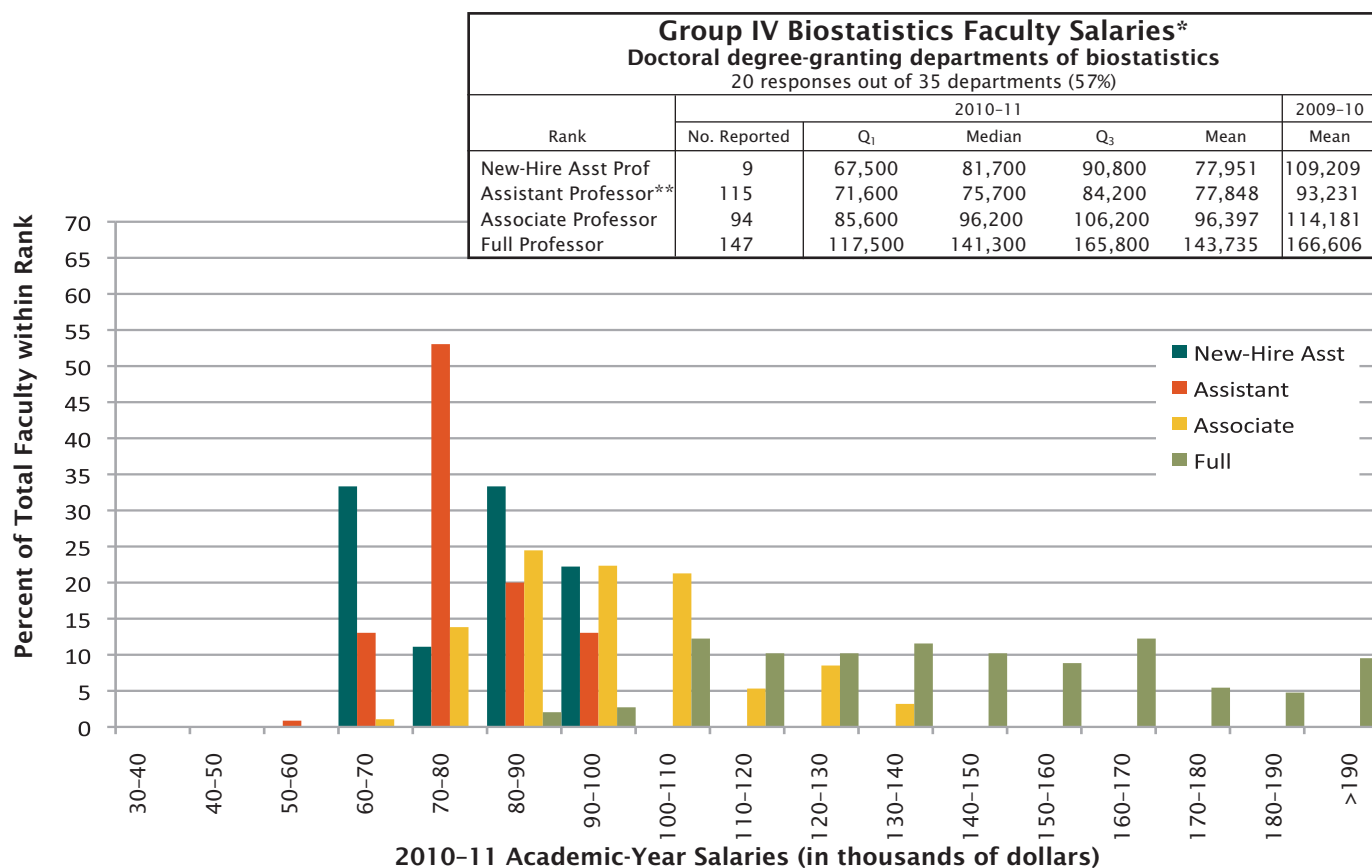
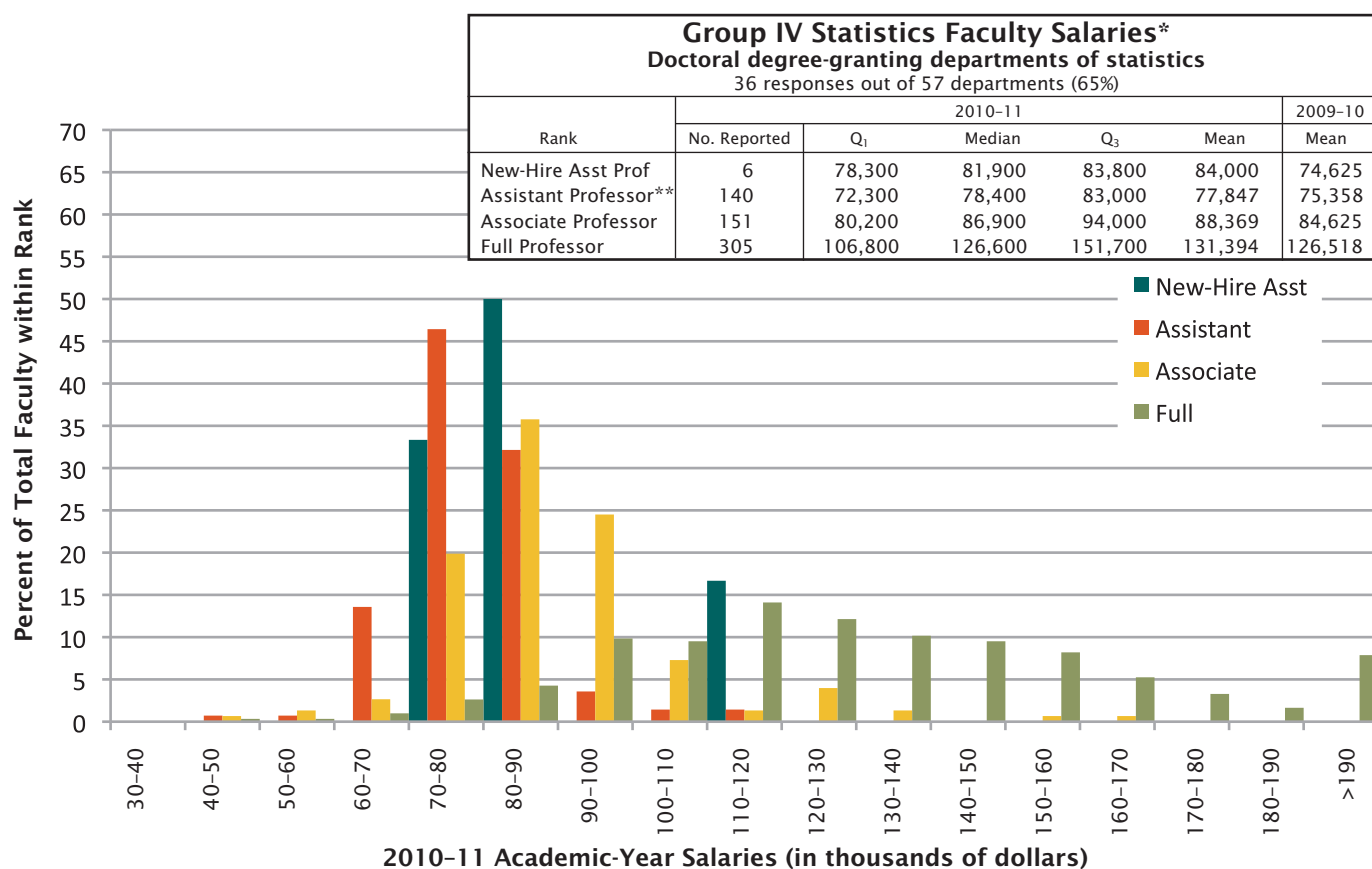


<b>Group Va Faculty Salaries</b> <b>Doctoral degree-granting departments of applied mathematics</b> 11 responses out of 17 departments (65%)						
Rank	2010-11					2009-10
	No. Reported	Q <sub>1</sub>	Median	Q <sub>3</sub>	Mean	Mean
New-Hire Asst Prof	5	55,800	58,300	71,300	59,800	78,600
Assistant Professor*	40	58,000	67,900	81,400	70,131	68,255
Associate Professor	34	67,500	82,500	97,000	86,231	77,617
Full Professor	92	96,600	123,600	152,500	126,285	117,041



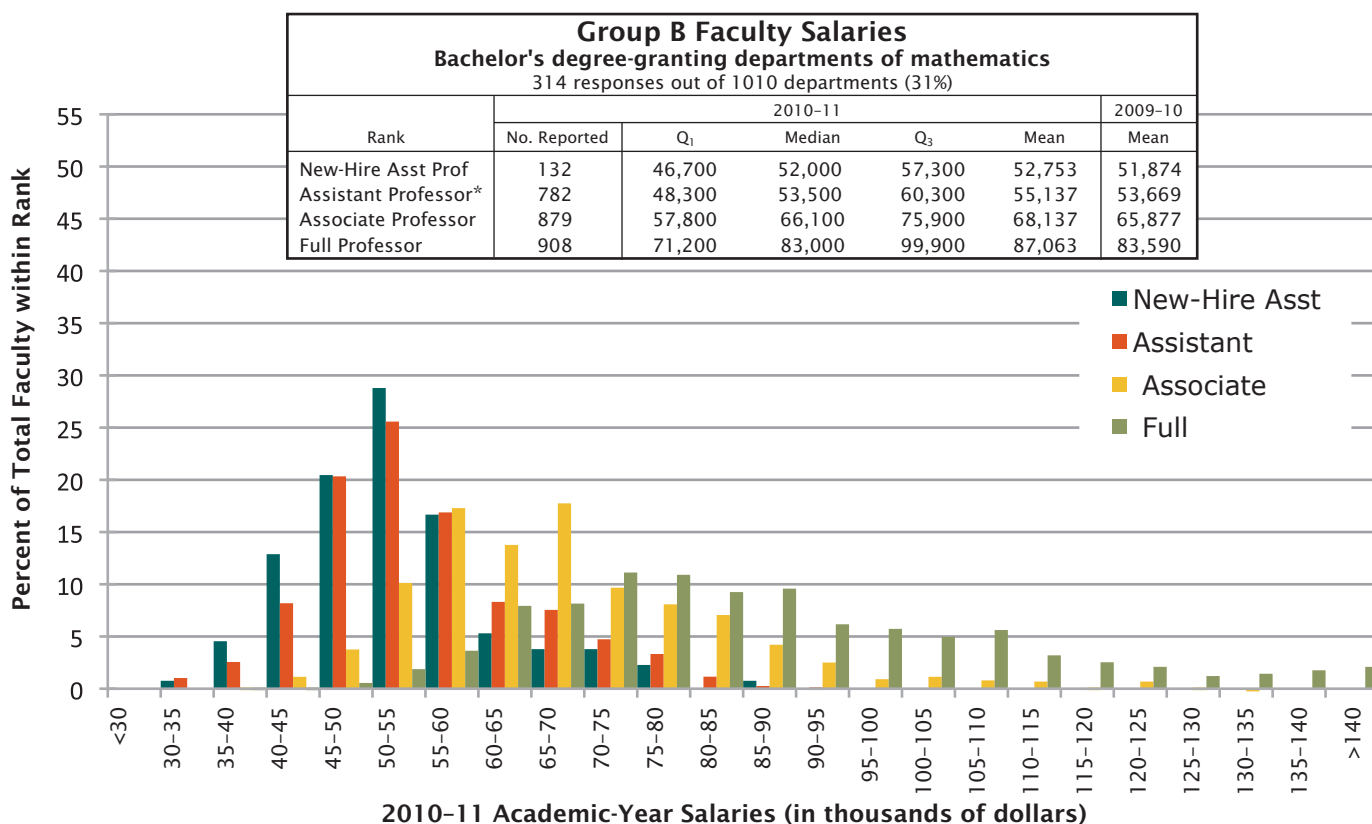
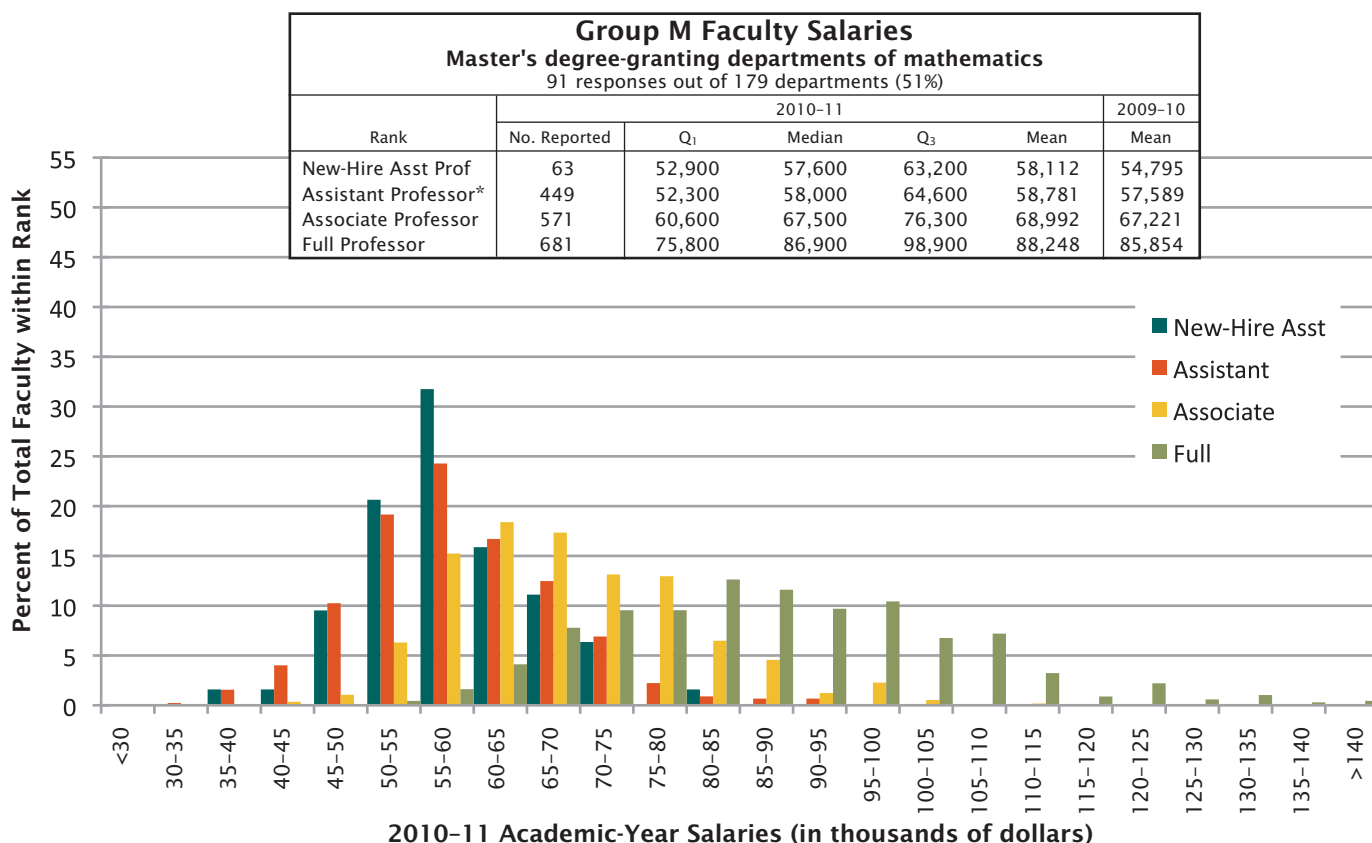
\*Includes new hires.





\*Faculty salary data provided by the American Statistical Association.

\*\*Includes new hires.



\*Includes new hires.

## Other Information

### Obtain a Special Faculty Salaries Analysis

See how the salaries of your department's tenured/tenure-track faculty compare to those in similar departments. The only requirement is that your department must have responded to our latest Faculty Salary survey.

Send a list of your peer institutions (a minimum of 12 institutions is required) to [ams-survey@ams.org](mailto:ams-survey@ams.org) along with the date needed. (If not enough of your peer group have responded to the salary survey you'll be asked to provide additional institutions.) A minimum of two weeks is needed to complete a special analysis.

The analysis produced includes a listing of your peer group institutions along their salary survey response status, a summary table including the rank (assistant, associate, and full professor), the number reported in each rank, the 1st quartile, median, 3rd quartile, and mean salaries for each along with bar graphs.

### Acknowledgements

The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the professional organizations. Every year, college and university departments in the United States are invited to respond. The Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments for the quality of its information. On behalf of the Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff members in the mathematical sciences departments for their cooperation and assistance in responding to the survey questionnaires.

### Previous Annual Survey Reports

The 2009 First, Second, and Third Annual Survey Reports were published in the *Notices of the AMS* in the February, August, and November 2009 issues respectively. These reports and earlier reports, as well as a wealth of other information from these surveys, are available on the AMS website at [www.ams.org/annual-survey/survey-reports](http://www.ams.org/annual-survey/survey-reports).

### Other Sources of Data

Visit the AMS website at [www.ams.org/annual-survey/other-sources](http://www.ams.org/annual-survey/other-sources) for a listing of additional sources of data on the Mathematical Sciences.

### Group Descriptions

**Group I** is composed of 48 departments with scores in the 3.00–5.00 range. Group I Public and Group I Private are Group I departments at public institutions and private institutions, respectively.

**Group II** is composed of 56 departments with scores in the 2.00–2.99 range.

**Group III** contains the remaining U.S. departments reporting a doctoral program, including a number of departments not included in the 1995 ranking of program faculty.

**Group IV** contains U.S. departments (or programs) of statistics, biostatistics, and biometrics reporting a doctoral program.

**Group V** contains U.S. departments (or programs) in applied mathematics/applied science, operations research, and management science which report a doctoral program.

**Group Va** is applied mathematics/applied science; **Group Vb**, which was no longer surveyed as of 1998–99, was operations research and management science.

**Group M** contains U.S. departments granting a master's degree as the highest graduate degree.

**Group B** contains U.S. departments granting a baccalaureate degree only.

Listings of the actual departments which compose these groups are available on the AMS website at [www.ams.org/annual-survey/groups\\_des](http://www.ams.org/annual-survey/groups_des).

### About the Annual Survey

The Annual Survey series, begun in 1957 by the American Mathematical Society, is currently under the direction of the Data Committee, a joint committee of the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society of Industrial and Applied Mathematics. The current members of this committee are Pam Arroway, Richard Cleary (chair), Steven R. Dunbar, Susan Geller, Abbe H. Herzig, Ellen Kirkman, Joanna Mitro, James W. Maxwell (ex officio), Bart S. Ng, Douglas Ravel, and Marie Vitulli. The committee is assisted by AMS survey analyst Colleen A. Rose. In addition, the Annual Survey is sponsored by the Institute of Mathematical Statistics. Comments or suggestions regarding this Survey Report may be directed to the committee.

# Interview with Abel Laureate John Tate

*Martin Raussen and Christian Skau*

John Tate is the recipient of the 2009 Abel Prize of the Norwegian Academy of Science and Letters. This interview took place on May 25, 2010, prior to the Abel Prize celebration in Oslo, and originally appeared in the September 2010 issue of the *Newsletter of the European Mathematical Society*.

## Education

**Raussen and Skau:** Professor Tate, you have been selected as this year's Abel Prize Laureate for your decisive and lasting impact on number theory. Before we start to ask you questions, we would like to congratulate you warmly on this achievement. You were born in 1925 in Minneapolis in the United States. Your father was a professor of physics at the University of Minnesota. We guess he had some influence on your attraction to the natural sciences and mathematics. Is that correct?

**Tate:** It certainly is. He never pushed me in any way, but on a few occasions he simply explained something to me. I remember once he told me how one could estimate the height of a bridge over a river with a stopwatch, by dropping a rock, explaining that the height in feet is approximately sixteen times the square of the number of seconds it takes until the rock hits the water. Another time he explained Cartesian coordinates and how one could graph an equation and, in particular, how the solution to two simultaneous linear equations is the point where two lines meet. Very rarely, but beautifully, he just explained something to me. He did not have to explain negative numbers—I learned about them from the temperature in the Minnesota winters.

But I have always, in any case, been interested in puzzles and trying to find the answers to

questions. My father had several puzzle books. I liked reading them and trying to solve the puzzles. I enjoyed thinking about them, even though I did not often find a solution.

**Raussen and Skau:** Are there other persons that have had an influence on your choice of fields of interest during your youth?

**Tate:** No. I think my interest is more innate. My father certainly helped, but I think I would have done something like physics or mathematics anyway.

**Raussen and Skau:** You started to study physics at Harvard University. This was probably during the Second World War?

**Tate:** I was in my last year of secondary school in December 1941 when Pearl Harbor was bombed. Because of the war Harvard began holding classes in the summer, and I started there the following June. A year later I volunteered for a Naval Officer Training Program in order to avoid being drafted into the army. A group of us was later sent to M.I.T. to learn meteorology, but by the time we finished that training and Midshipman School it was VE day.<sup>1</sup> Our campaign in the Pacific had been so successful that more meteorologists were not needed, and I was sent to do minesweeping research. I was in the Navy for three years and never aboard a ship! It was frustrating.

**Raussen and Skau:** Study conditions in those times must have been quite different from conditions today. Did you have classes regularly?

**Tate:** Yes, for the first year, except that it was accelerated. But then in the Navy I had specific classes to attend, along with a few others of my choice I could manage to squeeze in. It was a good program, but it was not the normal one.

It was not the normal college social life either, with parties and such. We had to be in bed or in a

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*Martin Raussen is associate professor of mathematics at Aalborg University, Denmark. His email address is raussen@math.aau.dk.*

*Christian Skau is professor of mathematics at the Norwegian University of Science and Technology, Trondheim, Norway. His email address is csk@math.ntnu.no.*

*This is a slightly edited version of an interview taken on the morning preceding the prize ceremony: May 25, 2010, at Oslo.*

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<sup>1</sup>Victory in Europe day: May 8, 1945.



study hall by ten and were roused at 6:30 A.M. by a recording of reveille, to start the day with calisthenics and running.

**Rausсен and Skau:** *Then you graduated in 1946 and went to Princeton?*

**Tate:** Yes, that's true. Harvard had a very generous policy of giving credit for military activities that might qualify—for instance, some of my navy training. This and the wartime acceleration enabled me to finish the work for my undergraduate degree in 1945. On my discharge in 1946, I went straight from the Navy to graduate school in Princeton.

**Rausсен and Skau:** *When you went to Princeton University, it was still with the intention to become a physicist?*

**Tate:** That's correct. Although my degree from Harvard was in mathematics, I entered Princeton graduate school in physics. It was rather silly, and I have told the story many times: I had read the book *Men of Mathematics* by Eric Temple Bell. That book was about the lives of the greatest mathematicians in history, people like Abel. I knew I wasn't in their league and I thought that unless I was, I wouldn't really be able to do much in mathematics. I didn't realize that a less talented person could still contribute effectively. Since my father was a physicist, that field seemed more human and accessible to me, and I thought that was a safer way to go, where I might contribute more. But after one term it became obvious that my interest was really in mathematics. A deeper interest, which should have been clear anyway, but I just was too afraid and thought I never would be able to do much research if I went into mathematics.

**Rausсен and Skau:** *Were you particularly interested in number theory from the very beginning?*

**Tate:** Yes. Since I was a teenager I had an interest in number theory. Fortunately, I came across a good number theory book by L. E. Dickson, so I knew a little number theory. Also I had been reading Bell's histories of people like Gauss. I liked number theory. It's natural, in a way, because many wonderful problems and theorems in number theory can be explained to any interested high-school student. Number theory is easier to get into in that sense. But of course it depends on one's intuition and taste also.

**Rausсен and Skau:** *Many important questions are easy to explain, but answers are often very tough to find.*

**Tate:** Yes. In number theory that is certainly true, but finding good questions is also an important part of the game.

## Teachers and Fellows

**Rausсен and Skau:** *When you started your career at Princeton you very quickly met Emil Artin, who became your supervisor. Emil Artin was born in Austria and became a professor in mathematics at the University of Hamburg. He had to leave*

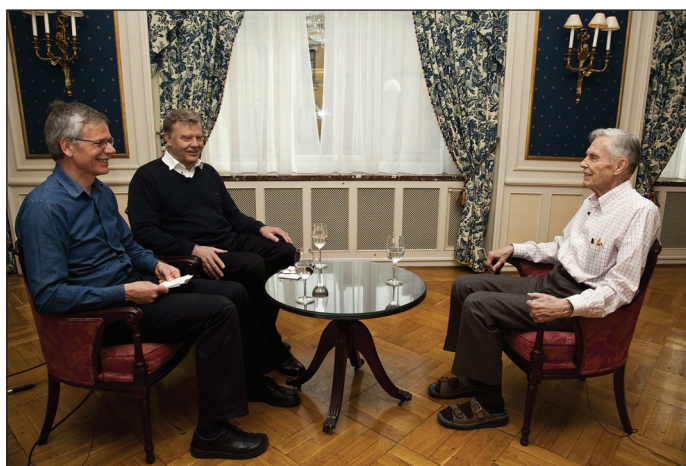


Photo by Eirik F. Baardsen/Abelprisen/Norwegian Academy of Sciences.

**Abel interview, from left to right: Martin Rausсен, Christian Skau, and John Tate.**

*Germany in 1937 and came to the United States. Can you tell us more about his background? Why did he leave his chair, and how did he adjust when he came to the States?*

**Tate:** His wife was half Jewish, and he eventually lost his position in Germany. The family left in '37, but at that time there weren't so many open jobs in the United States. He took a position at the University of Notre Dame in spite of unpleasant memories of discipline at a Catholic school he had attended in his youth. After a year or two he accepted an offer from Indiana University and stayed there until 1946. He and his wife enjoyed Bloomington, Indiana, very much. He told me it wasn't even clear that he would have accepted Princeton's offer in 1946 except that President H. B. Wells of Indiana University, an educational visionary, was on a world tour, and somehow Indiana didn't respond very well to Princeton's offer. Artin went to Princeton the same year I did.

**Rausсен and Skau:** *Artin had apparently a very special personality. First of all, he was an eminent number theorist, but also a very intriguing person; a special character. Could you please tell us a bit more about him?*

**Tate:** I think he would have made a great actor. His lectures were polished: He would finish at the right moment and march off the scene. A very lively individual with many interests: music, astronomy, chemistry, history.... He loved to teach. I had a feeling that he loved to teach anybody anything. Being his student was a wonderful experience; I couldn't have had a better start to my mathematical career. It was a remarkable accident. My favorite theorem, which I had first learned from Bell's book, was Gauss's law of quadratic reciprocity, and there, entirely by chance, I found myself at the same university as the man who had discovered the ultimate law of reciprocity. It was just amazing.

**Rausсен and Skau:** *What a coincidence!*

**Tate:** Yes, it really was.

**Raussen and Skau:** *You wrote your thesis with Artin, and we will certainly come back to it. After that you organized a seminar together with Artin on class field theory. Could you comment on this seminar: What was the framework and how did it develop?*

**Tate:** During his first two years in Princeton, Artin gave seminars in algebraic number theory, followed by class field theory. I did not attend the former, but one of the first things I heard about Artin concerned an incident in it. A young British student, Douglas Northcott, who had been captured when the Japanese trapped the British army in Singapore and barely survived in the Japanese prison camp, was in Princeton on a Commonwealth Fellowship after the war. Though his thesis was in analysis under G. H. Hardy, he attended Artin's seminar, and when one of the first speakers mentioned the characteristic of a field, Northcott raised his hand and asked what that meant. His question begot laughter from several students, whereupon Artin delivered a short lecture on the fact that one could be a fine mathematician without knowing what the characteristic of a field was. And, indeed, it turned out that Northcott was the most gifted student in that seminar.

But I'm not answering your question. I attended the second year, in which class field theory was treated, with Chevalley's nonanalytic proof of the second inequality, but not much cohomology. This was the seminar at the end of which Wang discovered that both published proofs of Grunwald's theorem, and in fact the theorem itself, were not correct at the prime 2.

At about that time, Gerhard Hochschild and Tadasi Nakayama were introducing cohomological methods in class field theory and used them to prove the main theorems, including the existence of the global fundamental class which A. Weil had recently discovered. In 1951–52 Artin and I ran another seminar giving a complete treatment of class field theory incorporating these new ideas. That is the seminar you are asking about. Serge Lang took notes, and thanks to his efforts they were eventually published, first as informal mimeographed notes and, in 1968, commercially, under the title *Class Field Theory*. A new edition (2008) is available from AMS-Chelsea.

**Raussen and Skau:** *Serge Lang was also a student of Emil Artin and became a famous number theorist. He is probably best known as author of many textbooks; almost every graduate student in mathematics has read a textbook by Serge Lang. He is also quite known for his intense temper, and he got into a lot of arguments with people. What can you tell us about Serge Lang? What are your impressions?*

**Tate:** He was indeed a memorable person. The memories of Lang in the May 2006 issue of the *Notices of the AMS*, written by about twenty of

his many friends, give a good picture of him. He started Princeton graduate school in philosophy, a year after I started in physics, but he, too, soon switched to math. He was a bit younger than I and had served a year and a half in the U.S. Army in Europe after the war, where he had a clerical position in which he learned to type at incredible speed, an ability which served him well in his later book writing.

He had many interests and talents. I think his undergraduate degree from Caltech was in physics. He knew a lot of history and he played the piano brilliantly.

He didn't have the volatile personality you refer to until he got his degree. It seemed to me that he changed. It was almost a discontinuity; as soon he got his Ph.D. he became more authoritative and asserted himself more.

It has been noted that there are many mathematical notions linked to my name. I think that's largely due to Lang's drive to make information accessible. He wrote voluminously. I didn't write easily and didn't get around to publishing; I was always interested in thinking about the next problem. To promote access, Serge published some of my stuff and, in reference, called things "Tate this" and "Tate that" in a way I would not have done had I been the author.

Throughout his life, Serge addressed great energy to disseminating information; to sharing where he felt it was important. We remained friends over the years.

## Research Contributions

**Raussen and Skau:** *This brings us to the next topic: Your Ph.D. thesis from 1950, when you were twenty-five years old. It has been extensively cited in the literature under the sobriquet "Tate's thesis". Several mathematicians have described your thesis as unsurpassable in conciseness and lucidity and as representing a watershed in the study of number fields. Could you tell us what was so novel and fruitful in your thesis?*

**Tate:** Well, first of all, it was not a new result, except perhaps for some local aspects. The big global theorem had been proved around 1920 by the great German mathematician Erich Hecke, namely the fact that all  $L$ -functions of number fields, abelian  $L$ -functions, generalizations of Dirichlet's  $L$ -functions, have an analytic continuation throughout the plane with a functional equation of the expected type. In the course of proving it Hecke saw that his proof even applied to a new kind of  $L$ -function, the so-called  $L$ -functions with Grössencharacter. Artin suggested to me that one might prove Hecke's theorem using abstract harmonic analysis on what is now called the adèle ring, treating all places of the field equally, instead of using classical Fourier analysis at the archimedean places and finite Fourier analysis with congruences

at the  $p$ -adic places as Hecke had done. I think I did a good job—it might even have been lucid and concise!—but in a way it was just a wonderful exercise to carry out this idea. And it was also in the air. So often there is a time in mathematics for something to be done. My thesis is an example. Iwasawa would have done it had I not.

**Raussen and Skau:** *What do you think of the fact that, after your thesis, all places of number fields are treated on an equal footing in analytic number theory, whereas the situation is very different in the classical study of zeta functions; in fact, gamma factors are very different from nonarchimedean local factors.*

**Tate:** Of course there is a big difference between archimedean and nonarchimedean places, in particular as regards the local factors, but that is no reason to discriminate. Treating them equally, using adeles and ideles, is the simplest way to proceed, bringing the local-global relationship into clear focus.

**Raussen and Skau:** *The title of your thesis was Fourier Analysis in Number Fields and Hecke's Zeta-Functions. Atle Selberg said in an interview five years ago that he preferred—and was most inspired by—Erich Hecke's approach to algebraic number theory, modular forms and  $L$ -functions. Do you share that sentiment?*

**Tate:** Hecke and Artin were both at Hamburg University for a long time before Artin left. I think Artin came to number theory more from an algebraic side, whereas Hecke and Selberg came more from an analytic side. Their basic intuition was more analytic and Artin's was more algebraic. Mine was also more algebraic, so the more I learned of Hecke's work, the more I appreciated it, but somehow I did not instinctively follow him, especially as to modular forms. I didn't know much about them when I was young.

I have told the story before, but it is ironic that being at the same university, Artin had discovered a new type of  $L$ -series and Hecke, in trying to figure out what kind of modular forms of weight one there were, said they should correspond to some kind of  $L$ -function. The  $L$ -functions Hecke sought were among those that Artin had defined, but they never made contact—it took almost forty years until this connection was guessed and ten more before it was proved, by Langlands. Hecke was older than Artin by about ten years, but I think the main reason they did not make contact was their difference in mathematical taste. Moral: Be open to all approaches to a subject.

**Raussen and Skau:** *You mentioned that Serge Lang had named several concepts after you, but there are lots of further concepts and conjectures bearing your name. Just to mention a few: Tate module, Tate curve, Tate cohomology group, Shafarevich-Tate group, Tate conjecture, Sato-Tate conjecture, etc. Good definitions and fruitful*

*concepts, as well as good problems, are perhaps as important as theorems in mathematics. You excel in all these categories. Did all or most of these concepts grow out of your thesis?*

**Tate:** No, I wouldn't say that. In fact, I would say that almost none of them grew out of my thesis. Some of them, like the Tate curve, grew out of my interest in  $p$ -adic fields, which were also very central in my thesis, but they didn't grow out of my thesis. They came from different directions. The Tate cohomology came from my understanding the cohomology of class field theory in the seminar that we discussed. The Shafarevich-Tate group came from applying that cohomology to elliptic curves and abelian varieties. In general, my conjectures came from an optimistic outlook, generalizing from special cases.

Although concepts, definitions, and conjectures are certainly important, the bottom line is to prove a theorem. But you do have to know what to prove, or what to try to prove.

**Raussen and Skau:** *In the introduction to your delightful book Rational Points on Elliptic Curves that you coauthored with your earlier Ph.D. student Joseph Silverman, you say, citing Serge Lang, that it is possible to write endlessly on elliptic curves. Can you comment on why the theory of elliptic curves is so rich and how it interacts and makes contact with so many different branches of mathematics?*

**Tate:** For one thing, they are very concrete objects. An elliptic curve is described by a cubic polynomial in two variables, so they are very easy to experiment with. On the other hand, elliptic curves illustrate very deep notions. They are the first nontrivial examples of abelian varieties. An elliptic curve is an abelian variety of dimension one, so you can get into this more advanced subject very easily by thinking about elliptic curves. On the other hand, they are algebraic curves. They are curves of genus one, the first example of a curve which isn't birationally equivalent to a projective line. The analytic and algebraic relations which occur in the theory of elliptic curves and elliptic functions are beautiful and unbelievably fascinating. The modularity theorem stating that every elliptic curve over the rational field can be found in the Jacobian variety of the curve which parametrizes elliptic curves with level structure its conductor is mind-boggling.

By the way, by my count about one quarter of Abel's published work is devoted to elliptic functions.

**Raussen and Skau:** *Among the Abel Prize laureates so far, you are probably the one whose contributions would have been closest to Abel's own interests. Could we challenge you to make a historical sweep, to put Abel's work in some perspective and to compare it to your research? In modern parlance, Abel studied the multiplication-by- $n$  map for elliptic equal parts and studied the algebraic equations that*



arose. He studied also complex multiplication and showed that, in this case, it gave rise to a commutative Galois group. These are very central concepts and observations, aren't they?

**Tate:** Yes, absolutely, yes. Well, there's no comparison between Abel's work and mine. I am in awe of what I know of it. His understanding of algebraic equations, and of elliptic integrals and the more general, abelian integrals, at that time in history is just amazing. Even more for a person so isolated. I guess he could read works of Legendre and other great predecessors, but he went far beyond. I don't really know enough to say more. Abel was a great analyst and a great algebraist. His work contains the germs of many important modern developments.

**Raussen and Skau:** Could you comment on how the concept of "good reduction" for an elliptic curve is so crucial, and how it arose?

**Tate:** If one has an equation with integer coefficients, it is completely natural, at least since Gauss, to consider the equation mod  $p$  for a prime  $p$ , which is an equation over the finite field  $F_p$  with  $p$  elements.

If the original equation is the equation of an elliptic curve  $E$  over the rational number field, then the reduced equation may or may not define an elliptic curve over  $F_p$ . If it does, we say  $E$  has "good reduction at  $p$ ". This happens for all but a finite set of "bad primes for  $E$ ", those dividing the discriminant of  $E$ .

**Raussen and Skau:** The Hasse principle in the study of Diophantine equations says, roughly speaking: If an equation has a solution in  $p$ -adic numbers, then it can be solved in the rational numbers. It does not hold in general. There is an example for this failure given by the Norwegian mathematician Ernst Selmer...

**Tate:** Yes. The equation  $3x^3 + 4y^3 + 5z^3 = 0$ .

**Raussen and Skau:** Exactly! The extent of the failure of the Hasse principle for curves of genus 1 is quantified by the Shafarevich-Tate group. The so-called Selmer groups are related groups, which are known to be finite, but as far as we know the Shafarevich-Tate group is not known to be finite. It is only a conjecture that it is always finite. What is the status concerning this conjecture?

**Tate:** The conjecture that the Shafarevich group Sha is finite should be viewed as part of the conjecture of Birch and Swinnerton-Dyer. That conjecture, BSD for short, involves the  $L$ -function of the elliptic curve, which is a function of a complex variable  $s$ . Over the rational number field,  $L(s)$  is known to be defined near  $s=1$ , thanks to the modularity theorem of A. Wiles, R. Taylor, et al. If  $L(s)$  either does not vanish or has a simple zero at  $s=1$ , then Sha is finite and BSD is true, thanks to the joint work of B. Gross and D. Zagier on Heegner points and the work of Kolyvagin on Euler systems. So, by three big results which are the work of many

people, we know a very special circumstance in which Sha is finite.

If  $L(s)$  has a higher order zero at  $s=1$ , we know nothing, even over the field of rational numbers. Over an imaginary quadratic field we know nothing, period.

**Raussen and Skau:** Do you think that this group is finite?

**Tate:** Yes. I firmly believe the conjecture is correct. But who knows? The curves of higher rank, or whose  $L$ -functions have a higher order zero—BSD says the order of the zero is the rank of the curve—one knows nothing about.

**Raussen and Skau:** What is the origin of the Tate conjecture?

**Tate:** Early on I somehow had the idea that the special case about endomorphisms of abelian varieties over finite fields might be true. A bit later I realized that a generalization fit perfectly with the function field version of the Birch and Swinnerton-Dyer conjecture. Also it was true in various particular examples which I looked at and gave a heuristic reason for the Sato-Tate distribution. So it seemed a reasonable conjecture.

**Raussen and Skau:** In the arithmetic theory of elliptic curves, there have been major breakthroughs like the Mordell-Weil theorem, Faltings' proof of the Mordell conjecture, using the known reduction to a case of the Tate conjecture. Then we have Wiles's breakthrough proving the Shimura-Taniyama-Weil conjecture. Do you hope the next big breakthrough will come with the Birch and Swinnerton-Dyer conjecture? Or the Tate conjecture, maybe?

**Tate:** Who knows what the next big breakthrough will be, but certainly the Birch and Swinnerton-Dyer conjecture is a big challenge; and also the modularity, i.e., the Shimura-Taniyama-Weil idea, which is now seen as part of the Langlands program. If the number field is not totally real, we don't know much about either of these problems. There has been great progress in the last thirty years, but it is just the very beginning. Proving these things for all number fields and for all orders of vanishing, to say nothing of doing it for abelian varieties of higher dimension, will require much deeper insight than we have now.

**Raussen and Skau:** Is there any particular work from your hand that you are most proud of, that you think is your most important contribution?

**Tate:** I don't feel that any one of my results stands out as most important. I certainly enjoyed working out the proofs in my thesis. I enjoyed very much proving a very special case of the so-called Tate conjecture, the result about endomorphisms of abelian varieties over finite fields. It was great to be able to prove at least one nontrivial case and not have only a conjecture! That's a case that is useful in cryptography, especially elliptic curves over finite fields. Over number fields, even finitely generated fields, that case of my conjecture was

proved by Faltings, building on work of Zarhin over function fields, as the first step in his proof of the Mordell conjecture. I enjoyed very much the paper which I dedicated to Jean-Pierre Serre on the  $K^2$  groups of number fields. I also had fun with a paper on residues of differentials on curves giving a new definition of residue and a new proof that the sum of the residues is zero, even though I failed to see a more important aspect of the construction.

## Applied Number Theory

**Raussen and Skau:** *Number theory stretches from the mysteries of the prime numbers to the way we save, transmit, and secure information on modern computers. Can you comment on the amazing fact that number theory, in particular the arithmetic of elliptic curves, has been put to use in practical applications?*

**Tate:** It certainly is amazing to me. When I first studied and worked on elliptic curves I had no idea that they ever would be of any practical use. I did not foresee that. It is the high-speed computers which made the applications possible, but of course many new ideas were needed also.

**Raussen and Skau:** *And now it's an industry: elliptic curves, cryptography, intelligence, and communication!*

**Tate:** It's quite remarkable. It often happens that things which are discovered just for their own interest and beauty later turn out to be useful in practical affairs.

**Raussen and Skau:** *We interviewed Jacques Tits a couple of years ago. His comment was that the Monster group, the biggest of all the sporadic simple groups, is so beautiful that it has to have some application in physics or whatever.*

**Tate:** That would be interesting!

## Collaboration and Teaching

**Raussen and Skau:** *You have been one of the few non-French members of the Bourbaki group, the group of mathematicians that had the endeavor to put all existing mathematics into a rigid format. Can you explain what this was all about and how you got involved?*

**Tate:** I would not say it was about putting mathematics in a rigid format. I view Bourbaki as a modern Euclid. His aim was to write a coherent series of books which would contain the fundamental definitions and results of all mathematics as of mid-twentieth century. I think he succeeded pretty well, though the books are somewhat unbalanced—weak in classical analysis and heavy on Lie theory. Bourbaki did a very useful service for a large part of the mathematics community just by establishing some standard notations and conventions.

The presentation is axiomatic and severe, with no motivation except for the logic and beauty of the development itself. I was always a fan of

Bourbaki. That I was invited to collaborate may have been at Serge Lang's suggestion, or perhaps Jean-Pierre Serre's also. As I mentioned, I am not a very prolific writer. I usually write a few pages and then tear them up and start over, so I never was able to contribute much to the writing. Perhaps I helped somewhat in the discussion of the material. The conferences were enjoyable, all over France, in the Alps, and even on Corsica. It was a lot of fun.

**Raussen and Skau:** *You mentioned Jean-Pierre Serre, who was the first Abel Prize laureate. He was one of the driving forces in the Bourbaki project after the Second World War. We were told that he was—as [was] Serge Lang—instrumental in getting some of your results published in the form of lecture notes and textbooks. Do you have an ongoing personal relation with Jean-Pierre Serre?*

**Tate:** Yes. I'm looking forward to meeting him next week when we will both be at Harvard for a conference in honor of Dick Gross on his sixtieth birthday. Gross was one of my Ph.D. students.

I think Serre was a perfect choice for the first Abel Prize laureate.

**Raussen and Skau:** *Another possible choice would have been Alexander Grothendieck. But he went into reclusion. Did you meet him while you were in Paris or maybe at Harvard?*

**Tate:** I met him in Paris. I had a wonderful year. Harvard had the enlightened policy of giving a tenure-track professor a year's sabbatical leave. I went to Paris for the academic year 1957–58, and it was a great experience. I met Serre, I met Grothendieck, and I was free from any duty. I could think and I could learn. Later, they both visited Harvard several times, so I saw them there too. It's great good fortune to be able to know such people.

**Raussen and Skau:** *Did you follow Grothendieck's program reconstructing the foundations of algebraic geometry closely?*

**Tate:** Well, yes, to the extent I could. I felt "ah, at last, we have a good foundation for algebraic geometry." It just seemed to me to be the right thing. Earlier I was always puzzled, do we have affine varieties, projective varieties? But it wasn't a category. Grothendieck's schemes, however, did form a category. And breaking away from a ground field to a ground ring, or even a ground scheme, so that the foundations could handle not only polynomial equations but also Diophantine equations and reduction mod  $p$ , was just what number theorists needed.

**Raussen and Skau:** *We have a question of a more general and philosophical nature: A great mathematician once mentioned that it is essential to possess a certain naiveté in order to be able to create something really new in mathematics. One can do impressive things requiring complicated techniques, but one rarely makes original discoveries without being a bit naive. In the same vein, André Weil claimed that breakthroughs in mathematics*



are typically not done by people with long experience and lots of knowledge. New ideas often come without that baggage. Do you agree?

**Tate:** I think it's quite true. Most mathematicians do their best work when they are young and don't have a lot of baggage. They haven't worn grooves in their brains that they follow. Their brains are fresher, and certainly it's important to think for oneself rather than just learning what others have done. Of course, you have to build on what has been done before or else it's hopeless; you can't rediscover everything. But one should not be prejudiced by the past work. I agree with the point of view you describe.

**Raussen and Skau:** Did you read the masters of number theory already early in your career?

**Tate:** I've never been such a good reader. My instincts have been to err on the side of trying to be independent and trying to do things myself. But as I said, I was very fortunate to be in contact with brilliant people, and I learned very much from personal conversations. I never was a great reader of the classics. I enjoyed that more as I got older.

**Raussen and Skau:** You have had some outstanding students who have made important contributions to mathematics. How did you attract these students in the first place, and how did you interact with them, both as students and later?

**Tate:** I think we were all simply interested in the same kind of mathematics. You know, with such gifted students there is usually no problem: After getting to know them and their interests you suggest things to read and think about, then just hear about progress and problems, offering support and encouragement as they find their way.

**Raussen and Skau:** Did you give them problems to work on or did they find the problems themselves?

**Tate:** It varies. Several found their own problems. With others I made somewhat more specific suggestions. I urged Dick Gross to think about a problem which I had been trying unsuccessfully to solve, but very sensibly he wrote a thesis on a quite different subject of his own choosing. I was fortunate to have such able students. I continued to see many of them later, and many are good friends.

**Raussen and Skau:** You have taught mathematics for more than sixty years, both at Harvard and at Austin, Texas. How much did you appreciate this aspect of your professional duties? Is there a particular way of teaching mathematics that you prefer?

**Tate:** I always enjoyed teaching at all levels. Teaching a subject is one of the best ways to learn it thoroughly. A few times, I've been led to a good new idea in preparing a lecture for an advanced course. That was how I found my definition of Neron's height, for example.

## Work Style

**Raussen and Skau:** Would you consider yourself mainly a theory builder or a problem solver?

**Tate:** I suppose I'm a theory builder or maybe a conjecture maker. I'm not a conjecture prover very much, but I don't know. It's true that I'm not good at solving problems. For example, I would never be good in the Math Olympiad. There speed counts and I am certainly not a speedy worker. That's one pleasant thing in mathematics: It doesn't matter how long it takes if the end result is a good theorem. Speed is an advantage, but it is not essential.

**Raussen and Skau:** But you are persistent. You have the energy to stay with a problem.

**Tate:** At least, I did at one time.

**Raussen and Skau:** May we ask you a question that we, in various ways, have asked almost everybody in previous interviews: Look back on how you came up with new concepts or made a breakthrough in an area you had been working on for some time. Did that usually happen when you were concentrated and working intensely on the problem, or did it happen in a more relaxed situation? Do you have concrete examples?

**Tate:** The first thing I did after my thesis was the determination of the higher-dimensional cohomology groups in class field theory. I had been working on that for several months, off and on. This was at the time of the seminar after my thesis at Princeton. One evening I went to a party and had a few drinks. I came home after midnight and thought I would think a little about the problem. About one or two in the morning I saw how to do it!

**Raussen and Skau:** So this was a "Poincaré moment"?

**Tate:** In a way. I think that, like him, I had put the work aside for a longer time when this happened. I remember what it was: I had been invited to give some talks at MIT on class field theory and I thought "what am I going to say?" So it was after a party, motivated by needing something to say at MIT, that this idea struck me. It was very fortunate.

But it varies. Sometimes I've had an idea after talking to someone and had the impression the person I was talking to had the idea and told me about it. The Ph.D. thesis of my student Jonathan Lubin was on what should be called the Lubin-Tate groups. They somehow have been called the Lubin-Tate groups. Incidentally, I think it's useful in math that theorems or ideas have two names so you can identify them. If I say Serre's theorem, my God, that doesn't say too much. But anyway, they are called Lubin-Tate groups, and it occurred to me, just out of the blue, that they might be useful in class field theory. And then we worked it out and indeed they were. One gets ideas in different ways, and it's a wonderful feeling for a few minutes, but then there is a letdown after you get used to the idea.

**Raussen and Skau:** Group cohomology had been studied in various guises, long before the

*notion of group cohomology was formulated in the 1940s. You invented what is called Tate cohomology groups, which are widely used in class field theory, for instance. Could you elaborate?*

**Tate:** In connection with class field theory it suddenly dawned on me that if the group is finite—the operating group  $G$ —then one could view the homology theory of that group as negative dimensional cohomology. Usually the homology and the cohomology are defined in nonnegative dimensions, but suddenly it became clear to me that for a finite group you could glue the two theories together. The  $i$ th homology group can be viewed as a  $(1-i)$ th cohomology group and then you can glue these two sequences together so that the cohomology goes off to plus infinity and the homology goes off, with renumbering, to minus infinity, and you fiddle a little with the joining point and then you have one theory going from negative infinity to plus infinity.

**Rausen and Skau:** *Was this a flash of insight?*

**Tate:** Perhaps. There was a clue from the finite cyclic case, where there is periodicity; a periodicity of length two. For example,  $H^3$  is isomorphic to  $H^1$ , the  $H^5$  is isomorphic to  $H^3$ , etc., and it's obvious that you could go on to infinity in both directions. Somehow it occurred to me that one could do that for an arbitrary finite group. I don't remember exactly how it happened in my head.

## The Roles of Mathematics

**Rausen and Skau:** *Can we speculate a little about the future development of mathematics? When the Clay Millennium Prizes for solving outstanding problems in mathematics were established back in the year 2000, you presented three of these problems to the mathematical public. Not necessarily restricting to those, would you venture a guess about new trends in mathematics: the twenty-first century compared to the twentieth century? Are there trends that are entirely new? What developments can we expect in mathematics and particularly in your own field, number theory?*

**Tate:** We certainly have plenty of problems to work on. One big difference in mathematics generally is the advent of high-speed computers. Even for pure math, that will increase the experimental possibilities enormously. It has been said that number theory is an experimental science, and until recently that meant experimenting by looking at examples by hand and discovering patterns that way. Now we have a zillionfold more powerful way to do that, which may very well lead to new ideas even in pure math, but certainly also for applications.

Mathematics somehow swings between the development of new abstract theories and the application of these to more concrete problems and from concrete problems to theories needed to solve them. The pendulum swings. When I was

young better foundations were being developed, things were becoming more functorial, if you will, and a very abstract point of view led to much progress. But then the pendulum swung the other way to more concrete things in the 1970s and 1980s. There were modular forms and the Langlands program, the proof of the Mordell conjecture and of Fermat's last theorem. In the first half of my career, theoretical physics and mathematics were not so close. There was the time when the development of mathematics went in the abstract direction, and the physicists were stuck. But now in the last thirty years they have come together. It is hard to tell whether string theory is math or physics. And noncommutative geometry has both sides.

Who knows what the future will be? I don't think I can contribute much in answering that question. Maybe a younger person would have a better idea.

**Rausen and Skau:** *Are you just as interested in mathematics now as you were when you were young?*

**Tate:** Well, not as intensely. I'm certainly still very much interested, but I don't have the energy to really go so deeply into things.

**Rausen and Skau:** *But you try to follow what is happening in your field?*

**Tate:** Yes, I try. I'm in awe of what people are doing today.

**Rausen and Skau:** *Your teacher Emil Artin, when asked about whether mathematics was a science, would rather say: "No. It's an art." On the other hand, mathematics is connected to the natural sciences, to computing and so on. Perhaps it has become more important in other fields than ever; the mutual interaction between science and engineering on one side and mathematics on the other has become more visible. Is mathematics an art, is it rather to be applied in science, or is it both?*

**Tate:** It's both, for heaven's sake! I think Artin simply was trying to make a point that there certainly is an artistic aspect to mathematics. It's just beautiful. Unfortunately it's only beautiful to the initiated, to the people who do it. It can't really be understood or appreciated much on a popular level the way music can. You don't have to be a composer to enjoy music, but in mathematics you do. That's a really big drawback of the profession. A nonmathematician has to make a big effort to appreciate our work; it's almost impossible.

Yes, it's both. Mathematics is an art, but there are stricter rules than in other arts. Theorems must be proved as well as formulated; words must have precise meanings. The happy thing is that mathematics does have applications which enable us to earn a good living doing what we would do even if we weren't paid for it. We are paid mainly to teach the useful stuff.

## Public Awareness of Mathematics

**Raussen and Skau:** *Have you tried to popularize mathematics yourself?*

**Tate:** When I was young I tried to share my enthusiasm with friends, but I soon realized that's almost impossible.

**Raussen and Skau:** *We all feel the difficulty of communicating with the general audience. This interview is one of the rare occasions providing public attention for mathematics!<sup>2</sup> Do you have any ideas about how mathematicians can make themselves and what they do more well known? How can we increase the esteem of mathematics among the general public and among politicians?*

**Tate:** Well, I think prizes like this do some good in that respect. And the Clay Prizes likewise. They give publicity to mathematics. At least people are aware. I think the appreciation of science in general and mathematics in particular varies with the country. What fraction of the people in Norway would you say have an idea about Abel?

**Raussen and Skau:** *Almost everyone in Norway knows about Abel, but they do not know anything about Lie. And not necessarily anything about Abel's work, either. They may know about the quintic.*

**Tate:** I see. And how about Sylow?

**Raussen and Skau:** *He is not known either. Abel's portrait has appeared on stamps and also on bills, but neither Lie's nor Sylow's.*

**Tate:** I think in Japan, people are more aware. I once was in Japan and eating alone. A Japanese couple came and wanted to practice their English. They asked me what I did. I said I was a mathematician but could not get the idea across until I said: "Like Hironaka". Wow! It's as though in America I'd said "Like Babe Ruth", or Michael Jordan, or Tiger Woods. Perhaps Hironaka's name is, like Abel's, the only one known, but in America I don't think any mathematician's name would get any response.

## Private Interests

**Raussen and Skau:** *Our last question: What other interests do you have in life? What are you occupied with when you are not thinking about mathematics? Certainly that happens once in a while, as well?*

**Tate:** I'm certainly not a Renaissance man. I don't have wide knowledge or interests. I have enjoyed very much the outdoors, hiking, and also sports. Basketball was my favorite sport. I played on the Southeast Methodist church team as a teenager and we won the Minneapolis church league championship one year. There were several of us who went to church three out of four Sundays during a certain period in the winter, in order to play on the team. In the Navy I coached a team from the minesweeping research base which beat

Coca-Cola for the Panama City league championship. Anyway, I have enjoyed sports and the outdoors.

I like to read a reasonable amount and I enjoy music, but I don't have a really deep or serious hobby. I think I'm more concentrated in mathematics than many people. My feeling is that to do some mathematics I just have to concentrate. I don't have the kind of mind that absorbs things very easily.

**Raussen and Skau:** *We would like to thank you very much for this interview; as well as on behalf of the Norwegian, Danish, and European mathematical societies. Thank you very much!*

**Tate:** Well, thank you for not asking more difficult questions! I have enjoyed talking with you.

<sup>2</sup>The interview was broadcast on Norwegian television; cf. <http://www.abelprisen.no/en/multimedia/2010/>.

# The Mathematical Work of the 2010 Fields Medalists

The *Notices* solicited the following articles about the works of the four individuals to whom Fields Medals were awarded at the International Congress of Mathematicians in Hyderabad, India, in August 2010. The International Mathematical Union also issued news releases about the medalists' work, and these appeared in the December 2010 *Notices*.

—Allyn Jackson

## *The Work of Ngô Bao Châu*

### Thomas C. Hales

In August 2010 Ngô Bao Châu was awarded a Fields Medal for his deep work relating the Hitchin fibration to the Arthur-Selberg trace formula, and in particular for his proof of the Fundamental Lemma for Lie algebras [27], [28].

### The Trace Formula

A function  $h : G \rightarrow \mathbb{C}$  on a finite group  $G$  is a *class function* if  $h(g^{-1}xg) = h(x)$  for all  $x, g \in G$ . A class function is constant on each conjugacy class. A basis of the vector space of class functions is the set of characteristic functions of conjugacy classes.

A *representation* of  $G$  is a homomorphism  $\pi : G \rightarrow GL(V)$  from  $G$  to a group of invertible linear transformations on a complex vector space  $V$ . It follows from the matrix identity

$$\text{trace}(B^{-1}AB) = \text{trace}(A)$$

that the function  $g \mapsto \text{trace}(\pi(g))$  is a class function. This function is called an *irreducible character* if  $V$  has no proper  $G$ -stable subspace. A basic theorem in finite group theory asserts that the set of irreducible characters forms a second basis of the vector space of class functions on  $G$ .

A *trace formula* is an equation that gives the expansion of a class function  $h$  on one side of the equation in the basis of characteristic functions of conjugacy classes  $C$  and on the other side in the

basis of irreducible characters

$$\sum_C b_C \text{char}_C = h = \sum_{\pi} a_{\pi} \text{trace } \pi$$

for some complex coefficients  $b_C$  and  $a_{\pi}$  depending on  $h$ . The side of the equation with conjugacy classes is called the *geometric side* of the trace formula, and the side with irreducible characters is called the *spectral side*.

When  $G$  is no longer assumed to be finite, some analysis is required. We allow  $G$  to be a Lie group or, more generally, a locally compact topological group. The vector space  $V$  may be infinite-dimensional so that a trace of a linear transformation of  $V$  need not converge. To improve convergence, the irreducible character is no longer viewed as a function but rather as a distribution

$$f \mapsto \text{trace} \int_G \pi(g)f(g) dg,$$

where  $f$  runs over smooth compactly supported test functions on the group, and  $dg$  is a  $G$ -invariant measure. Similarly, the characteristic function of the conjugacy class is replaced with a distribution that integrates a test function  $f$  over the conjugacy class  $C$  with respect to an invariant measure:

$$(1) \quad f \mapsto \int_C f(g^{-1}xg) dg.$$

The integral (1) is called an *orbital integral*. A trace formula in this setting becomes an identity that expresses a class distribution (called an *invariant distribution*) on the geometric side of the equation as a sum of orbital integrals and on the spectral side of the equation as a sum of distribution characters.

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The celebrated Selberg trace formula is an identity of this general form for the invariant distribution associated with the representation of  $SL_2(\mathbb{R})$  on  $L^2(SL_2(\mathbb{R})/\Gamma)$ , for a discrete subgroup  $\Gamma$ . Arthur generalized the Selberg trace formula to reductive groups of higher rank.

## History

The Fundamental Lemma (FL) is a collection of identities of orbital integrals that arise in connection with a trace formula. It takes several pages to write all of the definitions that are needed for a precise statement of the lemma [17]. Fortunately, the significance of the lemma and the main ideas of the proof can be appreciated without the precise statement.

Langlands conjectured these identities in lectures on the trace formula in Paris in 1980 and later put them in more precise form with Shelstad [21], [22]. Over time, supplementary conjectures were formulated, including a *twisted* conjecture by Kottwitz and Shelstad and a *weighted* conjecture by Arthur [20], [1]. Identities of orbital integrals on the group can be reduced to slightly easier identities on the Lie algebra [23]. Papers by Waldspurger rework the conjectures into the form eventually used by Ngô in his solution [35], [33]. Over the years, Chaudouard, Goresky, Kottwitz, Laumon, MacPherson, and Waldspurger, among others, have made fundamental contributions that led up to the proof of the FL or extended the results afterward [24], [14], [15], [9], [10], [11]. It is hard to do justice to all those who have contributed to a problem that has been intensively studied for decades, while giving special emphasis to the spectacular breakthroughs by Ngô.

With the exception of the FL for the special linear group  $SL(n)$ , which can be solved with representation theory, starting in the early 1980s all plausible lines of attack on the general problem have been geometric. Indeed, a geometric approach is suggested by direct computations of these integrals in special cases, which give their values as the number of points on hyperelliptic curves over finite fields [19], [16].

To motivate the FL, we must recall the bare outlines of the ambitious program launched by Langlands in the late 1960s to use representation theory to understand vast tracts of number theory. Let  $F$  be a finite field extension of the field of rational numbers  $\mathbb{Q}$ . The ring of adeles  $\mathbb{A}$  of  $F$  is a locally compact topological ring that contains  $F$  and has the property that  $F$  embeds discretely in  $\mathbb{A}$  with a compact quotient  $F\backslash\mathbb{A}$ . The ring of adeles is a convenient starting point for the analytic treatment of the number field  $F$ . If  $G$  is a reductive group defined over  $F$  with center  $Z$ , then  $G(F)$  is a discrete subgroup of  $G(\mathbb{A})$  and the quotient  $G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$  has finite volume. A representation  $\pi$  of  $G(\mathbb{A})$  that

appears in the spectral decomposition of

$$L^2(G(F)Z(\mathbb{A})\backslash G(\mathbb{A}))$$

is said to be an *automorphic representation*. The automorphic representations (by descending to the quotient by  $G(F)$ ) are those that encode the number-theoretic properties of the field  $F$ . The theory of automorphic representations just for the two linear groups  $G = GL(2)$  and  $GL(1)$  already encompasses the classical theory of modular forms and global class field theory.

There is a complex-valued function  $L(\pi, s)$ ,  $s \in \mathbb{C}$ , called an automorphic  $L$ -function, attached to each automorphic representation  $\pi$ . (The  $L$ -function also depends on a representation of a dual group, but we skip these details.) Langlands's philosophy can be summarized as two objectives:

- (1) Show that many  $L$ -functions that routinely arise in number theory are automorphic.
- (2) Show that automorphic  $L$ -functions have wonderful analytic properties.

There are two famous examples of this philosophy. In Riemann's paper on the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

he proved that it has a functional equation and meromorphic continuation by relating it to a  $\theta$ -series (an automorphic entity) and then using the analytic properties of the  $\theta$ -series. Wiles proved Fermat's Last Theorem by showing that the  $L$ -function  $L(E, s)$  of every semistable elliptic curve over  $\mathbb{Q}$  is automorphic. From automorphy follows the analytic continuation and functional equation of  $L(E, s)$ .

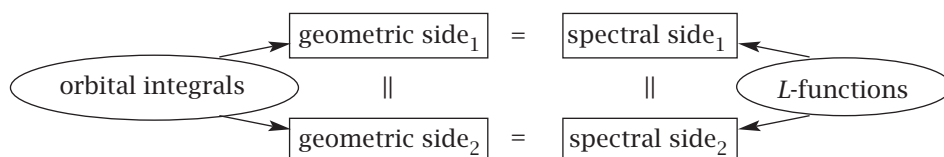
The Arthur-Selberg trace formula has emerged as a general tool to reach the first objective (1) of Langlands's philosophy. To relate one  $L$ -function to another, two trace formulas are used in tandem (Figure 1). An automorphic  $L$ -function can be encoded on the spectral side of the Arthur-Selberg trace formula. A second  $L$ -function is encoded on the spectral side of a second trace formula of a possibly different kind, such as a topological trace formula. By equating the geometric sides of the two trace formulas, identities of orbital integrals yield identities of  $L$ -functions. The value of the FL lies in its utility. The FL can be characterized as the minimal set of identities that must be proved in order to put the trace formula in a useable form for applications to number theory, such as those mentioned at the end of this report.

## The Hitchin Fibration

Ngô's proof of the FL is based on the Hitchin fibration [18].

Every endomorphism  $A$  of a finite-dimensional vector space  $V$  has a characteristic polynomial

$$(2) \quad \det(t - A) = t^n + a_1 t^{n-1} + \cdots + a_n.$$



**Figure 1.** A pair of trace formulas can transform identities of orbital integrals into identities of  $L$ -functions.

Its coefficients  $a_i$  are symmetric polynomials of the eigenvalues of  $A$ . This determines a *characteristic map*  $\chi : \text{end}(V) \rightarrow \mathfrak{c}$ , from the Lie algebra of endomorphisms of  $V$  to the vector space  $\mathfrak{c}$  of coefficients  $(a_1, \dots, a_n)$ . This construction generalizes to a characteristic map  $\chi : \mathfrak{g} \rightarrow \mathfrak{c}$  for every reductive Lie algebra  $\mathfrak{g}$ , by evaluating a set of symmetric polynomials on  $\mathfrak{g}$ .

Fix once and for all a smooth projective curve  $X$  of genus  $g$  over a finite field  $k$ .

In its simplest form, a *Higgs pair*  $(E, \phi)$  is what we obtain when we allow an element  $Z$  of the Lie algebra  $\text{end}(V)$  to vary continuously along the curve  $X$ . As we vary along the curve, the vector space  $V$  sweeps out a vector bundle  $E$  on  $X$ , and the element  $Z \in \text{end}(V)$  sweeps out a section  $\phi$  of the bundle  $\text{end}(E)$  or of the bundle  $\text{end}(E) \otimes \mathcal{O}_X(D)$  when the section acquires finitely many poles prescribed by a divisor  $D$  of  $X$ . Extending this construction to a general reductive Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , a Higgs pair  $(E, \phi)$  consists of a principal  $G$ -bundle  $E$  and a section  $\phi$  of the bundle  $\text{ad}(E) \otimes \mathcal{O}_X(D)$  associated with  $E$  and the adjoint representation of  $G$  on  $\mathfrak{g}$ . For each  $X$ ,  $G$ , and  $D$ , there is a moduli space  $\mathcal{M}$  (or more correctly, moduli stack) of all Higgs pairs  $(E, \phi)$ .

The *Hitchin fibration* is the morphism obtained when we vary the characteristic map  $\chi : \mathfrak{g} \rightarrow \mathfrak{c}$  along a curve  $X$ . For each Higgs pair  $(E, \phi)$ , we evaluate the characteristic map  $p \mapsto \chi(\phi_p)$  of the endomorphism  $\phi$  at each point  $p \in X$ . This function belongs to the set  $\mathcal{A}$  of a global sections of the bundle  $\mathfrak{c} \otimes \mathcal{O}_X(D)$  over  $X$ . The Hitchin fibration is this morphism  $\mathcal{M} \rightarrow \mathcal{A}$ .

*Abelian varieties* occur naturally in the Hitchin fibration. To illustrate, we return to the Lie algebra  $\mathfrak{g} = \text{end}(V)$ . For each section  $a = (a_1, \dots, a_n) \in \mathcal{A}$ , the characteristic polynomial

$$(3) \quad t^n + a_1(p)t^{n-1} + \dots + a_n(p) = 0,$$

defines an  $n$ -fold cover  $Y_a$  of  $X$  (called the *spectral curve*). By construction, each point of the spectral curve is a root of the characteristic polynomial at some  $p \in X$ . We consider the simple setting when  $Y_a$  is smooth and the discriminant of the characteristic polynomial is sufficiently generic. A Higgs pair  $(E, \phi)$  over the section  $a$  determines a line (a one-dimensional eigenspace of  $\phi$  with eigenvalue that root) at each point of the spectral curve, and hence a line bundle on  $Y_a$ . This establishes a map from Higgs pairs over  $a$  to  $\text{Pic}(Y_a)$ , the group of line bun-

dles on the spectral curve  $Y_a$ . Conversely, just as linear maps can be constructed from eigenvalues and eigenspaces, Higgs pairs can be constructed from line bundles on the spectral curve  $Y_a$ . The connected component  $\text{Pic}^0(Y_a)$  is an abelian variety. Even outside this simple setting, the group of symmetries of the Hitchin fiber over  $a \in \mathcal{A}$  has an abelian variety as a factor.

### The Proof of the FL

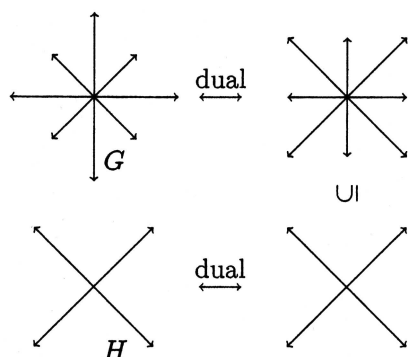
Shifting notation (as justified in [34], [11]), we let  $F$  be the field of rational functions on a curve  $X$  over a finite field  $k$ . One of the novelties of Ngô's work is to treat the FL as identities over the global field  $F$ , rather than as local identities at a given place of  $X$ . By viewing each global section of  $\mathcal{O}_X(D)$  as a rational function on  $X$ , each point  $a \in \mathcal{A}$  is identified with an  $F$ -valued point  $a \in \mathfrak{c}(F)$ . The preimage of  $a$  under the characteristic map  $\chi$  is a union of conjugacy classes in  $\mathfrak{g}(F)$ , and therefore corresponds to terms of the Arthur-Selberg trace formula for the Lie algebra. The starting point of Ngô's work is the following geometric interpretation of the trace formula.

**Theorem 1** (Ngô). *There is an explicit test function  $f_D$ , depending on the divisor  $D$ , such that for every anisotropic element  $a \in \mathcal{A}^{an}$ , the sum of the orbital integrals with characteristic polynomial  $a$  in the trace formula for  $f_D$  equals the number of Higgs pairs in the Hitchin fibration over  $a$ , counted with multiplicity.*

The proof is based on Weil's description of vector bundles on a curve in terms of the cosets of a compact open subgroup of  $G(\mathbb{A})$ . Orbital integrals have a similar coset description.

From this starting point, the past thirty years of research on the trace formula can be translated into geometrical properties of the Hitchin fibration. In particular, Ngô formulates and then solves the FL as a statement about counting points in Hitchin fibrations.

The identities of the FL are between the orbital integrals on two different reductive groups  $G$  and  $H$ . A root system is associated with each reductive group. There is a duality of every root system that interchanges its long and short roots. The two reductive groups of the FL are related only indirectly: the root system dual to that of  $H$  is a subset of the root system dual to that of  $G$  (Figure 2). Informally, the set of representations of a group is in duality



**Figure 2.** The two root systems in each row are in duality. The root system on the bottom right is a subset of the root system on the upper right.



**Figure 3.** After giving a direct proof of the FL under the assumption of transversality (left), Ngô obtains the general case (right) by continuity.

with the group itself, so by a double duality, when the dual root systems are directly related, we might also expect their representation theories to be directly related. This expectation is supported by an overwhelming amount of evidence.

By using the same curve  $X$  for both  $H$  and  $G$ , and by comparing the characteristic maps for the two groups, Ngô produces a map  $\nu : \mathcal{A}_H \rightarrow \mathcal{A}_G$  of the bases of the two Hitchin fibrations, but to kill unwanted monodromy he prefers to work with a base-change  $\tilde{\nu} : \tilde{\mathcal{A}}_H \rightarrow \tilde{\mathcal{A}}_G$ . The particular identities of the FL pick out a subspace  $\tilde{\mathcal{A}}_k$  of  $\tilde{\mathcal{A}}_G$  containing  $\tilde{\nu}(\tilde{\mathcal{A}}_H)$ . Restricting the Hitchin fibration to anisotropic elements, to prove the FL, he must compare fibers of the two (base-changed, anisotropic) Hitchin fibrations  $\tilde{\mathcal{M}}_G^{an} \rightarrow \tilde{\mathcal{A}}_k^{an}$  and  $\tilde{\mathcal{M}}_H^{an} \rightarrow \tilde{\mathcal{A}}_H^{an}$  over corresponding points of the base spaces.

The base  $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$  contains a dense open subset of elements that satisfy a transversality condition. For  $\mathfrak{g} = \text{end}(V)$  this condition requires the self-intersections of the spectral curve (Equation 3) to be transversal (Figure 3). For a particularly nice open subset  $\tilde{\mathcal{U}} \subset \tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$  of transversal elements, the number of points in a Hitchin fiber may be computed directly, and the FL can be verified in this case without undue difficulty.

To complete the proof, Ngô argues by continuity that because the identities of the FL hold on a dense open subset of  $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$ , the identities are also forced to hold on the closure of the subset, even without transversality. The justification of this continuity principle is the deepest part of his work.

Through the legacy of Weil and Grothendieck, we know the number of points on a variety (or even on a stack if you are brave enough) over a finite field to be determined by the action of the Frobenius operator on cohomology. To cohomology we turn. After translation into this language, the FL takes the form of a desired equality of (the semisimplifications of) two perverse sheaves over a common base space  $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$ . By the BBDG decomposition theorem, over the algebraic closure of  $k$ , the perverse sheaves break into direct sums of simple terms, each given as the intermediate extension of a local system on an open subset  $Z^0$  of its support  $Z$  [5]. The decomposition theorem already implies a weak continuity principle; each simple factor is uniquely determined by its restriction to a dense open subset of its support. This weak continuity is not sufficient, because it does not rule out the existence of supports  $Z$  that are disjoint from the open set of transverse elements.

To justify the continuity principle, Ngô shows that the support  $Z$  of each of these sheaves lies in  $\tilde{\nu}(\mathcal{A}_H^a)$  and intersects the open set  $\tilde{\mathcal{U}}$  of transverse elements. In rough terms, the continuity principle consists in showing that every cohomology class can be pushed out into the open. There are two parts to the argument: the cohomology class first is pushed into the top degree cohomology and then from there into the open. In the first part, the abelian varieties mentioned above enter in a crucial way. By taking cap product operations coming from the abelian varieties, and using Poincaré duality, a nonzero cohomology class produces a nonzero class in the top degree cohomology of a Hitchin fiber. This part of his proof uses a stratification of the base of the Hitchin fibration and a delicate inequality relating the dimension of the abelian varieties to the codimension of the strata.

In the second part of the argument, a set of generators of the top degree cohomology of the fiber is provided by the component group  $\pi_0$  of a Picard group that acts as symmetries on the fibers. Recall that the two groups  $G$  and  $H$  are related only indirectly through a duality of root systems. At this step of the proof, a duality is called for, and Ngô describes  $\pi_0$  explicitly, generalizing classical dualities of Kottwitz, Tate, and Nakayama in class field theory. With this dual description of the top cohomology, he is able to transfer information about the support  $Z$  on the Hitchin fibration for  $G$  to the Hitchin fibration on  $H$  and deduce the desired support and continuity theorems. With continuity in hand, the FL follows as described above.

Further accounts of Ngô's work and the proof of the FL appear in [26], [12], [2], [13], [8], [7], [29].

## Applications

Only in the land of giants does the profound work of a Fields medalist get called a lemma. Its name reminds us nonetheless that the FL was never intended as an end in itself. A lemma it is. Although proved only recently, it has already been put to use as a step in the proofs of the following major theorems in number theory:

- (1) The forthcoming classification of automorphic representations of classical groups [3].
- (2) The calculation of the cohomology of Shimura varieties and their Galois representations [25], [30].
- (3) The Sato-Tate conjecture for elliptic curves over a totally real number field [4].
- (4) Iwasawa's main conjecture for  $GL(2)$  [32], [31].
- (5) The Birch and Swinnerton-Dyer conjecture for a positive fraction of all elliptic curves over  $\mathbb{Q}$  [6].

The proof of the following recent theorem invokes the FL [4]. It is striking that this result in pure arithmetic ultimately relies on the Hitchin fibration, which was originally introduced in the context of completely integrable systems!

**Theorem 2.** *Let  $n_p$  be the number of ways a prime  $p$  can be expressed as a sum of twelve squares:*

$$n_p = \text{card} \{ (a_1, \dots, a_{12}) \in \mathbb{Z}^{12} \mid p = a_1^2 + \dots + a_{12}^2 \}.$$

*Then the real number*

$$t_p = \frac{n_p - 8(p^5 + 1)}{32p^{5/2}}$$

*belongs to the interval  $[-1, 1]$ , and as  $p$  runs over all primes, the numbers  $t_p$  are distributed within that interval according to the probability measure*

$$\frac{2}{\pi} \sqrt{1 - t^2} dt.$$

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## The Work of Elon Lindenstrauss

**Benjamin Weiss**

### Introduction

The citation that accompanied the awarding of the Fields Medal to Elon Lindenstrauss at the ICM2010 read: “For his results on measure rigidity in ergodic theory, and their applications to number theory.” My main goal in this survey is to explain this somewhat mysterious sentence without assuming any specific background on the part of the mathematically educated reader (beyond the first year of graduate studies). It will take a little time before I get to Elon’s spectacular contributions, and I call upon the reader to be patient. By the way, this is not the first award that Elon has received for his work. In 2001 he was awarded the Blumenthal Prize of the AMS. This award is given once every four years for the best Ph.D. thesis. In his thesis he extended the pointwise ergodic theorem to arbitrary amenable groups and made a deep study of the mean dimension, a new invariant introduced by M. Gromov to study systems with infinite topological entropy.

Returning to my main goal, I will begin by explaining what ergodic theory is. Work by L. Boltzmann on statistical mechanics in the nineteenth century led to the formulation of the “ergodic hypothesis”, which asserts that one may replace the time averages of evolving systems by their spatial averages. More precisely, suppose that  $X$  is some space and  $T_t$  a one-parameter family of transformations of  $X$  preserving some natural probability measure  $\mu$  on  $X$ . In Boltzmann’s situation  $X$  was a surface of constant energy in some high-dimensional state space of a mechanical system, and if the initial state of the system was  $x \in X$ , then  $T_t x$  gives the state of the system  $t$  time units later. The original ergodic hypothesis that was attributed to Boltzmann turned out to be false. However, J. von Neumann and G. D. Birkhoff in 1931 proved ergodic theorems that made precise the sense in which time averages of functions  $f$  defined on  $X$  do

converge and how they relate to the spatial average  $\int f d\mu$ . These theorems gave birth to what we call ergodic theory, which can be briefly described as the study of transformations preserving a probability measure.

It is better to start with just the dynamical system, and for simplicity we shall, for a while, talk about discrete time so that the system is given by a space  $X$  and a single transformation  $T$  of  $X$  to itself. The one-parameter family is now just the semigroup consisting of the iterates of  $T$ . Unless one imposes some structure on  $X$ , invariant probability measures needn’t exist (think about adding one as a transformation on the integers). If, however,  $X$  is a compact Hausdorff space and  $T$  is continuous, then it is not hard to see that at least one invariant measure will exist. Consider the following example:  $X = \{z \in \mathbb{C} \mid |z| = 1\}$  and  $Tz = \rho z$  with  $|\rho| = 1$ . The normalized arc length is clearly a probability measure that is invariant under  $T$ . The iterates of  $T$  are simply the powers  $\rho^n$  of  $\rho$ . If the argument of  $\rho$  is an irrational multiple of  $\pi$ , then a classical theorem (named after Kronecker, but known to N. Oresme in the Middle Ages) asserts that these powers are dense in  $X$ , from which it easily follows that the arc length is the only finite measure invariant under  $T$ . In this case the mapping  $T$  is said to be **uniquely ergodic**. The terminology comes from a general definition in which a system  $(X, T, \mu)$  with  $T\mu = \mu$  is said to be ergodic if the only invariant measurable sets have either measure zero or one. If there is a unique invariant measure, then it is easy to see that the system is ergodic, whereas if there is more than one invariant measure, then it can be shown that there is more than one ergodic invariant measure.

Keeping the same space  $X$ , if we replace this rotation by squaring,  $Sz = z^2$ , then arc length is once again invariant, but now there are many more invariant probability measures. The easiest way to see this is to open up the circle and think of squaring as the map of  $t \in [0, 1)$  to  $2t \pmod{1}$ . Now the dyadic expansion of numbers represents  $t$  by an infinite sequence of zeroes and ones, and the map becomes the shift. The arc length corresponds to having the digits being independent identically distributed random variables with equal probability of being zero or one, and replacing this distribution by unfair coins gives a whole family of distinct probability measures invariant under the shift.

Continuing this example, notice that arc length is also invariant under the map that takes  $z$  to  $z^k$  for any natural number  $k$ . If a probability measure,  $\mu$ , is invariant under all of these maps, then it is easy to see that it is a convex combination of arc length and the point masses concentrated at 0. Indeed, the invariance implies that all nonzero Fourier-Stieltjes coefficients are constant. Thus subtracting off a suitable multiple of the delta measure at zero will give rise to a measure all of whose nonzero Fourier-Stieltjes vanish, and this

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is a multiple of the arc length measure. It is a famous open problem raised by H. Furstenberg, as to whether or not this is still true if we restrict  $k$  to be of the form  $k = 2^r 3^s$ . **Measure rigidity** is not a formal concept but is a term used to refer to situations in which there are very few invariant measures and they can be explicitly described. We would say that irrational rotation is measure rigid, as is the full semigroup of maps  $z^k$ , whereas the measure rigidity of the sub-semigroup generated by two and three is an open problem.

## Homogeneous Spaces

The circle can be thought of as the real line modulo the integers, a discrete subgroup, and the mappings that we have considered are algebraically defined. The examples that Elon deals with also come from algebraically defined mappings associated with groups possessing a more complicated structure, which we proceed to describe. Let  $SL(2, \mathbb{R})$  denote the group of two-by-two matrices with real entries and determinant one. Geometrically this group can be identified with the group of orientation-preserving isometries of the upper half plane with the hyperbolic metric  $ds^2 = \frac{dx^2 + dy^2}{y^2}$ . The action is by the fractional linear transformation that maps  $z$  to  $\frac{az+b}{cz+d}$  where we write the upper half plane in complex notation  $z = x + iy$ . To be more precise we should mod out by minus the identity and identify  $PSL(2, \mathbb{R})$  with the isometries since the matrix  $-I$  in this correspondence is the identity mapping. The subgroup that fixes the point  $i$  is the group of rotations, and so we can actually identify the group with the unit tangent bundle of the upper half plane. If  $M$  is a two-dimensional Riemannian manifold with constant negative curvature, then its unit tangent bundle can be identified with  $SL(2, \mathbb{R})/\Gamma$ , where  $\Gamma$  is a discrete subgroup (the fundamental group of the manifold). The reader unfamiliar with the differential geometry language can simply think of this homogeneous space as an algebraic object inheriting the topology from the natural topology on the group of two-by-two matrices. For a concrete example of such a  $\Gamma$  take the subgroup  $SL(2, \mathbb{Z})$ , which is important in number theory. This homogeneous space can also be thought of as the space of two-dimensional lattices in the plane as follows. The action of  $SL(2, \mathbb{R})$  on the plane acts transitively on the space of lattices, and  $SL(2, \mathbb{Z})$  is the stability group of the integer lattice so that the space of lattices with a natural topology can be identified with  $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ .

We started with compact spaces, and although the space of lattices is not compact, it is easy to construct geometric examples of  $\Gamma$ 's such that  $SL(2, \mathbb{R})/\Gamma$  is compact. This can be done geometrically by looking at tilings of the hyperbolic plane by proper triangles (in contrast to the tiling that corresponds to  $SL(2, \mathbb{Z})$  which consists of trian-

gles having one vertex at infinity). Now any one parameter subgroup of  $SL(2, \mathbb{R})$  acts on these homogeneous spaces, and we get quite a rich family of examples. There are two kinds of one-parameter subgroups that exhibit rather different behavior. If we take the diagonal subgroup for our  $T_t$ , then geometrically this corresponds to the geodesic flow on the unit tangent bundle of a hyperbolic manifold with constant negative curvature. These geodesic flows have been extensively studied ever since the dawn of ergodic theory and have served as an important testing ground for the theory. It turns out that in many ways they behave like the multiplication maps of the circle. In particular, they have a plethora of invariant measures. This fact is not so easy to see and requires ideas which I will not take the time to explain. On the other hand, the shearing subgroup of transformations of the form  $U_t(z) = z + t$  behave more like the rotations of the circle in that they are uniquely ergodic in the compact case and have only algebraically defined measures in the case of manifolds with finite volume such as the space of lattices. This flow also has a geometrical interpretation and is called the horocycle flow. These horocycle flows are the archetypical examples of what is called measure rigidity. Indeed, this term was introduced by Marina Ratner in 1990 in her deep studies of the invariant measures of higher dimensional versions of these horocycle flows.

This example generalizes easily to  $SL(n, \mathbb{R})$ , where  $n \geq 3$  and  $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$  can once again be thought of as the space of lattices in  $\mathbb{R}^n$ . The diagonal matrices form now an abelian subgroup,  $A \cong \mathbb{R}^{n-1}$ , and the fact that now the dimension is greater than one changes the situation dramatically. It turns out that there are far fewer measures invariant under the entire action of  $A$ , and this has important number theoretical consequences. This can be seen already in the simpler example of the circle and multiplication maps, and we will go back to that example to see how entropy enters the story.

## Entropy and Hausdorff Dimension

The average entropy of a stationary stochastic process was a key tool in Shannon's development of a mathematical theory of communication, now known as information theory. We recall quickly the basic definitions. The entropy of a random variable  $X$  that takes values  $v_j$  with probabilities  $p_j$  is given by the formula  $H(X) = -\sum_j p_j \log p_j$ . A sequence of random variables  $\{X_n\}$  is said to be stationary if for any  $N$  and  $t$  the joint distribution of the random variables  $\{X_n : |n| \leq N\}$  equals that of the random variables  $\{X_{n+t} : |n| \leq N\}$ . Shannon's average entropy of the process  $\{X_n\}$  is given by the formula

$$h(\{X_n\}) = \lim_{n \rightarrow \infty} \frac{H(X_1, X_2, X_3 \dots X_n)}{n}.$$

Here the subadditivity of the entropy is used to show that the limit exists. If  $T$  is a measurable transformation of a probability space  $\{\Omega, \Sigma, \mathbf{P}\}$  that preserves  $\mathbf{P}$ , then any finite-valued random variable  $X$  defined on this space will define a stationary stochastic process by setting  $X_n(\omega) = X(T^n\omega)$ . This makes sense even if  $T$  is not invertible; the index set is then restricted to the nonnegative integers. Kolmogorov used this construction and Shannon's entropy to define the entropy of a probability preserving system  $\{\Omega, \Sigma, \mathbf{P}, T\}$  as the supremum of the Shannon entropy of the processes defined in this way. This is clearly an invariant under the natural notion of isomorphism, and it has played a very important role in classifying measure-preserving systems up to isomorphism.

There is another way of computing the entropy of a process based on a conditional version of the basic definition. From this one sees that zero entropy for a process  $\{X_n\}$  is equivalent to the assertion that  $X_0$  is measurable with respect to the  $\sigma$ -field generated by the  $\{X_i : i > 0\}$ , or reversing time, the  $\sigma$ -field generated by the  $\{X_i : i < 0\}$ . Such processes are called deterministic. On the other hand, positive entropy for a process means that the conditional distribution of  $X_0$  given the past is nontrivial. The positivity of entropy has played an important role in applications of measure rigidity ever since the pioneering work of Russell Lyons on Furstenberg's question. Lyons showed that if  $p$  and  $q$  are not powers of the same integer and if  $\mu$  is nonatomic and invariant under both maps,  $z^p$  and  $z^q$ , and is ergodic with respect to the joint action generated by the two maps, and if, furthermore, the measure has completely positive entropy (all nontrivial processes defined over the system have positive entropy) with respect to one of these maps, then  $\mu$  must be Lebesgue measure. The work of the late Dan Rudolph made this dichotomy even more evident. He showed that if  $p$  and  $q$  are relatively prime and if  $\mu$  is nonatomic and invariant under both maps,  $z^p$  and  $z^q$ , and is ergodic with respect to the joint action generated by the two maps, then either  $\mu$  is Lebesgue measure (arc length) or its Kolmogorov entropy is zero with respect to each of the maps. This latter work was the starting point of a whole series of works in which measures invariant under higher rank groups were successfully classified under the additional assumption that the entropy of some individual map was positive.

In order to formulate some of Elon's remarkable results, we will also need the notion of the **Hausdorff dimension** of a set of points in  $\mathbb{R}^d$ . Ever since the work of the late B. Mandelbrot on fractals, this is sometimes called **fractal dimension**, and since many expositions of this are available, we shall not give a formal definition. Suffice it to say that it is a more refined notion of the topological dimension that, in particular, captures the different sizes that sets of zero topological dimension might have. It is important to point out that it

is closely related to entropy. This can be seen, for example, in the case of the circle and our favorite map  $T(z) = z^2$ . If  $\nu$  is any ergodic  $T$ -invariant measure, then the Kolmogorov entropy of  $T$  is (up to a constant depending on the base of the logarithm in the definition of entropy) the infimum of the Hausdorff dimension of subsets of the circle that have full  $\nu$  measure. In the other direction, if  $E$  is a closed subset of the circle that is invariant under the map  $T$ , then  $E$  supports an invariant measure whose entropy is (again up to that constant) the Hausdorff dimension of the set  $E$ .

### Littlewood's Conjecture

The Littlewood conjecture concerns how well one can approximate irrational numbers by rational numbers. For conciseness we will denote by  $\|x\|$  the distance from a real number  $x$  to the nearest integer. The classical expansion of a real number  $x$  into a continued fraction easily shows that for any  $x$  the  $\limsup_{n \rightarrow \infty} n\|nx\|$  is finite. As for the  $\liminf$ , for Lebesgue almost every  $x$  the  $\liminf_{n \rightarrow \infty} n\|nx\| = 0$ , although for quadratic irrationals and in fact for any  $x$  with bounded continued fraction expansion the  $\liminf$  is strictly positive. Around eighty years ago J. E. Littlewood conjectured that if we take any two real numbers  $x$  and  $y$ , then  $\liminf_{n \rightarrow \infty} n\|nx\|\|ny\| = 0$ .

In 1955 Cassels and Swinnerton-Dyer showed that the Littlewood conjecture would follow from the following more general conjecture concerning linear forms:

**Conjecture 1.** Let  $F(x_1, \dots, x_d) = \prod_{i=1}^d (\sum_{j=1}^d g_{ij}x_j)$  be a product of  $d$ -linearly independent linear forms in  $d$  variables, not proportional to an integral form (as a homogeneous polynomial in  $d$  variables), with  $d \geq 3$ . Then

$$(4) \quad \inf \{ |F(v)| : v \in \mathbb{Z}^d \setminus \{0\} \} = 0.$$

G. Margulis pointed out that, in turn, this stronger conjecture is related to the action that we described above by the diagonal subgroup,  $A$ , on the space of lattices  $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ . Indeed, he showed that it is equivalent to the statement:

**Conjecture 2.** Any  $A$ -orbit  $A \cdot \xi$  in  $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$  for  $d \geq 3$  is either periodic or unbounded.

In this conjecture a periodic orbit means the  $L$ -orbit of some closed subgroup  $L$  of  $SL(n, \mathbb{R})$  that contains  $A$  and has finite volume in  $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ . The analogous notion for measures is called a homogeneous measure, that is to say,  $\mu$  is a homogeneous measure on the space of lattices if  $\mu$  is the  $L$ -invariant measure on a single, finite volume,  $L$ -orbit for some closed subgroup  $A \leq L \leq SL(n, \mathbb{R})$ . The corresponding conjecture concerning invariant measures for the action of  $A$  on the space of lattices is then:

**Conjecture 3.** Let  $\mu$  be an  $A$ -invariant and ergodic probability measure on  $X_d$  for  $d \geq 3$  (and  $A <$

$PGL(3, \mathbb{R})$  the group of diagonal matrices). Then  $\mu$  is homogeneous.

We come finally to one of the remarkable achievements of Elon's. While he did not settle this last conjecture completely, he was able to obtain a partial result analogous to what Rudolph proved for the circle. In joint work with M. Einsiedler and A. Katok he established:

**Theorem 3** ([EKL]). *Let  $A$  be the group of diagonal matrices and  $n \geq 3$ . Let  $\mu$  be an  $A$ -invariant and ergodic probability measure on  $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ . If for some  $a \in A$  the entropy  $h_\mu(a) > 0$ , then  $\mu$  is homogeneous.*

Due to the close connections between entropy and Hausdorff dimension, they were able to establish the following striking result:

**Theorem 4** ([EKL]). *The set of pairs of real numbers  $(x, y)$  for which*

$$\liminf_{n \rightarrow \infty} n \|nx\| \|ny\| = 0$$

*fails to hold has Hausdorff dimension zero.*

### Quantum Unique Ergodicity

To formulate Elon's marvelous contributions to the quantum unique ergodicity (QUE) conjecture, we shall need more concepts that we do not have the space to explain in detail. There is a recent article by Peter Sarnak, "Recent progress on the quantum unique ergodicity conjecture", to which we can refer the interested reader for more of the background.

Let  $M$  be a compact Riemannian manifold and denote by  $\Delta$  the Laplacian on  $M$ . Since  $M$  is compact,  $L^2(M)$  is spanned by the eigenfunctions of the Laplacian. Quantum ergodicity deals with the equidistribution properties of these eigenfunctions. To be precise let  $\phi_n$  be a complete orthonormal sequence of eigenfunctions of  $\Delta$  ordered by eigenvalue. These can be interpreted, for example, as the steady states for Schrödinger's equation

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi$$

describing the quantum mechanical motion of a free (spinless) particle on  $M$ . According to Bohr's interpretation of quantum mechanics  $|\phi_n(x)|^2$  integrated over a set  $A$  is the probability of finding a particle in the state  $\phi_n$  inside the set  $A$  at any given time. A. I. Šnirel'man, Y. Colin de Verdière, and S. Zelditch have shown that whenever the geodesic flow on  $M$  is ergodic, for example if  $M$  has negative curvature, there is a subsequence  $n_k$  of density one on which these probability measures converge in the weak\* topology to the normalized volume measure on  $M$ . This phenomenon is called quantum ergodicity. While these measures are on the manifold they also defined liftings of these measures to the unit tangent bundle,  $S^*M$ , which become more and more invariant under the geodesic flow

as the eigenvalue increases so that any weak\* limit of these lifts is invariant under the geodesic flow. Any such limiting measure is called a **quantum limit**. Z. Rudnick and P. Sarnak made the following conjecture:

**Conjecture 4.** (QUE) *If  $M$  is a compact manifold of negative curvature, the only quantum limit is the normalized volume measure on  $S^*M$ .*

There are also conjectures of this type in the case of manifolds that are not compact but do have a finite volume. In that case the Laplacian also has a nondiscrete spectrum, and that must be taken into account, but we won't go into that here. Elon's results pertain to a special class of manifolds that are called **arithmetic manifolds** since they are defined by number theoretical means. The easiest to describe is our space of lattices. For compact examples one has discrete subgroups of  $SL(2, \mathbb{R})$  that are defined by means of certain quaternionic division algebras. For these manifolds the eigenfunctions typically have additional symmetries that can be exploited. In fact, the quantum limits that appear below are limits of eigenfunctions that are also eigenfunctions of the Hecke operators. Elon, together with J. Bourgain [BL], showed that for quantum limits in this arithmetic case the entropy of the geodesic flow is positive. He was then able to combine this positivity with a number of highly original arguments in order to prove:

**Theorem 5.** ([L]) *If  $M$  is a compact arithmetic surface, then the only quantum limit is the normalized volume element.*

We should emphasize that what lurks behind this result is the fact that the quantum limit not only is invariant under the geodesic flow but also possesses additional symmetry. Otherwise, as we have pointed out, the geodesic flow has a wide variety of measures of positive entropy.

For the noncompact case in the same paper, Elon was able to show that any quantum limit must be a constant multiple of the volume element but didn't resolve the issue of whether the constant was actually equal to one. This was resolved in the affirmative in more recent work of K. Soundararajan.

I have tried to explain a few of the more outstanding results of Elon's. There are many more that I haven't touched on, some of which are described in [L2]. For example, in recent joint work with M. Einsiedler, P. Michel, and A. Venkatesh, he shows that the union of the periodic orbits of the diagonal subgroup acting on  $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$  with volume  $V_i$  become uniformly distributed as  $V_i$  tends to infinity. While a variety of analytic methods are involved, at the very heart of the proof lie the measure rigidity results. To sum up, Elon has taken the interaction between ergodic theory and number theory to new heights, and it is our hope to see even more in the future.



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## The Work of Stanislav Smirnov

### Wendelin Werner

Last August the Fields Medal was awarded to Stanislav Smirnov (“Stas” is the short version of his first name that is commonly used) for his proofs of conformal invariance of two of the most famous lattice models from statistical physics. Given the importance of these results and the influence that they already have on the subject, this award did not surprise anybody acquainted with this part of mathematical physics.

Smirnov was educated in the great analysis school of St. Petersburg. He grew up in this town (called Leningrad in those days), went to one of its elite classes in high school, and then to its university. His first steps in research were guided by Victor Havin, whose seminar had a stimulating and lasting influence on students. He then went to Caltech in the United States to write his Ph.D. thesis under the supervision of Nikolai Makarov (a student of Nikolskii in Leningrad, who in turn had been a student of Havin) on the spectral analysis of Julia sets. After a postdoc at Yale (where he interacted with Peter Jones) Smirnov took in 1998 a position at KTH in Stockholm, where he also had many natural collaborators. There, for instance, he started a series of important papers with Jacek Graczyk. His first papers dealt with complex analysis and complex dynamics, and they would deserve a detailed description as well. It was in Stockholm that he started to think seriously about probabilistic questions, encouraged also by Lennart Carleson. Since 2003 Smirnov has been professor at the Université de Genève.

### Conformal Invariance of Critical Percolation

In the early 1980s the British theoretical physicist John Cardy proposed—based on ideas of Belavin,

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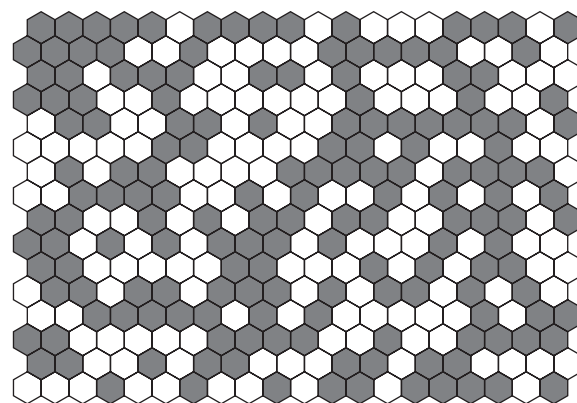


Figure 1. A horizontal crossing of  $R_L$ .

Polyakov, and Zamolodchikov, and more precisely on the symmetries of conformal field theories that were supposedly related to those particular lattice models—an explicit formula for the limit of crossing probabilities of conformal rectangles in critical percolation when the mesh of the lattice goes to 0. Let us immediately describe this model without explaining why this is a key question.

For each different hexagonal cell in the planar honeycomb lattice (as in Figure 1, in a black and white version), toss a fair coin to choose its color: with probability  $1/2$  it is blue, and with probability  $1/2$  it is yellow (these are the two colors used by Smirnov in his paper, to pay tribute to his Swedish colleagues). One is interested in the existence of long paths of the same color (for instance, consisting only of blue cells).

Let us first consider the rectangle  $R_L = (0, L) \times (0, 1)$ . In the honeycomb lattice with mesh  $\epsilon$ , choose a lattice-approximation  $R_L^\epsilon$  of this rectangle, and let  $p_\epsilon(L)$  denote the probability of existence of a blue left-to-right crossing of  $R_L^\epsilon$  (which joins the left and right boundary segments of the rectangle). The problem is to prove that  $p_\epsilon(L)$  converges as  $\epsilon \rightarrow 0$  and to identify its limit.

More generally, when  $D$  is a simply connected domain with a smooth boundary in which one chooses two disjoint arcs  $d_1$  and  $d_2$ , one can study the asymptotic behavior of the probability  $p_\epsilon(d_1 \leftrightarrow d_2; D)$  that there exists a blue path joining  $d_1$  and  $d_2$  in a lattice approximation of  $D$  with mesh size  $\epsilon$ .

In 2001 Smirnov showed that the quantities  $p_\epsilon(\cdot)$  do indeed converge when  $\epsilon \rightarrow 0$  and that their limits are those predicted by Cardy. In particular, these limits turn out to be conformally invariant: This means that if one chooses  $L$  in such a way that there exists a conformal (angle-preserving one-to-one) map from  $D$  onto  $R_L$  that maps  $d_1$  and  $d_2$  onto the two vertical sides of  $R_L$ , then the limits of  $p_\epsilon(d_1 \leftrightarrow d_2; D)$  and of  $p_\epsilon(L)$  are identical.

The most remarkable part of this result is not the explicit formula but rather the fact that the

limiting probabilities are conformally invariant quantities. The proof uses a simple combinatorial property that Smirnov is able to reformulate as the (almost)-discrete analyticity of a suitably generalized crossing probability (in order to get a complex function of a complex variable). With this observation in hand, it is possible to control the limiting behavior of this discrete almost-analytic function to its continuous counterpart. Conformal invariance is therefore itself part of the proof of the derivation of the explicit formula for the asymptotics of  $p_c(\cdot)$  (on other graphs, where no discrete analyticity has been detected, the existence of the limit of crossing probabilities itself is still an open problem).

The entire proof is essentially contained in a short, six-page note published in the *Comptes-rendus de l'Académie des Sciences*. Several prizes (including the Clay Research award) were awarded to Smirnov for this wonderful gem.

### Conformal Invariance of the Ising Model

The Ising model may be the most famous lattice model from statistical physics. Here one is again coloring at random a portion of a graph, but this time the colors (which are more often called “spins” in this context) of different cells are not independent. To fix ideas let us now consider the square lattice, but the results of Dmitry Chelkak and Smirnov on the Ising model are valid for a larger class of planar lattices.

The Ising model was first defined as a model for ferromagnetism. Intuitively, one can describe it by saying that neighboring cells prefer to have the same color. The smaller the number  $N$  of pairs of disagreeing neighboring cells in the configuration, the more likely the coloring will be. The most probable configurations are therefore those in which all colors are identical. The model is specified by a parameter  $x$  in  $(0, 1)$  that describes how much one penalizes configurations with additional disagreeing neighbors. More precisely, the probability of a configuration is  $x^N$ , modulo some multiplicative constant that ensures that all probabilities add up to 1; for instance, a configuration with a total of 4 disagreeing pairs of neighbors will have a probability that is equal to  $x^4$  times the probability of a configuration in which all sites choose the same spin.

It turns out that a special value  $x_c$  of  $x$  plays an important role: When  $x < x_c$ , at large scales, the typical systems are in an ordered phase: one color has a clear majority. By contrast, when  $x > x_c$ , the systems are likely to be very disordered and look somewhat like percolation on large scale. Critical phenomenon studies the large-scale behavior of the system when  $x$  is equal to this special value, called the critical or phase-transition point.

In the case of the Ising model, the most natural quantities to investigate are not the existence of long crossings but rather “correlations between

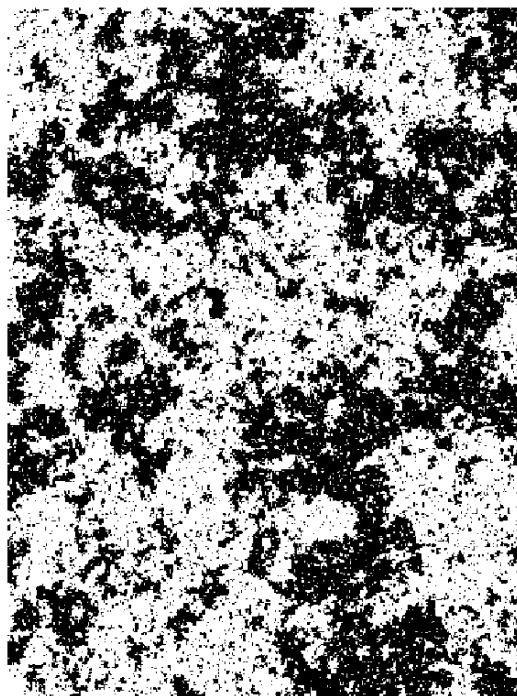


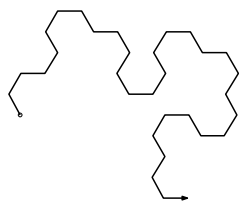
Figure 2. A configuration of the critical Ising model.

faraway spins” that give information on the total number of cells of a given color (the “global magnetization” in the ferromagnetic interpretation): What is the probability that two sites that are far away from each other have chosen the same color? But this type of question can be easily reformulated in terms of connectivity properties of a related model, so that there are similarities with the previous percolation model.

Smirnov (in part with Chelkak) showed in [8], [3] that in the case of the critical Ising model, it is possible to construct quantities (in fact the mean of complex-valued functions of the colorings) that are discrete analytic with respect to a site used to define them (this exact discrete analyticity differs from the approximate discrete analyticity of percolation—it is also what relates these questions to integrable systems). This allows him to control their behavior when the lattice-spacing tends to 0 for a fixed reference domain  $D$  and to prove their asymptotic conformal invariance. These results make it possible to give full and complete answers to questions studied and raised by physicists for more than sixty years (the name of Lars Onsager immediately comes to mind); see also [4].

### Related Works

The Schramm-Loewner evolution (SLE) processes are continuous random curves introduced in 1999 by Oded Schramm, who conjectured them to be the scaling limits of interfaces in various critical planar models for statistical physics. The work of



**Figure 3. A self-avoiding walk with 31 steps on the honeycomb lattice.**

Stas Smirnov in fact proves this conjecture (see also [5]) in the two previously described cases. This makes it possible to exploit the computations that are possible in the continuous SLE setting in order to deduce additional results for the discrete models and, more generally, to get a complete picture of the scaling limits of these two models. The case of critical percolation is for instance studied in the preprint by Schramm and Smirnov [6].

It appears that percolation (on the lattice described above) and the Ising model (on a wider class of lattices), together with the uniform spanning tree (which had been studied in a similar spirit by Richard Kenyon) play a particular role. Today, these are almost the only classical lattice models in which conformal invariance has been fully proved. Nevertheless, the ideas developed by Smirnov can be used to prove spectacular results for other models. For instance, with Hugo Duminil-Copin, Smirnov proved the famous conjecture of the Dutch theoretical physicist Bernhard Nienhuis about the asymptotic number of self-avoiding walks on the honeycomb lattice: the number of self-avoiding paths with  $N$  steps starting from the origin that one can draw on this lattice grows like  $\lambda^{n+o(n)}$  when  $\lambda = \sqrt{2} + \sqrt{2}$ .

Smirnov's work sheds yet another light on the power and beauty of complex analysis, this time in the context of probabilistic questions arising from physical lattice models. A recommended more detailed introduction to his work that we very briefly described here is Smirnov's contribution [9] to the Proceedings of the ICM.

### Acknowledgments

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## The Work of Cédric Villani

### Luigi Ambrosio

Cédric Villani combines in his work, at the highest level, mathematical rigor and elegance, physical intuition and depth. His energetic, enthusiastic, and friendly personality also contributed to making him a driving force in many fields of mathematics and a source of inspiration for younger mathematicians. I will describe in the next three sections his main achievements, following to some extent the chronological order.

### Boltzmann Equation

This fundamental equation was derived by L. Boltzmann in 1873 to describe the time evolution of the density  $f_t(x, v)$  in the phase space  $\mathbb{R}_x^3 \times \mathbb{R}_v^3$  of a sufficiently rarified gas. It can be written as

$$(5) \quad \frac{d}{dt} f_t(x, v) + v \cdot \nabla_x f_t(x, v) = Q(f_t, f_t)(x, v)$$

where  $Q(f, f)$  is a nonlinear operator describing the collision process between particles, usually representable as

$$Q(f, f)(x, v) = \int_{\mathbb{R}^3} \int_{S^2} [f(x, v') f(x, v'_*) - f(x, v) f(x, v_*)] B(v - v_*, \sigma) d\sigma dv_*.$$

Here  $v, v_*$  stand for the postcollisional velocities, and  $v', v'_*$  stand for the precollisional velocities, related to  $v$  and  $v_*$  and the impact direction  $\sigma$  by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

Boltzmann's  $H$  theorem states that the quantity

$$S(f_t) := - \int_{\mathbb{R}^6} f_t(x, v) \ln f_t(x, v) dx dv$$

always increases along solutions to (5). Here I shall adopt mathematicians' usage of the word "entropy", which is opposite to that of

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physicists, and call it  $H$  (this is the standard convention in probability and optimal transport, discussed in the next section). Then we can say that  $H(f_t) = -S(f_t) = \int f_t \ln f_t$  decreases in time. In addition, the dissipation term  $D(f) := -\frac{d}{dt}H(f)$  appearing in Boltzmann's  $H$  theorem vanishes if and only if  $f_t$  is locally (in  $x$ ) Maxwellian in  $v$ , namely

$$f_t(x, v) = \rho_t(x) \frac{1}{(2\pi T_t(x))^{3/2}} \exp\left[-\frac{|v - u_t(x)|^2}{2T_t(x)}\right].$$

Here  $\rho_t$ , the first marginal of  $f_t$ , is the local density;  $u_t$ , the first marginal of  $vf_t$ , is the local mean velocity; and  $T_t$ , the first marginal of  $|v|^2 f_t$ , is the local temperature.

Boltzmann's  $H$  theorem determines an arrow of time, and since Newton's equations describing collisions between gas particles are time-reversible, a long debated and basic question (Loschmidt's paradox) is to understand where, in Boltzmann's derivation of (5) from Newton's law, a time-asymmetric assumption enters. The interested reader can find in the recent book [7] a very good account of the state of the art on the mathematical and physical issues related to Boltzmann's equation.

Despite decades of research, many questions about the Boltzmann equation are still unanswered. Existence and regularity are known for initial data close to the equilibrium measure, while for general initial data R. DiPerna and P. L. Lions were able to prove existence of the so-called renormalized solutions, a suitable notion of weak solution. Villani contributed to the existence theory for Boltzmann's equation for several collision operators, but undoubtedly his main contributions concern Cercignani's conjecture and the understanding of the rate of decay of entropy and convergence to equilibrium.

Even in the spatially homogeneous case, i.e., when the  $f_t$  are independent of  $x$ , the analysis is far from trivial, due to the nonlinear character of the collision operator. In 1983 C. Cercignani conjectured that, for suitable kernels  $B$ , there is a constant  $K > 0$  dependent on the initial condition  $f_0$  such that the relative entropy-entropy dissipation inequality holds:

$$D(f_t) \geq K \int_{\mathbb{R}^3} \left( \frac{f_t}{M} \ln \frac{f_t}{M} \right) M dv.$$

Here  $M$  is the Maxwellian limit state, uniquely determined by  $f_0$ . By Gronwall's lemma, the validity of Cercignani's conjecture would imply exponential convergence to  $M$  as  $t \rightarrow \infty$ . L. Desvillettes's proof of a lower bound on the entropy production was later improved in joint work of E. Carlen and M. Carvalho, who also pointed out links between Cercignani's conjecture and information theory and Sobolev inequalities. Eventually the conjecture was settled by Villani in [16] (in collaboration with G. Toscani) and in [18]: Cercignani's

conjecture is not true, but the weaker inequality  $D(f_t) \geq K_\epsilon (\int f_t \ln(f_t/M) dv)^{1+\epsilon}$  holds for all  $\epsilon > 0$ . This suffices to provide polynomial rates of convergence to equilibrium.

The extension of these results to the spatially inhomogeneous case is immediately seen to be very demanding: indeed, since the collision operator depends only on  $f(x, \cdot)$ , we may consider (5) as a kind of system of homogeneous equations indexed by  $x$ , where the only coupling is given by the transport term  $v \cdot \nabla_x f$ . Being degenerate and first order, this term exhibits very poor regularizing properties. Nevertheless, in a joint paper with L. Desvillettes [6], Villani was able to exploit this term to show polynomial convergence to the Maxwellian even in the spatially inhomogeneous case, under suitable growth and smoothness assumptions on the solution. This remarkable result is the *first* convergence theorem for initial conditions not close to equilibrium, i.e., in a nonperturbative regime (so that linearization of the collision term around the Maxwellian is not useful). Later on, Villani developed a general theory [19], the so-called hypocoercivity, applicable also to the asymptotics of other classes of operators. His work in this area influenced many younger mathematicians, including C. Mouhot, C. Baranger, R. Strain, M. Gualdani, and S. Michler.

## Optimal Mass Transport

Villani's work in optimal mass transport has been extremely influential, as I will illustrate, for the development of connections between curvature, optimal transport, functional inequalities, and Riemannian geometry. Besides these contributions, his monumental work [20], containing a fairly complete and updated description of the state of the art in the theory of optimal transport, played a major role in indicating research directions and in spreading the new discoveries among different communities. As we continue to witness, this subject is still expanding so quickly that presumably Villani's treatise will be the last attempt to keep track of the whole theory in a single book.

The problem of optimal mass transport was proposed by G. Monge, one of the founders of the École Polytechnique, in a famous *memoire* in 1781. Despite its very natural formulation and a prize offered by the Académie des Sciences in Paris for its solution, the problem received very little attention in the mathematical literature (partly because the right mathematical tools to attack the problem were lacking) until the work of L. Kantorovich in 1942, who proposed a weak formulation of the problem and received, for related work, the Nobel Prize in Economics in 1975. Kantorovich's formulation became very popular in optimization and linear programming, but also in probability and information theory, one of the reasons being that the optimal transport problem provides a very



natural family of distances in the space of probability measures. In more recent years, starting with Brenier's seminal 1991 paper [2] on polar factorization and existence of optimal transport maps, connections have been discovered with many more areas, such as fluid mechanics, gradient flows and dissipative PDE's, shape optimization, irrigation networks, Riemannian geometry, and analysis in metric measure spaces.

A modern formulation of Monge's problem is the following: given two Borel probability measures  $\mu$  and  $\nu$  in a metric space  $X$ , and given a Borel cost function  $c = c(x, y) : X \times X \rightarrow [0, +\infty]$  (whose heuristic meaning is the cost of shipping a unit of mass from  $x$  to  $y$ ), one has to minimize the transport cost

$$\int_X c(x, T(x)) d\mu(x)$$

among all Borel transport maps  $T$  mapping the "mass distribution"  $\mu$  to  $\nu$  (i.e.,  $\mu(T^{-1}(E)) = \nu(E)$  for all  $E$  Borel). Monge proposed in his *memoire* the case  $X = \mathbb{R}^2$  and  $c$  equal to the Euclidean distance: in this case the transport cost has the physical meaning of work. However, it turns out that many other choices of  $c$  are possible, and definitely the "best" choice, in terms of connections with other fields and regularity of optimal maps, is the case in which  $c$  is the *square* of the distance, at least in Euclidean and Riemannian spaces.

Even when  $X$  has a linear structure, the class of admissible maps  $T$  is not stable under weak convergence, and this is the main technical difficulty in the proof of existence of optimal maps. Kantorovich's relaxation of the problem consists in minimizing the transport cost, now written as

$$\int_{X \times X} c(x, y) d\pi(x, y)$$

within the class of transport plans  $\pi$  from  $\mu$  to  $\nu$ , i.e., probability measures in  $X \times X$  whose first and second marginals are respectively  $\mu$  and  $\nu$  (i.e.,  $\mu(A) = \pi(A \times X)$ ,  $\nu(B) = \pi(X \times B)$ ). This formulation allows for general existence results (it suffices that  $c$  be lower semicontinuous and  $X$  be complete and separable) and powerful duality results. Heuristically, in this more general formulation we are allowing for splitting of mass, so that the mass at  $x$  need not be sent at a single point  $T(x)$ , but it can be distributed according to  $\pi_x$ , the conditional probability of  $\pi$  given  $x$ .

In some situations one can show that no mass splitting occurs and recover an optimal map  $T$ ; this was achieved independently by Y. Brenier [2] and S. T. Rachev-L. Rüschendorf [14] (building upon earlier work by M. Knott and C. S. Smith) in the Euclidean case, when  $c(x, y) = |x - y|^2$ . In this case optimal maps coincide precisely with gradients (or subgradients) of convex functions. Particularly relevant for the most recent developments in Riemannian geometry is the analogous

result obtained by R. McCann in 2001 on Riemannian manifolds, with  $c$  equal to the squared Riemannian distance.

From now on, I shall focus on the case in which the cost is the square of a distance  $d$  and consider the so-called *Wasserstein distance*  $W_2(\mu, \nu)$ , whose square is the minimum in Kantorovich's problem:

$$W_2^2(\mu, \nu) := \min \left\{ \int_{X \times X} d^2(x, y) d\pi(x, y) : \pi \text{ plan from } \mu \text{ to } \nu \right\}.$$

It is fairly easy to show that  $W_2$  is indeed a distance in the space

$$\mathcal{P}_2(X) := \left\{ \mu \in \mathcal{P}(X) : \int_X d^2(x, x_0) d\mu(x) < \infty \quad \forall x_0 \in X \right\}$$

of probability measures with finite quadratic moments and that  $\mathcal{P}_2(X)$  inherits many properties from the base space  $X$ , such as completeness, compactness, and the property of being a length space (i.e., existence of length-minimizing curves with length equal to the distance).

At the end of the 1990s a deeper and more geometric description of the relations between  $X$  and  $\mathcal{P}_2(X)$ , at first involving the differentiable structure and then curvature, started to emerge. This point of view could be traced back to the work of R. McCann on displacement convexity of the internal energy of a gas (now interpreted as convexity along geodesics in  $\mathcal{P}_2(\mathbb{R}^n)$ ) and to the work of R. Jordan, D. Kinderlehrer, and F. Otto, showing that the classical heat equation  $\partial_t f = \Delta f$  can be viewed as the gradient flow of the entropy functional  $H(f)$  in  $\mathcal{P}_2(\mathbb{R}^n)$ . The fact that  $\mathcal{P}_2(\mathbb{R}^n)$  is indeed some sort of infinite-dimensional Riemannian manifold is made explicit for the first time in the seminal paper by F. Otto [12] on the asymptotics of the porous medium equations; the formula,

$$W_2^2(\mu_0, \mu_1) := \min \left\{ \int_0^1 \int_{\mathbb{R}^n} |v_t|^2 d\mu_t dt : \frac{d}{dt} \mu_t + \nabla \cdot (v_t \mu_t) = 0 \right\}$$

independently discovered by J. D. Benamou and Y. Brenier, shows that  $W_2$  is indeed the induced Riemannian distance (because the right-hand side can be interpreted as the infimum of the action of all paths from  $\mu_0$  to  $\mu_1$ ). This interpretation of  $\mathcal{P}_2(X)$  is particularly useful for the study of the asymptotics and rate of contraction of large classes of PDEs of gradient flow type, and by now a complete theory is available [1].

This line of thought has been pursued by F. Otto and Villani [13] in an extremely influential paper, in which they use this geometric interpretation to extend and to provide a new proof of Talagrand's inequality involving transport distance and relative entropy with respect to the standard Gaussian  $\gamma_n$ :

$$\frac{1}{2} W_2^2(\rho \gamma_n, \gamma_n) \leq \int_{\mathbb{R}^n} \rho \ln \rho d\gamma_n.$$

In this abstract and fruitful perspective, Talagrand's inequality follows from the observation (with  $E = \mathcal{P}_2(\mathbb{R}^n)$ ,  $\Phi$  equal to the relative entropy with respect to  $\gamma_n$ ,  $x_{\min} = \gamma_n$ ) that for any 1-convex functional  $\Phi : E \rightarrow \mathbb{R} \cup \{+\infty\}$  the inequality

$$\Phi(x) \geq \Phi(x_{\min}) + \frac{1}{2}d^2(x, x_{\min})$$

holds, with  $x_{\min}$  equal to the ground state of  $\Phi$ . Also L. Gross's logarithmic Sobolev inequality can be extended and interpreted in this more general perspective.

It is in [13], and independently in the work [4] by D. Cordero Erasquin, McCann, and M. Schmuckenschläger, that the first link between Ricci curvature and optimal transport appears, with the observation (based on Bochner's identity) that in a Riemannian manifold  $M$  the relative entropy with respect to the volume measure is convex along Wasserstein geodesics of  $\mathcal{P}_2(M)$  if the Ricci tensor of  $M$  is nonnegative. The conjectured equivalence of the two properties was proved later on by K. T. Sturm and M. Von Renesse. These results paved the way to synthetic notions of lower bounds on Ricci curvature for metric measure spaces (a theory somehow parallel to Alexandrov's, which deals with triangle comparisons and sectional curvature) thoroughly explored by J. Lott and Villani in [8] and independently by Sturm [15]. In this very general framework Ricci bounds from below are stable under measured Gromov-Hausdorff convergence; in addition, the Poincaré inequality and other functional inequalities can be obtained. If we consider, instead of a fixed manifold  $(M, g)$ , a time-dependent family  $(M, g_t)$  evolving by Ricci flow, as in the celebrated work by R. Hamilton and G. Perelman, new connections emerge, as shown first by R. McCann and P. Topping and then by J. Lott [9].

Another very influential paper by Villani is his proof, in a joint paper [5] with D. Cordero-Erasquin and B. Nazaret, of the Sobolev inequality with optimal constant via optimal transportation. It is reminiscent of Gromov's idea of proving the isoperimetric inequality via transport maps, but the use of the Brenier map (instead of the so-called Knothe map) leads to sharper results. This has become apparent with the recent work by A. Figalli, F. Maggi, and A. Pratelli, inspired by [5], in which sharp quantitative versions of the isoperimetric inequality are obtained for general anisotropic surface energies.

One more deep connection between optimal transport and curvature arises when one studies the regularity theory of optimal maps between Riemannian manifolds, a theory pioneered by L. Caffarelli [3] in the flat Euclidean case (with  $c(x, y) = |x - y|^2$ ). X. Ma, N. Trudinger, and X. J. Wang devised in [10] a fourth-order differential condition on the cost function  $c$  sufficient to provide regularity. If  $c$  is the square of the

Riemannian distance, it turns out that this differential condition can be expressed as positivity of a new geometric tensor, now called the MTW tensor. The properties of this tensor, its stability under perturbations of the metric, and the regularity of optimal maps have been investigated in a series of papers by P. Delanoë, A. Figalli, Y. Ge, H. Y. Kim, G. Loeper, R. McCann, L. Rifford, and Villani. It is now understood that the MTW tensor is an object of independent interest, because of its implications on the geometry and the stability of the cut locus of a Riemannian manifold.

### Landau Damping

Villani's latest and most spectacular achievement is his proof, in a joint work with C. Mouhot [11], of the Landau damping for the Vlasov-Poisson equation,

$$(6) \quad \frac{d}{dt}f_t + v \cdot \nabla_x f_t + E_t \cdot \nabla_v f_t = 0,$$

the basic equation of plasma physics. Here  $f_t(x, v) \geq 0$  represents the time-dependent density in phase space of charged particles, and the electric field  $E_t$  is coupled to  $f_t$  by Poisson's equation, namely  $E_t = -\nabla \phi_t$  with  $-\Delta \phi_t = \rho_t - 1$ ,  $\rho_t$  being the first marginal of  $f_t$  (for the sake of simplicity I will not consider the gravitational case, included in [11]).

This equation describes collisionless dynamics, and it is time-reversible, so that no dissipation mechanism or Lyapunov functional can be invoked to expect or to prove convergence to equilibrium. Nevertheless, in 1946 L. Landau studied the behavior of the linearized Vlasov-Poisson equation, starting from a Gaussian distribution, and made the astonishing discovery that the electric field decays exponentially fast as  $|t| \rightarrow \infty$ .

The equation (6) has infinitely many stationary solutions, given by probability densities  $h(v)$  independent of  $x$ . For the linearized equation their stability analysis was initiated by O. Penrose in the 1960s, and the Landau damping was well understood in the same years thanks to the work of A. Saenz. Nevertheless, as pointed out by G. Backus in 1960, the linearization introduces cumulative errors that make it impossible to use the behavior of the linearized equation in order to predict the behavior of solutions of (6) for large times. For this reason, in the nonlinear regime the validity of the Landau damping was only conjectured, although it had been shown by E. Caglioti and C. Maffei to occur in a specific situation.

In the spatially periodic case (i.e., when  $x$  belongs to  $[0, L]^3$  and periodic boundary conditions are considered), C. Mouhot and Villani proved rigorously in [11] that the phenomenon occurs for all initial conditions sufficiently close to a linearly stable and analytic velocity profile  $h$ . Their statement provides even more, namely weak convergence of  $f_t$  as  $t \rightarrow \pm\infty$  to analytic profiles  $f_{\pm\infty}(v)$ . Their

proof is a technical masterpiece and a real *tour de force*, via the introduction of analytic norms in the space  $(k, \nu)$  (where  $k$  is the Fourier variable dual to  $x$ ) that incorporate the loss of regularity induced by the transport term. The growth in time of these norms is carefully estimated using a Newton scheme, which provides approximations of (6) on which the evolution of the norms is computable.

Since the limiting profiles  $f_{\pm\infty}$  can be stable as well, it is possible to describe the relaxation to equilibrium only in terms of weak convergence in phase space, or equivalently in terms of convergence of averaged quantities, like the position density  $\rho_t$  (whose decay is closely related to the decay of  $E_t$ ). On the other hand, since the Vlasov-Poisson equation is time-reversible, the initial datum  $f_0$  is uniquely determined by  $f_t$ , so that there is no loss of information in passing from  $f_0$  to  $f_t$ . Another way to see that weak convergence cannot be improved to strong convergence relies on the understanding that the initial information in  $f_0$  is “stored” in  $f_t$ , as  $|t|$  increases, at higher and higher energy modes, in analogy with the theory of turbulence in fluid mechanics. This transfer mechanism and the relation between relaxation and mixing are analyzed in great detail in [11].

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# On the Work of Louis Nirenberg

*Simon Donaldson*

Louis Nirenberg received the first Chern Medal Award at the International Congress of Mathematicians (ICM) in August 2010. Sponsored by the International Mathematical Union and the Chern Medal Foundation, the award will be given every four years at the ICM to an individual whose lifelong outstanding achievements in the field of mathematics warrant the highest level of recognition. The award consists of a medal and a monetary award of US\$500,000. There is a requirement that half of the award will be donated to organizations of the recipient's choice to support research, education, outreach or other activities to promote mathematics.

The *Notices* solicited the following article about the work of Louis Nirenberg. The International Mathematical Union also issued news releases about the Chern Medal Award, which appeared in the December 2010 *Notices*.

It is an honour to be asked to write this celebration of the award of the first Chern Prize of the International Mathematical Union to Louis Nirenberg, but the author's enthusiasm for the task is matched by considerable doubts about his adequacy for it. Nirenberg is a giant in the subject of partial differential equations and has made fundamental contributions to that area over more than half a century. The author is far from being a PDE specialist, and this account is of necessity highly selective, emphasising those topics (mainly connected with geometry) that the author knows a little better. I will say nothing about huge swathes of Nirenberg's work, which might well be the focus of other essays—for example, his work with Agmon and Douglis on very general elliptic boundary value problems, or his place as one of the fathers of pseudodifferential operators. Our small selection perhaps gives a glimpse of the range of his work, melding PDE theory with classical differential geometry, with the theory of complex manifolds, with harmonic analysis, even though this leaves out much else, such as his work on fluid mechanics.

## Isometric Embedding

Nirenberg's Ph.D. thesis [8] completed the solution of a famous problem in differential geometry, developing 1916 work of Weyl. The statement is very simple: an abstract Riemannian metric on the

2-sphere with positive curvature can be realised by an isometric embedding in  $\mathbf{R}^3$ . (Another proof was given by Pogorelov at about the same time.) This is a wonderful result for several reasons. The statement is easy to grasp and decisive. The strategy of proof, by the "continuity method", is a model for many other PDE problems in differential geometry and elsewhere. For example, the overall strategy in Yau's proof of the Calabi conjecture has just the same shape. Finally, while it might appear an elementary problem, some serious difficulties arise. Nearly one hundred years after Weyl's paper and nearly sixty years after Nirenberg's and having in mind the huge development in "geometric analysis" over this period, one might think that supplying a proof would nowadays be a straightforward exercise, but that is very far from the case.

Let  $g_1$  be a Riemannian metric of positive curvature on  $S^2$ . In the continuity method one first shows that  $g_1$  can be joined to a standard "round" metric  $g_0$  by a path  $g_t$ ,  $t \in [0, 1]$  of metrics of positive curvature. This is quite easy, using the fact that any metric is conformal to a round metric. Now one considers the set  $T \subset [0, 1]$  of parameter values  $t$  such that  $g_t$  can be realised by an isometric embedding. The task is to show that  $T$  is both open and closed, and hence must be the whole interval (since the round metric is certainly realised by an embedding and so  $T$  is not empty).

A first difficulty is that, if set up in the obvious way as a PDE for a map  $\iota : S^2 \rightarrow \mathbf{R}^3$ , the isometric embedding problem is degenerate. Infinitesimal

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variations of the map in the normal direction change the induced metric by an algebraic, rather than a differential, operator. Thus the usual approach to prove openness, through the inverse function theorem in Banach spaces, does not immediately apply. A more sophisticated machine, the Nash-Moser theory, can be applied [3], but this came later and Nirenberg used an intricate and ingenious argument, partly following Weyl, to get around the difficulty.

The closedness comes down to establishing a priori estimates. One estimate goes back to Weyl. The Gauss curvature  $K$  of an embedded surface is given by a quadratic expression in the second fundamental form  $B$ , and one finds that

$$\Delta K = b\Delta b - \langle B, \nabla \nabla b \rangle + |\nabla b|^2 - |\nabla B|^2 + K(b^2 - 4K),$$

where  $\Delta$  is the intrinsic Laplacian on the surface and  $b = \text{Tr}B$  is the mean curvature. Then an application of the maximum principle, considering the point where  $b$  attains its maximum, gives an a priori bound on  $b$  that translates into a  $C^2$  bound on the isometric embedding  $\iota : S^2 \rightarrow \mathbf{R}^3$ . The other major component in Nirenberg's proof is to promote this first to a  $C^{2,\alpha}$  bound and then to all higher derivatives. This follows from general theorems about elliptic equations in two variables that he had developed at about the same time [9], together with another ingenious differential geometric device, considering the equation satisfied by the distance on the surface to a fixed origin in  $\mathbf{R}^3$ .

Nirenberg has many other papers related to classical differential geometry and isometric embedding. There is a particular difficulty at points at which the sign of the Gauss curvature changes and the PDE changes from elliptic to hyperbolic type.

## Complex Geometry

Another of Nirenberg's renowned achievements in geometry is the Newlander-Nirenberg theorem on the integrability of almost-complex structures [10], which is a foundation stone for the Kodaira-Spencer-Nirenberg treatment of deformations of complex manifolds [5]. (In the context of this essay, note that the authors of [10] thank Chern for bringing the problem to their attention.) Recall that an almost-complex structure on a  $2n$ -dimensional manifold  $M$  is a bundle map  $I : TM \rightarrow TM$  with  $I^2 = -1$ . The question is: when does such an almost-complex structure come from a complex structure; that is, when can one find a diffeomorphism between the neighbourhood of any point in  $M$  and a polydisc in  $\mathbf{C}^n$  that takes  $I$  to the standard almost-complex structure on  $\mathbf{C}^n$ ? The eigenspaces of  $I$  yield a decomposition of the complexified tangent bundle  $TM \otimes \mathbf{C} = T' \oplus T''$ , and a necessary condition is that sections of  $T'$  are closed under the Lie bracket. The Newlander-Nirenberg theorem asserts that this condition is also sufficient. This is formally analogous to the Frobenius integrability

condition, for a real sub-bundle of the tangent bundle to define a foliation, and when the data is real-analytic one can derive the result from the Frobenius theorem by a complexification argument that goes back, in the case  $n = 1$ , to Gauss. But if the data is  $C^\infty$  or worse, different methods are required.

There are now many different approaches to the proof of this integrability theorem. We have to construct a diffeomorphism between some neighbourhood  $N \subset M$  and a polydisc  $B \subset \mathbf{C}^n$ . The character of the problem appears rather different depending on whether one seeks to construct a map  $f : N \rightarrow B$  or a map  $g : B \rightarrow N$ . Of course, at the end of the day, once one has shown that the constructed maps are diffeomorphisms, one follows from the other by inversion, but at the outset the problems look different. If we seek a map  $f$ , then the problem is *linear*, the condition is just that  $f$  be a vector of holomorphic functions and the PDE to be satisfied is  $\bar{\partial}_M f = 0$  where  $\bar{\partial}_M$  is the natural Cauchy-Riemann operator defined on  $M$ . In the classical case when  $n = 1$  the problem can easily be solved this way. One takes the neighbourhood  $N$  very small so that, in suitable coordinates  $\bar{\partial}_M$  can be written as a small perturbation of the standard Cauchy-Riemann operator which is essentially just  $\frac{\partial}{\partial \bar{z}}$ . Then a local holomorphic function can be constructed by perturbation methods, using the explicit integral operator inverting  $\frac{\partial}{\partial \bar{z}}$ . The essential difficulty that appears when  $n > 1$  is that the equation  $\bar{\partial}_M f = 0$  is *overdetermined*. Indeed, we have to use the integrability condition in some way, and in fact if this condition does not hold there will typically be *no* nonconstant local solutions of the equation  $\bar{\partial}_M f = 0$ .

Subsequent advances in several complex variables, due to Kohn and Hörmander, do lead to proofs of the integrability theorem via linear theory along the above lines, but the original Newlander-Nirenberg approach takes the other point of view, with the construction of a map  $g : B \rightarrow N \subset M$ . Taking  $n = 2$  for simplicity, we will have  $g(z, w) \in N$  for  $z, w$  in the standard disc  $D \subset \mathbf{C}$ . The condition we want to achieve requires that, for each fixed  $w$  the map  $\gamma_w : D \rightarrow M$  defined by  $\gamma_w(z) = g(z, w)$  is holomorphic, so we seek a family of *holomorphic curves*  $\gamma_w$  in  $M$ , parametrised by  $w$ . From the point of view of the local theory of holomorphic curves, almost-complex manifolds behave much like complex ones—in contrast to the situation with holomorphic functions above. Thus one can produce families of curves  $\gamma_w$  and the integrability condition can be brought in to show that such a family can be constructed so that  $\gamma_w(z)$  is also holomorphic in the  $w$  variable. Related ideas have become important in the context of Gromov's theory applying holomorphic curves in almost-complex manifolds to symplectic topology. One

area in which the Newlander-Nirenberg theorem has been crucial is “twistor theory”, which encodes equations satisfied by a metric on space-time in the integrability condition for an almost-complex structure on twistor space.

Now we turn to the work of Kodaira, Nirenberg, and Spencer [5]. Many readers will be familiar with the simplest example of variation of complex structure—the classification of Riemann surfaces of genus 1 by a discrete quotient of the upper half plane. The Kodaira-Nirenberg-Spencer theory gives a vast extension of this idea to a general compact complex manifold  $M$ . Deformations of the almost-complex structure can be parametrised by certain tensors  $\mu \in \Omega^{0,1}(TM)$ —the differential forms of type  $(0,1)$  with values in the tangent bundle of  $M$ —and the integrability condition takes the form of a first-order nonlinear PDE

$$\bar{\partial}\mu + [\mu, \mu] = 0.$$

The problem of deformation theory is, roughly speaking, to describe the small solutions  $\mu$  of this equation, modulo the action of the diffeomorphism group of the manifold. If we naively linearise the equation about  $\mu = 0$  by dropping the quadratic term, we simply have the equation  $\bar{\partial}\mu = 0$ , which states that  $\mu$  defines a class in  $H^1(TM)$  viewed as the Dolbeault cohomology defined by the complex

$$\Omega^0(TM) \xrightarrow{\bar{\partial}} \Omega^1(TM) \xrightarrow{\bar{\partial}} \Omega^{0,2}(TM) \xrightarrow{\bar{\partial}} \dots$$

Kodaira, Nirenberg, and Spencer considered the case when  $H^2(TM) = 0$ , which is just the statement that for any  $\rho \in \Omega^{0,2}(TM)$  with  $\bar{\partial}\rho = 0$  one can solve the equation  $\bar{\partial}v = \rho$ . They construct deformations parameterised by a neighbourhood of 0 in  $H^1(TM)$  with a universal property that implies that *any* sufficiently small deformation of  $M$  is isomorphic to one in their family. Their method is to construct  $\mu$  as a power series  $\mu = \mu_0 + t\mu_1 + t^2\mu_2 + \dots$  where  $\mu_0$  satisfies  $\bar{\partial}\mu_0 = 0$  and the subsequent terms are found by solving equations of the form  $\bar{\partial}\mu_i = \rho_i$  with  $\rho_i$  determined by  $\mu_0, \dots, \mu_{i-1}$ . The proof was an early application of Hodge theory, which gives a preferred solution to an equation  $\bar{\partial}v = \rho$ , admitting Hölder estimates. The theory was later extended by Kuranishi [6] to cases in which  $H^2(TM)$  does not vanish.

For a very simple example of these ideas, go back to a Riemann surface of genus 1 (a torus). Then  $TM$  is trivial, so the cohomology group  $H^1(TM)$  is the same as  $H^1(\mathcal{O})$ , which is isomorphic to  $\mathbb{C}$ . The theory gives a family of deformations of the complex structure parameterised by a small disc in  $\mathbb{C}$ , and of course this is just what we see in the familiar explicit representation using the upper half plane. A striking example of the power of the theory comes in the case of K3 surfaces. Here one gets certain “visible” deformations arising from algebraic geometry. For example, if we consider the family of K3 surfaces

defined by smooth surfaces of degree 4 in  $\mathbb{CP}^3$ , then one finds a nineteen-dimensional family of deformations (the quartic polynomial has thirty-five coefficients and the linear group has dimension sixteen). On the other hand, calculation shows that  $\dim H^1(TM) = 20$  and the theory tells us that there is actually a twenty-dimensional family of deformations. This is an analytical statement—the generic member is not an algebraic surface at all, and the picture is inaccessible from algebraic geometry.

## Analysis

One of Nirenberg’s most famous achievements is his introduction, with F. John, of the function space  $BMO$  [4]. The  $L^p$  norms of functions are a familiar tool in PDE theory and analysis generally, but often the information one has is limited to some fixed value of  $p$ . For example, the  $L^2$  norm of the derivative of a function appears as the Dirichlet integral in the theory of harmonic functions and harmonic maps. Rather than varying the exponent, one can vary the domain of the integrals considered. The Morrey spaces  $M_p$  on  $\mathbb{R}^n$  are defined by the finiteness of the norm

$$\|\phi\|_{M_p} = \sup_B \frac{1}{|B|^{1-1/p}} \int_B |\phi|.$$

Here  $B$  runs over the set of balls in  $\mathbb{R}^n$ , and we write  $|B|$  for the volume. Hölder’s inequality implies that  $L^p \subset M_p$ , and in fact many results about  $L^p$  extend to  $M_p$ . Thus one can get good information about a function  $f$  from consideration of the  $M_p$  norm of  $|\nabla f|^2$ —which is defined by the Dirichlet integral over balls—as opposed to the less accessible  $L^{2p}$  norm of the derivative.

The Morrey norm when  $p = \infty$  reduces to the  $L^\infty$  norm. John and Nirenberg’s  $BMO$  (bounded mean oscillation) norm is somewhat different:

$$\|\phi\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |\phi - \phi_B|,$$

where  $\phi_B$  is the mean value of  $\phi$  over  $B$ . A basic example of an unbounded function in  $BMO$  is  $\log|x|$  on  $\mathbb{R}$ . The famous John-Nirenberg inequality fills out the idea that  $BMO$  is “slightly larger” than  $L^\infty$  but smaller than any  $L^p$ . For a function  $\phi$  supported on the unit ball  $B \subset \mathbb{R}^n$ , with integral zero and with  $BMO$  norm 1 the inequality gives a fixed bound on the integral of  $e^{a\phi}$  for a certain definite value of  $a$ , depending on  $n$ . The proof uses the Calderón-Zygmund cube decomposition. It is related to the fact, discovered subsequently by Fefferman, that  $BMO$  is the dual of the Hardy space  $H^1$ . Indeed, a function in  $H^1$  has an “atomic decomposition” [11]  $f = \sum \lambda_i f_i$  (a.e.) where  $\lambda_i \in \mathbb{R}$ ,  $\sum |\lambda_i| < \infty$ , each function  $f_i$  is supported on a ball  $B_i$  has integral zero and is bounded in modulus by  $|B_i|^{-1}$ . Given this representation (which is related to wavelet expansions) it is clear, at least, that if  $\phi$  is in  $BMO$ , then the integral  $\int \phi f$  is well

defined, and this leads to the duality between the two spaces. Many fundamental results of harmonic analysis involving  $L^p$  spaces fail for the extremes  $p = 1, \infty$ , and the pair  $BMO, H^1$  provide the correct substitutes. For example, if a harmonic function on a half-plane has normal derivative on the boundary in  $L^\infty$ , then the tangential derivative need not be in  $L^\infty$  but it must be in  $BMO$ . In PDE theory the John-Nirenberg inequality often appears through a Sobolev-type estimate for functions at the critical exponent  $n/(n-1)$ . A compactly supported function  $f$  with derivative in  $L^{n/(n-1)}$  is not continuous, but it does lie in  $BMO$  and hence satisfies the corresponding exponential integral estimate.

### General PDE Theory

It is ridiculous to attempt to describe Nirenberg's massive contribution to PDE theory in the few lines here. We are looking at a period of more than a half century in which the literature on even, say, elliptic second-order equations runs to many thousands of pages and whose intricate developments are covered in texts such as [2]. The awesome number of citations to Nirenberg's papers is one measure of the central nature of his contributions to this huge field. Nevertheless, let us try to pick out some strands.

As a model for a nonlinear PDE consider the Monge-Ampère equation

$$\det \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right) = e^p,$$

where  $u$  is to be convex. Linear theory enters when one considers the equation satisfied by a derivative  $h = \frac{\partial u}{\partial x_k}$  which is  $L(h) = \rho_k$ , where  $L$  is the linear operator

$$L(h) = \sum a^{ij} \frac{\partial^2 h}{\partial x_i \partial x_j},$$

( $a^{ij}$ ) being the inverse matrix of the Hessian  $\frac{\partial^2 u}{\partial x_i \partial x_j}$ . The idea is that a good theory of linear operators with suitably general coefficients  $a^{ij}$  will allow bootstrapping to derive improved information about the derivatives. One important issue is the *uniform ellipticity* of the linear equation: a bound on the ratio of the maximum and minimum eigenvalues of ( $a^{ij}$ ). Another is the *continuity* of the coefficients  $a^{ij}$ . If these satisfy a fixed modulus of continuity, then the linear operator  $L$  can be approximated by a constant coefficient operator on balls of a fixed size, and the situation is rather well controlled.

The two-dimensional case is special because an operator  $L$  can be related to a generalised Cauchy-Riemann operator and thus to the theory of quasiconformal maps. We have mentioned in the first section above Nirenberg's early work [9] deriving regularity theorems, using this approach, for a very general class of elliptic PDE in two dimensions (related to work of Morrey). Here the crucial point is to show that ( $a^{ij}$ ) uniformly elliptic

implies a  $C^{1,\alpha}$  bound on  $\frac{\partial u}{\partial x_k}$  and hence a Hölder ( $C^\alpha$ ) modulus of continuity of  $a^{ij}$ . A renowned development in the years around 1960, due to de Giorgi, Nash, and Moser, was a general regularity theory for "quasilinear" elliptic equations in higher dimensions. The John-Nirenberg inequality was applied by Moser [7] to (quoting from [2]) "bridge a vital gap" in the proof of the fundamental Harnack inequality in the relevant linear theory. A general theory for fully nonlinear equations in higher dimensions, such as the Monge-Ampère equation, came later with the work of Caffarelli, Nirenberg, and Spruck [1]. This concerns the Dirichlet problem, where  $u$  is defined on a convex set  $\Omega \subset \mathbf{R}^n$  with prescribed value on the boundary. The crucial issue turns out to be deriving a modulus of continuity for the second derivatives of  $u$  along the boundary. Caffarelli, Nirenberg, and Spruck obtained a logarithmic modulus of continuity by an extremely delicate application of the maximum principle (construction of a barrier function). This crucial logarithmic bound, finer than any Hölder estimate, has some relation, in spirit and content (involving the relation between tangential and normal derivatives), with the ideas surrounding BMO.

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# Presidential Views: Interview with Eric Friedlander

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. What follows is an edited version of an interview with Eric M. Friedlander, whose two-year term as president begins on February 1, 2011. Friedlander is Dean's Professor of Mathematics at the University of Southern California. The interview was conducted in fall 2010 by *Notices* senior writer and deputy editor Allyn Jackson.

An interview with past president George Andrews appeared in the February 2011 issue of the *Notices*.

**Notices:** *Today young mathematicians, and in fact professionals in many different areas, are not joining professional societies in the numbers that they used to. Can you comment on this?*

**Friedlander:** In the past few months, topics related to this issue have been concerning me, and I am trying to suggest some changes that might help the AMS in this regard. At the moment the AMS has many Nominee Members—graduate students who receive free membership. A lot of them don't even know they are members. They are often puzzled by the arrival of the *Notices* and the *Bulletin* in their mailboxes. This Nominee Program should do more to sustain our pipeline. We have a working group that is examining the program and will suggest some changes.

I think the most important thing that's on the table, but that hasn't been approved and will require some funding, is putting into place a program of student chapters. SIAM [Society for Industrial and Applied Mathematics] already has such a program. The idea would be that a graduate program could apply to start a student chapter, and the AMS would provide a small amount of funding to invite speakers and maybe for student travel. This would be, of course, an expenditure by the AMS, but I think it would be well worth it. We could also identify students in graduate programs to represent some of the things the AMS is interested in. So, for example, rather than reaching students through mass mailings or relying on them to look at certain blogs, we would have contacts who could encourage other students to participate in various activities. They might provide a way to digest and personalize some of the information that is available about how to apply for jobs, how to write a thesis, where to submit papers, things like that. So part of the answer is to engage the newest generation directly in activities that are in their own interest and to do it through the AMS.

**Notices:** *Publishing and communication in mathematics, and in the sciences more generally, is changing rapidly. What do you see as the main issues in publishing that the AMS needs to address?*

**Friedlander:** Most of the financial activity of the AMS is involved with publishing. *Math Reviews* is the AMS's most vital publication, and it seems to be thriving. It is the one that produces the most revenue and also costs the most. There is also the AMS book program, which is good but is not one of the big players—the AMS is not doing things like producing calculus books. We're about to institute e-books, and it is not so evident what is the future of books versus e-books, especially in a subject like mathematics. But there is a future for mathematics books in print, and I think the AMS will try hard to play a significant role in that future. Continuing the AMS book program requires a lot of participation by members. For example, I am on the editorial boards for two of the AMS book series, and that takes time.

This brings me to journals. The AMS has anticipated, or feared, a loss of journal subscriptions for many years. And that never really happened. Journal subscriptions have been fairly stable, despite the changes in journal publishing brought about by electronic communications. It costs almost as much to produce an electronic journal as a print journal. The AMS will continue to produce journals in a traditional way, at least for the foreseeable future. But it is harder to see what that future is. There are at least two things that can change. One is the possibility of introducing a more interactive aspect to journal publication, so that readers can comment on articles—sort of like a blog for each



Eric Friedlander

Photograph by Tom Stio Photography.



article. That's not something that's happening now in journals. A second influence on the future is mathematicians' concern about the cost of journals. There is quite a bit of disquiet prompted by lack of funds in university library budgets. Some mathematicians are starting up journals on their own, and others are moving journals from one publisher to another to keep prices down. The AMS is one of the less expensive publishers, probably the least expensive of the mainstream publishers. But I can see that many journals might not thrive in the future, and with financial pressures, the result will be a couple of very large publishers bundling many journals. Some of us view bundling as an unfair marketing trick: You bundle journals that are good with journals that are not as good, and the university has to buy the whole bundle.

**Notices:** *How well is mathematics research funded in the United States? What role can the AMS play in ensuring that funding is adequate?*

**Friedlander:** It's very hard to find the right frame of reference to answer this question. The United States does have some funding for mathematical research, which I think is extremely important. Crass as it may seem, I think many people go into mathematics because there is the chance of some funding and some recognition, and they continue because of this recognition and funding. The National Science Foundation [NSF] is essentially the only government agency that does large-scale funding in mathematics, and the AMS has very little influence with the NSF. The individual mathematics program directors, who deal directly with mathematicians, are accessible and seem to have their funding priorities similar to ours. But the political pressure at the top of the NSF creates some disconnect. So many of us aren't very pleased with the directions of funding, but there is very little it seems we can do to influence it.

**Notices:** *What directions are you talking about specifically?*

**Friedlander:** One thing some of us tried to achieve was to increase the number of grants and make them smaller. Others argued that a) this was expensive, b) it looked bad compared to the other sciences to say that mathematics could be funded cheaply, and c) the mathematics that is really excellent should be fully funded, whatever "fully funded" meant. Many from the mathematical community wanted a more widespread funding mechanism, but the NSF said this wasn't going to be the case. Possibly the NSF reacted constructively, by creating more NSF institutes to support more people in that way, rather than directly funding principal investigators. Outside the NSF, a recent positive development is the new small grants program launched by the Simons Foundation [see "New Program at the Simons Foundation", *Notices*, November 2010, page 1324].

Also, one feels that the NSF is putting more emphasis on applications of mathematics and interdisciplinarity and is always seeking new ways to spend its money. Perhaps mathematics is somewhat conservative, in that many of us like to go off on our own and just do mathematics, maybe with a couple of other people. We are not working on some new, hot program. I am really speaking here more from the point of view of the mathematical community than from the point of view of the AMS. There is a great sense that we are very fortunate that the NSF supports us. There is also a great sense that we need a lot more money.

**Notices:** *Raising public awareness of mathematics is an important goal of the AMS. Perhaps things have improved a bit, with movies like A Beautiful Mind and TV shows like NUMB3RS. How well do you think the general public understands the field?*

**Friedlander:** A mathematician told me the following anecdote. A college student had to take the average of 1.2 and 1.2. He got out his calculator and plugged in 1.2 plus 1.2 and divided by 2, and got 1.8. He didn't believe the mathematician that the average of 1.2 and 1.2 is 1.2. So finally the mathematician showed the student how to use the calculator by putting in parentheses. And then of course the student was convinced.

I'm not sure the general public does have much basic sense for numbers and mathematics. One thing that disturbs me is that some mathematicians despair of conveying any sense of mathematics, which is very sad. I think essentially everyone can get some sense of what mathematics is and how to use it and what it is important for. The AMS can do some good by raising the visibility of mathematics and helping in education. The AMS does reach out to a broad audience, for example, on its website, which is improving and has some information about what is happening in mathematics. I think the best thing the AMS does by far is the *Notices*. It could easily be in all doctors' and dentists' offices.

**Notices:** *Really?*

**Friedlander:** It could be. The *Notices* is great. Maybe I am biased because my wife Susan is a long-term participant in the *Notices*, although she is now the editor of the *Bulletin* and less involved in the *Notices*. The AMS also produces educational materials such as Mathematical Moments, it has a Washington presence, it has people in its Public Awareness Office trying to put out the word about prizes, the Math Olympiad, things like that. All these things help. But I think the general lack of numeracy in society is an overwhelming problem.

**Notices:** *This brings us to mathematics education, which in the United States is a big, complicated enterprise with lots of players. How do you see the role of the AMS here?*

**Friedlander:** I just came back from a meeting of the Committee on Education. It was the first time

I had attended. I don't know enough to know who is playing on which side of the "Math Wars". I am really an innocent, and I was trying to understand some of the controversy. I made a presentation at the meeting saying that I would like the AMS, quite possibly through a working subgroup of the Committee on Education, to increase attention toward graduate education. That is the aspect of education I know best, and I think it is certainly an aspect the AMS could address better than other organizations. I am not saying the AMS should withdraw or put less effort into K-12 education, but I would like to direct some AMS effort into Ph.D. programs especially, and also Master's programs. This ties in with our first topic of graduate student mentors. There are things the AMS can do to help graduate programs and help training of Ph.D. and master's degree students. It is past time the AMS got more involved in such matters.

**Notices:** *What other specific projects do you want to work on as president of the AMS?*

**Friedlander:** We have for some years been talking about having more prizes. I don't have data on this, but many of us feel strongly that mathematicians are not very generous to each other in terms of giving out prizes, compared to other sciences. The AMS has increased the number of prizes somewhat, but we could do more to recognize good mathematics.

Related to this is the AMS Fellows program, which would recognize outstanding achievement in mathematics by naming a certain number of members as Fellows. SIAM has launched a fellows program, and various prestigious mathematicians—including former AMS president Jim Glimm—immediately became fellows of SIAM and were greatly recognized by their universities. An AMS Fellows program would stimulate the same kind of recognition within our universities and garner some credit for mathematics, and we would be honoring the good performance and quality of a select group—though not too select—of mathematicians. I think to recognize and boost mathematics any way we can is very important. Giving out prizes is one way, but that reaches a very small group; the Fellows program would reach a larger group. I suspect the Fellows program will be enacted soon, and I will be glad to help it go forward.

Another issue I am interested in—though I don't know how to confront it—is the global economic situation and its impact on the job market for mathematicians. The AMS is trying to help in some ways, but it is not clear exactly what it can or should do. The AMS has been trying to help young mathematicians get good employment, not necessarily in academia but in places that utilize their mathematical knowledge and talents. The problem is very high on our list of priorities, but it's an overwhelming problem that is far too big for the AMS to solve.

I would like to make one last comment. It's difficult to know how to state this exactly. I would like to think the AMS's first, primary role is to support and promote research mathematics. We have discussed many things today, and this is the first time research mathematics has even been mentioned. I am very interested in somehow raising the involvement of the AMS in research mathematics. This translates into prizes, it translates into more involvement in publications, it translates into the AMS Web presence. For example, I and others made suggestions for changes to the AMS website, to emphasize aspects that are at a higher mathematical level. This is a good thing. The AMS should have research mathematics as its central focus.

## About the Cover

### Martin Gardner 1914 - 2010

The cover accompanies the memorial article for Martin Gardner in this issue. According to the records available from the Stanford archive of Gardner's papers (which you can locate by running "Stanford archives Martin Gardner" through Google), his principal correspondent was Don Knuth (followed by Solomon Golomb), so when we contemplated using one of Gardner's letters as the basis of the cover we asked Knuth what he could suggest. He very generously spent quite a bit of time and effort in producing the image we have used. Knuth tells us:

I doubt that I received more mail from Martin than anybody else. Possibly Stan Isaacs listed me more often in the database, when he did the first pass over Martin's papers, because he knows me and we see each other fairly often. He didn't have time to index everything.

Anyway, I do have copies of more than 100 letters that I wrote to Martin since 1970, and most of them start with "Thanks for your letter"! Looking through the files just now, I found one from him dated "8 Jan 76", when he was in full stride, and I think ... it perfectly reveals his working style and, in addition, refers to many other people.

When you see it, you'll know why I forwarded a copy of it at the time to John McCarthy and Patrick Suppes and also to Karel de Leeuw (who as you probably know was shockingly murdered two-and-a-half years later).

We are extremely grateful to Knuth for his efforts, and also to Martin's son Jim Gardner for help in locating photographs of his father for our use in the memorial article as well as for permission to reproduce the cover image.

—Bill Casselman  
Graphics Editor  
([notices-covers@ams.org](mailto:notices-covers@ams.org))

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# Mathematics People

## Lewis Awarded CRM-Fields-PIMS Prize

MARK LEWIS of the University of Alberta has been awarded the 2011 CRM-Fields-PIMS Prize by the Centre de Recherches Mathématiques (CRM), the Fields Institute, and the Pacific Institute for the Mathematical Sciences (PIMS). According to the prize citation, Lewis's research "involves mathematical modeling of biological processes and is an example of the best interplay of science and mathematics, where ideas from each lead to advances in the other." His work "develops techniques in stochastic processes, dynamical systems and partial differential equations and has led to significant advances, for example, in modeling territorial pattern formation in wolf populations, predicting population spread in biological invasions like the West Nile virus, and assessing the effect of habitat fragmentation on species survival."

The CRM-Fields-PIMS Prize is awarded annually for research achievements in the mathematical sciences. The winner's research should have been conducted primarily in Canada or in affiliation with a Canadian university.

—From a CRM-Fields-PIMS announcement

## 2010 ICTP Prize Awarded

SHIRAZ MINWALLA of the Tata Institute of Fundamental Research has been awarded the 2010 ICTP Prize of the Abdus Salam International Centre for Theoretical Physics (ICTP) for his work in theoretical physics and mathematics. According to the prize citation, Minwalla was honored "for his many outstanding contributions in the field of string theory and gauge/gravity duality" that have greatly influenced research in many areas. His contributions include work in ultraviolet/infrared mixing in noncommutative field theories, noncommutative solitons, gauge/string theory duality in the Penrose limit, closed string tachyon condensation, Hagedorn-like phase transition in weakly

coupled Yang-Mills theories, relating plasma balls in large  $N$ -gauge theories to black holes in the gravity duals, and, most recently, obtaining equations of nonlinear fluid dynamics in  $3+1$  dimensions from Einstein's equation for black-branes in 5-dimensional anti-de Sitter space, as well as a study of weak-field black hole formation in asymptotically anti-de Sitter space-times and its relation to the thermalization process in the dual conformal field theory.

The prize recognizes outstanding and original contributions in physics and mathematics by scientists under forty years old from developing countries. It includes a sculpture, a certificate, and a cash award of 3,000 euros (approximately US\$4,000).

—From an ICTP announcement

## Gualtieri and Tang Awarded Lichnerowicz Prize

MARCO GUALTIERI of the University of Toronto and XIANG TANG of Washington University in St. Louis were awarded the 2010 André Lichnerowicz Prize for notable contributions to Poisson geometry. The prize is awarded every two years at the International Conference on Poisson Geometry in Mathematics and Physics to researchers who have completed their doctorates at most eight years before the year of the conference.

Gualtieri has conducted pioneering work on generalized geometry. His Ph.D. thesis developed the basic structure theory of generalized complex geometry, as well as of generalized Kähler geometry. He has done research in generalized geometry and its applications to physics on D-branes in generalized complex manifolds and their relation to noncommutative geometry, as well as further generalizations of classical geometries.

Tang's work has involved index theorems on singular spaces using the tools of noncommutative geometry (cyclic cohomology,  $K$ -theory, the general index theorems of Connes-Moscovici and Nest-Tsygan) in combination with

algebraic and geometric structures arising from Poisson geometry. His contributions include a new proof of the Atiyah-Weinstein conjecture on the index of Fourier integral operators and the relative index of CR structures, the study of noncommutative Poisson structures on orbifolds, the study of various Hopf-like structures, and the index theory on orbifolds.

The prize was named in memory of André Lichnerowicz (1915–1998), whose work was fundamental in establishing Poisson geometry as a branch of mathematics.

—Elaine Kehoe

## Coates Receives Leverhulme Prize

THOMAS COATES of Imperial College, London, has been awarded a Philip Leverhulme Prize by the Leverhulme Trust. His work involves the study of the geometry of manifolds, or curved spaces, using techniques inspired by theoretical physics. The prize consists of a research grant of 70,000 pounds (approximately US\$109,000). The prize money may be used in any way that enhances the recipient's research over a two- to three-year period.

—From a Leverhulme Trust announcement

## Montalban Awarded Packard Fellowship

ANTONIO MONTALBAN of the University of Chicago has been awarded a Packard Fellowship by the David and Lucile Packard Foundation. The fellowship will allow him to analyze the notion of robust construction or robust complexity level, using computability theory and proof theory, in order to address important questions on the foundations of mathematics.

—From a Packard Foundation announcement

## Dillon Awarded Marshall Sherfield Fellowship

JOSHUA DILLON of the Georgia Institute of Technology has been awarded a two-year Marshall Sherfield Fellowship at the University of Cambridge. Dillon is a doctoral student in computational science and engineering who works in machine learning, computational statistics, and information visualization. At Cambridge he will work in the department of engineering. The fellowships, of which two are awarded each year, are funded by the Marshall Sherfield Fellowship Foundation and administered by the Marshall Commission; they enable American scientists or engineers to undertake postdoctoral research for a period

of one to two academic years at a British university or research institute.

—Elaine Kehoe

## 2010 Siemens Competition

Several high school students whose work involves the mathematical sciences have won prizes in the Siemens Competition in Math, Science, and Technology.

BENJAMIN CLARK, a fifteen-year-old senior at Penn Manor High School in Millersville, Pennsylvania, won the US\$100,000 grand prize scholarship in the individual category for his project, "The Close Binary Fraction: A Bayesian Analysis of SDSS M Dwarf Spectra". The detailed process by which stars form is a major unanswered question in astrophysics. Although many theories have been proposed, it has been difficult to determine which is most likely. One method used to differentiate between theories is the predicted frequency of binary stars. In his project, Clark used data from the Sloan Digital Sky Survey (SDSS) to look for systems with two M stars orbiting each other and determined the close binary fraction of M stars. By showing that observational data does not support one of the main theories of star formation, Clark's research is a step toward a more thorough understanding of how stars form. Clark is a National Merit semifinalist, a Model United Nations head delegate, and a member of the National Honor Society. He is active in the Boy Scouts of America and has participated in the USA Mathematical Olympiad, the USA Physics Olympiad, the Princeton University Mathematics Competition, and the Pennsylvania Math League. He plans to major in physics/astrophysics and pursue a career at a major research institution.

ALLEN YUAN, a senior at Detroit Country Day School, Beverly Hills, Michigan, received a US\$30,000 scholarship for his project, "Linearly Many Faults in  $(n, k)$ -Star Graphs". His work uses graph theory to explore parallel computing networks. He examines a network's ability to function with the presence of faults due to broken computers. His project may help overcome network challenges that arise from faults. He also creates alternative network structures, which he establishes as viable structures for future parallel computing networks. He is an accomplished pianist and won first prize in the Eastman International Piano Competition. In his free time, he plays recitals at local senior centers and enjoys playing tennis. He plans to study mathematics and piano performance in college and hopes to become a mathematics professor.

The team of JEFFREY SHEN (Park Tudor School, Indianapolis, Indiana) and YOUNG HONG and LYNDON JI (Carmel High School, Carmel, Indiana) were awarded US\$50,000 scholarships for their project, "A Study of Nearest Neighbor Distances on a Circle: Multidimensional Case". They take a mathematical approach to the fundamental problem in quantum theory of the quantum harmonic oscillator, the adaptation to an atomic scale of the classical harmonic oscillator, which describes the movement of a weight on a spring. The motions of both



of these oscillators are controlled by a physical parameter, the spring constant. The team studied the behavior of the oscillator with spring constants forming a mathematically special set of numbers, called badly approximable. Their results provide deeper insight into the behavior of the quantum harmonic oscillator. Shen, a senior, is a National AP Scholar who has participated in many math and science competitions, including the U.S. Mathematical, Physics, and Computing Olympiads. He is a 2010 Research Science Institute scholar, winner of the 2010 Rensselaer Medal, a math tutor, and an active member of his church youth group. He is also an accomplished pianist and two-time gold medalist in the Indiana State Piano Competition. He plans to study mathematics and computer science and hopes to become a professor and research scientist. Homma, a junior, is an AP Scholar, a member of student government, and vice president of the Math Club. He is fluent in Japanese and plays the flute in his school wind ensemble. He has participated in MathCounts, the USA Mathematics Olympiad, and the Mathematical Olympiad Summer Program. He plans to study mathematics and physics in college. Ji is a junior whose favorite subjects are economics and physics. He has participated in MathCounts and the USA Mathematics Olympiad and enjoys Ping-Pong. He plays the piano and violin and is a member of his school orchestra and Share the Music club.

The team of JAMES PINKERTON and RAFAEL SETRA of Montgomery Blair High School, Silver Spring, Maryland, was awarded US\$40,000 scholarships for their project, "The Duplicator-Spoiler Game for an Ordinal Number of Turns". They investigated duplicator-spoiler games, a method of comparing the properties of structures defined by base sets and relations. They chose to investigate these games because of their connection to logic and logical expressibility. In their project, the game length was delayed to allow more freedom to distinguish structures, giving previously dead-end games new interest. Their work could be applicable to graph theory and other areas of mathematics, as well as to comparisons between different computer programs to determine their equivalency. Pinkerton, a senior, is the president of the Chess Club and a member of the National Honor Society and French Honor Society. He single-sculls on the Potomac and plays chess and Go competitively. He teaches chess as a volunteer in several programs in his county and in inner-city Washington, DC. He also teaches mathematics to underclassmen. He credits his father, who taught him "fun mathematics, not the dreary algebra of secondary school", with nurturing his love for the subject. He would like to study mathematics in college and to become a university professor. Setra, a senior, was born in Sao Paulo, Brazil, and moved to the United States when he was eight years old. He speaks Portuguese and Spanish. He participates in Operation Fly (a student-led organization that serves the inner-city population), is a member of the National Honor Society and the Martial Arts Club, and volunteers at Viers Mill Elementary School. He plays Starcraft 2 and noncompetitive football. He would like to study mathematics, engineering, and computer science and to become a college professor.

The team of SITAN (STAN) CHEN of Northview High School, Johns Creek, Georgia, and TIANQI (TIM) WU of Parkview High School, Lilburn, Georgia, received US\$20,000 scholarships for their project, "Cellular Automata to More Efficiently Compute the Collatz Map". They explored the Collatz Conjecture, a famous unsolved mathematical claim regarded by mathematicians as deceptively simple looking yet notoriously difficult. They addressed the problem in an unconventional manner by using computer programs to simulate its mathematical processes. Their research has potential applications in exposing flaws in the security of Internet-based financial transactions and may provide solutions to major open problems in number theory. Chen, a junior, is an accomplished pianist and violinist who has performed twice at Carnegie Hall. He enjoys fencing and volunteering at his local library and has won awards at the Georgia Science and Engineering Fair and the FBLA State and National Leadership Conferences. He plans to study mathematics, aerospace engineering, and music in college and aspires to become a university professor. Wu, a senior, is president of his school math team and ranks first in his class of 575 students. He was born in Shanghai, China, where he lived for fourteen years before coming to the United States. He enjoys Choi Kwang Do martial arts, discussing philosophy, and reading fantasy fiction. He plans to major in mathematics and would like to become a research mathematician.

—From a Siemens Competition announcement

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# Mathematics Opportunities

## AMS-Simons Travel Grants for Early-Career Mathematicians

With support provided by the Simons Foundation, the AMS is launching a new program, the AMS-Simons Travel Grants. Each grant will provide an early-career mathematician with \$2,000 per year for two years to reimburse travel expenses related to research. A mentor will help guide the awardee through the two years of the grant. The department of the awardee will also receive a small amount of funding to help enhance the research atmosphere of the department. A selection committee composed of research mathematicians will choose individuals who are not more than four years past the completion of the Ph.D. One application cycle will be conducted in each of the next three years (2011, 2012, and 2013), and sixty new awards will be made each year.

The first round of applications will be accepted in the spring of 2011. The deadline is **March 31, 2011**. Applicants must be located in the United States or be U.S. citizens. For complete details of eligibility and application instructions, visit: <http://www.ams.org/programs/travel-grants/AMS-SimonsTG>.

—AMS announcement

## Call for Nominations for Prizes of the Academy of Sciences for the Developing World

The Academy of Sciences for the Developing World (TWAS) prizes will be awarded to individual scientists in developing countries in recognition of outstanding contributions to knowledge in eight fields of science.

Eight awards are given each year in the fields of mathematics, medical sciences, biology, chemistry, physics, agricultural sciences, earth sciences, and engineering sciences. Each award consists of a prize of US\$15,000 and a plaque. Candidates for the awards must be scientists who have been working and living in a developing country for at least ten years.

The deadline for nominations for the 2011 prizes is **March 31, 2011**. Nomination forms should be sent to: TWAS Prizes, International Centre for Theoretical Physics (ICTP) Campus, Strada Costiera 11, 1-34151 Trieste, Italy; fax: 39 040 2240 7387; email: [prizes@twas.org](mailto:prizes@twas.org). Further information is available on the World Wide Web at <http://www.twas.org/>.

—From a TWAS announcement

## News from the CRM

The Centre de Recerca Matemàtica (CRM) in Bellaterra, Spain, has several programs planned for the year 2011. All will be held at CRM. Titles, dates, and scientific committee members are listed below.

**March 14–19, 2011:** Short Courses and Workshop on Spectral Functions. *Scientific Committee:* Hakan Hedenmalm, Nikolai Makarov, Joaquim Ortega-Cerdà, Mikhail Sodin. Information: <http://www.crm.cat/wkspectral>.

**April 6–8, 2011:** Workshop on Dynamics and  $C^*$ -Algebras. *Scientific Committee:* Nathaniel P. Brown, Joachim Cuntz, Marius Dardarlat, George A. Elliott, Mikael Rordam, Andrew S. Toms. Information: <http://www.crm.cat/wkalgebras>.

**May 30–June 4, 2011:** Short Courses and Workshop on Hilbert Spaces of Entire Functions and Spectral Theory of Self-Adjoint Differential Operators. *Scientific Committee:* Hakan Hedenmalm, Nikolai Makarov, Joaquim Ortega-Cerdà, Mikhail Sodin. Information: <http://www.crm.cat/wkentire>.

**June 6–10, 2011:** Conference on Structure and Classification of  $C^*$ -Algebras. *Scientific Committee:* Nathaniel P. Brown, Joachim Cuntz, Marius Dardarlat, George A. Elliott, Mikael Rordam, Andrew S. Toms. Information: <http://www.crm.cat/calgebras>.

**July 11–15, 2011:** Exploratory Workshop on Emerging Infectious Diseases and Mathematical Modeling. *Scientific Committee:* Bedr'Eddine Ainseba, Tomás Alarcón, Glenn F. Webb. Information: <http://www.crm.cat/diseases>.

—From a CRM announcement

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# Inside the AMS

## From the AMS Public Awareness Office



### Highlights of the 2011 Joint Mathematics Meetings

Approximately 6,000 mathematicians attended the 2011 Joint Mathematics Meetings, January 6–9. More than 2,000 papers—a record number—were presented at the meeting, which took place in New Orleans, Louisiana. In addition, there were poster presentations by undergraduate students, several sessions on improving math education, the second national contest of *Who Wants to Be a Mathematician*, receptions, a screening of *Between the Folds*, the annual Mathematical Art Exhibition, and the 2011 Prize Ceremony. See photographs and write-ups about JMM 2011 at <http://www.ams.org/meetings/national/jmm11-highlights.html>.

—Annette Emerson and Mike Breen  
AMS Public Awareness Officers  
[paoffice@ams.org](mailto:paoffice@ams.org)

## Deaths of AMS Members

BOGDAN BAISHANSKI, professor, Ohio State University, died on August 16, 2010. Born on January 1, 1930, he was a member of the Society for 16 years.

GEORGE EPSTEIN, of Plano, Texas, died on November 19, 2010. Born on July 4, 1934, he was a member of the Society for 51 years.

L. TERRELL GARDNER, professor, University of Toronto, died on December 22, 2010. Born on September 22, 1926, he was a member of the Society for 51 years.

JACK K. HALE, professor, Georgia Institute of Technology, died on December 9, 2009. Born on October 3, 1928, he was a member of the Society for 57 years.

SAMUEL KAPLAN, of Marshfield, Wisconsin, died on October 10, 2010. Born in September, 1916, he was a member of the Society for 71 years.

JOSEPH B. KRUSKAL, of Maplewood, New Jersey, died on September 19, 2010. Born on January 29, 1928, he was a member of the Society for 58 years.

MICHEL LAZARUS, of Paris, France, died in September, 2010. Born on February 21, 1945, he was a member of the Society for 20 years.

CHARLES J. PARRY, professor, Virginia Polytech Institute and State University, died on December 25, 2010. Born in 1942, he was a member of the Society for 39 years.

ANNA RYCERZ, of Cracow, Poland, died on November 7, 2010. Born on September 24, 1948, she was a member of the Society for 18 years.

KEITH W. SCHRADER, of Columbia, Missouri, died on December 27, 2010. Born on April 22, 1938, he was a member of the Society for 45 years.

JOHN F. WILKINSON, of San Jose, California, died on August 16, 2010. Born on July 13, 1940, he was a member of the Society for 17 years.

JONAS STASYS ZMUIDZINAS, of Glendale, California, died on April 23, 2010. Born on June 29, 1931, he was a member of the Society for 7 years.

# Reference and Book List

The **Reference** section of the *Notices* is intended to provide the reader with frequently sought information in an easily accessible manner. New information is printed as it becomes available and is referenced after the first printing. As soon as information is updated or otherwise changed, it will be noted in this section.

## Contacting the *Notices*

The preferred method for contacting the *Notices* is electronic mail. The editor is the person to whom to send articles and letters for consideration. Articles include feature articles, memorial articles, communications, opinion pieces, and book reviews. The editor is also the person to whom to send news of unusual interest about other people's mathematics research.

The managing editor is the person to whom to send items for "Mathematics People", "Mathematics Opportunities", "For Your Information", "Reference and Book List", and "Mathematics Calendar". Requests for permissions, as well as all other inquiries, go to the managing editor.

The electronic-mail addresses are [notices@math.wustl.edu](mailto:notices@math.wustl.edu) in the case of the editor and [ams.org](mailto:ams.org) in the case of the managing editor. The fax numbers are 314-935-6839 for the editor and 401-331-3842 for the managing editor. Postal addresses may be found in the masthead.

## Upcoming Deadlines

**February 21, 2011:** Applications for EDGE for Women Summer Program. See [http://www.edgeforwomen.org/?page\\_id=5](http://www.edgeforwomen.org/?page_id=5).

**February 27, 2011:** Entries for Association for Women in Mathematics (AWM) Essay Contest. See <http://www.awm-math.org/biographies/contest.html>.

**March 1, 2011:** Applications for Summer Program for Women in Mathematics (SPWM2011). Contact the director, Murli M. Gupta, email: [mmg@gwu.edu](mailto:mmg@gwu.edu); telephone: 202-994-4857;

or visit the program's website at <http://www.gwu.edu/~spwm/>.

**March 31, 2011:** Applications for AMS-Simons Travel Grants for Early-Career Mathematicians. See "Mathematics Opportunities" in this issue.

**March 31, 2011:** Nominations for the Academy of Sciences for the Developing World (TWAS) prizes. See "Mathematics Opportunities" in this issue.

**May 1, 2011:** Applications for May review for National Academies Research Associateship Programs. See the National Academies website at [http://sites.nationalacademies.org/PGA/RAP/PGA\\_050491](http://sites.nationalacademies.org/PGA/RAP/PGA_050491) or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email [rap@nas.edu](mailto:rap@nas.edu).

**May 1, 2011:** Applications for National Academies Christine

Mirzayan Graduate Fellowship Program for fall 2011. See <http://sites.nationalacademies.org/PGA/policyfellows/index.htm>.

**May 1, 2011:** Applications for AWM Travel Grants. See <http://www.awm-math.org/travelgrants.html#standard>.

**August 1, 2011:** Applications for August review for National Academies Research Associateship Programs. See the National Academies website at [http://sites.nationalacademies.org/PGA/RAP/PGA\\_050491](http://sites.nationalacademies.org/PGA/RAP/PGA_050491) or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email [rap@nas.edu](mailto:rap@nas.edu).

**October 1, 2011:** Applications for AWM Travel Grants. See <http://www.awm-math.org/travelgrants.html#standard>.

## Where to Find It

A brief index to information that appears in this and previous issues of the *Notices*.

**AMS Bylaws**—November 2009, p. 1320

**AMS Email Addresses**—February 2011, p. 326

**AMS Ethical Guidelines**—June/July 2006, p. 701

**AMS Officers 2008 and 2009 Updates**—May 2010, p. 670

**AMS Officers and Committee Members**—October 2010, p. 1152

**Conference Board of the Mathematical Sciences**—September 2010, p. 1009

**IMU Executive Committee**—December 2010, page 1488

**Information for *Notices* Authors**—June/July 2010, p. 768

**Mathematics Research Institutes Contact Information**—August 2010, p. 894

**National Science Board**—January 2011, p. 77

**New Journals for 2008**—June/July 2009, p. 751

**NRC Board on Mathematical Sciences and Their Applications**—March 2011, p. 482

**NRC Mathematical Sciences Education Board**—April 2010, p. 541

**NSF Mathematical and Physical Sciences Advisory Committee**—February 2011, p. 329

**Program Officers for Federal Funding Agencies**—October 2010, p. 1148 (DoD, DoE); December 2010, page 1488 (NSF Mathematics Education)

**Program Officers for NSF Division of Mathematical Sciences**—November 2010, p. 1328



**October 1, 2011:** Nominations for the 2012 Emanuel and Carol Parzen Prize. Contact Thomas Wehrly, Department of Statistics, 3143 TAMU, Texas A&M University, College Station, Texas 77843-3143.

**November 1, 2011:** Applications for November review for National Academies Research Associateship Programs. See the National Academies website at [http://sites.nationalacademies.org/PGA/RAP/PGA\\_050491](http://sites.nationalacademies.org/PGA/RAP/PGA_050491) or contact Research Associateship Programs, National Research Council, Keck 568, 500 Fifth Street, NW, Washington, DC 20001; telephone 202-334-2760; fax 202-334-2759; email [rap@nas.edu](mailto:rap@nas.edu).

### Board on Mathematical Sciences and Their Applications, National Research Council

The Board on Mathematical Sciences and Their Applications (BMSA) was established in November 1984 to lead activities in the mathematical sciences at the National Research Council (NRC). The mission of BMSA is to support and promote the quality and health of the mathematical sciences and their benefits to the nation. Following are the current BMSA members.

**Tanya Styblo Beder**, SB Consulting Corporation

**Philip Bernstein**, Microsoft Corporation

**Patricia Brennan**, University of Wisconsin

**Emery N. Brown**, Massachusetts Institute of Technology, Harvard Medical School

**Gerald G. Brown**, Naval Postgraduate School

**Ricardo Caballero**, Massachusetts Institute of Technology

**L. Anthony Cox**, Cox Associates, Inc.

**Brenda Dietrich**, IBM Thomas J. Watson Research Center

**Susan Friedlander**, University of Southern California

**Peter Wilcox Jones**, Yale University

**Kenneth L. Judd**, Stanford University

**C. David Levermore** (Chair), University of Maryland

**Charles M. Lucas**, Deer Isle Consulting

**James C. McWilliams**, University of California, Los Angeles

**Vijayan N. Nair**, University of Michigan

**Claudia Neuhauser**, University of Minnesota

**J. Tinsley Oden**, University of Texas at Austin

**Donald Saari**, University of California at Irvine

**J. B. Silvers**, Case Western Reserve University

**George Sugihara**, University of California, San Diego

**Karen L. Vogtmann**, Cornell University

**Bin Yu**, University of California, Berkeley

The postal address for BMSA is: Board on Mathematical Sciences and Their Applications, National Academy of Sciences, Room K974, 500 Fifth Street, NW, Washington, DC 20001; telephone: 202-334-2421; fax: 202-334-2422; email: [bms@nas.edu](mailto:bms@nas.edu); website: [http://sites.nationalacademies.org/DEPS/BMSA/DEPS\\_047709](http://sites.nationalacademies.org/DEPS/BMSA/DEPS_047709).

### Book List

*The Book List highlights books that have mathematical themes and are aimed at a broad audience potentially including mathematicians, students, and the general public. When a book has been reviewed in the Notices, a reference is given to the review. Generally the list will contain only books published within the last two years, though exceptions may be made in cases where current events (e.g., the death of a prominent mathematician, coverage of a certain piece of mathematics in the news) warrant drawing readers' attention to older books. Suggestions for books to include on the list may be sent to [notices-booklist@ams.org](mailto:notices-booklist@ams.org).*

*\*Added to "Book List" since the list's last appearance.*

*Apocalypse When?: Calculating How Long the Human Race Will Survive*, by Willard Wells. Springer Praxis, June 2009. ISBN-13: 978-03870-983-64.

*The Best Writing on Mathematics: 2010*, edited by Mircea Pitici. Princeton University Press, December 2010. ISBN-13: 978-06911-484-10.

*\*The Black Swan: The Impact of the Highly Improbable*, by Nassim Nicholas Taleb. Random House Trade Paperbacks, second edition, May 2010. ISBN-13: 978-08129-738-15. (First edition reviewed in this issue.)

*Bright Boys: The Making of Information Technology*, by Tom Green. A K Peters, April 2010. ISBN-13: 978-1-56881-476-6.

*The Calculus of Friendship: What a Teacher and Student Learned about Life While Corresponding about Math*, by Steven Strogatz. Princeton University Press, August 2009. ISBN-13: 978-0-691-13493-2. (Reviewed June/July 2010.)

*The Cult of Statistical Significance: How the Standard Error Costs Us Jobs, Justice, and Lives*, by Stephen T. Ziliak and Deirdre N. McCloskey. University of Michigan Press, February 2008. ISBN-13: 978-04720-500-79. (Reviewed October 2010.)

*Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics*, by Amir Alexander. Harvard University Press, April 2010. ISBN-13: 978-06740-466-10. (Reviewed November 2010.)

*Euler's Gem: The Polyhedron Formula and the Birth of Topology*, by David S. Richeson. Princeton University Press, September 2008. ISBN-13: 97-80691-1267-77. (Reviewed December 2010.)

*\*The Grand Design*, by Stephen Hawking and Leonard Mlodinow. Bantam, September 2010. ISBN-13: 978-05538-053-76.

*Here's Looking at Euclid: A Surprising Excursion through the Astonishing World of Math*, by Alex Bellos. Free Press, June 2010. ISBN-13: 978-14165-882-52.

*The Housekeeper and the Professor*, by Yoko Ogawa. Picador, February 2009. ISBN-13: 978-03124-278-01. (Reviewed May 2010.)

*How to Read Historical Mathematics*, by Benjamin Wardhaugh. Princeton University Press, March 2010. ISBN-13: 978-06911-401-48.

*Isaac Newton on Mathematical Certainty and Method*, by Niccolò Guicciardini. MIT Press, October 2009. ISBN-13: 978-02620-131-78.

*Logicomix: An Epic Search for Truth*, by Apostolos Doxiadis and Christos Papadimitriou. Bloomsbury USA, September 2009. ISBN-13: 978-15969-145-20. (Reviewed December 2010.)

*The Math Book: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics*, by Clifford A. Pickover. Sterling, September 2009. ISBN-13: 978-14027-579-69.

*A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form*, by Paul Lockhart. Bellevue Literary Press, April 2009. ISBN-978-1-934137-17-8.

*Mathematicians: An Outer View of the Inner World*, by Mariana Cook. Princeton University Press, June 2009. ISBN-13: 978-0-691-13951-7. (Reviewed August 2010.)

*Mathematicians Fleeing from Nazi Germany: Individual Fates and Global Impact*, by Reinhard Siegmund-Schultze. Princeton University Press, July 2009. ISBN-13: 978-0-691-14041-4. (Reviewed November 2010.)

*A Motif of Mathematics: History and Application of the Mediant and the Farey Sequence*, by Scott B. Guthery. Docent Press, September 2010. ISBN-13 978-4538-105-76.

*Mrs. Perkins's Electric Quilt: And Other Intriguing Stories of Mathematical Physics*, Paul J. Nahin, Princeton University Press, August 2009. ISBN-13: 978-06911-354-03.

*Naming Infinity: A True Story of Religious Mysticism and Mathematical Creativity*, by Loren Graham and Jean-Michel Kantor. Belknap Press of Harvard University Press, March 2009. ISBN-13: 978-06740-329-34.

*Nonsense on Stilts: How to Tell Science from Bunk*, by Massimo Pigliucci. University of Chicago Press, May 2010. ISBN-13: 978-02266-678-67.

*Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present*, by George G. Szpiro. Princeton University Press, April 2010. ISBN-13: 978-06911-399-44. (Reviewed in January 2011.)

*The Numerati*, by Stephen Baker. Houghton Mifflin, August 2008. ISBN-13: 978-06187-846-08. (Reviewed October 2009.)

*Our Days Are Numbered: How Mathematics Orders Our Lives*, by Jason Brown. Emblem Editions, April 2010. ISBN-13: 978-07710-169-74.

*Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century*, by Masha Gessen. Houghton Mifflin Harcourt, November 2009. ISBN-13: 978-01510-140-64. (Reviewed in January 2011.)

*Pioneering Women in American Mathematics: The Pre-1940 Ph.D.'s*, by Judy Green and Jeanne LaDuke. AMS, December 2008. ISBN-13: 978-08218-4376-5.

*Plato's Ghost: The Modernist Transformation of Mathematics*, by Jeremy Gray. Princeton University Press, September 2008. ISBN-13: 978-06911-361-03. (Reviewed February 2010.)

*Probabilities: The Little Numbers That Rule Our Lives*, by Peter Olofsson. Wiley, March 2010. ISBN-13: 978-04706-244-56.

*Proofs from THE BOOK*, by Martin Aigner and Günter Ziegler. Expanded fourth edition, Springer, October 2009. ISBN-13: 978-3-642-00855-9.

*Pythagoras' Revenge: A Mathematical Mystery*, by Arturo Sangalli. Princeton University Press, May 2009. ISBN-13: 978-06910-495-57. (Reviewed May 2010.)

*Recountings: Conversations with MIT Mathematicians*, edited by Joel Segel. A K Peters, January 2009. ISBN-13: 978-15688-144-90.

*Roger Boskovich*, by Radoslav Dimitric (Serbian). Helios Publishing Company, September 2006. ISBN-13: 978-09788-256-21.

*The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions*, by Shing-Tung Yau (with Steve Nadis). Basic Books, September 2010. ISBN-13: 978-04650-202-32. (Reviewed February 2011.)

*The Solitude of Prime Numbers*, by Paolo Giordano. Pamela Dorman Books, March 2010. ISBN-13: 978-06700-214-82. (Reviewed September 2010.)

*Solving Mathematical Problems: A Personal Perspective*, by Terence Tao. Oxford University Press, September 2006. ISBN-13: 978-0-199-20560-8. (Reviewed February 2010.)

*The Strangest Man*, by Graham Farmelo. Basic Books, August 2009. ISBN-13: 978-04650-182-77.

*Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving*, by Sanjoy Mahajan. MIT Press, March 2010. ISBN-13: 978-0-262-51429-3.

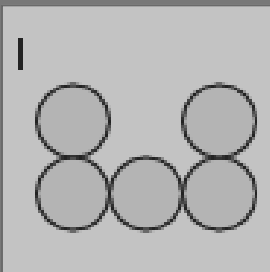
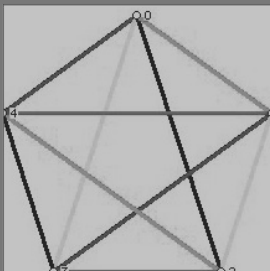
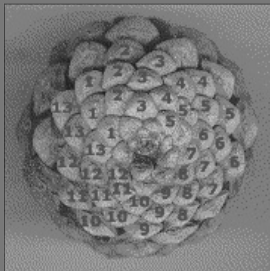
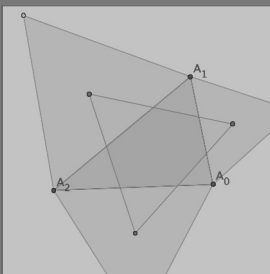
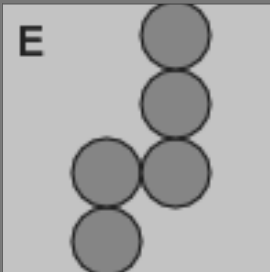
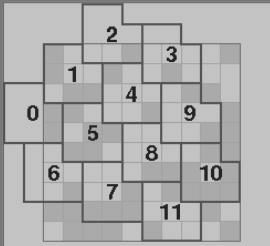
*Survival Guide for Outsiders: How to Protect Yourself from Politicians, Experts, and Other Insiders*, by Sherman Stein. BookSurge Publishing, February 2010. ISBN-13: 978-14392-532-74.

*Symmetry: A Journey into the Patterns of Nature*, by Marcus du Sautoy. Harper, March 2008. ISBN: 978-00607-8940-4. (Reviewed February 2011.)

*Symmetry in Chaos: A Search for Pattern in Mathematics, Art, and Nature*, by Michael Field and Martin Golubitsky. Society for Industrial and Applied Mathematics, second revised edition, May 2009. ISBN-13: 978-08987-167-26.

*Teaching Statistics Using Baseball*, by James Albert. Mathematical Association of America, July 2003. ISBN-13: 978-08838-572-74. (Reviewed April 2010.)

*What's Luck Got to Do with It? The History, Mathematics and Psychology of The Gambler's Illusion*, by Joseph Mazur. Princeton University Press, July 2010. ISBN: 978-0-691-13890-9.



# THE FEATURE COLUMN

*monthly essays on mathematical topics*

[www.ams.org/featurecolumn](http://www.ams.org/featurecolumn)

Each month, the Feature Column provides an online in-depth look at a mathematical topic. Complete with graphics, links, and references, the columns cover a wide spectrum of mathematics and its applications, often including historical figures and their contributions. The authors—David Austin, Bill Casselman, Joe Malkevitch, and Tony Phillips—share their excitement about developments in mathematics.

Recent essays include:

Geometry and the Discrete Fourier Transform

Farey Numbers and the Magnetic Cactus

Who Won!

Multiplication Is Easier When It's Complex

How Did Escher Do It?

From Pascal's Triangle to the Bell-shaped Curve

Mathematics and Sports

Moving Remy in Harmony: Pixar's Use of Harmonic Functions

Crypto Graphics

Keep on Trucking

Puzzling Over Exact Cover Problems



**AMS members:** Sign up for the AMS members-only *Headlines & Deadlines* service at [www.ams.org/enews](http://www.ams.org/enews) to receive email notifications when each new column is posted.

## SECRETARY



### Position

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The American Mathematical Society is seeking candidates for the position of Secretary, one of the most important and influential positions within the Society. The Secretary participates in formulating policy for the Society, participates actively in governance activities, plays a key role in managing committee structures, oversees the scientific program of meetings, and helps to maintain institutional memory.

The first term of the new AMS Secretary will begin February 1, 2013, with initial appointment expected in Fall 2011 in order that the Secretary-designate may observe the conduct of Society business for a full year before taking office.

All necessary expenses incurred by the Secretary in performance of duties for the Society are reimbursed, including travel and communications. The Society is prepared to negotiate a financial arrangement with the successful candidate and his/her employer in order that the new Secretary be granted sufficient release time to carry out the many functions of the office.

### Qualifications

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The Secretary should be a research mathematician and must have substantial knowledge of Society activities. Although the AMS Secretary is appointed by the Council for a term of two years, candidates should be willing to make a long-term commitment, for it is expected that the new Secretary will be reappointed for subsequent terms pending successful performance reviews.

### Duties of the office include:

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Organizing and coordinating the Council and its committees. Serving as ex officio member of the Council, the Executive Committee, the Agenda and Budget Committee, the Liaison Committee, the Long Range Planning Committee, the Committee on Meeting and Conferences, the Committee on the Profession, and the Committee on Publications. The Secretary also serves as a non-voting member of the Committee on Education and the Committee on Science Policy. Working closely with the President to coordinate and administer the activities of committees. Overseeing, together with the Associate Secretaries, the scientific program of all Society meetings.

### Applications

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A Search Committee with Eric Friedlander as chair has been formed to seek and review applications. Persons wishing to be considered or to make a nomination are enthusiastically encouraged to inform

AMS Secretary Search Committee  
c/o Robert J. Daverman  
Department of Mathematics  
University of Tennessee  
Knoxville, TN 37996-1320

**For full consideration, nominations and supporting documentation should be received before April 15, 2011.**



# AMS Award for Mathematics Programs *That Make a Difference*

Deadline: September 15, 2011

This award was established in 2005 in response to a recommendation from the AMS's Committee on the Profession that the AMS compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are replicable models.

Preference will be given to programs with significant participation by underrepresented minorities.

Two programs are highlighted annually.

Nomination process: Letters of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements that make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages.

Send nominations to:  
Programs That Make a Difference  
c/o Ellen Maycock  
American Mathematical Society  
201 Charles Street  
Providence, RI 02904  
or via email to [ejm@ams.org](mailto:ejm@ams.org)

Recent Winners:

2010: Department of Computational and Applied Mathematics (CAAM), Rice University; Summer Program in Quantitative Sciences, Harvard School of Public Health

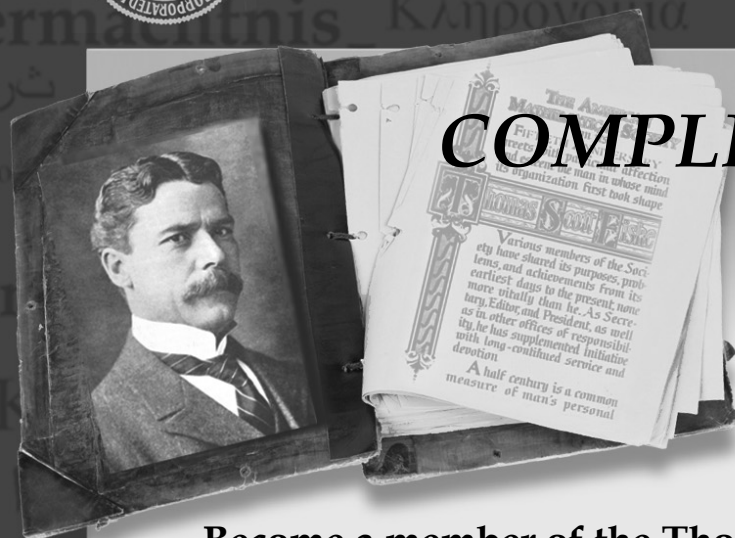
2009: Department of Mathematics at the University of Mississippi; Department of Statistics at North Carolina State University.

2008: Summer Undergraduate Mathematical Science Research Institute (SUMSRI), Miami University (Ohio); Mathematics Summer Program in Research and Learning (Math SPIRAL), University of Maryland, College Park.

2007: Enhancing Diversity in Graduate Education (EDGE), Bryn Mawr College and Spelman College; and Mathematical Theoretical Biology Institute (MTBI), Arizona State University.



AMERICAN MATHEMATICAL SOCIETY



# COMPLETE YOUR *Legacy ...*

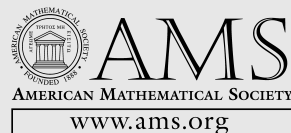
## Become a member of the Thomas S. Fiske Society

You have been building your legacy through research, education, and service to the profession. The **Thomas S. Fiske Society** provides an opportunity to complete your legacy by supporting the future of mathematics through planned giving.

In 1888, Thomas S. Fiske, along with two friends, founded the organization we know as the American Mathematical Society. The **Thomas S. Fiske Society** (Fiske Society) honors individuals who have included a gift to the AMS in their will, living trust, life insurance policy, retirement plan, or other planned-giving vehicle.

For more information about planned giving and Thomas S. Fiske, please visit [www.ams.org/giving-to-ams](http://www.ams.org/giving-to-ams).

Development Office  
Email: [development@ams.org](mailto:development@ams.org)  
Phone: (401) 455-4000  
Toll free in the US and Canada (800) 321-4267  
Postal mail: 201 Charles Street,  
Providence, RI 02904-2294



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# Mathematics Calendar

Please submit conference information for the Mathematics Calendar through the Mathematics Calendar submission form at <http://www.ams.org/cgi-bin/mathcal-submit.pl>.

The most comprehensive and up-to-date Mathematics Calendar information is available on the AMS website at <http://www.ams.org/mathcal/>.

## March 2011

2–5 **Integration, Vector Measures and Related Topics IV. Dedicated to Joe Diestel**, University of Murcia, Murcia, Spain. (Dec. 2010, p. 1495)

\* 5–6 **Great Lakes Geometry Conference 2011**, Ohio State University, Columbus, Ohio.

**Description:** The 2011 edition of the Great Lakes Geometry Conference will feature topics related to Gromov-Witten theory, topological field theory, and geometric analysis. If you plan to attend the conference, please register by sending an e-mail to: [glgc2011@yahoo.com](mailto:glgc2011@yahoo.com) with your name, title, and affiliation. Pending NSF awards, partial support will be available for graduate students, post docs, and unsupported faculties.

**Organizers:** Contact the organizers for information: Bo Guan (OSU), Hsian-Hua Tseng (OSU), Damin Wu (OSU).

**Speakers:** Dan Abramovich (Brown), Ezra Getzler (Northwestern), Chiu-Chu Melissa Liu (Columbia), William Minicozzi II (Johns Hopkins), Xiaochun Rong (Rutgers), Richard Schoen (Stanford), A. J. Tolland (Stony Brook).

**Information:** [http://www.math.ohio-state.edu/~hhtseng/glgc11\\_main.html](http://www.math.ohio-state.edu/~hhtseng/glgc11_main.html).

7–11 **Free Boundary Problems, Theory and Applications Workshop**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)

7–11 (NEW DATE) **IMA Workshop: Computing in Image Processing, Computer Graphics, Virtual Surgery, and Sports**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

12–13 **AMS Southeastern Section Meeting**, Georgia Southern University, Statesboro, Georgia. (Sept. 2010, p. 1034)

\* 14–18 **Instantons in complex geometry**, Laboratory of Algebraic Geometry, Higher School of Economics, Moscow, Russia.

**Description:** The Kobayashi-Hitchin correspondence, proven by Donaldson and Uhlenbeck-Yau, allows one to use the methods of theoretical physics and geometric analysis to study the classical structures of algebraic geometry. We aim to bring together specialists in complex algebraic geometry and related gauge theory to review the recent advances in the theory of stable bundles and their moduli spaces.

**Organizers:** Dimitri Markushevich (Univ. de Lille), email: [markushe@math.univ-lille1.fr](mailto:markushe@math.univ-lille1.fr); Alexander Tikhomirov (YSPU, Yaroslavl), email: [astikhomirov@mail.ru](mailto:astikhomirov@mail.ru); Misha Verbitsky (HSE, Moscow), email: [verbit@verbit.ru](mailto:verbit@verbit.ru).

**Information:** <http://bogomolov-lab.ru/INST/>.

14–18 **Workshop 4: Insect Self-organization and Swarming**, Mathematical Biosciences Institute, The Ohio State University, Columbus, Ohio. (Jan. 2011, p. 82)

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**This section** contains announcements of meetings and conferences of interest to some segment of the mathematical public, including ad hoc, local, or regional meetings, and meetings and symposia devoted to specialized topics, as well as announcements of regularly scheduled meetings of national or international mathematical organizations. A complete list of meetings of the Society can be found on the last page of each issue.

**An announcement** will be published in the *Notices* if it contains a call for papers and specifies the place, date, subject (when applicable), and the speakers; a second announcement will be published only if there are changes or necessary additional information. Once an announcement has appeared, the event will be briefly noted in every third issue until it has been held and a reference will be given in parentheses to the month, year, and page of the issue in which the complete information appeared. Asterisks (\*) mark those announcements containing new or revised information.

**In general**, announcements of meetings and conferences carry only the date, title of meeting, place of meeting, names of speakers (or sometimes a general statement on the program), deadlines for abstracts or contributed papers, and source of further information. If there is any application deadline with respect to participation in the meeting, this fact should be noted. All communications on meetings and conferences

in the mathematical sciences should be sent to the Editor of the *Notices* in care of the American Mathematical Society in Providence or electronically to [notices@ams.org](mailto:notices@ams.org) or [mathcal@ams.org](mailto:mathcal@ams.org).

**In order** to allow participants to arrange their travel plans, organizers of meetings are urged to submit information for these listings early enough to allow them to appear in more than one issue of the *Notices* prior to the meeting in question. To achieve this, listings should be received in Providence **eight months** prior to the scheduled date of the meeting.

**The complete listing** of the Mathematics Calendar will be published only in the September issue of the *Notices*. The March, June/July, and December issues will include, along with new announcements, references to any previously announced meetings and conferences occurring within the twelve-month period following the month of those issues. New information about meetings and conferences that will occur later than the twelve-month period will be announced once in full and will not be repeated until the date of the conference or meeting falls within the twelve-month period.

**The Mathematics Calendar**, as well as Meetings and Conferences of the AMS, is now available electronically through the AMS website on the World Wide Web. To access the AMS website, use the URL: <http://www.ams.org/>.

\* 14-19 **Short Courses and Workshop on Spectral Function Theory**, Centre de Recerca Matemàtica, Bellaterra, Spain.

**Description:** Spectral complex analysis was created in the classical works by Carleman and Wiener, and then developed by Beurling, Krein, Levinson and many other prominent analysts of the 20th century. The unifying theme of these works is the complex Fourier transform, which translates various problems of real harmonic analysis into the language of complex analysis. This field, which originally studied problems such as sampling, interpolation and uniqueness in Paley-Wiener spaces, the uncertainty principle, various notions of spectrum of a function and related questions of spectral analysis and synthesis, has been expanding in several directions. A series of mini-courses will provide a perspective of the areas in which the field is developing. This will be continued with a two-day work-shop where some of the latest advances in the area will be presented.

**Information:** <http://www.crm.cat/wkspectral>.

14-June 10 **Probability and Discrete Mathematics in Mathematical Biology**, Institute for Mathematical Sciences, National University of Singapore, Singapore. (Nov. 2010, p. 1348)

14-June 17 **Navigating Chemical Compound Space for Materials and BioDesign**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California. (Jan. 2010, p. 76)

16-18 **IAENG International Conference on Operations Research 2011**, Royal Garden Hotel, Kowloon, Hong Kong. (Aug. 2010, p. 904)

17-18 **Finitely presented solvable groups**, The City College of New York, New York, New York. (Jan. 2011, p. 82)

17-19 **The 45th Annual Spring Topology and Dynamical Systems Conference**, University of Texas at Tyler, Tyler, Texas. (Oct. 2010, p. 1165)

18-20 **AMS Central Section Meeting**, University of Iowa, Iowa City, Iowa. (Sept. 2010, p. 1034)

21-24 **SIAM Conference on Mathematical & Computational Issues in the Geosciences**, Hilton Long Beach & Executive Meeting Center, Long Beach, California. (Dec. 2010, p. 1495)

21-25 **AIM Workshop: Hypergraph Turán Problem**, American Institute of Mathematics, Palo Alto, California. (Aug. 2010, p. 905)

21-25 **Young Set Theory Workshop 2011**, A.-Z. Königswinter near Bonn, Germany. (Feb. 2011, p. 334)

22-26 **Current Topic Workshop: New Developments in Dynamical Systems Arising from the Biosciences**, Mathematical Biosciences Institute, The Ohio State University, Columbus, Ohio. (Jan. 2011, p. 82)

\* 23-26 **SETIT'11**, Sousse, Tunisia.

**Extended Deadline and Financial Support:** At the request of a number of potential contributors, we have decided to extend the deadline for receipt of papers to be presented to The 6th International Conference: Sciences of Electronics, Technologies of Information and Telecommunications SETIT 2011. This deadline is extended to December 31, 2010. The paper submission is on-line at: <http://www.setit.rnu.tn/?pg=submission>. In this conference; 300 participants will benefit from financial support. This financial support, of amount 250 euro, will be available to help participants from developing or emerging countries as well as young researchers to attend SETIT'2011. The financial support form is available on the web site: <http://www.setit.rnu.tn/FinancialSupport.dot>. It should be filled in in detail and sent by e-mail to [financialsupport.setit@gmail.com](mailto:financialsupport.setit@gmail.com). The SETIT organization committee will examine on a case by case basis all requests and provide a reply in a one week period. Please note that financial support requests must be sent before January 15, 2011. Online registration can be found at: [http://www.setit.rnu.tn/?main=1&pg=registration\\_aut](http://www.setit.rnu.tn/?main=1&pg=registration_aut).

**Deadlines:** Extended Deadline: December 31, 2010. Notification of acceptance: January 31, 2011. Final manuscript due: February 15, 2011. Main conference: March 23-26, 2011.

**Information:** You can find more details in: <http://www.setit.rnu.tn/>

25-27 **The Seventh International Conference on Number Theory and Smarandache Notions**, Weinan Teachers University, Weinan, Shaanxi, People's Republic of China. (Jan. 2011, p. 82)

28-April 1 **International Conference on Homotopy and Non-Commutative Geometry**, Batumi State University, Batumi, Republic of Georgia. (Jun./Jul. 2010, p. 785)

28-April 1 **International workshop: Unlikely intersections in algebraic groups and Shimura varieties**, Scuola Normale Superiore, Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy. (Oct. 2010, p. 1165)

29-30 **International Workshop on Mathematical and Physical Foundations of Discrete Time Quantum Walk**, Oh-Okayama Campus, Tokyo Institute of Technology, Meguro, Tokyo, Japan. (Jan. 2011, p. 82)

## April 2011

1-3 **Underrepresented Students in Topology and Algebra Research Symposium (USTARS): Research symposium for underrepresented graduate students in Algebra and Topology**, University of Iowa, Iowa City, Iowa. (Jan. 2011, p. 82)

2-3 **Midwest Graduate Student Topology and Geometry Conference**, Michigan State University, East Lansing, Michigan. (Oct. 2010, p. 1165)

\* 6-8 **Workshop on Dynamics and  $C^*$ -Algebras**, Centre de Recerca Matemàtica, Bellaterra, Spain.

**Description:** Classification is a central theme in mathematics. It has driven some of the most exciting developments of the 20th century, and is particularly prominent in the theory of operator algebras. The Elliott Program seeks classification of simple, separable and amenable  $C^*$ -algebras by means of  $K$ -theoretical data. Following a remarkable success in the nineties, it has enjoyed a resurgence of late, owing to the discovery by M. Rordam (and later, in a different direction, A. Toms) that  $K$ -theory alone, at least in its naive topological formulation, does not suffice for the classification of all simple separable amenable  $C^*$ -algebras. Rather than being a detracting factor, this opens up two ways forward: restrict the class of  $C^*$ -algebras considered, or enlarge the proposed invariant.

**Information:** <http://www.crm.cat/wkalgebras>.

\* 7-9 **Conformal Differential Geometry and its Interaction with Representation Theory**, University of Arkansas, Fayetteville, Arkansas.

**Description:** The 36th Annual Spring Lecture Series in Mathematics at the University of Arkansas titled "Conformal Differential Geometry and its Interaction with Representation Theory" will be held at the University of Arkansas from April 7-9, 2011. Professor Michael Eastwood will give a series of five lectures, and there will be an additional ten invited talks. The conference is partially supported by the NSF and participants are welcome to give a contributed talk. Funding for travel and lodging is available for junior participants.

**Information:** <http://math.uark.edu/3723.php>.

7-9 **ICMCS'11 - IEEE co-sponsored Conference 2nd International Conference on Multimedia Computing and Systems**, Ouarzazate, Morocco. (Aug. 2010, p. 905)

9-10 **AMS Eastern Section Meeting**, College of the Holy Cross, Worcester, Massachusetts. (Sept. 2010, p. 1034)

11-15 **Arithmetic Statistics**, Mathematical Sciences Research Institute, Berkeley, California. (Jun./Jul. 2010, p. 785)



11–15 **IMA Workshop: Societally Relevant Computing**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

11–16 **The Fifth de Brun Workshop: Groups, Combinatorics, Computing**, National University of Ireland, Galway, Ireland. (Jan. 2011, p. 83)

12–14 **International Conference on Mathematical and Computational Biology 2011(ICMCB2011)**, Malacca, Malaysia. (Jan. 2011, p. 83)

14–16 **The First International Conference on Multidimensional Finance, Insurance and Investment: ICMFII 2011**, Hammamet, Tunisia. (Nov. 2010, p. 1348)

17–19 **7th IMA Modelling in Industrial Maintenance and Reliability**, Sidney Sussex College, University of Cambridge, United Kingdom. (Sept. 2010, p. 1035)

18–22 **Computational Statistical Methods for Genomics and Systems Biology**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

27–28 **Third Conference on Mathematical Sciences (CMS'2011)**, Department of Mathematics, Faculty of Science and Information Technology, Zarqa Private University, Zarqa, Jordan. (Aug. 2010, p. 905)

28–30 **SIAM International Conference on Data Mining**, Hilton Phoenix East/Mesa, Mesa, Arizona. (Dec. 2010, p. 1495)

\* 29–May 1 **Graduate Student Probability Conference at Georgia Tech**, Georgia Institute of Technology, Atlanta, Georgia.

**Description:** The conference is open to all graduate students and post-doctoral fellows who are interested in probability. It is organized by students from Georgia Tech under the supervision of Professor Christian Houdré.

**Keynote speakers:** Professor Nathalie Eisenbaum (Université Pierre et Marie Curie) and Professor Philip Protter (Columbia University). They will each give talks daily about areas of interest within their research.

**Organizers:** The student organizers are Allen Hoffmeyer (head organizer), Huy Huynh (head organizer), Ruoting Gong, Linwei Xin, Jinyong Ma, and Ruodu Wang

**Information:** This year, we are expecting over 100 participants from all over the country to join the conference. This conference is a great opportunity to give a talk in a friendly environment, so we encourage you to register to speak if you are interested. The website will be the source of all information for a variety of topics (funding, travel info, etc.), and the page will be updated frequently. <http://gspc.math.gatech.edu/>.

30–May 1 **AMS Western Section Meeting**, University of Nevada, Las Vegas, Nevada. (Sept. 2010, p. 1035)

## May 2011

\* 1–7 **Talbot Workshop in Non-Abelian Hodge Theory**, Salt Lake City, Utah.

**Description:** The workshop, mentored by Carlos Simpson, will constitute a weeklong retreat with talks and organized discussions during the mornings and evenings, informal discussions and collaborations during the afternoon. The workshop will generally focus on understanding the representations of fundamental groups of algebraic varieties. Topics developed may include the harmonic theory relating the moduli of Higgs bundles to the moduli of G-bundles with flat connection, deformation theory, the topology of character varieties and symplectic/hyperkahler structures on moduli spaces, and further directions of research. The workshop will have an expository character and is aimed at graduate students and junior

faculty interested in this area. Most talks will be given by participants of the workshop.

**Information:** For more information, please visit the website listed below. Interested participants should fill out the application form on the website by January 31, 2011; <http://math.mit.edu/talbot>.

1–August 31 **MITACS International Focus Period on Advances in Network Analysis and its Applications**, Locations throughout Canada. (Apr. 2010, p. 552)

\* 2 **Info-Metrics across the Sciences**, American University, Washington, District of Columbia.

**Description:** The objective of this workshop is to continue the exploration into the basics of info-Metrics and entropic inference. In that workshop we focus on Info-Metrics in the natural sciences, study the state of information-theoretic estimation and data analysis in some areas of the natural sciences, and then discuss the implications of info-metrics in the natural sciences to the social sciences in general and to economics, econometrics, statistics and finance in particular.

**Hosts:** The workshop is hosted by the Info-Metrics Institute. The co-chairs are Amos Golan (AU), and Ariel Caticha (SUNY Albany).

**Information:** <http://www.american.edu/cas/economics/info-metrics/workshop/index.cfm>.

\* 2–3 **Present your research work in ICQI'2011**, Lahore, Pakistan.

**Description:** 2nd Regional Asian Network for Quality Conference: ICQI Final 20 Dec 2010.gif.

**Information:** You may use the following links directly also: <http://www.easychair.org/account/signin.cgi?conf=icqi2011>; email: [icqi@piqc.com.pk](mailto:icqi@piqc.com.pk).

2–4 **Statistical Issues in Forest Management**, Université Laval, Québec City, Canada. (Aug. 2010, p. 905)

5–7 **Workshop Discrete, Tropical and Algebraic Geometry**, Goethe University, Frankfurt am Main, Germany. (Feb. 2011, p. 334)

9–13 **Causal Inference in Health Research**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

10–13 **Second Buea International Conference on the Mathematical Sciences**, University of Buea, Cameroon. (Jan. 2011, p. 83)

14–15 **Methodological Aspects of Teaching Mathematics**, Faculty of Education in Jagodina, Jagodina, Serbia. (Jan. 2011, p. 83)

16–19 **Analysis of Survival and Event History Data**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

16–19 **SIAM Conference on Optimization**, Darmstadtium Conference Center, Darmstadt, Germany. (Dec. 2010, p. 1496)

16–20 **IMA Hot Topics Workshop: Strain Induced Shape Formation: Analysis, Geometry and Materials Science**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Jan. 2011, p. 83)

\* 16–20 **NSF/CBMS Conference on “Deformation Theory of Algebras and Modules”**, North Carolina State University, Raleigh, North Carolina.

**Principal Speaker:** Professor Martin Markl.

**Organizers:** Kailash C. Misra and Thomas J. Lada.

**Information:** email: [misra@math.ncsu.edu](mailto:misra@math.ncsu.edu); [lada@math.ncsu.edu](mailto:lada@math.ncsu.edu); <http://www4.ncsu.edu/~lada/nsfcbms.htm>.

\* 20–22 **Workshop: “The method to yield analytical solutions for dynamic systems”**, Laboratory SELF, Ochakov, Ukraine.

**Description:** We have developed a new method to yield exact and complete analytical solutions. We have solved a broad amount of problems for mechanical systems and electric circuits, distributed, lumped and combined; under matched and mismatched load and for unsteady processes; when the force affects the first either inner element of the system, along, across the system and when the force

is inclined; ideal and resistant (with attenuation), linear and nonlinear, bended, closed-loop and spider-like, with resonance subsystems etc. This method allows applying numerical techniques well after we reveal features of the model making possible to correct the numerical calculation, achieving better correspondence to the modelled process. The possibility to model with analytical functions makes the final numerical modelling less laborious. The experimental check of our solutions showed full coincidence of results. To disseminate this method, we will conduct this second workshop.

**Information:** <http://selflab.com.ua/index2.htm>.

22–26 **SIAM Conference on Applications of Dynamical Systems (DS11)**, Snowbird Ski and Summer Resort, Snowbird, Utah. (Mar. 2010, p. 434)

22–27 **Progress on Difference Equations 2011**, Dublin City University, Dublin, Ireland. (Dec. 2010, p. 1496)

23–26 **MAMERN'11: 4th International Conference on Approximation Methods and Numerical Modeling in Environment and Natural Resources**, Saidia, Morocco.

\* 23–27 **The 81st European Study Group with Industry**, ISEG, Technical University of Lisbon, Lisbon, Portugal.

**Description:** This meeting is part of the series of European Study Groups and will count on the participation of several experts with a large experience in this type of events. The purpose of these meetings is to strengthen the links between Mathematics and Industry by using Mathematics to tackle industrial problems which are proposed by industrial partners.

**Information:** <http://cemapre.iseg.utl.pt/esgi81/>.

\* 25–26 **Geometric Topology of Knots**, Centro di Ricerca Matematica "Ennio De Giorgi", Collegio Puteano, Piazza dei Cavalieri 3, Pisa, Italy.

**Description:** As other branches of 3-dimensional topology, the theory of knots and links was deeply revolutionized by the geometric approach first developed by Thurston. This workshop will focus on the geometry of knots, and particularly on aspects of hyperbolic geometry and the knot invariants associated with it, most notably the volume. Strong emphasis will be put on the relationships between these geometric invariants and the more classical ones, such as the crossing number. The algorithmic and computational methods now available to construct hyperbolic structures, and to compute invariants, will also be considered as central topics.

**Invited Speakers:** Michel Boileau (Univ. Toulouse), Cameron Gordon (Univ. Texas, Austin), Marc Lackenby (Univ. Oxford), Feng Luo (Rutgers Univ.), Walter Neumann (Columbia Univ.), Jessica Purcell (Brigham Young Univ.)

**Information:** <http://www.crm.sns.it/event/205/>.

\* 25–27 **Operator theory and boundary value problems**, University Paris-Sud, Orsay, France.

**Description:** We are organizing a conference on the applications of the operator theory to the study of the boundary value problems arising in quantum mechanics and other areas. The boundary value problems are understood not only in the classical sense of PDE, but include also systems with singularities or structures composed of pieces of different nature and involving boundary conditions at the interaction set. We are planning to have three days of talks, May 24 being the arrival date and May 28 being the departure date. The aim of the conference is to bring together experts working on various aspects of the topic and to present the state of art to the local mathematical community.

**Information:** <http://www.math.u-psud.fr/~pankrash/ot11/>.

25–28 **Sixth International Conference on Dynamic Systems and Applications**, Morehouse College, Atlanta, Georgia. (Dec. 2010, p. 1496)

\* 27–29 **10th Panhellenic Geometry Conference (with international participation)**, University of Patras, Conference and Cultural Center, Rion, Greece.

**Description:** Following the tradition of the previous Panhellenic Geometry Conferences, we invite researchers and postgraduate students

who work on Geometry in a wide sense (Differential geometry-applications, algebraic geometry, convex geometry, geometric analysis, algebraic topology) to participate in this conference.

**Organizers:** A. Arvanitoyeorgos, A. Cotsiolis, G. Kaimakamis, V. Papanioniou (Chairman).

**Invited Speakers:** G. Calvaruso (Lecce), H. Tamaru (Hiroshima).

**Information:** <http://www.math.upatras.gr/~geomconf11>.

27–June 3 **12th Mathematical Theory in Fluid Mechanics**, Paseky/Kacov School, Kacov, Czech Republic.

29–June 4 **Function spaces—Approximation, Inequalities and Lineability**, Paseky nad Jizerou, Czech Republic. (Jan. 2011, p. 83)

30–June 2 **Discrete Groups and Geometric Structures, with Applications IV**, This workshop will be held at Hotel Royal Astrid in Oostende, Belgium. (Jan. 2011, p. 83)

30–June 3 **International Conference on Asymptotics and Special Functions**, City University of Hong Kong, Hong Kong. (Jan. 2011, p. 83)

\* 30–June 3 **MEGA 2011: Effective Methods in Algebraic Geometry - ESF-EMS-ERCOM/IML Conference**, Stockholm University, Stockholm, Sweden.

**Description:** MEGA is the acronym for Effective Methods in Algebraic Geometry (and its equivalent in Italian, French, Spanish, German, Russian, etc.), a series of roughly biennial conferences on computational and application aspects of Algebraic Geometry and related topics with very high standards. As in previous conferences, we plan to publish selected papers from the conference in a special issue of the Journal of Symbolic Computation.

**Information:** <http://www.esf.org/conferences/11372>.

\* 30–June 4 **Short Courses and Workshop on Hilbert Spaces of Entire Functions and Spectral Theory of Self-adjoint Differential Operators**, Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.

**Description:** Spectral complex analysis was created in the classical works by Carleman and Wiener, and then developed by Beurling, Krein, Levinson and many other prominent analysts of the 20th century. The unifying theme of these works was the complex Fourier transform, which translates various problems of harmonic analysis in the real domain into the language of complex analysis. Originally this circle of ideas and problems included sampling, interpolation and uniqueness in Paley-Wiener spaces of entire functions and related properties of exponential systems in  $L^2$ -spaces; later it expanded to the uncertainty principle, to various notions of spectrum of a function and to related questions of spectral analysis and synthesis.

**Information:** <http://www.crm.cat/wkentire>.

31–June 3 **Nonlinear Modeling and Analysis International Conference: 4th Chaotic Modeling and Simulation International Conference (CHAOS2011)**, Agios Nikolaos, Crete, Greece. (Jan. 2011, p. 83)

31–June 3 **Poster of the 4th Chaotic Modeling and Simulation International Conference (CHAOS2011)**, Agios Nikolaos, Crete, Greece.

## June 2011

\* 1–3 **5th Global Conference on Power Control and Optimization**, Dubai, United Arab Emirates

**Description:** The scope of the conference is contemporary and original research and educational development in the area of mechanical, electrical, communication engineering, sustainable energy, controllers, robotics, wireless sensors, biomedicine, computing, management, environment, continuous and hybrid optimization. Prospective authors from universities or other educational institutes and industry are invited to submit a full paper by email before the deadline. All papers will be peer reviewed by independent specialists. The conference proceedings will be published in the PCO CD. Selected papers will be published online by AIP proceeding. Some other selected papers will be submitted to Elsevier, Springer, InderScience, GJTO, T&F, PE and others for special issue journal publication. Proposals for holding special session, tutorial session, exhibition and

workshops are invited from prospective authors, industrial bodies and academicians, and should be addressed to the conference secretariat. The conference organizing committee is currently looking for financial sponsors from industry, academia, and professional bodies. **Speakers:** Dobrica Milovanovic (Serbia), Sergej Čelikovský (Czech Republic), Klaus-Dieter (Germany), Felix Blyakhman (Russia), Rana Abdul Jabbar Khan (Pakistan).

**Organizing Committee:** Professopr Nader Barsoum (Australia), Dr. P. Vasant (Malaysia), Dr. Rabi Habash (Iraq), Dr. J. F. Webb (UK).

**Organizing Committee:** <http://www.pcoglobal.com>.

**Information:** <http://www.pcoglobal.com>; email: [pcoglobal@gmail.com](mailto:pcoglobal@gmail.com); [icpco.20@gmail.com](mailto:icpco.20@gmail.com).

**5-7 National Conference On Nonlinear Analysis and Applications**, Department of Mathematics, H.N.B. Garhwal University, Campus Pauri, Pauri Garhwal, Uttarakhand. (Jun./Jul. 2010, p. 786)

\* **5-10 16th Workshop on Stochastic Geometry, Stereology and Image Analysis**, Sandbjerg Estate, Soenderborg, Denmark.

**Description:** This interdisciplinary workshop will bring together scientists from integral geometry, stereology, stochastic geometry, applied probability, spatial statistics and bioimaging. An important scope of the workshop is to promote the advance of stochastic geometry. The workshop will have longer talks by invited speakers and shorter contributed talks by the participants as well as a poster session.

**Information:** <http://csgb.au.dk/events/2011/sgsia11/>.

**5-12 Symmetry and Perturbation Theory 2011**, Otranto, near Lecce, Italy. (Feb. 2011, p. 335)

**6-8 The International Conference on Numerical Analysis and Optimization (IceMATH 2011)**, Universitas Ahmad Dahlan, Yogyakarta, Indonesia. (Feb. 2011, p. 335)

**6-9 Copula Models and Dependence**, Centre de recherches mathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal, (Québec) H3T 1J4 Canada. (Aug. 2010, p. 905)

\* **6-10 Conference on Structure and Classification of  $C^*$ -Algebras**, Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.

**Description:** Classification is a central theme in mathematics. It has driven some of the most exciting developments of the 20th century, and is particularly prominent in the theory of operator algebras. The Elliott Program seeks classification of simple, separable and amenable  $C^*$ -algebras by means of  $K$ -theoretical data. Following a remarkable success in the nineties, it has enjoyed a resurgence of late, owing to the discovery by M. Rordam (and later, in a different direction, A. Toms) that  $K$ -theory alone, at least in its naive topological formulation, does not suffice for the classification of all simple separable amenable  $C^*$ -algebras. Rather than being a detracting factor, this opens up two ways forward: restrict the class of  $C^*$ -algebras considered, or enlarge the proposed invariant. Both of these courses have been pursued vigorously over the past three or four years, leading to several breakthroughs in Elliott's program.

**Information:** <http://www.crm.cat/calgebras>.

**6-10 Faces of Geometry: 3-manifolds, Groups and Singularities**, Columbia University/Barnard College, New York, New York. (Feb. 2011, p. 335)

**6-10 IMA Workshop: Large-scale Inverse Problems and Quantification of Uncertainty**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Apr. 2010, p. 552)

**6-10 Low-dimensional manifolds and high-dimensional categories**, UC Berkeley, Berkeley, California. (Dec. 2010, p. 1496)

**6-10 Toric geometry and applications**, Catholic University of Leuven, Leuven (Heverlee), Belgium. (Feb. 2011, p. 335)

\* **7-9 4th International Workshop on Symbolic-Numeric Computation (SNC 2011)**, San Jose, California.

**Description:** Algorithms that combine techniques from symbolic and numeric computation have been of increasing importance and interest over the past decade. The necessity to work reliably with imprecise and noisy data, and for speed and accuracy within algebraic and hybrid-numerical problems, has encouraged a new synergy between the numerical and symbolic computing fields. Novel and exciting problems from industrial, mathematical and computational domains are now being explored and solved.

**Goal:** The goal of the present workshop is to support the interaction and integration of symbolic and numeric computing. Earlier meetings in this series include the SNAP 96 Workshop, held in Sophia Antipolis, France, the SNC 2005 meeting, held in Xi'an, China, SNC 2007 which was held in London, Canada, and SNC 2009, held in Kyoto, Japan. The 4th International Workshops on Symbolic-Numeric Computation will be held on June 7-9 at San Jose, California, as a member of the ACM Federated Computing Research Conference (FCRC). **Information:** <http://www.cargo.wlu.ca/SNC2011/>.

**7-10 9th International Conference on Applied Cryptography and Network Security (ACNS 2011)**, Nerja, Malaga, Spain. (Feb. 2011, p. 335)

**8-11 2011 International Symposium on Symbolic and Algebraic Computation (ISSAC 2011)**, San Jose Convention Center, 150 West San Carlos St., San Jose, California. (Jan. 2011, p. 84)

**12-17 Geometric and nonlinear analysis, meeting in Lorraine**, Université Henri Poincar'e, Nancy, France. (Jan. 2011, p. 84)

**12-18 International Conference on Waves and Stability in Continuous Media WASCOM XVI**, Brindisi, Italy. (Feb. 2011, p. 335)

**13-16 2011 International Conference on Applied Mathematics and Interdisciplinary Research**, Nankai University, Tianjin, China. (Sept. 2010, p. 1166)

**13-16 FPP6: Foundations of Probability and Physics-6**, Linnaeus University, Vaxjo, Sweden. (Jan. 2011, p. 84)

**13-17 AIM Workshop: The Cohen-Lenstra heuristics for Class Groups**, American Institute of Mathematics, Palo Alto, California. (Nov. 2010, p. 1349)

\* **13-17 Formal Power Series and Algebraic Combinatorics (conference)**, Reykjavik, Iceland.

**Description:** The conference will feature invited lectures, contributed presentations, poster session, and software demonstrations. There will be no parallel sessions. Topics include all aspects of combinatorics and their relations with other parts of mathematics, physics, computer science, and biology. The official languages of the conference are English and French.

**Information:** <http://combinatorics.is>.

**13-19 Strobl2011 - From Abstract to Computational Harmonic Analysis**, Bifeb, Strobl, Salzburg, Austria. (Nov. 2010, p. 1349)

**14-17 2011 World Conference on Natural Resource Modeling**, Ottawa, Canada. (Sept. 2010, p. 1035)

**15-17 2011 Usenix Annual Technical Conference (USENIX ATC '11)**, Portland Marriott, Downtown Waterfront, 1401 SW Naito Parkway, Portland, Oregon. (Feb. 2011, p. 335)

**16-18 Lorentz Geometry in Mathematics and in Physics**, Institut de Recherche Mathématique Avancée, University of Strasbourg, France. (Dec. 2010, p. 1496)

**17-23 The International Conference "Painleve Equations and related topics"**, Euler Institute, Saint Petersburg, Russia. (Feb. 2011, p. 335)

**19-25 49th International Symposium on Functional Equations**, Graz, Austria. (Dec. 2010, p. 1496)



\* 20-24 **AIM Workshop: Careers in academia**, American Institute of Mathematics, Palo Alto, California.

**Description:** This workshop, sponsored by AIM and the NSF, will focus on preparing participants to start and maintain a successful career as a mathematician at a college or university. Applicants should have received a Ph.D. in 2010 or earlier and currently hold a postdoctoral appointment which will end in 2012 or 2013.

**Information:** <http://www.aimath.org/ARCC/workshops/jobskills2.html>.

\* 20-24 **Journées de Probabilités 2011**, Institut Elie Cartan Nancy, Nancy, France.

**Information:** <http://jp2011.iecn.u-nancy.fr/>.

\* 20-24 **NSF-CBMS Conference on Radial Basis Functions: Mathematical Developments and Applications**, University of Massachusetts Dartmouth, Dartmouth, Massachusetts.

**Description:** The mission of our RBF conference is to educate and motivate researchers (at all levels) and students in RBF methods, and to stimulate and inspire research in this field.

**Talks:** The conference will feature ten talks by two leading researchers in this field, Bengt Fornberg at the Applied Mathematics Department of the University of Colorado at Boulder and Natasha Flyer at the Institute for Mathematics Applied to Geosciences of National Center for Atmospheric Research (NCAR). The talks will be designed to appeal to both experts and novices, and to stimulate discussion and collaboration between the speakers and attendees about recent advances and open problems in RBF. This conference will provide an environment to communicate the latest research and development of RBF methods in recent years, and will attract a wider and more diverse group of researchers to undertake research in RBF methods and build a supportive community of RBF researchers.

**Information:** <http://www.umassdcomputing.org/conference/rbfcbms2011/index.html>.

20-24 **Permutation Patterns 2011**, California Polytechnic State University, San Luis Obispo, California. (Jan. 2011, p. 84)

20-25 **3rd Conference of the Euro-American Consortium for Promoting the Application of Mathematics in Technical and Natural Sciences**, Resort of Albena, Bulgaria. (Jan. 2011, p. 84)

\* 20-July 1 **IMA New Directions Short Course: Invariant Objects in Dynamical Systems and their Applications**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota.

**Description:** The IMA will offer an intensive short course on modern mathematical tools for study of dynamical systems and their applications. The course is designed for researchers in the mathematical sciences and related disciplines.

**Lecturers:** The main lecturers for the course are Peter Bates, Department of Mathematics, Michigan State University, and Rafael de la Llave, Department of Mathematics, University of Texas at Austin. Additional short introduction to computational methods will be provided by Alex Haro Provinciale, Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Spain, and Gemma Huguet, Centre de Recerca Matemàtica, Universitat Politècnica de Catalunya, Spain. Other guest lecturers include Martin Lo, Jet Propulsion Lab, and Stephen Schecter, Department of Mathematics, North Carolina State University.

**Information:** <http://www.ima.umn.edu/2010-2011/ND6.20-7.1.11/>.

20-July 1 (NEW DATE) **Polyhedral Geometry and Algebraic Combinatorics**, University of Wyoming, Laramie, Wyoming. (Jan. 2011, p. 84)

\* 20-July 2 **School on D-modules and applications in Singularity Theory**, Mathematical Research Institute, University of Sevilla, Spain (first week); Mathematical Sciences Institute, CSIC, Madrid, Spain (second week).

**Description:** There will be a summer school on D-modules and applications in singularity theory organized by the Mathematical Research

Institute of the University of Sevilla (IMUS) and the Mathematical Sciences Institute of CSIC (ICMAT, Madrid). It will take place at Sevilla on June 20-25, 2011 and at Madrid on June 27 to July 2, 2011. The School is addressed to Ph.D. students and young post-doc researchers working on algebraic geometry, singularity theory and related areas. The first week will be devoted to the general theory of D-modules, from the basic definitions to the Riemann-Hilbert correspondence. The second week will be centered on the interactions of D-module theory with Hodge theory and with applications to singularity theory.

**Lecturers:** F. J. Castro-Jiménez (Sevilla), E. Cattani (Amherst), M. Granger (Angers), Ph. Maisonobe (Nice), Z. Mebkhout (Paris), L. Narváez Macarro (Sevilla), C. Sabbah (Paris), C. Sevenheck (Mannheim).

**Information:** <http://www.imus.us.es/ACT/dmod2011/>.

\* 20-July 8 **RTG Summer School on Inverse Problems & Partial Differential Equations**, Seattle, Washington.

**Description:** The Research Training Group in the Department of Mathematics at the University of Washington will host a summer school for advanced undergraduates and beginning graduate students on Inverse Problems & Partial Differential Equations. Students will attend lectures in the morning and problem sessions in small groups with mentors in the afternoon. Two mini-courses will be given: Gunther Uhlmann, Guillaume Bal, Steve McDowell: Inverse Transport and the X-Ray Transform; Randall LeVeque, Donna Calhoun: Finite Volume Methods and the Clawpack Software: <http://www.math.washington.edu/ipde/summer>.

**Accommodations:** On-campus accommodations and meals will be provided, plus a travel allowance of up to \$500. Apply online by: April 1. (The Summer School is supported by an NSF Research Training Grant. Support is restricted to U.S. citizens/permanent residents. Applications from international students may be considered, but international students must provide their own support for travel, accommodation, and meals.)

**Information:** <http://www.math.washington.edu/ipde/summer/>.

22-24 **3rd IMA International Conference Mathematics in Sport**, The Lowry, Salford Quays, United Kingdom. (Sept. 2010, p. 1035)

\* 26-July 2 **New developments in noncommutative algebra and its applications**, Sabhal Mòr Ostaig, a Gaelic college in Sleat, Isle of Skye, Scotland.

**Description:** A meeting honoring the 60th birthdays of Ken Brown and Toby Stafford. The meeting will focus on noncommutative algebra in its broadest sense, and will emphasize the most recent developments within the field as well as its most exciting interactions with other topics of mathematics including symplectic geometry, noncommutative geometry and representation theory.

**Information:** <http://www.icms.org.uk/workshop.php?id=163>.

\* 26-July 2 **XXX Workshop on Geometric Methods in Physics**, Białowieża, Poland.

**Description:** Workshop on Geometric Methods in Physics is the annual meeting in the field of mathematical physics which offers a good opportunity for exchange of ideas between physicists and mathematicians. The programme consists of invited plenary lectures, as well as contributed talks, which are also open for graduate students. During the Workshop three special sessions will be organized. These sessions will be dedicated to outstanding mathematical physicists whose areas of scientific interest were always in center of the Białowieża Workshops.

**Topics:** The topics of 2011 Workshop will include among others: Quantization and supersymmetry (Berezin Memorial Session), foundations of quantum mechanics (Session devoted to B. Mielnik), operator algebras and quantum groups (Session devoted to S. L. Woronowicz), infinite dimensional Lie groups and Lie algebras, integrable systems, noncommutative geometry, Poisson and symplectic geometry.

**Information:** <http://wgmp.uwb.edu.pl>.



27–July 8 **Metric Measure Spaces: Geometric and Analytic Aspects**, Université de Montréal, Pavillon André-Aisenstadt, Montréal, Québec, Canada. (Feb. 2011, p. 335)

\* 29–July 2 **International Conference on Applied Analysis and Algebra-ICAA2011**, Yildiz Technical University, Davutpasa Campus, Davutpasa, Istanbul, Turkey.

**Description:** The aim of this conference is to bring together mathematicians working in the new trends of applications of math in a wonderful city of the world, Istanbul. This conference marks the 100th year of the establishment of the Yildiz Technical University which is the hosting university of this conference.

**Topics:** Are but not limited to: Topics related to the applications of math: Applied mathematics and modeling analysis and its applications; applied algebra and its applications; geometry and its applications.

**Information:** <http://www.icaa1.yildiz.edu.tr>.

\* 29–July 5 **The Seventh Congress of Romanian Mathematicians**, Transilvania University, Brasov, Romania.

**Description:** This meeting is intended to resume an old tradition of holding congresses of Romanian mathematicians and it is largely open to international participation. Six such congresses were organized in Cluj (1929), Turnu Severin (1932), Bucharest (1945, 1956, and 2007), and Pitesti (2003).

**Organizers:** Section of Mathematical Sciences of the Romanian Academy, “Simion Stoilow” Institute of Mathematics of the Romanian Academy, Transilvania University of Brasov, the Faculty of Mathematics and Computer Science of the University of Bucharest, and the Romanian Mathematical Society, in partnership with the European Mathematical Society.

**Information:** <http://imar.ro/diverse/congmatro7.htm>.

30–July 2 **IDOTA—Integral and Differential Operators and their Applications**, Department of Mathematics, University of Aveiro, Aveiro, Portugal. (Feb. 2011, p. 335)

## July 2011

3–9 **22nd International Workshop on Operator Theory and Applications (IWOTA 2011)**, Universidad de Sevilla, Spain. (Dec. 2010, p. 1497)

4–8 **Conference on Several Complex Variables on the Occasion of Professor Jozef Siciak’ 80th birthday**, Jagiellonian University, Krakow, Poland. (Dec. 2010, p. 1497)

\* 4–8 **International Conference on Differential & Difference Equations and Applications: Conference in honour of Professor Ravi P. Agarwal**, Azores University, Ponta Delgada, Portugal.

**Description:** The main aim of the conference is to promote, encourage, cooperate, and bring together researchers in the fields of differential & difference equations. All areas of differential & difference equations will be represented with special emphasis on applications. It will be mathematically enriching and socially exciting event.

**Information:** <http://www.spinelas.uac.pt/AzoresConference.htm>.

4–10 **International Conference on Topology and its Applications (ICTA), 2011**, Department of Mathematics, COMSATS Institute of Information Technology (CIIT), Islamabad, Pakistan. (Oct. 2010, p. 1166)

\* 4–15 **The 2011 Gene Golub SIAM Summer School on Waves and Imaging**, University of British Columbia, Vancouver, Canada.

**Description:** Applications are encouraged for the 2011 Gene Golub SIAM Summer School, on the topic of Waves and Imaging, to take place in Vancouver on July 4–15, 2011. The summer school is for graduate students (studying toward a master’s or Ph.D.) with interests in computational mathematics.

**Deadline:** For application is February 1, 2011. The website is <http://g2s3.org/>. The event poster (<http://g2s3.org/g2s3.pdf>) looks great on notice boards.

**Information:** <http://g2s3.org>.

5–7 **The 4th Congress of the Turkic World Mathematical Society (TWMS)**, Baku, Azerbaijan. (Jan. 2011, p. 84)

5–8 **International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA-2011)**, Babeş-Bolyai University, Cluj-Napoca, Romania. (Dec. 2010, p. 1497)

6–8 **The 2011 International Conference of Applied and Engineering Mathematics**, Imperial College London, London, United Kingdom. (Nov. 2010, p. 1349)

6–8 **IMA Conference on Nonlinearity and Coherent Structures**, University of Reading, United Kingdom. (Sept. 2010, p. 1035)

\* 6–16 **Graduate Summer School: Probabilistic Models of Cognition**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.

**Description:** This summer school is motivated by recent advances which offer the promise of building rigorous models for human cognition by applying the mathematical and computational tools developed for designing artificial systems. In turn, the complexity of human cognitive abilities offers challenges which test current theories and drive the development of more advanced tools. The goal is to develop a common mathematical framework for all aspects of cognition, and review how it explains empirical phenomena in the major areas of cognitive science-including vision, memory, reasoning, learning, planning, and language. More information and an application are available online. Applications are due by March 1, 2011.

**Information:** <http://www.ipam.ucla.edu/programs/gss2011/>.

10–16 **International Conference on Rings and Algebras in Honor of Professor Pjek-Hwee Lee**, National Taiwan University, Taipei, Taiwan. (Sept. 2010, p. 1035)

\* 11–15 **Exploratory Workshop on Emerging Infectious Diseases and Mathematical Modelling**, Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain.

**Description:** Dedicated to Michel Langlais. Mathematical modeling can help in the context of emerging infectious diseases to understand the dynamical properties of emerging diseases. These questions are directly related to serious public health problems. There are many kind of diseases (ex. Nosocomial infection, Influenza, Malaria, West Niles Valley virus, SARS, etc.) which require some further analysis. Differential Equations have been extensively used in the context of mathematical biology, and the goal of this school is to focus on applications coming from epidemiology.

**Information:** <http://www.crm.cat/diseases>.

11–15 **The 10th International Conference on Finite Fields and their Applications**, Ghent, Belgium. (Jun./Jul. 2010, p. 786)

12–15 **The 6th SEAMS-GMU 2011 International Conference on Mathematics and Its Applications; Workshop on Financial Mathematics and Workshop on Dynamical System in Biology**, Gadjah Mada University, Yogyakarta, Indonesia. (Oct. 2010, p. 1166)

15–30 **XIV Summer Diffiety School**, Levi-Civita Institute, Santo Stefano del Sole (AV), Italy. (Dec. 2010, p. 1497)

18–22 **7th International Congress on Industrial and Applied Mathematics - ICIAM 2011**, Vancouver, BC, Canada. (Jun./Jul. 2010, p. 786)

\* 18–22 **AIM Workshop: Research experiences for undergraduate faculty**, American Institute of Mathematics, Palo Alto, California.

**Description:** This workshop, sponsored by AIM and the NSF, will introduce undergraduate faculty to research opportunities in several

fields of mathematics that will equip them with the tools to mentor students in undergraduate research in mathematics.

**Information:** <http://www.aimath.org/ARCC/workshops/reuf3.html>.

18–22 **Geometry & Topology Down Under – A Conference in Honour of Hyam Rubinstein**, The University of Melbourne, Australia. (Sept. 2010, p. 1035)

19–22 **The SUMMER 9th International Conference on Computing, Communications and Control Technologies: CCCT 2011**, Orlando, Florida. (Jan. 2011, p. 84)

21–23 **The 7th IMT-GT International Conference on Mathematics, Statistics, and its Applications (ICMSA 2011)**, Bangkok, Thailand. (Feb. 2011, p. 335)

21–27 **Loops' 11**, Trest, Czech Republic. (Dec. 2010, p. 1497)

25–27 **SIAM Conference on Control and Its Applications (CT11)**, Hyatt Regency Baltimore, Baltimore, Maryland. (Sept. 2010, p. 1035)

25–29 **AIM Workshop: Branching Problems for Unitary Representations**, Max Planck Institute for Mathematics, Bonn, Germany. (Nov. 2010, p. 1349)

25–29 **IMA Special Workshop: Macaulay2**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Feb. 2011, p. 336)

\* 25–29 **International Conference on Applied Mathematics, Modeling and Computational Science (Laurier Centennial Conference: AMMCS-2011)**, Waterloo, Ontario, Canada.

**Description:** The conference provides a unique opportunity for in-depth technical discussions and exchange of ideas in mathematical and computational sciences, as well as their applications in natural and social sciences, engineering and technology, industry and finance. It offers to researchers, industrialists, engineers and students to present their latest research, to interact with the experts in the field, and to foster interdisciplinary collaborations required to meet the challenges of modern science, technology, and society. The conference is a satellite meeting of the ICIAM-2011. Located just west of Toronto in the Canada Research Triangle area, the city of Waterloo is home to two universities, the Perimeter Institute for Theoretical Physics, Research in Motion, and many other innovative institutions.

**Information:** <http://www.wlu.ca/ammcs2011>.

\* 25–29 **International Conference on Groups and Semigroups: Interactions and Computations**, University of Lisbon, Portugal.

**Description:** The aim of this conference is to deepen the existing interactions between group theory and semigroup theory. The main themes of the conference include, not exclusively: the application of permutation group theory in the theory of transformation semigroups; computational techniques in group theory and semigroup theory; and combinatorial methods in group theory and semigroup theory.

**Organizer:** The Centro de Álgebra da Universidade de Lisboa (CAUL), the Centro Internacional de Matemática (CIM), and the Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa (DM-FCUL).

**Information:** <http://caul.cii.fc.ul.pt/GSConf2011>.

25–29 **Non-Associative Algebras and Related Topics**, Department of Mathematics, University of Coimbra, Coimbra, Portugal. (Feb. 2011, p. 336)

26–29 **Conference in Harmonic Analysis and Partial Differential Equations in honour of Eric Sawyer**, Fields Institute, Toronto, Canada. (Jun./Jul. 2010, p. 786)

28–30 **International Conference on Special Functions & their Applications (ICSFA 2011)**, Department of Mathematics & Statistics, J. N. Vyas University, Jodhpur (Rajasthan) 342 005, India. (Jan. 2011, p. 84)

## August 2011

1–5 **Categories, Geometry and Physics**, Santa Marta, Colombia. (Jun./Jul. 2010, p. 786)

1–5 **Conference in Honour of Søren Asmussen—New Frontiers in Applied Probability**, Sandbjerg Estate, Sønderborg, Denmark. (Jun./Jul. 2010, p. 786)

\* 1–5 **Equadiff 2011**, Loughborough University, Leicestershire, United Kingdom.

**Description:** The Equadiff is a series of biannual European conferences on theoretical aspects of differential equations held in rotation in Eastern and Western Europe. Recent locations in the Western series include Berlin (1999), Hasselt (2003) and Vienna (2007), all of which attracted in excess of 400 participants. We are very pleased to announce that the next Western series Equadiff will be held at Loughborough University in the heart of England from August 1–5, 2011. It will be the first time that the meeting has taken place in the United Kingdom. The conference will be organised around fifteen plenary lectures, twenty-five minisymposia and additional sessions for contributed papers. A number of special events will also be organised.

**Information:** <http://www.lboro.ac.uk/equadiff>.

8–13 **Formal and analytic solutions of differential and difference equations**, Mathematical Research and Conference Center in Bedlewo, Poland. (Sept. 2010, p. 1036)

8–13 **Toposym 2011: 11th Prague Topological symposium**, Prague, Czech Republic. (Jan. 2011, p. 85)

10–12 **USENIX Security 2011: 20th USENIX Security Symposium**, The Westin St. Francis, 335 Powell Street, San Francisco, California. (Feb. 2011, p. 336)

14–20 **Special Functions and Orthogonal Polynomials of Lie Groups and their Applications**, Czech Technical University in Prague, Děčín, Czech Republic. (Feb. 2011, p. 336)

15–19 **AIM Workshop: Graph and Hypergraph Limits**, American Institute of Mathematics, Palo Alto, California. (Sept. 2010, p. 1036)

17–20 **2011 Shanghai International Conference on Social Sciences (SICSS 2011)**, Shanghai, China. (Feb. 2011, p. 335)

22–24 **The 3rd International Conference on Control and Optimization with Industrial Applications: COIA 2011**, Bilkent University, Ankara, Turkey. (Jan. 2011, p. 85)

22–27 **10th International Symposium on Generalized Convexity and Monotonicity (GCM10)**, Babes-Bolyai University, Cluj-Napoca, Romania. (Jan. 2011, p. 85)

\* 29–31 **14th International Conference on Computer Analysis of Images and Patterns (CAIP 2011)**, Escuela Técnica Superior de Ingeniería Informática, University of Seville, Seville, Spain.

**Description:** CAIP2011 is the fourteenth in the CAIP series of biennial international conferences devoted to all aspects of computer vision, image analysis and processing, pattern recognition and related fields. The scope of CAIP'11 includes, but not limited to, the following areas: 3D Vision; 3D TV; biometrics; color and texture; document analysis; graph-based methods; image and video indexing and database retrieval; image and video processing; image-based modeling; kernel methods; medical imaging; mobile multimedia; model-based vision approaches; motion analysis; non-photorealistic animation and modeling; object recognition; performance evaluation; segmentation and grouping; shape representation and analysis; structural pattern recognition; tracking; applications.

**Information:** <http://congreso.us.es/caip2011/>.

29–September 2 **11th International Workshop on Orthogonal Polynomials, Special Functions, and Applications**, Universidad Carlos III de Madrid, c/Universidad, 30 28911 Leganes, Madrid, Spain. (Nov. 2010, p. 1349)

29–September 2 **European Conference on Combinatorics, Graph Theory and Applications 2011**, Alfred Renyi, Institute of Mathematics, Budapest, Hungary. (Jan. 2011, p. 85)

## September 2011

\* 1–3 **Algebraic Representation Theory Conference**, Uppsala University, Uppsala, Sweden.

**Invited speakers:** Henning Haahr Andersen (Århus); Igor Burban (Bonn); Maud De Visscher (London); Bernhard Keller (Paris); Paul Martin (Leeds); Vanessa Miemietz (UEA); Sergey Mozgovoy (Oxford); Steffen Oppermann (Trondheim).

**Organizer:** Volodymyr Mazorchuk (Uppsala)

**Information:** <http://www.math.uu.se/Conference/>.

7–9 **IMA Hot Topics Workshop: Instantaneous Frequencies and Trends for Nonstationary Nonlinear Data**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Oct. 2010, p. 1166)

10–11 **AMS Eastern Section Meeting**, Cornell University, Ithaca, New York. (Sept. 2010, p. 1036)

\* 10–17 **International Conference “Harmonic Analysis and Approximations, V”**, Tsaghkadzor, Armenia.

**Description:** The conference will be held at Yerevan State University’s guesthouse, Tsaghkadzor (Armenia). The conference is dedicated to the 75th anniversary of academician of NAS of Armenia Norair Arakelian.

**Talks:** The following mathematicians have agreed to give a plenary lecture at the conference: David Drasin (USA), Sergey Konyagin (Russia), Michael Lacey (USA), Svitlana Mayboroda (USA), Jurgen Muller (Germany), Konstantin Oskolkov (Russia), Wieslaw Plesniak (Poland), Alexei Shadrin (UK), Winfried Sickel (Germany), Mikhail Sodin (Israel), Vilmos Totik (Hungary and USA), Przemyslaw Wojtaszczyk (Poland)

**Information:** <http://math.sci.am/conference/sept2011/conf.html>.

11–17 (NEW DATE) **Turning Dreams into Reality: Transformations and Paradigm Shifts in Mathematics Education**, Rhodes University, Grahamstown, South Africa. (Feb. 2010, p. 307)

12–16 **8th International Conference on Combinatorics on Words, WORDS 2011**, Czech Technical University in Prague, Prague, Czech Republic. (Dec. 2010, p. 1498)

12–16 **25th IFIP TC 7 Conference on System Modeling and Optimization**, University of Technology, Berlin, Germany. (Nov. 2010, p. 1349)

12–16 **AIM Workshop:  $L^2$  invariants and their relatives for finitely generated groups**, American Institute of Mathematics, Palo Alto, California. (Aug. 2010, p. 905)

12–16 **Mathematical and Computational Approaches in High-Throughput Genomics**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California. (Sept. 2010, p. 1036)

26–30 (NEW DATE) **IMA Workshop: High Dimensional Phenomena**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Oct. 2010, p. 1166)

19–23 **The Sixteenth Asian Technology Conference in Mathematics (ATCM 2011)**, Abant izzet Baysal University, Bolu, Turkey. (Dec. 2010, p. 1498)

19–25 **ICNAAM2011 Symposium: Semigroups of Linear Operators and Applications**, G-Hotels, Halkidiki, Greece. (Feb. 2011, p. 336)

24–25 **AMS Western Section Meeting**, Wake Forest University, Winston Salem, North Carolina. (Sept. 2010, p. 1036)

## October 2011

\* 3 **Philosophy of Information**, American University, Washington, District of Columbia.

**Description:** The overall objective of this workshop is to study some of the open questions within philosophy of information. Interest in the philosophy and meaning of information goes back half a century but has rapidly increased recently with many new directions of research into the meaning, quantification and measures of information and complexity as well as a vast range of applications across the scientific spectrum. In this conference we will focus on just one aspect of the philosophy of information: the different techniques to measure information and to identify meaningful information.

**Hosts:** The workshop is hosted by the Info-Metrics Institute. The co-chairs are Amos Golan (AU), and Pieter Adriaans (Univ. Amsterdam).

**Information:** <http://www.american.edu/cas/economics/info-metrics/workshop/workshop-2011-fall.cfm>.

\* 10–14 **International Conference on Scientific Computing 2011 (SC2011)** dedicated to Claude Brezinski and Sebastiano Seatzu on the occasion of their 70th birthday, S. Margherita di Pula, Sardinia, Italy.

**Description:** We are organizing an international conference in October 2011 to celebrate the 70th birthday of Claude Brezinski and Sebastiano Seatzu, and, at the same time, the 20th anniversary of the Springer journal *Numerical Algorithms*. The themes of the conference will cover all aspects of numerical analysis and applied mathematics. Special sessions will be devoted to selected topics. The conference will be held at Hotel Flamingo (<http://www.hotelflamingo.it/>), a tourist resort located in S. Margherita di Pula, Sardinia, Italy. An agreement has been reached with the Hotel to obtain special reduced prices for full board accommodation. We are asking people interested in attending the Conference and receiving more information, to fill out the preregistration form available at the web page <http://bugs.unica.it/SC2011/preregistration/>. Those requesting more information can contact us at the email address: [sc2011@bugs.unica.it](mailto:sc2011@bugs.unica.it). The web site of the event is <http://bugs.unica.it/SC2011/>.

**Information:** <http://bugs.unica.it/SC2011/>.

14–16 **AMS Central Section Meeting**, University of Nebraska-Lincoln, Lincoln, Nebraska. (Sept. 2010, p. 1036)

22–23 **AMS Western Section Meeting**, University of Utah, Salt Lake City, Utah. (Sept. 2010, p. 1036)

24–28 **IMA Workshop: Large Graphs: Modeling, Algorithms and Applications**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Oct. 2010, p. 1166)

## November 2011

\* 1–5 **International conference of Settat on Operator algebras and applications**, Faculty of Sciences and Techniques, University Hassan I. Settat, Morocco.

**Description:** As a continuation of the first and the second international conferences on operator algebras and applications in Morocco, the 3rd conference ICSOAA2011, will be held in Settat November 1–5, 2011, with a similar structure. It would be a big pleasure for us to meet you there. ICSOAA 2011 in Morocco is intended to be a comprehensive, inclusive conference covering all aspects of theoretical and applied operator algebras.

**Information:** <http://www.math.ist.utl.pt/~elharti/3rd/settat2011.htm>.

\* 12 **Information Theory and Shrinkage Estimation**, American University, Washington, District of Columbia.

**Description:** Interest in shrinkage estimators goes back half a century but has rapidly increased recently with many new directions of research that cover a vast range of applications in different disciplines. Ongoing research on Information-Theoretic estimation and



inference methods is similarly inter-disciplinary, involving information theory, engineering, mathematical statistics, econometrics and the natural sciences. This one day conference will address the various themes of shrinkage estimation, the inter-connections between shrinkage estimation methodology and info-metrics, and explore recent advances in shrinkage methods and applications.

**Information:** <http://www.american.edu/cas/economics/info-metrics/workshop/workshop-2011-november.cfm>.

14-18 **IMA Workshop: Large Data Sets in Medical Informatics**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota. (Nov. 2010, p. 1349)

19-21 **International Conference on Analysis and its Applications**, Department of Mathematics, Aligarh Muslim University, Aligarh, India. (Dec. 2010, p. 1498)

## December 2011

\* 1-April 1 **Call for papers: A special issue of *Symmetry*** (ISSN 2073-8994) "Symmetry in Probability and Inference", *Symmetry Journal*, MDPI Publishing, Basel, Switzerland.

**Guest Editor:** M. Viana.

**Deadline:** April 1st, 2011.

**Description:** Papers should address any aspects of symmetry arguments in probability and statistical inference, such as, but not limited to: Constructive rules of probability and inference derived from symmetry arguments; relative probabilities; symmetric probability measures, symmetry in probability distributions, symmetry-related arguments in entropy (probabilistic) laws; epistemic probabilities and symmetry principles, symmetry arguments in the cognitive foundations of probability, statistical inference under symmetry, quantum statistical inference, asymmetric inference (in Markov processes), exchangeability and symmetry. Group-theoretic approaches to probability and inference, including those discussing aspects of symmetry invariance derived from symmetry arguments will be considered. Papers discussing covariance structures derived from symmetry arguments, for example, will also be considered. Annotated reviews may also be considered.

**Information:** [http://www.mdpi.com/journal/symmetry/special\\_issues/probability/](http://www.mdpi.com/journal/symmetry/special_issues/probability/).

\* 2-4 **Introduction to Neutrosophic Physics: Unmatter & Unparticle**, The University of New Mexico, Mathematics & Sciences Department, 200 College Rd., Gallup, New Mexico.

**Description:** This idea of unparticle was first considered by F. Smarandache in 2004, 2005 and 2006, when he uploaded a paper on CERN web site and he published three papers about what he called 'unmatter', which is a new form of matter formed by matter and antimatter that bind together. In 2006 E. Goldfain introduced the concept of "fractional number of field quanta" and he conjectured that these exotic phases of matter may emerge in the near or deep ultraviolet sector of quantum field theory. H. Georgi proposed the theory of unparticle physics in 2007 that conjectures matter that cannot be explained in terms of particles using the Standard Model of particle physics, because its components are scale invariant. Fragments from Wikipedia Papers on current trends in High Energy Physics about exotic matter, about connections between unmatter and unparticle, about Neutrosophic Logic as new research in Theoretical Physics, should be sent to the organizer preferably by email.

**Information:** <http://www.gallup.unm.edu/~smarandache/unmatter.htm>.

4-9 **LISA'11: 25th Large Installation System Administration Conference**, Sheraton Boston Hotel at 39 Dalton St., Boston, Massachusetts. (Feb. 2011, p. 336)

\* 14-16 **5th Indian International Conference on Artificial Intelligence**, Tumkur (near Bangalore), India.

**Description:** 5th Indian International Conference on Artificial Intelligence (IICAI-11) will be held during December 14-16, 2011, in

Tumkur (near Bangalore), India. IICAI is a series of high quality technical events in Artificial Intelligence (AI) and is also one of the major AI events in the world.

**Information:** <http://www.iiconference.org/>.

16-18 **The International Congress on Science and Technology**, Allahabad, U.P., INDIA (Oct. 2009, p. 1148)

## February 2012

27-March 2 **IMA Workshop: Network Links: Connecting Social, Communication and Biological Network Analysis**, Institute for Mathematics and its Applications (IMA), University of Minnesota, Minneapolis, Minnesota.

**The following new announcements will not be repeated until the criteria in the next to the last paragraph at the bottom of the first page of this section are met.**

## March 2012

\* 12-15 **Computational Methods in High Energy Density Plasmas**, Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California.

**Description:** High energy density physics (HEDP) is a rapidly growing field. HEDP conditions are typically from Mbar to tens of Gbar pressures and temperatures ranging from eV to GeV. This long program will focus on the computational approaches to the modeling of these extreme states of matter. It will address the scientific challenges facing the computational HEDP community and discuss the successes and failures of various methods. The long program will establish an interdisciplinary forum for researchers in HEDP. Funding for participants is available at all academic levels, though recent PhD's, graduate students, and researchers in the early stages of their career are especially encouraged to apply. Enhancing the careers of women and minority mathematicians and scientists is an important component of IPAM's mission and we welcome their applications. An application is available online. Applications will be accepted through November 12, 2011 but decisions will be made starting in June.

**Information:** <http://www.ipam.ucla.edu/programs/pl2012/>.

\* 30-31 **Information and Econometrics of Networks**, American University, Washington, District of Columbia.

**Description:** Social and economic networks are everywhere: from Facebook to the more complex global financial network or to networks connecting economic agents or to other complex and dynamic economic networks. The study of these networks is crucial for both academics and policy makers and presents a host of new theoretical and econometric challenges. This conference will concentrate on studying the information and econometrics of networks.

**Hosts:** The two-day conference is hosted by the Info-Metrics Institute and supported (partially) by the *Journal of Applied Econometrics* and will be organized jointly by Amos Golan (AU) and Essie Maasoumi (Emory).

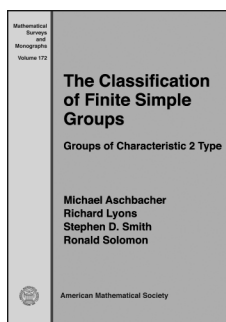
**Information:** <http://www.american.edu/cas/economics/info-metrics/workshop/workshop-2012-spring.g.cfm>.



# New Publications Offered by the AMS

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please go to <http://www.ams.org/bookstore-email>.

## Algebra and Algebraic Geometry



### The Classification of Finite Simple Groups

Groups of Characteristic 2  
Type

**Michael Aschbacher**, *California Institute of Technology, Pasadena, CA*, **Richard Lyons**, *Rutgers University, Piscataway, NJ*, **Stephen D. Smith**, *University of Illinois at Chicago, IL*, and **Ronald Solomon**, *The Ohio State University, Columbus, OH*

The book provides an outline and modern overview of the classification of the finite simple groups. It primarily covers the “even case”, where the main groups arising are Lie-type (matrix) groups over a field of characteristic 2. The book thus completes a project begun by Daniel Gorenstein’s 1983 book, which outlined the classification of groups of “noncharacteristic 2 type”.

However, this book provides much more. Chapter 0 is a modern overview of the logical structure of the entire classification. Chapter 1 is a concise but complete outline of the “odd case” with updated references, while Chapter 2 sets the stage for the remainder of the book with a similar outline of the “even case”. The remaining six chapters describe in detail the fundamental results whose union completes the proof of the classification theorem. Several important subsidiary results are also discussed. In addition, there is a comprehensive listing of the large number of papers referenced from the literature. Appendices provide a brief but valuable modern introduction to many key ideas and techniques of the proof. Some improved arguments are developed, along with indications of new approaches to the entire classification—such as the second and third generation projects—although there is no attempt to cover them comprehensively.

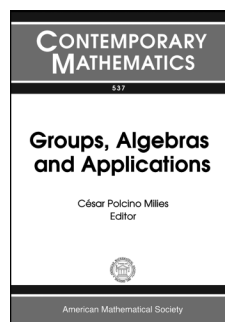
The work should appeal to a broad range of mathematicians—from those who just want an overview of the main ideas of the classification, to those who want a reader’s guide to help navigate

some of the major papers, and to those who may wish to improve the existing proofs.

**Contents:** *Background and overview:* Introduction; Overview: The classification of groups of Gorenstein-Walter type; Overview: The classification of groups of characteristic 2 type; *Outline of the classification of groups of characteristic 2 type:*  $e(G) \leq 2$ : The classification of quasithin groups;  $e(G) = 3$ : The classification of rank 3 groups;  $e(G) \geq 4$ : The pretrichotomy and trichotomy theorems; The classification of groups of standard type; The classification of groups of  $GF(2)$  type; The final contradiction: Eliminating the Uniqueness Case; *Appendices:* Some background material related to simple groups; Overview of some techniques used in the classification; *References and index:* References used for both  $GW$  type and characteristic 2 type; References mainly for  $GW$  type; References used primarily for characteristic 2 type; Expository references mentioned; Index.

**Mathematical Surveys and Monographs**, Volume 172

March 2011, 347 pages, Hardcover, ISBN: 978-0-8218-5336-8, LC 2010048011, 2010 *Mathematics Subject Classification*: 20D05; 20C20, **AMS members US\$75.20**, List US\$94, Order code SURV/172



### Groups, Algebras and Applications

**César Polcino Milles**, *University of São Paulo, Brazil*, Editor

This book contains the proceedings of the XVIII Latin American Algebra Colloquium, held from August 3–8, 2009, in São Paulo, Brazil.

It includes research articles as well as up-to-date surveys covering several directions of current research in algebra, such as Asymptotic Codimension Growth, Hopf Algebras, Structure Theory of both Associative and Non-Associative Algebras, Partial Actions of Groups on Rings, and contributions to Coding Theory.

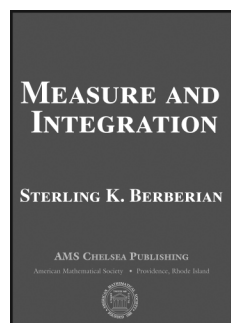
**Contents:** **R. Alfaro**, Linear codes over  $\mathbb{F}_q[u]/(u^t)$ ; **M. M. S. Alves** and **E. Batista**, Globalization theorems for partial Hopf (co)actions, and some of their applications; **N. Andruskiewitsch**, **F. Fantino**, **G. A. García**, and **L. Vendramin**, On Nichols algebras associated to simple racks; **I. Angiono** and **A. G. Iglesias**, Pointed Hopf algebras with standard braiding are generated in degree one; **A. Berele** and **A. Regev**, Asymptotics of Young tableaux in the  $(k, \ell)$

hook; **C. Boyallian** and **J. I. Liberati**, Classification of irreducible representations over finite simple Lie conformal superalgebras; **C. Carvalho**, Pure gaps and bounds for the generalized Hamming weights of Goppa codes; **F. N. Castro**, **L. A. Medina**, and **I. M. Rubio**, Exact divisibility of exponential sums over the binary field via the covering method; **W. Cortes**, **V. Rodrigues**, and **A. Sant'Ana**, All hereditary torsion theories are higher differential; **A. Davydov** and **A. Molev**, A categorical approach to classical and quantum Schur-Weyl duality; **M. Dokuchaev**, Partial actions: A survey; **V. O. Ferreira** and **L. S. I. Murakami**, On free associative algebras linearly graded by finite groups; **V. Futorny**, **S. Ovsienko**, and **M. Saorin**, Gelfand-Tsetlin categories; **A. Giambruno** and **E. Zelmanov**, On growth of codimensions of Jordan algebras; **L. Gutiérrez-Frez**, **J. Pantoja**, and **J. Soto-Andrade**, Geometric Weil representations for star-analogues of  $SL(2, k)$ ; **K. Igusa**, Exceptional sequences, braid groups and clusters; **F. Levstein** and **L. Saal**, Spherical distributions of some generalized Gelfand pairs attached to the Heisenberg group; **C. A. López-Andrade** and **H. Tapia-Recillas**, On the linearity and quasi-cyclicity of the Gray image of codes over a Galois ring; **M. Lorenz**, Some applications of Frobenius algebras to Hopf algebras; **J. Tirao**, The algebra of differential operators associated to a weight matrix: A first example.

**Contemporary Mathematics**, Volume 537

April 2011, 324 pages, Softcover, ISBN: 978-0-8218-5239-2, LC 2010045183, 2010 *Mathematics Subject Classification*: 05A16, 11T71, 16D90, 16T05, 16S34, 17B69, 20C15; 16L60, 20C33, 20F36, **AMS members US\$84**, List US\$105, Order code CONM/537

## Analysis



### Measure and Integration

**Sterling K. Berberian**

This highly flexible text is organized into two parts: Part I is suitable for a one-semester course at the first-year graduate level, and the book as a whole is suitable for a full-year course.

Part I treats the theory of measure and integration over abstract measure spaces.

Prerequisites are a familiarity with epsilon-delta arguments and with the language of naive set theory (union, intersection, function). The fundamental theorems of the subject are derived from first principles, with details in full. Highlights include convergence theorems (monotone, dominated), completeness of classical function spaces (Riesz-Fischer theorem), product measures (Fubini's theorem), and signed measures (Radon-Nikodym theorem).

Part II is more specialized; it includes regular measures on locally compact spaces, the Riesz-Markoff theorem on the measure-theoretic representation of positive linear forms, and Haar measure on a locally compact group. The group algebra of a locally compact group is constructed in the last chapter, by an especially transparent method that minimizes measure-theoretic difficulties. Prerequisites for Part II include Part I plus a course in general topology.

To quote from the Preface:

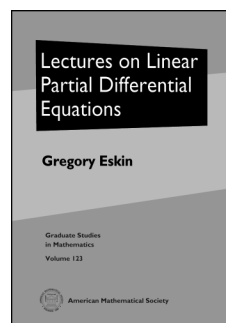
"Finally, I am under no illusions as to originality, for the subject of measure theory is an old one which has been worked over by many experts. My contribution can only be in selection, arrangement, and emphasis. I am deeply indebted to Paul R. Halmos, from whose textbook I first studied measure theory; I hope that these pages may reflect their debt to his book without seeming to be almost everywhere equal to it."

**Contents:** Measures; Measurable functions; Sequences of measurable functions; Integrable functions; Convergence theorems; Product measures; Finite signed measures; Integration over locally compact spaces; Integration over locally compact groups; References and notes; Bibliography; Index.

**AMS Chelsea Publishing**, Volume 241

April 2011, 312 pages, Hardcover, ISBN: 978-0-8218-5328-3, LC 74-128871, 2010 *Mathematics Subject Classification*: 28-01, **AMS members US\$45.90**, List US\$51, Order code CHEL/241.H

## Differential Equations



### Lectures on Linear Partial Differential Equations

**Gregory Eskin**, *University of California, Los Angeles, CA*

This book is a reader-friendly, relatively short introduction to the modern theory of linear partial differential equations. An effort has been made to present complete proofs in an accessible and self-contained form.

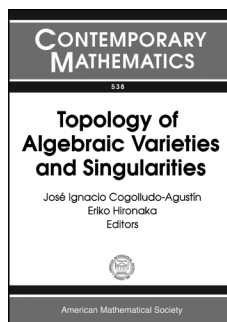
The first three chapters are on elementary distribution theory and Sobolev spaces with many examples and applications to equations with constant coefficients. The following chapters study the Cauchy problem for parabolic and hyperbolic equations, boundary value problems for elliptic equations, heat trace asymptotics, and scattering theory. The book also covers microlocal analysis, including the theory of pseudodifferential and Fourier integral operators, and the propagation of singularities for operators of real principal type. Among the more advanced topics are the global theory of Fourier integral operators and the geometric optics construction in the large, the Atiyah-Singer index theorem in  $\mathbb{R}^n$ , and the oblique derivative problem.

**Contents:** Theory of distributions; Fourier transforms; Applications of distributions to partial differential equations; Second order elliptic equations in bounded domains; The scattering theory; Pseudodifferential operators; Elliptic boundary value problems and parametrices; Fourier integral operators; Index.

**Graduate Studies in Mathematics**, Volume 123

May 2011, approximately 411 pages, Hardcover, ISBN: 978-0-8218-5284-2, 2010 *Mathematics Subject Classification*: 35J25, 35L40, 35K30, 35L05, 35L30, 35P20, 35P25, 35S05, 35S30, **AMS members US\$59.20**, List US\$74, Order code GSM/123

# Geometry and Topology



## Topology of Algebraic Varieties and Singularities

José Ignacio Cogolludo-Agustín,  
*Universidad de Zaragoza, Spain,*  
and Eriko Hironaka, *Florida*  
*State University, Tallahassee, FL,*  
Editors

This volume contains invited expository and research papers from the conference Topology of Algebraic Varieties, in honor of Anatoly Libgober's 60th birthday, held June 22–26, 2009, in Jaca, Spain.

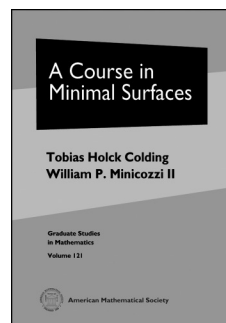
This volume contains four parts corresponding to the four main focal points of the conference: algebraic geometry and fundamental groups, braids and knots, hyperplane arrangements, and singularities. Together, the papers provide an overview of the current status of a broad range of topological questions in Algebraic Geometry.

*This item will also be of interest to those working in algebra and algebraic geometry.*

**Contents:** *A historical note:* A. Libgober, Development of the theory of Alexander invariants in algebraic geometry; *Algebraic geometry and fundamental groups:* P. Aluffi and M. Marcolli, Feynman motives and deletion-contraction relations; M. Amram, R. Lehman, R. Shwartz, and M. Teicher, Classification of fundamental groups of Galois covers of surfaces of small degree degenerating to nice plane arrangements; D. Arapura, Homomorphisms between Kähler groups; E. A. Artal-Bartolo, J. I. Cogolludo-Agustín, and D. Matei, Quasi-projectivity, Artin-Tits groups, and pencil maps; A. Degtyarev, Topology of plane algebraic curves: The algebraic approach; L. Maxim and J. Schürmann, Hirzebruch invariants of symmetric products; A. I. Suciu, Fundamental groups, Alexander invariants, and cohomology jumping loci; *Braids and knots:* J. González-Meneses, On reduction curves and Garside properties of braids; L. H. Kauffman, Topological quantum information, Khovanov homology and the Jones polynomial; L. Paris, HOMFLYPT skein module of singular links; *Hyperplane arrangements:* N. Budur, A. Dimca, and M. Saito, First Milnor cohomology of hyperplane arrangements; G. Gaiffi, F. Mori, and M. Salvetti, Minimal CW-complexes for complements to line arrangements via discrete Morse theory; R. Randell, The topology of hyperplane arrangements; *Singularities:* E. Artal-Bartolo, P. Cassou-Noguès, I. Luengo, and A. M. Melle-Hernández, On  $v$ -quasi-ordinary power series: Factorization, Newton trees and resultants; J. L. Cisneros-Molina, J. Seade, and J. Snoussi, Milnor fibrations for real and complex singularities; J. F. de Bobadilla, On homotopy types of complements of analytic sets and Milnor fibres; D. Kerner and A. Némethi, The Milnor fibre signature is not semi-continuous; A. Némethi, W. D. Neumann, and A. Pichon, Principal analytic link theory in homology sphere links; M. Oka, On mixed Brieskorn variety; P. Popescu-Pampu, Introduction to Jung's method of resolution of singularities; S. S.-T. Yau, L. Zhao, and H. Zuo, Biggest sharp polynomial estimate of integral points in right-angled simplices.

Contemporary Mathematics, Volume 538

April 2011, approximately 484 pages, Softcover, ISBN: 978-0-8218-4890-6, LC 2010047841, 2010 *Mathematics Subject Classification:* 00B99, 14B05, 32S99, 57M99, 14-02, 32-02, 51-02, 53-02, 55-02, 57-02, **AMS members US\$108**, List US\$135, Order code CONM/538



## A Course in Minimal Surfaces

Tobias Holck Colding,  
*Massachusetts Institute of*  
*Technology, Cambridge, MA,* and  
William P. Minicozzi II, *Johns*  
*Hopkins University, Baltimore,*  
*MD*

Minimal surfaces date back to Euler and Lagrange and the beginning of the calculus

of variations. Many of the techniques developed have played key roles in geometry and partial differential equations. Examples include monotonicity and tangent cone analysis originating in the regularity theory for minimal surfaces, estimates for nonlinear equations based on the maximum principle arising in Bernstein's classical work, and even Lebesgue's definition of the integral that he developed in his thesis on the Plateau problem for minimal surfaces.

This book starts with the classical theory of minimal surfaces and ends up with current research topics. Of the various ways of approaching minimal surfaces (from complex analysis, PDE, or geometric measure theory), the authors have chosen to focus on the PDE aspects of the theory. The book also contains some of the applications of minimal surfaces to other fields including low dimensional topology, general relativity, and materials science.

The only prerequisites needed for this book are a basic knowledge of Riemannian geometry and some familiarity with the maximum principle.

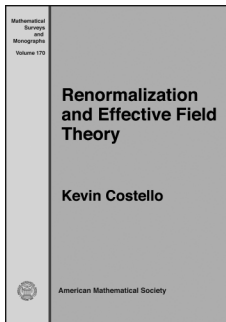
*This item will also be of interest to those working in differential equations.*

**Contents:** The variation formulas and some consequences; Curvature estimates and consequences; Weak Bernstein-type theorems; Existence results; Min-max constructions; Embedded solutions of the Plateau problem; Minimal surfaces in three-manifolds; The structure of embedded minimal surfaces; Exercises; Bibliography; Index.

Graduate Studies in Mathematics, Volume 121

April 2011, approximately 320 pages, Hardcover, ISBN: 978-0-8218-5323-8, LC 2010044373, 2010 *Mathematics Subject Classification:* 49Q05, 53A10, 53C42, 58E12, 57M50, 57N10, 35J15, 35J60, 83C57, **AMS members US\$50.40**, List US\$63, Order code GSM/121

## Mathematical Physics



### Renormalization and Effective Field Theory

**Kevin Costello**, *Northwestern University, Evanston, IL*

*This book tells mathematicians about an amazing subject invented by physicists and it tells physicists how a master mathematician must proceed in order to understand it. Physicists who know quantum field theory can learn the powerful methodology of mathematical*

*structure, while mathematicians can position themselves to use the magical ideas of quantum field theory in "mathematics" itself. The retelling of the tale mathematically by Kevin Costello is a beautiful tour de force.*

—**Dennis Sullivan**

*This book is quite a remarkable contribution. It should make perturbative quantum field theory accessible to mathematicians. There is a lot of insight in the way the author uses the renormalization group and effective field theory to analyze perturbative renormalization; this may serve as a springboard to a wider use of those topics, hopefully to an eventual nonperturbative understanding.*

—**Edward Witten**

Quantum field theory has had a profound influence on mathematics, and on geometry in particular. However, the notorious difficulties of renormalization have made quantum field theory very inaccessible for mathematicians. This book provides complete mathematical foundations for the theory of perturbative quantum field theory, based on Wilson's ideas of low-energy effective field theory and on the Batalin-Vilkovisky formalism. As an example, a cohomological proof of perturbative renormalizability of Yang-Mills theory is presented.

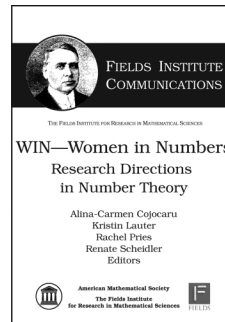
An effort has been made to make the book accessible to mathematicians who have had no prior exposure to quantum field theory. Graduate students who have taken classes in basic functional analysis and homological algebra should be able to read this book.

**Contents:** Introduction; Theories, Lagrangians and counterterms; Field theories on  $\mathbb{R}^n$ ; Renormalizability; Gauge symmetry and the Batalin-Vilkovisky formalism; Renormalizability of Yang-Mills theory; Asymptotics of graph integrals; Nuclear spaces; Bibliography.

**Mathematical Surveys and Monographs**, Volume 170

March 2011, 251 pages, Hardcover, ISBN: 978-0-8218-5288-0, LC 2010047463, 2010 *Mathematics Subject Classification*: 81T13, 81T15, 81T17, 81T18, 81T20, 81T70, **AMS members US\$67.20**, List US\$84, Order code SURV/170

## Number Theory



### WIN—Women in Numbers

**Research Directions in Number Theory**

**Alina-Carmen Cojocaru**, *University of Illinois at Chicago, IL, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania*, **Kristin Lauter**, *Microsoft Research, Redmond, WA*, **Rachel Pries**, *Colorado State University, Fort Collins, CO*, and **Renate Scheidler**, *University of Calgary, AB, Canada*, Editors

This volume is a collection of papers on number theory which evolved out of the workshop WIN—Women In Numbers, held November 2–7, 2008, at the Banff International Research Station (BIRS) in Banff, Alberta, Canada. It includes articles showcasing outcomes from collaborative research initiated during the workshop as well as survey papers aimed at introducing graduate students and recent PhDs to important research topics in number theory.

The contributions in this volume span a wide range of topics in arithmetic geometry and algebraic, algorithmic, and analytic number theory. Clusters of papers center around the four topics of moduli spaces and Shimura curves, curves and Jacobians over finite fields, Galois covers of function fields in positive characteristic, and zeta functions of graphs, with a fifth group of three individual articles on modular forms, Iwasawa theory, and Galois representations, respectively.

The workshop and this volume are part of a broader WIN initiative, whose goals are to highlight and increase the research activities of women in number theory and to train female graduate students in number theory and related fields.

*This item will also be of interest to those working in algebra and algebraic geometry.*

Titles in this series are co-published with the Fields Institute for Research in Mathematical Sciences (Toronto, Ontario, Canada).

**Contents:** Part I. *Moduli spaces and Shimura curves*: **H. Grundman**, Hilbert modular variety compositions; **P. Bayer**, Contributions to Shimura curves; **H. Grundman**, **J. Johnson-Leung**, **K. Lauter**, **A. Salerno**, **B. Viray**, and **E. Wittenborn**, Igusa class polynomials, embeddings of quartic CM fields, and arithmetic intersection theory; **E. Mantovan**,  $\ell$ -adic étale cohomology of PEL type Shimura varieties with non-trivial coefficients; Part II. *Curves and Jacobians over finite fields*: **J. Balakrishnan**, **J. Belding**, **S. Chisholm**, **K. Eisenträger**, **K. E. Stange**, and **E. Teske**, Pairings on hyperelliptic curves; **A. Bucur**, **C. David**, **B. Feigon**, and **M. Lalin**, Biased statistics for traces of cyclic  $p$ -fold covers over finite fields; **L. Berger**, **J. L. Hoelscher**, **Y. Lee**, **J. Paulhus**, and **R. Scheidler**, The  $\ell$ -rank structure of a global function field; Part III. *Galois covers of function fields in positive characteristic*: **R. Pries** and **K. Stevenson**, A survey of Galois theory of curves in characteristic  $p$ ; **I. I. Bouw**, Covers of

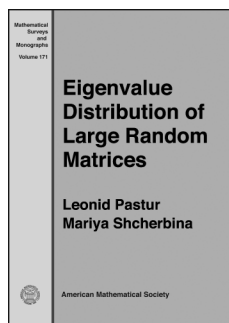


the affine line in positive characteristic with prescribed ramification; **L. Gruendken**, **L. Hall-Seelig**, **B.-H. Im**, **E. Ozman**, **R. Pries**, and **K. Stevenson**, Semi-direct Galois covers of the affine line; Part IV. *Zeta functions of graphs*: **A. Terras**, Looking into a graph theory mirror of number theoretic zetas; **W.-C. W. Li**, Zeta functions of group based graphs and complexes; **B. Malmskog** and **M. Manes**, Ramified covers of graphs and the Ihara zeta functions of certain ramified covers; Part V. *Other topics*: **S. A. Garthwaite**, **L. Long**, **H. Swisher**, and **S. Treneer**, Zeros of classical Eisenstein series and recent developments; **S. Ramdorai**, On the  $\mu$ -invariant in Iwasawa theory; **S. Arias-de-Reyna** and **N. Vila**, Galois representations and the tame inverse Galois problem.

**Fields Institute Communications**, Volume 60

March 2011, 288 pages, Hardcover, ISBN: 978-0-8218-5226-2, 2010 *Mathematics Subject Classification*: 11-06, 11-02, 14-06, 11Gxx, 11Mxx, 11Rxx, 11Yxx, 14Hxx, 14Kxx, 14Qxx, **AMS members US\$85.60**, List US\$107, Order code FIC/60

## Probability and Statistics



### Eigenvalue Distribution of Large Random Matrices

**Leonid Pastur** and **Mariya Shcherbina**, *Ukrainian National Academy of Sciences, Kharkov, Ukraine*

Random matrix theory is a wide and growing field with a variety of concepts, results, and techniques and a vast range of applications in mathematics and the related sciences. The book, written by well-known experts, offers beginners a fairly balanced collection of basic facts and methods (Part 1 on classical ensembles) and presents experts with an exposition of recent advances in the subject (Parts 2 and 3 on invariant ensembles and ensembles with independent entries).

The text includes many of the authors' results and methods on several main aspects of the theory, thus allowing them to present a unique and personal perspective on the subject and to cover many topics using a unified approach essentially based on the Stieltjes transform and orthogonal polynomials. The exposition is supplemented by numerous comments, remarks, and problems. This results in a book that presents a detailed and self-contained treatment of the basic random matrix ensembles and asymptotic regimes.

This book will be an important reference for researchers in a variety of areas of mathematics and mathematical physics. Various chapters of the book can be used for graduate courses; the main prerequisite is a basic knowledge of calculus, linear algebra, and probability theory.

**Contents**: Introduction; *Classical ensembles*: Gaussian ensembles: Semicircle law; Gaussian ensembles: Central limit theorem for linear eigenvalue statistics; Gaussian ensembles: Joint eigenvalue distribution and related results; Gaussian unitary ensemble; Gaussian orthogonal ensemble; Wishart and Laguerre ensembles; Classical compact groups ensembles: Global regime; Classical

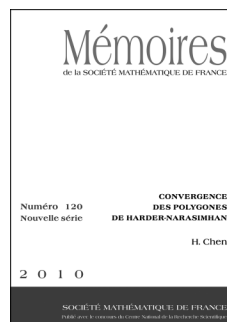
compact groups ensembles: Local regime; Law of addition of random matrices; *Matrix models*: Matrix models: Global regime; Bulk universality for hermitian matrix models; Universality for special points of hermitian matrix models; Jacobi matrices and limiting laws for linear eigenvalue statistics; Universality for real symmetric matrix models; Unitary matrix models; *Ensembles with independent and weakly dependent entries*: Matrices with Gaussian correlated entries; Wigner ensembles; Sample covariance and related matrices; Bibliography; Index.

**Mathematical Surveys and Monographs**, Volume 171

May 2011, approximately 634 pages, Hardcover, ISBN: 978-0-8218-5285-9, LC 2010047315, 2010 *Mathematics Subject Classification*: 60F05, 60B20, 15B52, 15B57, **AMS members US\$84**, List US\$105, Order code SURV/171

## New AMS-Distributed Publications

### Algebra and Algebraic Geometry



### Convergence des Polygones de Harder-Narasimhan

**Huayi Chen**, *Université Paris Diderot, France*

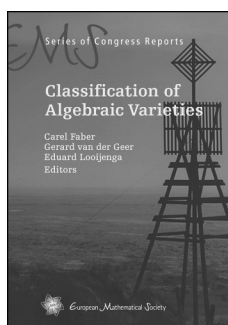
The author interprets the theory of Harder-Narasimhan polygons by the language of  $\mathbb{R}$ -filtrations. By using a variant version of Fekete's lemma and a combinatoric argument on monomials, he establishes the uniform convergence of polygons associated to a graded algebra equipped with filtrations. This leads to the existence of several arithmetic invariants, a very particular case of which is the sectional capacity. Two applications in Arakelov geometry are developed: the arithmetic Hilbert-Samuel theorem and the existence and the geometric interpretation of the asymptotic maximal slope.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Contents**: Introduction; Rappels et préliminaires; Filtrations de Harder-Narasimhan; Convergence des polygones; Applications; Bibliographie.

**Mémoires de la Société Mathématique de France**, Number 120

October 2010, 120 pages, Softcover, ISBN: 978-2-85629-296-9, 2010 *Mathematics Subject Classification*: 14G40, 14F05, **Individual member US\$37.80**, List US\$42, Order code SMFMEM/120



## Classification of Algebraic Varieties

**Carel Faber**, *Royal Institute of Technology, Stockholm, Sweden*,  
**Gerard van der Geer**, *University of Amsterdam, The Netherlands*,  
 and **Eduard Looijenga**, *Utrecht University, The Netherlands*,  
 Editors

Fascinating and surprising developments are taking place in the classification of algebraic varieties. The work of Hacon and McKernan and many others is causing a wave of breakthroughs in the minimal model program: we now know that for a smooth projective variety the canonical ring is finitely generated. These new results and methods are reshaping the field.

Inspired by this exciting progress, the editors organized a meeting at Schiermonnikoog and invited leading experts to write papers about the recent developments. The result is the present volume, a lively testimony to the sudden advances that originate from these new ideas.

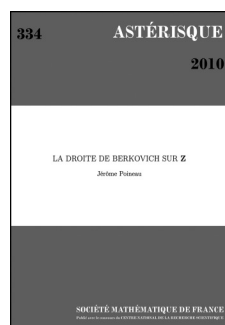
This volume will be of interest to a wide range of pure mathematicians, but will appeal especially to algebraic and analytic geometers.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**Contents:** D. Abramovich and B. Hassett, Stable varieties with a twist; F. Ambro, Basic properties of log canonical centers; I. Bauer and F. Catanese, Burniat surfaces I: Fundamental groups of moduli of primary Burniat surfaces; C. Birkar and M. Păun, Minimal models, flip and finite generation: A tribute to V. V. Shokurov and Y.-T. Siu; F. Campana, Remarks on an example of K. Ueno; F. Campana, Special orbifolds and birational classification: A survey; J. A. Chen, Birational geometry of threefolds; C. Ciliberto, O. Dumitrescu, R. Miranda, and J. Roé, Emptiness of homogeneous linear systems with ten general base points; A. Corti, Finite generation of adjoint rings after Lazić: An introduction; T. de Fernex, L. Ein, and M. Mustață, Log canonical thresholds on varieties with bounded singularities; G. Farkas, Brill-Noether geometry on moduli spaces of spin curves; K. Frantzen and T. Peternell, On the bimeromorphic geometry of compact complex contact threefolds; O. Fujino, Introduction to the theory of quasi-log varieties; O. Fujino, On Kawamata's theorem; Y. Kawamata, Remarks on the cone of divisors; Y. Prokhorov,  $p$ -elementary subgroups of the Cremona group of rank 3.

EMS Series of Congress Reports, Volume 3

January 2011, 346 pages, Hardcover, ISBN: 978-3-03719-007-4, 2010 *Mathematics Subject Classification*: 14Exx, 14E30, 14Dxx, 14J10, AMS members US\$78.40, List US\$98, Order code EMSSCR/3



## La Droite de Berkovich Sur Z

**Jérôme Poineau**, *Institut de Recherche Mathématique Avancée, Strasbourg, France*

This text is devoted to the study of the Berkovich line over the ring of integers of a number field. It is a geometric object which naturally contains complex analytic lines (or their quotient by conjugation),

associated to the infinite places, and classical Berkovich lines over complete valued fields, associated to the finite places. The authors prove that it satisfies nice properties, both from the topological and algebraic points of view. They also provide a few examples of Stein spaces that are contained in this line. The authors explain how this theory may be used to address various questions about convergent arithmetic power series: prescribing zeroes and poles, proving that global rings are Noetherian or constructing Galois groups over them. Typical examples of such series are given by holomorphic functions on the complex open unit disc whose Taylor developments in 0 have integer coefficients.

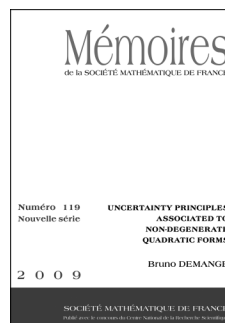
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Contents:** Espaces analytiques sur un anneau de Banach; Algèbres de séries convergentes; Espace affine analytique au-dessus d'un anneau d'entiers de corps de nombres; Droite affine analytique au-dessus d'un anneau d'entiers de corps de nombres; Morphismes finis; Espaces de Stein; Applications; Bibliographie; Glossaire des notations; Index.

Astérisque, Number 334

November 2010, 284 pages, Softcover, ISBN: 978-2-85629-294-5, 2010 *Mathematics Subject Classification*: 14G22, 14G25, 30B10, 13E05, 12F12, **Individual member US\$94.50**, List US\$105, Order code AST/334

## Analysis



## Uncertainty Principles Associated to Non-Degenerate Quadratic Forms

**Bruno Demange**, *Institut Fourier, St. Martin d'Heres, France*

This volume is devoted to several generalizations of the classical Hardy uncertainty principle on Euclidean spaces.

Instead of comparing functions and their Fourier transforms to a Gaussian, the author compares them to the exponential of general nondegenerate quadratic forms like the Lorentz form, for example. Using the Bargmann transform, he translates the problem into the description of several classes of analytic functions of several

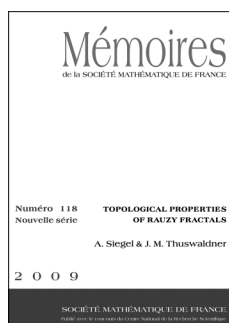
variables and at the same time simplifies and unifies proofs of results presented in several previous papers.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Contents:** Introduction; Hardy's uncertainty principle and its generalizations; Further results; Critical and non critical pairs; Critical pairs; Lorentz quadratic form; Bibliography.

**Mémoires de la Société Mathématique de France**, Number 119

October 2010, 102 pages, Softcover, ISBN: 978-2-85629-297-6, 2010 *Mathematics Subject Classification*: 30H99, 32A15, 42B10, **Individual member US\$37.80**, List US\$42, Order code SMFMEM/119



## Topological Properties of Rauzy Fractals

Anne Siegel, *Université de Rennes, France*, and Jörg M. Thuswaldner, *University of Leoben, Austria*

Substitutions are combinatorial objects (one replaces a letter by a word), which produce sequences by iteration. They occur in many mathematical fields,

roughly as soon as a repetitive process appears. In this monograph the authors deal with topological and geometric properties of substitutions; in particular, they study properties of the *Rauzy fractals* associated to substitutions.

To be more precise, let  $\sigma$  be a substitution over the finite alphabet  $\mathcal{A}$ . The authors assume that the incidence matrix of  $\sigma$  is primitive and that its dominant eigenvalue is a unit Pisot number (i.e., an algebraic integer greater than one whose norm is equal to one and all of whose Galois conjugates are of modulus strictly smaller than one). It is well known that one can attach to  $\sigma$  a set  $\mathcal{T}$  which is called *central tile* or *Rauzy fractal* of  $\sigma$ . Such a central tile is a compact set that is the closure of its interior and decomposes in a natural way in  $n = |\mathcal{A}|$  subtiles  $\mathcal{T}(1), \dots, \mathcal{T}(n)$ . The central tile, as well as its subtiles, are graph directed self-affine sets that often have fractal boundary.

Pisot substitutions and central tiles are of high relevance in several branches of mathematics such as tiling theory, spectral theory, Diophantine approximation, the construction of discrete planes and quasicrystals as well as in connection with numeration like generalized continued fractions and radix representations. The questions raised in all these domains can often be reformulated in terms of questions related to the topology and the geometry of the underlying central tile.

After a thorough survey of important properties of unit Pisot substitutions and their associated Rauzy fractals, the authors investigate a variety of topological properties of  $\mathcal{T}$  and its subtiles. Their approach is an algorithmic one. In particular, they address the question whether  $\mathcal{T}$  and its subtiles induce a tiling, calculate the Hausdorff dimension of their boundary, give criteria for their connectivity and homeomorphy to a closed disk, and derive properties of their fundamental group.

The basic tools for the authors' criteria are several classes of graphs built from the description of the tiles  $\mathcal{T}(i)$  ( $1 \leq i \leq n$ ) as the

solution of a graph directed iterated function system and from the structure of the tilings induced by these tiles. These graphs are of interest in their own right. For instance, they can be used to construct the boundaries  $\partial\mathcal{T}$  as well as  $\partial\mathcal{T}(i)$  ( $1 \leq i \leq n$ ) and all points where two, three, or four different tiles of the induced tilings meet.

When working with central tiles in one of the above-mentioned contexts it is often useful to know such intersection properties of tiles. In this sense this monograph aims to provide tools for "everyday life" when dealing with topological and geometric properties of substitutions.

Throughout the text, the authors give many examples to illustrate their results and also offer suggestions for further research.

*This item will also be of interest to those working in number theory.*

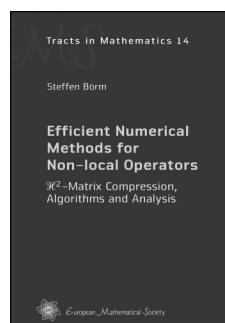
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Contents:** Introduction; Substitutions, central tiles and beta-numeration; Multiple tilings induced by the central tile and its subtiles; Statement of the main results: Topological properties of central tiles; Graphs that contain topological information on the central tile; Exact statements and proofs of the main results; Technical proofs and definitions; Perspectives; Bibliography.

**Mémoires de la Société Mathématique de France**, Number 118

November 2010, 144 pages, Softcover, ISBN: 978-2-85629-290-7, 2010 *Mathematics Subject Classification*: 28A80, 11A63, 54F65, **Individual member US\$37.80**, List US\$42, Order code SMFMEM/118

## Applications



## Efficient Numerical Methods for Non-local Operators

$\mathcal{H}^2$ -Matrix Compression, Algorithms and Analysis

Steffen Börm, *Kiel University, Germany*

Hierarchical matrices present an efficient way of treating dense matrices that arise in the context of integral equations, elliptic partial differential equations, and control theory.

While a dense  $n \times n$  matrix in standard representation requires  $n^2$  units of storage, a hierarchical matrix can approximate the matrix in a compact representation requiring only  $O(nk \log n)$  units of storage, where  $k$  is a parameter controlling the accuracy. Hierarchical matrices have been successfully applied to approximate matrices arising in the context of boundary integral methods, to construct preconditioners for partial differential equations, to evaluate matrix functions, and to solve matrix equations used in control theory.  $\mathcal{H}^2$ -matrices offer a refinement of hierarchical matrices: Using a multilevel representation

of submatrices, the efficiency can be significantly improved, particularly for large problems.

This book gives an introduction to the basic concepts and presents a general framework that can be used to analyze the complexity and accuracy of  $\mathcal{H}^2$ -matrix techniques. Starting from basic ideas of numerical linear algebra and numerical analysis, the theory is developed in a straightforward and systematic way, accessible to advanced students and researchers in numerical mathematics and scientific computing. Special techniques are required only in isolated sections, e.g., for certain classes of model problems.

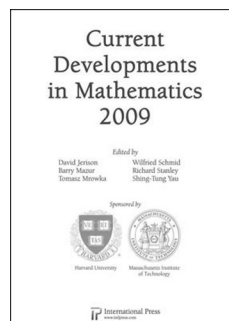
A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**Contents:** Introduction; Model problem; Hierarchical matrices; Application to integral operators; Orthogonal cluster bases and matrix projections; Compression; A priori matrix arithmetic; A posteriori matrix arithmetic; Application to elliptic partial differential operators; Applications; Bibliography; Algorithm index; Subject index.

**EMS Tracts in Mathematics, Volume 14**

December 2010, 441 pages, Hardcover, ISBN: 978-3-03719-091-3, 2010 *Mathematics Subject Classification*: 65-02, 65F05, 65F30, 65N22, 65N38, 65R20, **AMS members US\$62.40**, List US\$78, Order code EMSTM/14

## General Interest



### Current Developments in Mathematics, 2009

**Barry Mazur, Wilfried Schmid, and Shing-Tung Yau, Harvard University, Cambridge, MA, and David Jerison, Tomasz Mrowka, and Richard Stanley, Massachusetts Institute of Technology, Cambridge, MA, Editors**

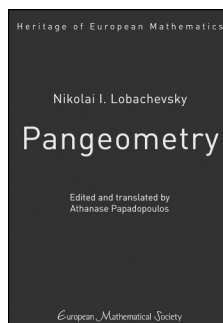
The papers in this volume are based on selected lectures given at the Current Development Mathematics Conference held in November 2009 at Harvard University.

A publication of International Press. Distributed worldwide by the American Mathematical Society.

**Contents:** N. B. Châu, Survey on the fundamental lemma; M. A. Hill, M. J. Hopkins, and D. C. Ravenel, The Arf-Kervaire problem in algebraic topology: An introduction; O. Savin, Phase transitions, minimal surfaces, and a conjecture of de Giorgi; S. Zelditch, Recent developments in mathematical quantum chaos.

**International Press**

November 2010, 204 pages, Hardcover, ISBN: 978-1-57146-146-9, 2010 *Mathematics Subject Classification*: 53C44, **AMS members US\$46.80**, List US\$58.50, Order code INPR/94



### Nikolai I. Lobachevsky, *Pangeometry*

**Athanase Papadopoulos, Université de Strasbourg, France, Editor**  
Translated by Athanase Papadopoulos

Lobachevsky wrote *Pangeometry* in 1855, the year before his death. This memoir is a résumé of his work on non-Euclidean

geometry and its applications and can be considered his clearest account on the subject. It is also the conclusion of his life's work and the last attempt he made to acquire recognition. The treatise contains basic ideas of hyperbolic geometry, including the trigonometric formulae, the techniques of computation of arc length, of area and of volume, with concrete examples. It also deals with the applications of hyperbolic geometry to the computation of new definite integrals. The techniques are different from those found in most modern books on hyperbolic geometry since they do not use models.

Besides its historical importance, Lobachevsky's *Pangeometry* is a beautiful work, written in a simple and condensed style. The material that it contains is still very alive, and reading this book will be most useful for researchers and for students in geometry and in the history of science. It can be used as a textbook, as a sourcebook, and as a repository of inspiration.

The present edition provides the first complete English translation of *Pangeometry* available in print. It contains facsimiles of both the Russian and the French original versions. The translation is accompanied by notes, followed by a biography of Lobachevsky and an extensive commentary.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**Contents:** I. *Pangeometry*; II. Lobachevsky's biography; III. A commentary on Lobachevsky's *Pangeometry*; Bibliography.

**Heritage of European Mathematics, Volume 4**

November 2010, 322 pages, Hardcover, ISBN: 978-3-03719-087-6, 2010 *Mathematics Subject Classification*: 01-02, 51-02, 01A55, 51-01, 51-03, 53-02, 53-01, 53-03, 53A35, **AMS members US\$78.40**, List US\$98, Order code EMSHEM/4



The American Mathematical Society announces:

## The AMS Graduate Student Blog

Now in its second year, this blog will serve as a tool for graduate students in mathematics, providing them with information from fellow graduate students.

AMS Vice President Frank Morgan (Williams College) is managing the blog. He will be assisted by the Graduate Student Editorial Board, comprised of current graduate students, in content control of the blog.

The blog covers topics of importance to graduate students, offering advice on subject matter relevant to each stage of their development. Each writer brings a personal perspective based on experience, while keeping content broad enough to deliver valuable points to all those seeking assistance.

From the entry **"Finding an Advisor"** ...

*"After passing my qualifying exams, I went to a couple professors and asked them, if I were to be their advisee, what kinds of problems would I work on. They gave me papers and books to read on a variety of topics and we set up additional meetings so I could tell them if any of these subjects interested me or ask them more questions."*

From the entry **"Navigating Seminars—A First Year's Perspective"** ...

*"The student seminars are often the most fun because they are talks given by your peers. Also you often get to see some of the intuition or 'how I think about it' that is sometimes left out in other seminars ... If your afternoon seminars don't involve dinner afterward, try to get a group together yourself. It's a lot of fun."*

From the entry **"Stick to the Content"** ...

*"A common pitfall I've seen among speakers—especially student speakers—is to apologize during the talk for such choices, or to make self-deprecating jokes. This is nearly always a bad idea, as it distracts from the point of your talk."*

Student readers are invited to join the discussion by posting questions, comments, and further advice on each entry. Further, they may nominate themselves or a fellow graduate student to the Graduate Student Editorial Board. Please visit the blog at:

**<http://mathgradblog.williams.edu/>**

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# Classified Advertisements

*Positions available, items for sale, services available, and more*

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## GEORGIA

### GEORGIA STATE UNIVERSITY Mathematics and Statistics

Georgia State University Mathematics and Statistics Department, located in Atlanta, GA, invites applications for the following positions: four assistant/associate professor positions—requires a Ph.D.; a lecturer position—requires a Ph.D.; an academic professional position—requires a Master's degree. For more information about these positions and to submit applications go to: <http://www.mathjobs.org>. An offer of employment will be conditional upon background verification. Georgia State University is a Research University of the University System of Georgia and an EEO/AAA institution that values the diversity of its faculty, staff, and student body.

000024

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## DISTRICT OF COLUMBIA

### HOWARD UNIVERSITY Mathematics Department Associate Professor

The Mathematics Department of Howard University invites applications for a tenure-track position at the associate

professor level. Applications in all areas of mathematics will be considered with a priority given to statistics. We seek a mathematician with an already established research program, who has directed Ph.D. theses and received external funding in support of his/her research. Outstanding recent Ph.D. recipients may also apply for a position at the assistant professor level. A history of good teaching is required.

Application must include the following: curriculum vitae, abstract of current research, and three letters of recommendations.

Consideration of applications will begin on December 31, 2010. Applications received after that date will be considered until the position is filled. Applications and supporting materials should be directed electronically to: [djames@howard.edu](mailto:djames@howard.edu) or hardcopy to: Dr. Davis James, Associate Chair, Howard University Department of Mathematics, College of Arts & Sciences, Academic Support Building B, Room 204, 2441 6th Street, N.W., Washington, D.C. 20059.

Equal Employment Opportunity: Howard University does not discriminate on the basis of race, color, national and ethnic origin, sex, marital status, religion, or disability.

000025

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## VIRGINIA

### UNIVERSITY OF VIRGINIA Department of Mathematics

The Department of Mathematics at the University of Virginia invites applications for a Whyburn Instructorship beginning August 25, 2011. This position carries a three-year appointment. Preference will be given to candidates who have received their Ph.D. within the last three years. Candidates must have a Ph.D. by the date of hire, an outstanding research record, and demonstrated teaching success.

Preference will be given to researchers working in an area of algebra or topology covered by the department. In the cover letter, it will be very helpful to indicate which members of our department are closest to your research interests. See <http://artsandsciences.virginia.edu/mathematics/research/researchguide/index.html>.

To apply, submit the following required documents electronically through <http://www.MathJobs.org>: A cover letter, an AMS Standard Cover Sheet, a curriculum vitae, a publication list, a description of research, and a statement about teaching interests and experience. The applicant must also have four letters of recommendation submitted, of which

**Suggested** uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services.

**The 2010 rate is** \$3.25 per word. No discounts for multiple ads or the same ad in consecutive issues. For an additional \$10 charge, announcements can be placed anonymously. Correspondence will be forwarded.

Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.

There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

**Upcoming deadlines** for classified advertising are as follows: April 2011 issue—January 30, 2011; May 2011 issue—February 28, 2011; June/July 2011 issue—April 28, 2011;

August 2011 issue—May 27, 2010; September 2011 issue—June 28, 2011; October 2011 issue—July 28, 2011.

**U.S. laws prohibit** discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the U.S. cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to U.S. laws. Details and specific wording may be found on page 1373 (vol. 44).

**Situations wanted advertisements** from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the U.S. and Canada or 401-455-4084 worldwide for further information.

**Submission:** Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02940; or via fax: 401-331-3842; or send email to [classifieds@ams.org](mailto:classifieds@ams.org). AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 02904. Advertisers will be billed upon publication.

one must support the applicant's effectiveness as a teacher.

In addition, all candidates must complete the Candidate Profile through the University of Virginia's employment system, which is: <http://jobs.virginia.edu>; posting number 0606827. Your application process will not be complete until all required documents are available on MathJobs and you receive a confirmation number for your Candidate Profile from Jobs@UVA.

Priority consideration will be given to applications received by March 1, 2011; however, the position remains open to applications until filled.

Additional information about our department is available at: <http://artsandsciences.virginia.edu/mathematics/>.

For more information about the position: <http://artsandsciences.virginia.edu/mathematics/aboutus/employment/index.html>.

Women and members of underrepresented groups are encouraged to apply. The University of Virginia is an Affirmative Action/Equal Opportunity Employer and is strongly committed to building diversity within its community.

000026

## CHILE

### PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE Departamento de Matemáticas

The Department of Mathematics invites applications for three tenure-track positions at the assistant professor level beginning either March or August 2012. Applicants should have a Ph.D. in mathematics, proven research potential either in pure or applied mathematics, and a strong commitment to teaching and research. The regular teaching load for assistant professors consists of three one-semester courses per year, reduced to two during the first two years. The annual salary will be US\$45,000 (calculated at the current exchange rate of 500 Chilean pesos per dollar).

Please send a letter indicating your main research interests, potential collaborators in our department (<http://www.mat.puc.cl>), detailed curriculum vitae, and three letters of recommendation to:

Director Departamento de Matemáticas,  
Pontificia Universidad Católica de Chile,  
Av. Vicuña Mackenna 4860 Santiago,  
Chile;  
fax: (56-2) 552-5916;  
email: [mmusso@mat.puc.cl](mailto:mmusso@mat.puc.cl)

For full consideration, complete application materials must arrive by June 30, 2011.

000027

## CHINA

### East China Normal University Center for Partial Differential Equations of East China Normal University Postdoctoral Positions Shanghai, China

The successful candidates are expected to be young researchers with Ph.D. degrees in mathematics or related areas, with a strong research record in at least one of the following areas: analysis/computation/modeling.

More information about the positions as well as the introduction of the Center are available at: <http://postdoctor.ecnu.edu.cn/details.aspx?id=31>.

000111

## KOREA

### KOREA INSTITUTE FOR ADVANCED STUDY (KIAS) Postdoctoral Research Fellowships

The School of Mathematics at the Korea Institute for Advanced Study (KIAS) invites applicants for the positions at the level of postdoctoral research fellows in pure and applied mathematics. KIAS, founded in 1996, is committed to the excellence of research in basic sciences (mathematics, theoretical physics, and computational sciences) through high-quality research programs and a strong faculty body consisting of distinguished scientists and visiting scholars. Applicants are expected to have demonstrated exceptional research potential, through the doctoral dissertation and beyond. The annual salary ranges from approximately ₩30,000,000–₩70,000,000 (equivalent to US\$27,000–\$64,000). In addition, research funds in the amount of approximately ₩10,000,000–₩15,000,000 (equivalent to US\$9,000–\$14,000) are provided each year. Appointments may start as early as September 1, 2011. The initial appointment will be for two years with a possibility of renewal for two additional years. Those interested are encouraged to contact a faculty member in their research areas at: <http://www.kias.re.kr/en/about/members.jsp>. Also, for more information please visit [http://www.kias.re.kr/en/notice/job\\_opportunity.jsp](http://www.kias.re.kr/en/notice/job_opportunity.jsp). Applicants should send a cover letter specifying the research area, a curriculum vitae with a list of publications, and a summary of research plans, and arrange three recommendation letters to be sent to:

School of Mathematics:  
Mr. Kang Won Lee (email: [kwlee@kias.re.kr](mailto:kwlee@kias.re.kr));  
KIAS, 207-43 Cheongnyangni-dong,  
Dongdaemun-gu, Seoul, 130-722,  
Korea

Email applications are strongly encouraged. We review the applications twice

a year; the deadlines are June 30 and December 31.

000022

## RIO DE JANEIRO

### PUC-RIO (Catholic University of Rio de Janeiro) Department of Mathematics Assistant Professorship

The Mathematics Department of PUC-Rio has an opening for an assistant professorship starting in August of 2011. This is a tenure-track contract, initially lasting one year but renewable (see "Hiring procedures" below). The university expects candidates to have a consistent record of high-level research and also to be qualified for undergraduate and graduate teaching.

PUC-Rio offers a stimulating research-oriented environment with plenty of academic freedom and excellent working conditions, such as a pleasant, well-located, and safe campus; administrative flexibility; research fellowships funded by the university as well as by the Brazilian government; and a Bachelors program that historically attracts excellent students. Pay is internationally competitive (currently around \$50,000 plus benefits per year, keep in mind that living expenses in Brazil are relatively low).

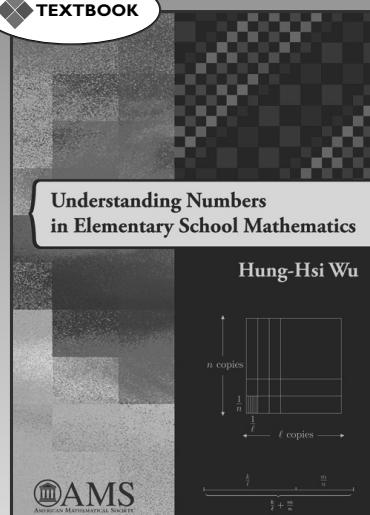
Language Skills: Knowledge of Portuguese is highly desirable as courses are usually taught in Portuguese; in the absence of that, knowledge of a similar Romance language (e.g., Spanish, French, or Italian) is a plus. A successful candidate who is not ready to teach in Portuguese by August 2011 will be given an English-language course load during the August–November 2011 semester under the condition that he or she commits to learning sufficient Portuguese by March 2012.

Full details of the submission process and hiring procedures can be found by clicking on the "New Faculty Position" link in the Announcements section of our homepage at: <http://www.mat.puc-rio.br/pagina.php?id=anuncios>.

Candidates must send the required documents to [candidato@mat.puc-rio.br](mailto:candidato@mat.puc-rio.br) by April 15, 2011.

000023

## TOOLS FOR EDUCATORS



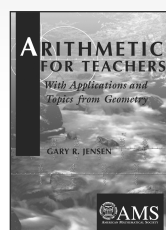
## Understanding Numbers in Elementary School Mathematics

Hung-Hsi Wu, *University of California, Berkeley, CA*

This is a textbook for pre-service elementary school teachers and for current teachers who are taking professional development courses. By emphasizing the precision of mathematics, the exposition achieves a logical and coherent account of school mathematics at the appropriate level for the readership. Wu provides a comprehensive treatment of all the standard topics about numbers in the school mathematics curriculum: whole numbers, fractions, and rational numbers. Assuming no previous knowledge of mathematics, the presentation develops the basic facts about numbers from the beginning and thoroughly covers the subject matter for grades K through 7.

Every single assertion is established in the context of elementary school mathematics in a manner that is completely consistent with the basic requirements of mathematics. While it is a textbook for pre-service elementary teachers, it is also a reference book that school teachers can refer to for explanations of well-known but hitherto unexplained facts. For example, the sometimes-puzzling concepts of percent, ratio, and rate are each given a treatment that is down to earth and devoid of mysticism. The fact that a negative times a negative is a positive is explained in a leisurely and comprehensible fashion.

2011; approximately 542 pages;  
Hardcover; ISBN: 978-0-8218-5260-6;  
List US\$79; AMS members US\$63.20;  
Order code MBK/79



## Arithmetic for Teachers

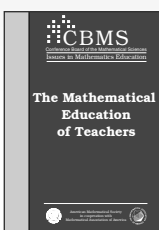
With Applications and Topics from Geometry

Gary R. Jensen,

*Washington University, St. Louis, MO*

This book helps prospective teachers achieve the necessary expertise by presenting topics from the K-6 mathematics curriculum at a greater depth than is found in most classrooms.

2003; 383 pages; Hardcover; ISBN: 978-0-8218-3418-3; List US\$62; AMS members US\$49.60;  
Order code FOA



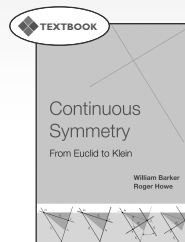
## The Mathematical Education of Teachers

*The writers of this material clearly are knowledgeable about the current problems with preparing teachers to have special, pedagogical understanding of mathematics. ... highly recommend the book.*

—*Teaching Children Mathematics*

This series is published in cooperation with the Mathematical Association of America.

**CBMS Issues in Mathematics Education,**  
Volume 11; 2001; 145 pages; Softcover; ISBN: 978-0-8218-2899-1; List US\$26; AMS members US\$20.80;  
All individuals US\$20.80; Order code CBMATH/11



## Continuous Symmetry

From Euclid to Klein

William Barker,  
*Bowdoin College,*

*Brunswick, ME,* and Roger Howe,  
*Yale University, New Haven, CT*

This text is for a one-semester course on geometry for undergraduates training to become high school teachers.

*... a very nice book [that is] worth reading... It should be in every library, and [would] be useful to students and teachers alike.*

—*Mathematical Reviews*

2007; 544 pages; Hardcover; ISBN: 978-0-8218-3900-3; List US\$69; AMS members US\$55.20;  
Order code MBK/47



www.ams.org/bookstore





# Meetings & Conferences of the AMS

**IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS:** AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See <http://www.ams.org/meetings/>. Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL and in an electronic issue of the *Notices* as noted below for each meeting.

## Statesboro, Georgia

*Georgia Southern University*

**March 12–13, 2011**

*Saturday – Sunday*

### Meeting #1068

Southeastern Section

Associate secretary: Matthew Miller

Announcement issue of *Notices*: January 2011

Program first available on AMS website: January 27, 2011

Program issue of electronic *Notices*: March 2011

Issue of *Abstracts*: Volume 32, Issue 2

### Deadlines

For organizers: Expired

For consideration of contributed papers in Special Sessions: Expired

For abstracts: Expired

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

### Invited Addresses

**Gordana Matic**, University of Georgia, *Contact structures, open books, and contact invariants in Floer homology*.

**Jason A Behrstock**, Lehman College and Graduate Center CUNY, *Quasi-isometric classification of 3-manifold groups*.

**Jeremy T. Tyson**, University of Illinois at Urbana-Champaign, *Sobolev mappings into metric spaces*.

**Brett D. Wick**, Georgia Institute of Technology, *The corona problem*.

### Special Sessions

*Advances in Biomedical Mathematics*, **Yangbo Ye**, University of Iowa, and **Jiehua Zhu**, Georgia Southern University.

*Advances in Optimization (in honor of Florian Potra's 60th Birthday)*, **Goran Lesaja**, Georgia Southern University.

*Algebraic Geometry*, **Jing Zhang**, State University of New York at Albany, **Roya Beheshti Zavareh**, Washington University in St Louis, and **Qi Zhang**, University of Missouri at Columbia.

*Algebraic and Geometric Combinatorics*, **Drew Armstrong**, University of Miami, and **Benjamin Braun**, University of Kentucky.

*Applied Combinatorics*, **Hua Wang**, Georgia Southern University, **Miklos Bona**, University of Florida, and **Laszlo Szekely**, University of South Carolina.

*Categorical Topology*, **Frederic Mynard**, Georgia Southern University, and **Gavin Seal**, EPFL, Lausanne.

*Control Systems and Signal Processing*, **Zhiqiang Gao**, Cleveland State University, **Frank Goforth**, Georgia Southern University, **Thomas Yang**, Embry-Riddle Aeronautical University, and **Yan Wu**, Georgia Southern University.

*Dynamic Equations on Time Scales with Applications*, **Billur Kaymakçalan**, Georgia Southern University, and **Bonita Lawrence**, Marshall University.

*Fractals and Tilings*, **Ka-Sing Lau**, The Chinese University of Hong Kong, **Sze-Man Ngai**, Georgia Southern University, and **Yang Wang**, Michigan State University.

*Geometric Group Theory*, **Xiangdong Xie**, Georgia Southern University, **Jason A. Behrstock**, Lehman College, CUNY, and **Denis Osin**, Vanderbilt University.

*Geometric Mapping Theory in Euclidean and Non-Euclidean Spaces*, **Jeremy Tyson**, University of Illinois at Urbana-Champaign, **David A. Herron**, University of Cincinnati, and **Xiangdong Xie**, Georgia Southern University.

*Harmonic Analysis and Applications*, **Dmitriy Bilyk**, University of South Carolina, **Laura De Carli**, Florida International University, **Alex Stokolos**, Georgia Southern University, and **Brett Wick**, Georgia Institute of Technology.

*Harmonic Analysis and Partial Differential Equations*, **Paul A. Hagelstein**, Baylor University, **Alexander Stokolos**, Georgia Southern University, **Xiaoyi Zhang**, IAS Princeton and University of Iowa, and **Shijun Zheng**, Georgia Southern University.

*Homological Methods in Commutative Algebra*, **Alina C. Jacob**, Georgia Southern University, and **Adela N. Vraciu**, University of South Carolina.

*Low Dimensional Topology and Contact and Symplectic Geometry*, **Gordana Matic**, University of Georgia, and **John Etnyre**, Georgia Institute of Technology.

*Matrix Theory and Numerical Linear Algebra*, **Richard S. Varga**, Kent State University, and **Xie Zhang Li**, Georgia Southern University.

*Nonlinear Analysis of PDEs*, **Ronghua Pan**, Georgia Institute of Technology, **Tristan Roy**, Institute for Advanced Study, and **Shijun Zheng**, Georgia Southern University.

*Set-theoretic Topology*, **Frederic Mynard**, Georgia Southern University, and **Peter Nyikos**, University of South Carolina.

*Sparse Data Representations and Applications*, **Alexander Petukhov** and **Alex Stokolos**, Georgia Southern University, **Ahmed Zayed**, DePaul University, and **Inna Kozlov**, Holon Institute of Technology, Department of Computer Science.

*Symplectic and Poisson Geometry*, **Yi Lin**, Georgia Southern University, **Alvaro Pelayo**, Washington University, St. Louis, and **Francois Ziegler**, Georgia Southern University.

## Iowa City, Iowa

*University of Iowa*

**March 18–20, 2011**

*Friday – Sunday*

### Meeting #1069

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: January 2011

Program first available on AMS website: February 5, 2011

Program issue of electronic *Notices*: March 2011

Issue of *Abstracts*: Volume 32, Issue 2

### Deadlines

For organizers: Expired

For consideration of contributed papers in Special Sessions: Expired

For abstracts: Expired

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

### Invited Addresses

**Mihai Ciucu**, Indiana University, *Gaps in dimer systems and beyond*.

**David Damanik**, Rice University, *The Hofstadter butterfly, uniform hyperbolicity, and gap labeling*.

**Kevin B. Ford**, University of Illinois Urbana-Champaign, *Prime chains, arithmetic functions and branching random walks*.

**Chiu-Chu Liu**, Columbia University, *Open and closed Gromov-Witten invariants of toric Calabi-Yau 3-folds*.

### Special Sessions

*Algebraic Combinatorics*, **Mihai Ciucu**, Indiana University.

*Algebraic K-Theory and Homotopy Theory*, **Teena Gerhardt**, Michigan State University, and **Daniel Ramras**, New Mexico State University.

*Analytic and Algebraic Number Theory*, **Ling Long**, Iowa State University, and **Yangbo Ye**, University of Iowa.

*Commutative Ring Theory*, **Daniel D. Anderson**, University of Iowa, and **David F. Anderson**, University of Tennessee Knoxville.

*Computational Medical Imaging*, **Jun Ni** and **Lihe Wang**, University of Iowa.

*Geometric Commutative Algebra and Applications*, **David Anderson**, University of Washington, and **Julianna Tymoczko**, University of Iowa.

*Global and P-adic Representation Theory*, **Muthukrishnan Krishnamurthy**, **Philip Kutzco**, and **Yangbo Ye**, University of Iowa.

*Graph Theory*, **Maria Axenovich**, **Lale Ozkahya**, and **Michael Young**, Iowa State University.

*History of Mathematics*, **Colin McKinney**, Bradley University.

*Modelling, Analysis and Simulation in Contact Mechanics*, **Weimin Han**, University of Iowa, and **Mircea Sofonea**, University of Perpignan.

*Nonlinear Partial Differential Equations*, **Hongjie Dong**, Brown University, and **Dong Li**, **Lihe Wang**, and **Xiaoyi Zhang**, University of Iowa.

*Numerical Analysis and Scientific Computing*, **Kendall E. Atkinson**, **Bruce P. Ayati**, **Weimin Han**, **Laurent O. Jay**, **Suely Oliveira**, and **David Stewart**, University of Iowa.

*Recent Advances in Hyperbolic and Kinetic Problems*, **Tong Li**, University of Iowa, and **Hailing Liu**, Iowa State University.

*Recent Developments in Nonlinear Evolution Equations*, **Yinbin Deng**, Central China Normal University, **Yong Yu** and **Yi Li**, University of Iowa, and **Shuangjie Peng**, Central China Normal University.

*Recent Developments in Schubert Calculus*, **Leonardo Mihalcea**, Baylor University.

*Representations of Algebras*, **Frauke Bleher**, University of Iowa, and **Calin Chindris**, University of Missouri.

*Spectral Theory*, **David Damanik**, Rice University, and **Christian Remling**, University of Oklahoma.

*Stochastic Processes with Applications to Mathematical Finance*, **Igor Cialenco**, Illinois Institute of Technology, and **José E. Figueroa-López**, Purdue University.

*Thin Position*, **Jesse Johnson**, Oklahoma State University, and **Maggie Tomova**, University of Iowa.

*Topological Problems in Molecular Biology*, **Isabel K. Darcy**, University of Iowa, **Stephen D. Levene**, University of Texas at Dallas, and **Jonathan Simon**, University of Iowa.

*Universal Algebra and Order*, **John Snow**, Concordia University, **Jeremy Alm**, Illinois College, **Clifford Bergman**, Iowa State University, and **Kristi Meyer**, Wisconsin Lutheran College.

# Worcester, Massachusetts

*College of the Holy Cross*

**April 9–10, 2011**

*Saturday – Sunday*

## Meeting #1070

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: February 2011

Program first available on AMS website: March 10, 2011

Program issue of electronic *Notices*: April 2011

Issue of *Abstracts*: Volume 32, Issue 3

## Deadlines

For organizers: Expired

For consideration of contributed papers in Special Sessions: Expired

For abstracts: February 15, 2011

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Vitaly Bergelson**, Ohio State University, *Title to be announced*.

**Kenneth M. Golden**, University of Utah, *Title to be announced*.

**Walter D. Neumann**, Columbia University, *What does a complex surface really look like?*.

**Natasa Sesum**, University of Pennsylvania, *Title to be announced*.

## Special Sessions

*Celestial Mechanics* (Code: SS 16A), **Glen R. Hall**, Boston University, and **Gareth E. Roberts**, College of the Holy Cross.

*Combinatorial Representation Theory* (Code: SS 14A), **Cristina Ballantine**, College of the Holy Cross, and **Rosa Orellana**, Dartmouth College.

*Combinatorics of Coxeter Groups* (Code: SS 19A), **Dana C. Ernst**, Plymouth State University, and **Matthew Macauley**, Clemson University.

*Complex Analysis and Banach Algebras* (Code: SS 1A), **John T. Anderson**, College of the Holy Cross, and **Alexander J. Izzo**, Bowling Green State University.

*Computability Theory and Applications* (Code: SS 18A), **Brooke Andersen**, Assumption College.

*Dynamics of Rational Systems of Difference Equations with Applications* (Code: SS 3A), **M. R. S. Kulenovic** and **O. Merino**, University of Rhode Island.

*Geometric and Topological Problems in Curvature* (Code: SS 17A), **Megan Kerr** and **Stanley Chang**, Wellesley College.

*Geometry and Applications of 3-Manifolds* (Code: SS 13A), **Abhijit Champanerker** and **Ilya Kofman**, College of Staten Island, CUNY, and **Walter Neumann**, Barnard College, Columbia University.

*Geometry of Nilpotent Lie Groups* (Code: SS 11A), **Rachelle DeCoste**, Wheaton College, **Lisa DeMeyer**, Central Michigan University, and **Maura Mast**, University of Massachusetts-Boston.

*History and Philosophy of Mathematics* (Code: SS 5A), **James J. Tattersall**, Providence College, and **V. Frederick Riskey**, United States Military Academy.

*Interactions between Dynamical Systems, Number Theory, and Combinatorics* (Code: SS 9A), **Vitaly Bergelson**, The Ohio State University, and **Dmitry Kleinbock**, Brandeis University.

*Mathematical and Computational Advances in Interfacial Fluid Dynamics* (Code: SS 15A), **Burt S. Tilley**, Worcester Polytechnic Institute, and **Lou Kondic**, New Jersey Institute of Technology.

*Mathematics and Climate* (Code: SS 8A), **Kenneth M. Golden**, University of Utah, **Catherine Roberts**, College of the Holy Cross, and **MaryLou Zeeman**, Bowdoin College.

*Modular Forms, Elliptic Curves, L-functions, and Number Theory* (Code: SS 20A), **Sharon Frechette** and **Keith Ouellette**, College of the Holy Cross.

*New Trends in College and University Faculty Engagement in K–12 Education* (Code: SS 21A), **Jennifer Beineke**, Western New England College, and **Corri Taylor**, Wellesley College.

*Number Theory, Arithmetic Topology, and Arithmetic Dynamics* (Code: SS 10A), **Michael Bush**, Smith College, and **Farshid Hajir**, University of Massachusetts, Amherst.

*Physically Inspired Higher Homotopy Algebra* (Code: SS 4A), **Thomas J. Lada**, North Carolina State University, and **Jim Stasheff**, University of North Carolina, Chapel Hill.

*Random Processes* (Code: SS 7A), **Andrew Ledoan**, Boston College, and **Steven J. Miller** and **Mihai Stoiciu**, Williams College.

*The Algebraic Geometry and Topology of Hyperplane Arrangements* (Code: SS 6A), **Graham Denham**, University of Western Ontario, and **Alexander I. Suciu**, Northeastern University.

*Topics in Partial Differential Equations and Geometric Analysis* (Code: SS 12A), **Maria-Cristina Caputo**, University of Arkansas, and **Natasa Sesum**, Rutgers University.

*Topological, Geometric, and Quantum Invariants of 3-manifolds* (Code: SS 2A), **David Damiano**, College of the Holy Cross, **Scott Taylor**, Colby College, and **Helen Wong**, Carleton College.

*Undergraduate Research* (Code: SS 22A), **David Damiano**, College of the Holy Cross, **Giuliana Davidoff**, Mount Holyoke College, **Steve Levandosky**, College of the Holy Cross, and **Steven J. Miller**, Williams College.



# Las Vegas, Nevada

University of Nevada

April 30 – May 1, 2011

Saturday – Sunday

## Meeting #1071

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: February 2011

Program first available on AMS website: March 17, 2011

Program issue of electronic *Notices*: April 2011

Issue of *Abstracts*: Volume 32, Issue 3

## Deadlines

For organizers: Expired

For consideration of contributed papers in Special Sessions: Expired

For abstracts: March 8, 2011

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Elizabeth Allman**, University of Alaska, *Evolutionary trees and phylogenetics: An algebraic perspective*.

**Danny Calegari**, California Institute of Technology, *Stable commutator length in free groups*.

**Hector Ceniceros**, University of California Santa Barbara, *Immersed boundaries in complex fluids*.

**Tai-Ping Liu**, Stanford University, *Hilbert's sixth problem*.

## Special Sessions

*Advances in Modeling, Numerical Analysis and Computations of Fluid Flow Problems* (Code: SS 2A), **Monika Neda**, University of Nevada, Las Vegas.

*Computational Algebra, Groups and Applications* (Code: SS 7A), **Benjamin Fine**, Fairfield University, **Gerhard Rosenberger**, University of Hamburg, Germany, and **De-laram Kahrobaei**, City University of New York.

*Discrete Dynamical Systems in Graph Theory, Combinatorics, and Geometry* (Code: SS 15A), **Eunjeong Yi** and **Cong X. Kang**, Texas A&M University at Galveston.

*Extremal Combinatorics* (Code: SS 6A), **Jozsef Balogh**, University of California San Diego, and **Ryan Martin**, Iowa State University.

*Flow-Structure Interaction* (Code: SS 9A), **Paul Atzberger**, University of California Santa Barbara.

*Geometric Group Theory and Dynamics* (Code: SS 12A), **Matthew Day**, **Danny Calegari**, and **Joel Louwsma**, California Institute of Technology, and **Andy Putnam**, Rice University.

*Geometric PDEs* (Code: SS 1A), **Matthew Gursky**, Notre Dame University, and **Emmanuel Hebey**, Université de Clermont-Pontoise.

*Knots, Surfaces and 3-manifolds* (Code: SS 18A), **Stanislav Jabuka**, **Swatee Naik**, and **Chris Herald**, University of Nevada, Reno.

*Lie Algebras, Algebraic Transformation Groups and Representation Theory* (Code: SS 16A), **Andrew Douglas** and **Bart Van Steirteghem**, City University of New York.

*Multilevel Mesh Adaptation and Beyond: Computational Methods for Solving Complex Systems* (Code: SS 4A), **Pengtao Sun**, University of Nevada, Las Vegas, and **Long Chen**, University of California Irvine.

*Nonlinear PDEs and Variational Methods* (Code: SS 11A), **David Costa** and **Hossein Tehrani**, University of Nevada, Las Vegas, and **Zhi-Qiang Wang**, Utah State University.

*Partial Differential Equations Modeling Fluids* (Code: SS 5A), **Quansen Jiu**, Capital Normal University, Beijing, China, and **Jiahong Wu**, Oklahoma State University.

*Recent Advances in Finite Element Methods* (Code: SS 3A), **Jichun Li**, University of Nevada, Las Vegas.

*Recent Developments in Stochastic Partial Differential Equations* (Code: SS 8A), **Igor Cialenco**, Illinois Institute of Technology, and **Nathan Glatt-Holtz**, Indiana University, Bloomington.

*Set Theory* (Code: SS 14A), **Douglas Burke** and **Derrick DuBose**, University of Nevada, Las Vegas.

*Special Session in Arithmetic Dynamics* (Code: SS 17A), **Arthur Baragar**, University of Nevada, Las Vegas, and **Patrick Ingram**, University of Waterloo.

*Special Session on Computational and Mathematical Finance* (Code: SS 13A), **Hongtao Yang**, University of Nevada, Las Vegas.

*Topics in Modern Complex Analysis* (Code: SS 10A), **Zair Ibragimov**, California State University, Fullerton, **Zafar Ibragimov**, Urgench State University, and **Hrant Hako-byan**, Kansas State University.

# Ithaca, New York

Cornell University

September 10–11, 2011

Saturday – Sunday

## Meeting #1072

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: June 2011

Program first available on AMS website: July 28, 2011

Program issue of electronic *Notices*: September 2011

Issue of *Abstracts*: Volume 32, Issue 4

## Deadlines

For organizers: Expired

For consideration of contributed papers in Special Sessions: May 24, 2011

For abstracts: July 5, 2011

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*



## Invited Addresses

**Mladen Bestvina**, University of Utah, *Title to be announced.*

**Nigel Higson**, Pennsylvania State University, *Title to be announced.*

**Gang Tian**, Princeton University, *Title to be announced.*

**Katrin Wehrheim**, Massachusetts Institute of Technology, *Title to be announced.*

## Special Sessions

*Difference Equations and Applications* (Code: SS 1A), **Michael Radin**, Rochester Institute of Technology.

*Parabolic Evolution Equations of Geometric Type* (Code: SS 4A), **Xiaodong Cao**, Cornell University, and **Bennett Chow**, University of California, San Diego.

*Partial Differential Equations of Mixed Elliptic-Hyperbolic Type and Applications* (Code: SS 3A), **Marcus Khuri**, Stony Brook University, and **Dehua Wang**, University of Pittsburgh.

*Set Theory* (Code: SS 2A), **Paul Larson**, Miami University, Ohio, **Justin Moore**, Cornell University, and **Ernest Schimmerling**, Carnegie Mellon University.

*Species and Hopf Algebraic Combinatorics* (Code: SS 6A), **Marcelo Aguiar**, Texas A & M University, and **Samuel Hsiao**, Bard College.

*Symplectic Geometry and Topology* (Code: SS 5A), **Tara Holm**, Cornell University, and **Katrin Wehrheim**, M.I.T.

# Winston-Salem, North Carolina

*Wake Forest University*

**September 24–25, 2011**

*Saturday – Sunday*

## Meeting #1073

Southeastern Section

Associate secretary: Matthew Miller

Announcement issue of *Notices*: June 2011

Program first available on AMS website: August 11, 2011

Program issue of electronic *Notices*: September 2011

Issue of *Abstracts*: Volume 32, Issue 4

## Deadlines

For organizers: February 24, 2011

For consideration of contributed papers in Special Sessions: June 7, 2011

For abstracts: August 2, 2011

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Benjamin B. Brubaker**, Massachusetts Institute of Technology, *Title to be announced.*

**Shelly Harvey**, Rice University, *Title to be announced.*  
**Allen Knutson**, Cornell University, *Title to be announced.*

**Seth M. Sullivan**, North Carolina State University, *Title to be announced.*

# Lincoln, Nebraska

*University of Nebraska-Lincoln*

**October 14–16, 2011**

*Friday – Sunday*

## Meeting #1074

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: August 2011

Program first available on AMS website: September 1, 2011

Program issue of electronic *Notices*: October 2011

Issue of *Abstracts*: Volume 32, Issue 4

## Deadlines

For organizers: March 14, 2011

For consideration of contributed papers in Special Sessions: June 28, 2011

For abstracts: August 23, 2011

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Lewis Bowen**, Texas A&M University, *Title to be announced.*

**Emmanuel Candes**, Stanford University, *Title to be announced* (Erdős Memorial Lecture).

**Alina Cojocaru**, University of Illinois at Chicago, *Title to be announced.*

**Michael Zieve**, University of Michigan, *Title to be announced.*

## Special Sessions

*Association Schemes and Related Topics* (Code: SS 1A), **Sung Y. Song**, Iowa State University, and **Paul Terwilliger**, University of Wisconsin, Madison.

*Asymptotic Behavior and Regularity for Nonlinear Evolution Equations* (Code: SS 4A), **Petronela Radu** and **Lorena Bociu**, University of Nebraska - Lincoln.

*Dynamic Systems on Time Scales with Applications* (Code: SS 3A), **Lynn Erbe** and **Allan Peterson**, University of Nebraska - Lincoln.

*Extremal and Probabilistic Combinatorics* (Code: SS 5A), **Stephen Hartke** and **Jamie Radcliffe**, University of Nebraska - Lincoln.

*Quantum Groups and Representation Theory* (Code: SS 2A), **Jonathan Kujawa**, University of Oklahoma, and **Natasha Rozhkovskaya**, Kansas State University.

# Salt Lake City, Utah

*University of Utah*

**October 22–23, 2011**

*Saturday – Sunday*

## Meeting #1075

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: August 2011

Program first available on AMS website: September 8, 2011

Program issue of electronic *Notices*: October 2011

Issue of *Abstracts*: Volume 32, Issue 4

## Deadlines

For organizers: March 22, 2011

For consideration of contributed papers in Special Sessions: July 5, 2011

For abstracts: August 30, 2011

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Invited Addresses

**Graeme Milton**, University of Utah, *Title to be announced.*

**Lei Ni**, University of California San Diego, *Title to be announced.*

**Igor Pak**, University of California Los Angeles, *Title to be announced.*

**Monica Visan**, University of California Los Angeles, *Title to be announced.*

## Special Sessions

*Geometric Evolution Equations and Related Topics.* (Code: SS 2A), **Andrejs Treibergs**, University of Utah, Salt Lake City, **Lei Ni**, University of California, San Diego, and **Brett Kotschwar**, Arizona State University.

*Special Session on Geometric, Combinatorial, and Computational Group Theory* (Code: SS 1A), **Eric Freden**, Southern Utah University, and **Eric Swenson**, Brigham Young University.

# Port Elizabeth, Republic of South Africa

*Nelson Mandela Metropolitan University*

**November 29 – December 3, 2011**

*Tuesday – Saturday*

## Meeting #1076

*First Joint International Meeting between the AMS and the South African Mathematical Society.*

Associate secretary: Matthew Miller

Announcement issue of *Notices*: June 2011

Program first available on AMS website: Not applicable

Program issue of electronic *Notices*: Not applicable

Issue of *Abstracts*: Not applicable

## Deadlines

For organizers: February 23, 2011

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/internmtgs.html](http://www.ams.org/amsmtgs/internmtgs.html).*

## Invited Addresses

**Mark J. Ablowitz**, University of Colorado, *Title to be announced.*

**James Raftery**, University of Kwazulu Natal, *Title to be announced.*

**Daya Reddy**, University of Cape Town, *Title to be announced.*

**Peter Sarnak**, Princeton University, *Title to be announced.*

**Robin Thomas**, Georgia Institute of Technology, *Title to be announced.*

**Amanda Weltman**, University of Cape Town, *Title to be announced.*

# Boston, Massachusetts

*John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel*

**January 4–7, 2012**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 118th Annual Meeting of the AMS, 95th Annual Meeting of the Mathematical Association of America, annual meetings of the*

*Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2011

Program first available on AMS website: November 1, 2011

Program issue of electronic *Notices*: January 2012

Issue of *Abstracts*: Volume 33, Issue 1

### Deadlines

For organizers: April 1, 2011

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## Honolulu, Hawaii

*University of Hawaii*

**March 3–4, 2012**

*Saturday – Sunday*

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: March 2012

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: August 3, 2011

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## Tampa, Florida

*University of South Florida*

**March 10–11, 2012**

*Saturday – Sunday*

Southeastern Section

Associate secretary: Matthew Miller

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: March 2012

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: August 10, 2011

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

Washington, District of Columbia

## Washington, District of Columbia

*George Washington University*

**March 17–18, 2012**

*Saturday–Sunday*

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: March 2012

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: August 17, 2011

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## Lawrence, Kansas

*University of Kansas*

**March 30 – April 1, 2012**

*Friday – Sunday*

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: April 2012

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## Rochester, New York

*Rochester Institute of Technology*

**September 22–23, 2012**

*Saturday – Sunday*

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: February 22, 2012

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

# New Orleans, Louisiana

*Tulane University*

**October 13–14, 2012**

*Saturday – Sunday*

Southeastern Section

Associate secretary: Matthew Miller

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: October 2012

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: January 13, 2012

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

# San Diego, California

*San Diego Convention Center and San Diego Marriott Hotel and Marina*

**January 9–12, 2013**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 119th Annual Meeting of the AMS, 96th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association for Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: October 2012

Program first available on AMS website: November 1, 2012

Program issue of electronic *Notices*: January 2012

Issue of *Abstracts*: Volume 34, Issue 1

## Deadlines

For organizers: April 1, 2012

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

# Ames, Iowa

*Iowa State University*

**April 27–28, 2013**

*Saturday – Sunday*

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: April 2013

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: September 27, 2012

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see [www.ams.org/amsmtgs/sectional.html](http://www.ams.org/amsmtgs/sectional.html).*

## Special Sessions

*Operator Algebras and Topological Dynamics* (Code: SS 1A), **Andrejs Treibergs**, University of Utah, Salt Lake City, **Lei Ni**, University of California, San Diego, and **Brett Kotschwar**, Arizona State University.

# Alba Iulia, Romania

**June 27–30, 2013**

*Thursday – Sunday*

*First Joint International Meeting of the AMS and the Romanian Mathematical Society, in partnership with the “Simion Stoilow” Institute of Mathematics of the Romanian Academy.*

Associate secretary: Robert J. Daverman

Announcement issue of *Notices*: To be announced

Program first available on AMS website: Not applicable

Program issue of electronic *Notices*: Not applicable

Issue of *Abstracts*: Not applicable

## Deadlines

For organizers: To be announced

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

# Riverside, California

*University of California, Riverside*

**November 2–3, 2013**

*Saturday – Sunday*

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: April 2, 2013

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced



## Baltimore, Maryland

*Baltimore Convention Center, Baltimore Hilton, and Marriott Inner Harbor*

**January 15–18, 2014**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 120th Annual Meeting of the AMS, 97th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Matthew Miller

Announcement issue of *Notices*: October 2013

Program first available on AMS website: November 1, 2013

Program issue of electronic *Notices*: January 2013

Issue of *Abstracts*: Volume 35, Issue 1

### Deadlines

For organizers: April 1, 2013

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## San Antonio, Texas

*Henry B. Gonzalez Convention Center and Grand Hyatt San Antonio*

**January 10–13, 2015**

*Saturday – Tuesday*

*Joint Mathematics Meetings, including the 121st Annual Meeting of the AMS, 98th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: October 2014

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: January 2015

Issue of *Abstracts*: Volume 36, Issue 1

### Deadlines

For organizers: April 1, 2014

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## Seattle, Washington

*Washington State Convention & Trade Center and the Sheraton Seattle Hotel*

**January 6–9, 2016**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 122nd Annual Meeting of the AMS, 99th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2015

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: January 2016

Issue of *Abstracts*: Volume 37, Issue 1

### Deadlines

For organizers: April 1, 2015

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

## Atlanta, Georgia

*Hyatt Regency Atlanta and Marriott Atlanta Marquis*

**January 4–7, 2017**

*Wednesday – Saturday*

*Joint Mathematics Meetings, including the 123rd Annual Meeting of the AMS, 100th Annual Meeting of the Mathematical Association of America, annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic, with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: October 2016

Program first available on AMS website: To be announced

Program issue of electronic *Notices*: January 2017

Issue of *Abstracts*: Volume 38, Issue 1

### Deadlines

For organizers: April 1, 2016

For consideration of contributed papers in Special Sessions: To be announced

For abstracts: To be announced

# Meetings and Conferences of the AMS

## Associate Secretaries of the AMS

**Western Section:** Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; e-mail: [lapidus@math.ucr.edu](mailto:lapidus@math.ucr.edu); telephone: 951-827-5910.

**Central Section:** Georgia Benkart, University of Wisconsin-Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; e-mail: [benkart@math.wisc.edu](mailto:benkart@math.wisc.edu); telephone: 608-263-4283.

**Eastern Section:** Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18105-3174; e-mail: [steve.weintraub@lehigh.edu](mailto:steve.weintraub@lehigh.edu); telephone: 610-758-3717.

**Southeastern Section:** Matthew Miller, Department of Mathematics, University of South Carolina, Columbia, SC 29208-0001, e-mail: [miller@math.sc.edu](mailto:miller@math.sc.edu); telephone: 803-777-3690.

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited in the table of contents on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. **Information in this issue may be dated. Up-to-date meeting and conference information can be found at [www.ams.org/meetings/](http://www.ams.org/meetings/).**

## Meetings:

### 2011

March 12–13	Statesboro, Georgia	p. 510
March 18–20	Iowa City, Iowa	p. 511
April 9–10	Worcester, Massachusetts	p. 512
April 30–May 1	Las Vegas, Nevada	p. 513
September 10–11	Ithaca, New York	p. 513
September 24–25	Winston-Salem, North Carolina	p. 514
October 14–16	Lincoln, Nebraska	p. 514
October 22–23	Salt Lake City, Utah	p. 515
November 29–December 3	Port Elizabeth, Republic of South Africa	p. 515

### 2012

January 4–7	Boston, Massachusetts	p. 515
	Annual Meeting	
March 3–4	Honolulu, Hawaii	p. 516
March 10–11	Tampa, Florida	p. 516
March 17–18	Washington, DC	p. 516
March 30–April 1	Lawrence, Kansas	p. 516
September 22–23	Rochester, New York	p. 516
October 13–14	New Orleans, Louisiana	p. 517

### 2013

January 9–12	San Diego, California	p. 517
	Annual Meeting	
April 27–28	Ames, Iowa	p. 517
June 27–30	Alba Iulia, Romania	p. 517
November 2–3	Riverside, California	p. 517

### 2014

January 15–18	Baltimore, Maryland	p. 518
	Annual Meeting	

### 2015

January 10–13	San Antonio, Texas	p. 518
	Annual Meeting	

### 2016

January 6–9	Seattle, Washington	p. 518
	Annual Meeting	

### 2017

January 4–7	Atlanta, Georgia	p. 518
	Annual Meeting	

## Important Information Regarding AMS Meetings

Potential organizers, speakers, and hosts should refer to page 100 in the January 2011 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

## Abstracts

Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of  $\text{\LaTeX}$  is necessary to submit an electronic form, although those who use  $\text{\LaTeX}$  may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in  $\text{\LaTeX}$ . Visit <http://www.ams.org/cgi-bin/abstracts/abstract.pl>. Questions about abstracts may be sent to [abs-info@ams.org](mailto:abs-info@ams.org). Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

**Conferences:** (see <http://www.ams.org/meetings/> for the most up-to-date information on these conferences.)

February 17–21, 2011: AAAS Meeting in Washington, DC (Please see [www.aaas.org/meetings](http://www.aaas.org/meetings) for more information.)

June 12–July 2, 2011: Mathematics Research Communities Conferences, Snowbird, Utah. (Please see <http://www.ams.org/amsmrgs/mrc.html> for more information.)

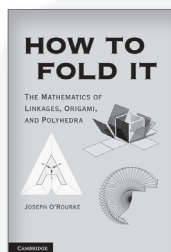
July 4–7, 2011: von Neumann Symposium on Multimodel and Multialgorithm Coupling for Multiscale Problems, Snowbird, Utah. (Please see <http://www.ams.org/meetings/amsmrgs/symposia/symposia-2011> for more information.)

CAMBRIDGE

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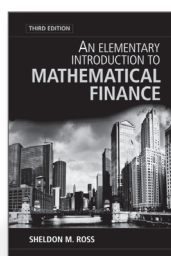
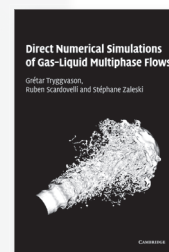
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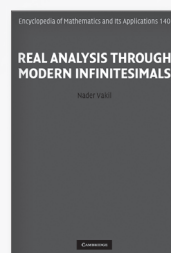


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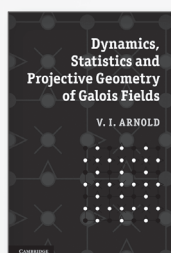
*Encyclopedia of Mathematics and its Applications*

\$115.00: Hb: 978-1-107-00202-9: 592 pp.

**Dynamics, Statistics and Projective Geometry of Galois Fields**

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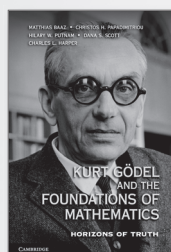
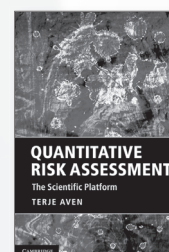


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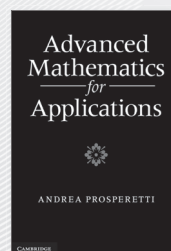


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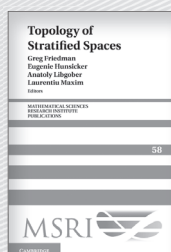
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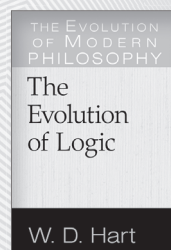
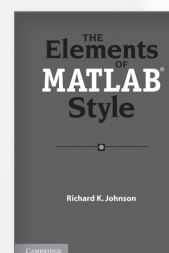
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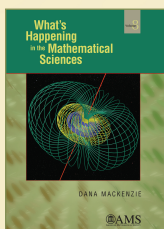
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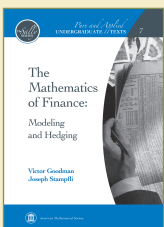
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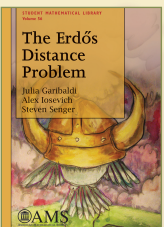
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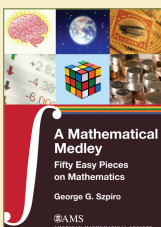
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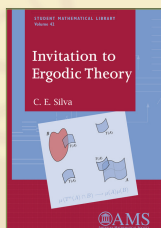
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