Modelling the Journey from Elementary Word Problems to Mathematical Research

Chris Sangwin

This article argues that traditional "word problems" provide an important early encounter with, and form of, mathematical modelling. Through a number of examples we will illustrate difficulties with the modelling process, both for mathematical and psychological reasons. Hence we argue that word problems, as simple proto-modelling exercises, play an important part in mathematics education.

Mathematics first developed over four thousand years ago in ancient Iraq to solve practical problems associated with metrology and accounting [17]. Such problems motivated the need for the development from arithmetic to algebra, and some of these "word problems" survive today in a recognizable form as traditional algebra story problems. Other examples occur in geometry and later in mechanics.

Example 1. A ladder stands with one end on the floor and the other against a wall. It slides along the floor and down the wall. If a cat is sitting in the middle of the ladder, what curve does it move along? [8, Ex 0.1]

To model this situation we need to make many assumptions that are not explicitly stated—for example, a smooth vertical wall and horizontal floor. We also need to imagine a thin ladder and even a point-sized cat! Unfortunately these assumptions can appear contrived and even silly, although we argue that the confidence needed to be able to take the initiative when making such assumptions and the experience to do so successfully are important parts of mathematical enculturation. When modelling a situation such as this, we represent it mathematically, we operate on the mathematical abstractions, but we periodically check for a meaningful correspondence between the real situation and symbolic abstraction. This is all part of *making sense*.

Engineering and science, particularly physics, all make use of mathematics in this way. Pure mathematics is less concerned with "real world" problems than with patterns and consequences and connections. These patterns are important, indeed even beautiful and intriguing, but of less interest to us here. Applied mathematics sits between explicit applications and pure intellectual pursuits with no clear boundaries between them. Rather than attempt to distinguish between pure and applied mathematics, we highlight the difference between deductive and empirical justification for truth. The mathematician justifies his or her work deductively from stated hypotheses, whereas the experimental scientist looks for empirical evidence. Therefore the applied mathematician needs to use and acknowledge both: checking the validity of an approximation or modelling assumption against an observation of a physical system is an empirical process. However, even pure mathematicians often do not *discover* their theorems by working deductively from axioms.

Mathematics can be used very effectively for solving *certain specific types of problems*; but change something, even in apparently trivial ways, and the resulting equations can become very difficult indeed to solve, perhaps even requiring mathematical research. Hence the skill of effective mathematical modelling is to represent a system in a way which (i) is not too bad, ideally *exact*, and (ii) results in an equation/system *which is still solvable*. Based on this observation we will consider the following four kinds of situations.

(1) Exact models, with exact solutions.

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(2) Exact models with approximate solutions. Ideally the accuracy of the approximation can be established.
(3) Approximate models with exact solutions. Here we have few ways of knowing with any certainty how well those things we have modelled fit the real system. More seriously, we may not have modelled something which is important and don't know it. At least the mathematical solution is exact.
(4) Approximate models with approximate solutions. This situation combines the problems of (2) and (3) above.

The experienced mathematician may reserve the verb "to model" to describe situations in which relevant aspects of a complex problem are selected and other parts of the physics are ignored. For example, we might choose to ignore friction or assume rigidity of components. Under this interpretation, an "exact model" is an oxymoron. However, we take the word *modelling* in its broadest sense so that the techniques can be valued and included at all levels of the curriculum, from the research mathematician to the elementary school student.

To see why this discussion is needed, let us look at two examination questions from United Kingdom GCE Advanced Level examinations. Example 2 was set in a paper titled "Mathematics" in June 1971, while Example 3 was set in June 2003 in a paper titled "Mechanics 1".

Example 2. A projectile is fired with an initial velocity of magnitude V inclined at an angle α above the horizontal. Find the equation of the trajectory referred to the horizontal and vertical axes through the point of projection.

A projectile is fired horizontally from a point O, which is at the top of a cliff, so as to hit a fixed target in the water, and it is observed that the time of flight is T. It is found that, with the same initial speed, the target can also be hit by firing at an angle α above the horizontal. Show that the distance of the target from the point at sea level vertically below O is $\frac{1}{2}gT^2 \tan\alpha$.

Example 3. Air resistance should be neglected in this question.

A bottle of champagne is held with its cork 1.5m above a level floor. The cork leaves the bottle at 60^o to the horizontal. The cork has *vertical* component of velocity of 9ms^{-1}, as shown in Figure 1.

(1) Show that the initial horizontal component of velocity is 5.20ms^{-1}, correct to three significant figures. [2 marks]
(2) Find the maximum height above the floor reached by the cork. [3 marks]
(3) Write down an expression in terms of t for the height of the cork above the floor t seconds after projection. [2 marks]

After projection, the cork is in the air for T seconds before it hits the floor.

(4) Show that T satisfies the equation $49T^2 - 90T - 15 = 0$.
Hence show that the cork is in the air for 1.99s, correct to three significant figures.
Calculate the horizontal distance travelled by the cork before it hits the floor. [5 marks]
(5) Calculate the speed with which the cork hits the floor. [3 marks]

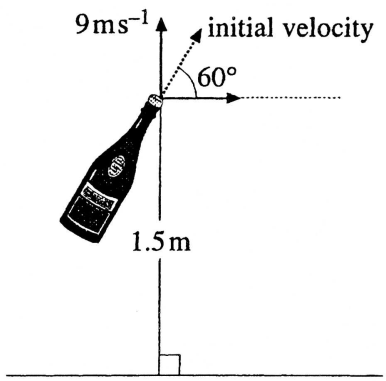

Figure 1.

These questions are on substantially the same topic, that of projectiles without air resistance. It is true they are quoted here without explaining any teaching context or the familiarity students have with the subject, but there is a striking difference. In Example 2, there is a modelling phase, and the student must read the question, draw and label an appropriate diagram, and link together multiple steps. All this is absent in Example 3: the diagram has been drawn and the question broken up into parts. For more comments on the contemporary habit of reducing problems to "single piece jigsaws", see [6]. There are other differences, such as the use of a specific and "nice" angle and the use of numerical approximations. But this is less important to our theme here than the total lack of the modelling phase in Example 3. We emphasize that these examples are from *final examinations*. Example 3 may be very useful at formative stages in providing some structure. Ultimately, eliminating the modelling stage reduces the activity to only imitation and practice, which is sterile and dull. More seriously, it is a short-term strategy for success in contrived examinations, which does not leave the student as an independent and confident mathematician.

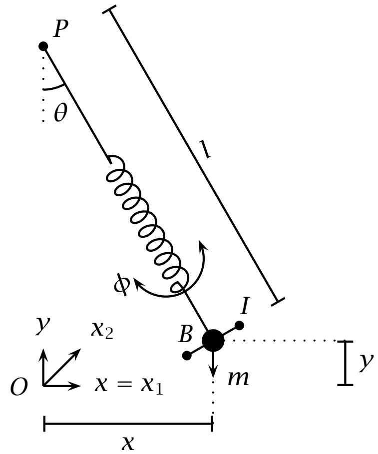

Figure 2. The general elastic pendulum.

Approximate Models with Exact Solutions

To motivate our discussion of more elementary problems, we shall examine, in some detail, the problem of modelling the pendulum. This iconic problem is solved with the help of various approximations, some of which are made *only* to ensure that exact solutions can be found to the resulting equations.

Consider the general *elastic pendulum* system shown in Figure 2. This consists of a heavy bob B attached to a point P by means of an elastic spring. The bob has a number of degrees of freedom relative to its rest position O: it may move horizontally ($x = x_1$) and vertically (y) in the plane of the page. It may perhaps also move in a direction perpendicular to the plane, (x_2). We describe any rotation along the axis PB by the coordinate ϕ. Although the full system is rather complex, we highlight these degrees of freedom to examine the modelling process.

The first modelling assumption is that the pendulum is rigid so that it does not vibrate vertically or rotate as a torsion spring and that all mass is concentrated at B. We further assume that the motion occurs only in one plane. This reduces the independent variables effectively to just the angle θ. Assuming that there is no air resistance or friction in the bearing at P, the only forces on the pendulum are contact forces at P and force due to gravity, assumed to act vertically downwards and to be constant, since the pendulum is small relative to the Earth. Resolving this perpendicular to PB gives a force of $-mg\sin(\theta)$ acting on B. Acceleration is given by $l\ddot{\theta}$, and by Newton's second law we have

$$(1) \quad ml\ddot{\theta} = -mg\sin(\theta), \quad \theta(0) = \theta^0,\ \dot{\theta}(0) = \dot{\theta}^0.$$

Even under all these assumptions we have a differential equation that does not yield an exact solution in terms of elementary functions! We shall make a further assumption that θ is small so that $\sin(\theta) \approx \theta$. Then $x = l\sin(\theta) \approx l\theta$, and (1) becomes

$$(2) \qquad \ddot{x}(t) = -\frac{g}{l}x(t),$$

the well-known equation for *simple harmonic motion.* The assumption that θ is small is made only so that the equation becomes linear and hence has an exact and explicit solution in terms of elementary functions. Taking $\omega_p = \sqrt{g/l}$, the unique solution of this equation is

$$(3) \qquad x(t) = a\cos(\omega_p t + \rho).$$

Here ω_p is the *natural frequency.* The *amplitude*, a, and *phase angle*, ρ, depend on the initial conditions. Notice the period of the oscillation is independent of the amplitude. However, for the full rigid pendulum, the period of the swing depends on the amplitude. So, how accurately does (3) approximate solutions to (1)? For a clock maker this is a key consideration. After all, there are $60 \times 60 \times 24 = 86,400$ seconds in a day. Any errors between the model and real system are cumulative, so that even a tiny error per oscillation may become significant. To the horologist this is known as *circular error.* To calculate the actual period of (1) we are quickly, and unavoidably, led to elliptic integrals of the first kind. The analysis needed to establish the accuracy of the approximation (3) to the true solution (1) is rather difficult. The search for a mechanism to avoid circular error motivated Huygens to investigate the path along which the pendulum bob would have to travel to ensure the period is independent of its amplitude. It is known as an *isochrone* (or *tautochrone*) curve and is first considered in [10].

Imagine you are teaching simple harmonic motion or applications of differential equations. We could argue that, to introduce simple harmonic motion, we can avoid the approximation $\sin(\theta) \approx \theta$ if we choose the vertically displaced mass and spring system. In terms of the situation shown in Figure 2, this corresponds to ignoring any horizontal movement but dropping the assumption of rigidity. Let us assume that the spring obeys Hooke's law, so that extension is proportional to the applied force. We shall continue to ignore damping, so for our system we derive

$$(4) \qquad m\ddot{y}(t) = -ky(t)$$

where k is the extension spring constant. We immediately recover (2) for $y(t)$ with natural frequency $\omega_s = \sqrt{k/m}$, and so we can apply standard techniques to solve the equation exactly. Here we have ignored some features, for example, air resistance, but under our assumptions the approximate model results in equation (4), which has an exact solution.

A very similar system is a *torsion spring.* Here there is no vertical or horizontal displacement; instead, the spring is twisted and released. The

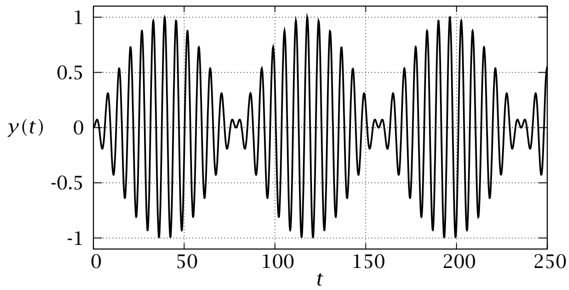

Figure 3. A time series for *y(t)* showing beating behavior.

displacement variable is ϕ, and now the moment of inertia, I, replaces the mass in the counterpart of (4). Such spiral springs are important in clocks and watches, but they are essentially linear models, as before.

When a spring is extended or compressed, there is a small winding or unwinding of the spring, and so we might expect the vertical compression/extension system to be weakly coupled to the torsion system. Can this be ignored? Including this within the model, the resulting system has *two degrees of freedom*: torsional (ϕ) and vertical (y) displacements. The resulting equations turn out to be weakly coupled linear ordinary differential equations, which can be solved exactly.

If the frequencies of the vertical and torsional modes are the same, then we have *resonance*, and the system will transfer its energy back and forth completely between the two modes of oscillation. This is quite a striking practical demonstration. One configuration is known as a Wilberforce spring, after its inventor Lionel Robert Wilberforce. See [23] or [7, pp. 205–212]. A time series for $y(t)$ showing beating behavior is illustrated in Figure 3. Notice how the amplitude changes over time. In practice the vertical oscillation appears to stop and start, out of phase with the twisting oscillation, which also appears to stop and start. With hindsight it was a mistake to ignore the torsion effect in the model in the special case of resonance. The exact solution to the full system is strikingly different from the exact solution to (4).

Approximate Models with Approximate Solutions

Our last class of systems is those in which approximate models are derived but only approximate solutions can be found. In fact, this is the usual situation in applied mathematics! Only very special cases can be solved explicitly. Most current work in fluid mechanics, for example, relies on numerical simulations to equations which approximate some of the physics. It is perhaps surprising quite how effective such approximations are in practice.

Even with the example illustrated in Figure 2 there are unexpected problems. We shall again ignore torsion, but this time consider the effect of horizontal displacement, resulting in y and $x = x_1$ coordinates. This allows a combination of a vertical spring mass and traditional planar pendulum to be considered. Let us assume we are trying to demonstrate (4) experimentally, with a traditional spring mass system. So we pull the mass vertically downwards and release it. In any experiment there will be a small horizontal displacement. What is its role? Perhaps, with intuitions guided by the independence of vertical and horizontal components during projectile motion, we might assume these motions are independent of each other? But if we pull the spring horizontally to one side, then an upward component of force is generated by the spring: they are not really decoupled.

Defining l_0 to be the unstretched length of the spring and l_1 to be the rest length of the spring under the mass m, we have

$$mg = k(l_1 - l_0), \quad \text{i.e.,} \quad l_1 = l_0 + \frac{mg}{k}. \tag{5}$$

For initial conditions close to those stated, that is, releasing from rest with $|x|$ small, the equations of motion relative to the rest position can [14] be *approximated by*

$$\ddot{x} = -\omega_p^2 x + \lambda x y, \tag{6}$$

$$\ddot{y} = -\omega_s^2 y + \frac{\lambda}{2} x^2, \tag{7}$$

where

$$\omega_p^2 = \frac{g}{l_1}, \quad \omega_s^2 = \frac{k}{m}, \quad \lambda = \omega_s^2 \frac{l_0}{l_1^2} = \frac{k}{m}\frac{l_0}{l_1^2}. \tag{8}$$

Note that if $x(0) = 0$, then from (6), $x(t) = 0$ for all time and (7) reduces to a simple harmonic oscillator with frequency ω_s, as expected. Although we are not free to choose λ, we note that if $\lambda = 0$, then we have decoupled linear oscillators. We might expect a strong coupling between these two equations to occur when the two frequencies resonate, and this really happens when $l_1 = \frac{4}{3} l_0$.

To illustrate a resonant system with $l_0 = 0.3$m, we release the mass from rest at $x = 0.0001$m (i.e., 0.1mm) and $y = 0.02$m. The resulting simulation is shown in Figure 4. The figure shows what appears, from long-term numerical simulations, to be one period of a periodic solution. Notice that for a significant initial period of time the system appears to be oscillating vertically, just as might be expected. However, for a brief period of time the vertical motion appears to all but stop, and a significant pendulum mode dominates. Literally, the mass stops going up and down and swings from side to side. Furthermore, the horizontal amplitude is double that of the vertical displacement.

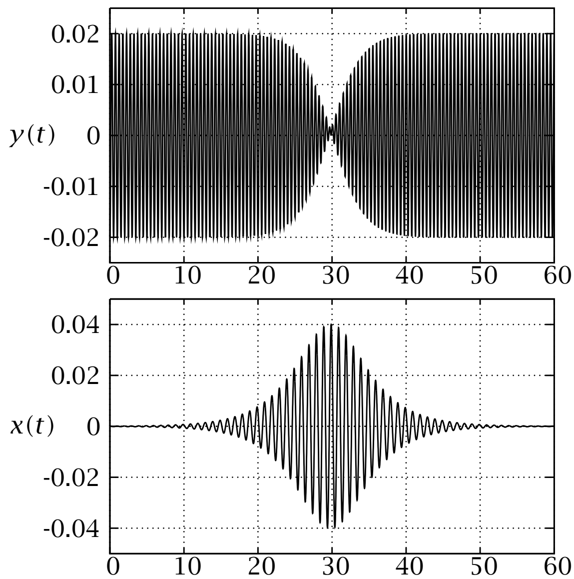

Figure 4. Time series *y(t)* (top) and *x(t)* (bottom) for the elastic pendulum at resonance.

Experimentally, this is difficult to achieve. The spring and mass have to be *very* carefully matched to see this nonlinear effect. Choosing random springs and masses, one is very unlikely to ever see this occur. And yet it happens and can be observed experimentally. You might just be unlucky, as Olsson in [14]:

> I first noticed the mode coupling in the swinging pendulum when I was a Teaching Assistant as a graduate student when (by chance) a student could not make the lab experiment work. This classical experiment, as you know, first measures the spring constant and then predicts the period. After stewing about this for a while I was able to work it out. (Olsson, M., 2008, private communication)

Unfortunately, the contemporary state of mathematical research only allows approximations to the equations to be derived, and these nonlinear differential equations cannot currently be solved explicitly; see [11].

By ignoring the horizontal component in the model (4), we failed to capture a significant and observable feature of the dynamics. In this case comparison of the mathematical solution with observations failed, and we were forced to include extra features in the model to accommodate the observations. In addition, when included, we only have approximate equations and numerical solutions. While the model assumes the motion to be planar, the oscillations are observed experimentally to occur in perpendicular horizontal directions alternately. It should also be possible to combine both the effects of the Wilberforce spring and misbehaving pendulums.

Whether a set of equations is tractable depends both on choosing the right type of model and on how this model is applied. In particular, a crucial aspect of modelling is the *ability to choose coordinates*. In many cases a particular choice of variables and coordinates significantly simplifies the resulting equations, ideally decoupling systems of equations so that they may be solved individually. Perhaps the most important example of the effect of the choice of coordinates is the way we view our solar system. If we put the Earth at the origin of our coordinate system, then describing the motion of the planets becomes terrible: they appear to "wander" on curves similar to epicycloids. If, however, the Sun is placed at the center, the resulting mathematical description becomes much simpler. The planets now all follow circles. Or the more accurate, but still approximate, model of Kepler is to fit the orbits to ellipses with the Sun at one focus. As another example, we turn to matrices which represent transformations. Matrices that can be "diagonalized" have a basis of eigenvectors. This basis is a coordinate system that reveals the structure of the transformation. The Jordan normal form does the same for more general matrices. The ability to choose a coordinate system which simplifies a problem requires sophistication, experience, and perhaps even some luck.

> For many years I used to give tutorials (and sometimes the lectures as well) on "Classical Mechanics" in the 2nd year at Oxford, largely about Lagrangian dynamics. The subject of "small oscillations" was always on the syllabus, and a typical question involved something like a double pendulum hanging down—find the (two) natural frequencies of oscillation and describe the type of motion involved in each. Such questions would *sometimes* end with "Find the normal coordinates", to which my reaction was often, "What's the point? What are the students going to do with them?"
>
> The point—whether clear to all the students or not—is that if you use normal coordinates, rather than the original "physical" ones, the linearized differential equations for small amplitude

oscillations (two, in the double pendulum case) become *decoupled*.

When I sat down to try and prove what turned out to be my upside-down pendulum theorem [1] on a rainy, windswept Sunday afternoon in November 1992, I therefore started out with the downward state, with N general pendulums, and wrote down N uncoupled equations using normal coordinates. I worried about what would happen when I turned my system upside-down, by changing the sign of g, and, even worse, how the equations would get vaguely Mathieu-like but hideously coupled when I added the effects of pivot vibration. I hadn't a clue what I would do in those circumstances.

After a few minutes, and half a page of work, it suddenly became clear that with my particular system (and more or less that particular system only!) the equations *were never going to become coupled*, so I found myself dealing with N uncoupled Mathieu equations.

The whole thing took 45 minutes, though my computer models of inverted 2 and 3 pendulum systems, generated over the previous few weeks, had set the scene. And so it was that I will always be grateful for hours of (otherwise pointless, I suspect) teaching of normal coordinates, over many years. (Acheson, D., 2008, private communication)

The pendulum is iconic. Renaissance science hypothesized "Divine order", and what better to represent this than the pendulum clock, studied through the geometrical physics of [10]? Small oscillations of a double pendulum, or the Wilberforce spring, are examined using the classical mechanics of the eighteenth and nineteenth centuries. Hysteresis and catastrophe theory can be demonstrated by the driven rigid pendulum. Chaos theory is aptly illustrated by the full double pendulum. Indeed, the pendulum in its many forms has served applied mathematics as the paradigmatic example for the concerns of each successive age. It continues to yield surprises, such as the pendulum theorem of [1]. There is every reason to suppose that it, and related systems, will continue to provide interesting mathematics. Here it illustrates the fact that selecting a mathematical model for the very simplest mechanical system is genuinely difficult. Under certain conditions the model may not be rich enough to illustrate significant qualitative features of the underlying physics/dynamics, that is, that the model is "bad" and that exact solutions are the exception rather than the norm. However, given a mathematical system, it is often far from clear to which of the following classes it belongs. Those for which

(1) a solution can be found exactly with a standard technique;
(2) a solution is known to exist but is difficult to establish explicitly, and hence it is approximated;
(3) a solution is known not to exist;
(4) the very question of existence of a solution is an open problem.

Mathematicians should, perhaps, be more honest about the range of problems that are solvable, but the culture of mathematics is very much to report only *theory* (1) and *counterexamples* (3). For the nonexpert, classes (2) and (4) remain shrouded in mystery. Perhaps this should be so: after all, innovations might be stifled if a student knows that others have struggled and failed.

Selection of Models by Students

We have set up a dichotomy between exact/approximate models and exact/approximate solutions to the resulting mathematical abstractions. The first class are problems that have exact models which can be solved exactly. While these are the exception in mechanics, they do occur in simpler problems, particularly in school in the form of word problems, e.g., [9].

Example 4. A dog starts in pursuit of a hare at a distance of thirty of his own leaps from her. He takes five leaps while she takes six but covers as much ground in two as she in three. In how many leaps of each will the hare be caught?

Or the ubiquitous

Example 5. A rectangle has length 8cm greater than its width. If it has an area of 33cm^2, find the dimensions of the rectangle.

Generally these word problems reduce to standard algebraic techniques, such as (i) a linear equation of a single variable, (ii) a quadratic equation in a single variable, (iii) simultaneous linear equations, or (iv) problems involving harmonic sums, for example, $\frac{1}{n} + \frac{1}{m}$. A systematic analysis of story problems is given in [13]. However, reducing problems to these forms is far from easy: problems involving rates are particularly difficult.

Example 6. Alice and Bob take two hours to dig a hole together. Bob and Chris take three hours to dig the hole. Chris and Alice would take four hours to dig the same hole. How long would all three of them take working together?

The temptation is to model the work of Alice and Bob incorrectly using literal transcription to rewrite "Alice and Bob take 2 hours" as $A + B = 2$. In one correct interpretation we use A to be Alice's *rate of work*, and so write $2A + 2B = 1$, where the right-hand side represents "one hole". This assumes rate of work is independent of any collaboration, which may or may not be realistic. The real psychological difficulty of these kinds of interpretations is well established, but the challenge of how to overcome them remains.

Example 7. Write an equation for the following statement: *"There are six times as many students as professors at this university."* Use S for the number of students and P for the number of professors.

When Clement et al. [4] gave the Students-and-Professors Problem to 150 calculus-level students, 37 percent answered incorrectly with $6S = P$, accounting for two-thirds of all errors. There are genuine difficulties in reaching a correct interpretation, and hence in moving from a word problem to a mathematical system which represents it. Similar conceptual difficulties occur with concentration and dilution problems. Consider a problem related to, but different from, Example 6, in which pairs of people "walk into town" rather than "dig a hole". We argue here that moving from such word problems to mathematical systems constitutes the beginning of mathematical modelling. We also note that such problems are available at all levels of mathematical sophistication, including the most elementary numeracy and algebra.

In Example 4, let l be the number of leaps taken by the dog. The problem can reduce to solving

$$l = \frac{6}{5} \times \frac{2}{3} l + 30.$$

This is a rather trivial linear equation, but it can only be arrived at by careful work on the part of the student. In Example 5 the student is free to choose either the length or width of the rectangle as a variable. Ignoring the particular letter used for the variable, this choice results in one of two different equations, that is, $x(x + 8) = 33$ or $x(x - 8) = 33$. One solution must be discarded as "unrealistic": a valuable critical judgment by the student.

Being able to select and correctly use standard techniques presupposes a certain level of fluency, which only comes with practice. This includes geometry puzzles for which a student builds up a range of approaches, "tricks", and useful general results. Seeing the practice of mathematics itself as solving real problems through modelling, and thus understanding the satisfaction of dispatching the routine steps accurately and efficiently, may act as motivation for students to undertake the repetitive work needed for the acquisition of skills.

Traditional word problems assume a certain level of cultural knowledge. For example, we take for granted that the problem solver is familiar with metrological systems or the rules of a sports game. This has always been the case since the first recorded use of such word problems in the mathematical training of scribes from around 2500 BCE in ancient Iraq [17]. In a mathematics class it is unlikely that the purpose of such problems is a test of such cultural knowledge, and this obviously has implications for how they are used with students, particularly if assessed. Problems need to be carefully selected for their intended students.

Given that modelling is, in the larger mathematical sense, very difficult, how are students supposed to solve problems? By "problems" we mean questions for which the modelling step is very much part of the process. For us something that is technically complex but that can be solved by a routine, well-established technique is not called a problem; rather we call it an *exercise*. For example,

$$\text{show that } \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \mathrm{d}x = \frac{22}{7} - \pi$$

is an exercise in polynomial long division and basic integration. Here there is no modelling step. In fact it can be done automatically with a computer algebra system. For us, a problem requires the student to make choices about how to represent a problem in a mathematical form, even when there is a simple (e.g., linear) equation that represents the situation exactly. An excellent source of problems and puzzles for which there exists an exact model and exact solution is plane geometry, and many examples are given in [8].

The following example, proposed recently as a school project, illustrates how difficult apparently simple problems can actually be to design and solve.

Example 8. What is the greatest number of people who may comfortably stand on a soccer field?

Here choices have to be made. Soccer fields may vary in size (90–120m × 45–90m), so for argument's sake we might take Wembley Stadium at 105m × 68m. Such arguments can be excellent opportunities to waste valuable time discussing essentially arbitrary things. Similar problems arise with the size of the people, but much more seriously are those caused by their *shape*. Here the choices that are made have a direct bearing on the difficulty of the mathematics.

Let us assume that a person occupies a rectangle or square and that everybody occupies the same area. Then the problem becomes somewhat trivial: people tessellate the field in a natural way. There is an unambiguous correct answer to this problem under these modelling assumptions (i.e., exact model, exact solution). Given the minimum amount of space required by each person, we can find the maximum number of rectangles that fit inside the rectangular area. But a rectangular packing is not the most efficient. Choosing the shape occupied by people to be hexagons results in more people being packed into the area but without the distance between any two people being reduced. However, a more difficult analysis results, as hexagons tessellate an infinite plane but not a finite rectangle. This problem is still tractable, and results in an answer. What about modelling each person as occupying a circle? Furthermore, it is not true that everyone occupies the same space. Packing different-sized circles into a rectangle is a classical problem, and in general it is unsolvable. But how does the novice know in advance the consequences of choosing a particular model? What does the teacher say to a student who chooses a model the teacher knows will result in intractable mathematics? Arguing, or more ambitiously proving, a solution does not exist can require very advanced ideas. If the rectangular football field is replaced with an athletics track, a shape with half circles instead of the two short straight sides, then the whole problem becomes intractable. Even slight variations cause problems to fundamentally change in character.

For this reason problems for which an exact model results in an exact solution constitute safe territory for both the student and teacher. This does not make such problems easy for students to solve, of course. To prevent confusion and wasted time, learning in mathematics often proceeds in the following phases, which might be termed "traditional teaching".

Imitation

The student is shown something new. He or she then *imitates* this by solving exercises which differ only slightly from that shown and in ways which do not fundamentally matter. This imitation shows the student new examples. Here an example might be a new mathematical object, for example, rational function instead of polynomial, or an "example" could be a technique. The importance of this has been acknowledged by many writers, for example

> (1) Concepts of a higher order than those which a person already has cannot be communicated to him by definition, but only by arranging for him to encounter a suitable collection of examples.
> [...]
> (2) Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.
> [19, p. 32]

Problem Solving

Next the student is given a problem and must choose which technique should be used and how. This includes the modelling phase described above. In modelling the student has *autonomy*, and with this *responsibility* to pursue a line of thought to a conclusion. While there is some autonomy, normally the required techniques will involve those imitated most recently. This helps the students. It gives them confidence they do have the required techniques, if only they can recognize which ones to use and how.

> One of the fundamental contributions of modern *didactique* consists of showing the importance of the rôle played in the teaching process by the learning phases in which the student works almost alone on a problem or in a situation for which she assumes the maximum responsibility. [2, p. 229]

When the teacher proposes an activity in a mathematics class, the student trusts both that this will be interesting and will lead to mathematical insights. This is fundamental to what Brousseau [2] calls the *didactic contract*. The teacher has the responsibility of choosing problems that are sufficiently novel to be a worthwhile challenge but that students still have a realistic prospect of solving. The phrase *zone of proximal development* is used to refer to problem-solving processes that have not yet matured but are in the process of maturation. It is

> ...the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers. [21, p. 86]

A fundamental part of the didactic contract is that a correct answer exists and that the problem is finished when this has been found and justified. This is why applied mathematics adopts standing assumptions such as "light inextensible strings" and "smooth frictionless pulleys". This characterizes the difference between mathematics and science, where experimental evidence suffices. Further work, such as generalizing the techniques used, abstracting, and so on form further separate problems. A student should have the satisfaction of recognizing this for him- or herself. For further comments on problem solving, see, for example, [18].

Practice

Practice of technique often follows. Ericsson et al. [5] stress that mere repetition is not sufficient to develop expertise; *deliberate practice* is required. Deliberate practice is *effortful* and can only be sustained for a limited period of time without leading to exhaustion.

> Drill in mere manipulation is necessary at every stage in school algebra. That this should be thorough, so far as it goes, will be admitted by all teachers, but it should in the main be given *after* its necessity in applications has been perceived by the pupil and not *before*; also, it should not be carried further than is needed to ensure facility in these applications. [20, p. 10]

Furthermore, carefully structured task design can help expose the *domains of variation* and *ranges of permissible change* within which a property holds or a technique remains valid; see [22]. Effective practice is far from just repetition. The teacher is essential in sequencing these tasks and monitoring performance to decide how to create a synopsis of what has just been done and when to introduce more complex and challenging topics. For this reason we are not proposing specific measures for particular stages of a student's education, but rather arguing for the importance of modelling in its broadest sense throughout. Many teachers do this already, but some examinations do not seem to reward it.

Some teachers omit the imitation phase and use problem solving to introduce new techniques; see, for example, [24] or [3]. Letting students work out solutions to problems themselves, without being given answers, is also the basis of the "Moore method" of teaching; see, for example, [15].

When trying to solve such problems we argue that *guess and check* or *trial and improvement* are only rarely appropriate techniques. Relevant details should be extracted to abstract the problem to a mathematical system of a known type in a conscious and deliberate way. Such questions may well provide choices of model, but the solution must involve a correctly applied standard technique, for example, solving a system of simultaneous linear equations or finding the roots of a quadratic equation. An important aspect of modelling is the selection of a particular model based on knowledge of how difficult it is going to be to solve. Once this is done, any solutions should be interpreted in the terms originally supplied by the problem. This modelling process immediately turns a rather trivial single-step mathematical exercise into a multistep chain of reasoning.

Conclusion

This article argues that word problems are a valuable form of mathematical modelling. Word problems are often seen as motivating a bridge from arithmetic to algebra, and in this role their use is controversial [12]. In arguing for their importance we do not trivialize the difficulties in teaching them, either as proto-algebra or proto-modelling. Other authors acknowledge the controversies but also assert their value.

> I hope I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary school is to teach the setting up of equations to solve word problems. [...] And so the future engineer, when he learns in the secondary school to set up equations to solve "word problems" has a first taste of, and has an opportunity to acquire the attitude essential to, his principal professional use of mathematics. [16, Vol. I, p. 59]

The point of applied mathematics is to solve real problems. And so, ultimately, one purpose of mathematics education is to allow students to solve problems independently and to appreciate the significance and accuracy of any solutions. Ideally, students should be in a position to criticize and compare various models, just as we have done in the case of the pendulum and Example 8. One purpose of practice and the acquisition of fluency in techniques is to recognize when a problem has a solution. This is crucial when moving from contrived word problems for which there is an exact model with exact solutions to approximations. Unless the student is intimately familiar with the *details of solving the standard problems*, he or she will simply not be able to make informed choices that result in an approximate model that (i) is within the desired level of accuracy and (ii) results in an equation/system *that is still solvable*.

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About the Cover

Home economics

This month's cover illustrates in rough graphical terms the Earth's energy transactions, in so far as they affect the Earth's temperature. The theme was suggested by Goong Chen, co-author of an article in this issue on the mathematics of greenhouse gases. The data used in the cover image and its basic design are taken from the article "Earth's annual global mean energy budget" in the *Bulletin of the American Meteorologial Society* of February 1997, written by J. T. Kiehl and Kevin Trenberth.

The heat transport represented in solid colors represents radiation, while the two dashed paths indicate the transfer of latent heat caused by water vapor condensing at altitude to form rain, and the direct transfer of heat by thermals rising from the Earth's surface. The ultimate radiation from the Earth is with the spectrum of a black body at 254° K, which is also what the Earth would emit if it had no atmosphere. The effect of the atmosphere, which contains molecules of water, carbon dioxide, and methane among others, is to absorb and emit low-frequency radiation, hence heat, back to the surface, whose temperature is therefore a considerably warmer 288° K.

Our first idea for a cover was something more specifically mathematical, but deciding which particular theme to illustrate reminded us of the proverbial blind men exploring the proverbial elephant. It didn't seem to do justice to the magnitude of the problems involved, whether in mathematics, physics, economics, or politics. So we decided to try to give an overall view.

The Wikipedia article on the greenhouse effect begins by telling you:

> *The greenhouse effect is a process by which thermal radiation from a planetary surface is absorbed by atmospheric greenhouse gases, and is re-radiated in all directions.*

This is true enough, but gives almost no idea of what is really going on, or how complex the issues are. There is a huge literature on the subject, but little of it seems to strike the right balance between clarity and accuracy. We have found the textbook *Principles of Planetary Climate* by Raymond Pierrehumbert enlightening, as well as *The Warming Papers*, edited by Pierrehumbert and David Archer. The Wikipedia article on black bodies is also instructive.

—Bill Casselman
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