Henri Poincaré. A Scientific Biography

Reviewed by John Stillwell

Henri Poincaré. A Scientific Biography
Jeremy Gray
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I think I can say, without fear of contradiction, that it takes a brave mathematician to write a scientific biography of Poincaré. It is remarkable that Jeremy Gray has dared to do it and even more remarkable that he has succeeded so brilliantly.

Poincaré was, with the possible exception of Hilbert, the deepest, most prolific, and most versatile mathematician of his time. His collected works fill eleven large volumes, and that does not include several volumes on mathematical physics and another several volumes of essays on science and philosophy for the educated reader. For most people it would be a life's work simply to read his output, let alone understand it well enough to write a clear and absorbing account. We are very fortunate to have this book.

Poincaré is probably best known to modern mathematicians for his contributions to non-Euclidean geometry, his discovery of chaos (in celestial mechanics), and his creation of algebraic topology (in which the "Poincaré conjecture" was the central unsolved problem for almost a century). These topics also belong to the three main areas of Poincaré's research that have been translated into English, and I discuss them further below. But they are merely some highlights, and they cannot be properly understood without knowing how they fit into the big picture of Poincaré's scientific work and philosophy. Gray, who has written and edited many books on nineteenth-century mathematics, particularly geometry and complex analysis, is the ideal guide to this big picture.

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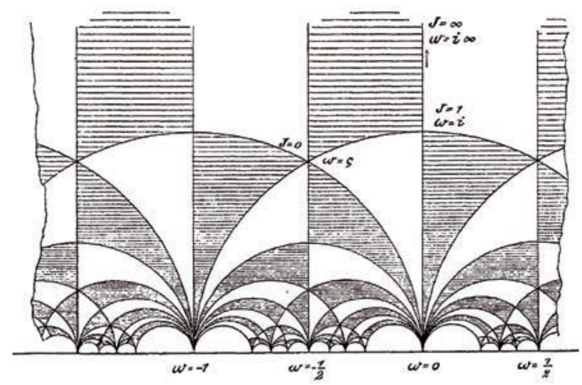

Figure 1. The modular tessellation.

Non-Euclidean Geometry

How did Poincaré find himself in non-Euclidean geometry? Bolyai and Lobachevskii developed non-Euclidean geometry in the 1820s, and Beltrami put it on a firm foundation (using Riemann's differential geometry) in 1868. So non-Euclidean geometry was already old news, in some sense, when Poincaré began his research in the late 1870s. But in another sense it wasn't. Non-Euclidean geometry was still a fringe topic in the 1870s, and Poincaré brought it into the mainstream by noticing that non-Euclidean geometry was *already present* in classical mathematics.

Specifically, there are classical functions with "non-Euclidean periodicity", such as the modular function j in the theory of elliptic functions and number theory. The periodicity of the modular function was observed by Gauss, and beautiful pictures illustrating this periodicity were produced in the late nineteenth century at the instigation of Felix Klein. Figure 1 is one such picture.

The function $j(z)$ repeats its values in the pattern shown: namely, at corresponding points in each of the curved black and white triangles. More precisely, $j(z+1) = j(z)$ (corresponding to the obvious symmetry of the tessellation under horizontal translation), and less obviously $j(-1/z) = j(z)$ (which corresponds to a half turn around the vertex $z = i$). It turns out that non-Euclidean periodicity is actually more common than the Euclidean periodicity that nineteenth-century mathematicians knew well from their study of elliptic functions.

At least, that is how it looks with hindsight. Poincaré found his way into non-Euclidean geometry by a more roundabout route, partly due to his ignorance of the existing literature (a common occurrence with Poincaré). He began by studying some functions defined by complex differential equations in response to a question posed in an essay competition by his former teacher Charles Hermite. The equations in question were first studied by Lazarus Fuchs, so Poincaré gave the name "Fuchsian" to the equations, to the functions they defined, and later to the symmetry groups he found those functions to possess.

Just as an elliptic function $f(z)$ repeats its value when z is replaced by $z + \omega_1$ or $z + \omega_2$ (where ω_1 and ω_2 are the so-called "periods" of f), Poincaré discovered that a Fuchsian function $g(z)$ repeats its value when z is replaced by $\frac{\alpha z+\beta}{\gamma z+\delta}$ for certain quadruples $\alpha, \beta, \gamma, \delta$ of real numbers. The periodicity of a Fuchsian function can be illustrated by tessellating the upper half-plane by "polygons" whose sides are arcs of semicircles with centers on the real axis. The polygons are mapped onto each other by the transformations $z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$.

At some point (Poincaré tells us it was as he was stepping onto an omnibus), he had the epiphany that *such transformations are the same as those of non-Euclidean geometry*. In particular, the polygonal cells tessellating the half-plane are *congruent* in the sense of non-Euclidean geometry, and the transformations mapping one cell onto another are non-Euclidean isometries. Poincaré had in fact rediscovered the *half-plane model* of the non-Euclidean (hyperbolic) plane first found by Beltrami [2], but in a mainstream mathematical context. He also rediscovered Beltrami's related *disk model*.

His non-Euclidean tessellations were also a rediscovery. The modular tessellation was one (though it had not yet been accurately drawn), and so too was a beautiful tessellation of the disk given by Schwarz [19] (see Figure 2) in connection with hypergeometric functions. What was new, and crucial, was Poincaré's realization of their non-Euclidean nature.

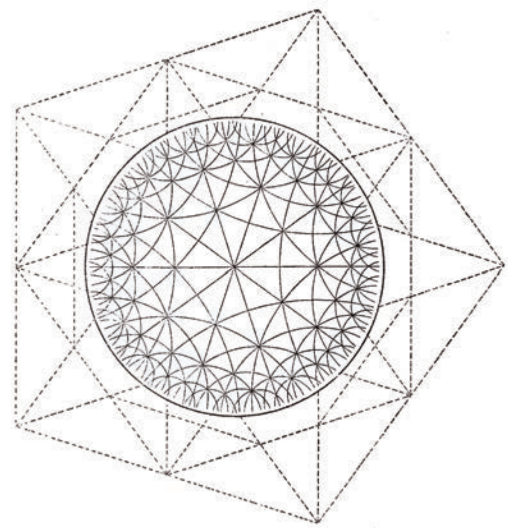

Figure 2. The Schwarz triangle tessellation.

Poincaré was not at first even aware of Schwarz's paper, which led to some awkward repercussions, related by Gray on page 229. In a letter to Klein, Poincaré admitted that he would not have called the functions "Fuchsian" had he known of Schwarz's work. Indeed, the triangle functions and their symmetry groups are now, belatedly, named after Schwarz. But at the time Schwarz got no satisfaction and was said (in a letter from Mittag-Leffler to Poincaré) to be very angry with himself "for having had an important result in his hand and not profiting from it."

Thus Poincaré's first steps in non-Euclidean geometry were mainly a matter of finding a better language to describe known situations: the non-Euclidean interpretation made it possible to describe linear fractional transformations $z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$ in traditional geometric terms such as congruence and rigid motions. His next step was more profound: he used this language to describe a situation which, until then, had been incomprehensible. This situation arises when the transformations are of the form $z \mapsto \frac{az+b}{cz+d}$, where a, b, c, d are *complex*. In this case one again has a tessellation of $\mathbb{C}$ by curvilinear cells mapped onto each other by the given transformations, but the tessellation can be enormously complicated.

The limit points, where the (Euclidean) size of the cells tends to zero, can form a nondifferentiable curve ("if one can call it a curve," said Poincaré) or other highly complicated sets. Poincaré's inspiration was to view this tessellated plane as the boundary of the *upper half-space*, which turns out to be none other than *non-Euclidean 3-dimensional space*. The upper half-space is nicely tessellated with 3-dimensional cells, bounded by portions of hemispheres whose centers lie in the plane we started with. Thus the nasty tessellation of the plane can be replaced by the nice tessellation of space, and all is well again. In this way Poincaré was able to make progress with functions that were previously intractable: the so-called *Kleinian functions*, whose values repeat under transformations of the form $z \mapsto \frac{az+b}{cz+d}$. (The name "Kleinian" was mischievously suggested by Poincaré after Klein had complained to him about the name "Fuchsian". It had some basis in Klein's mathematics, but not much.)

English translations of Poincaré's papers on Fuchsian and Kleinian groups may be found in [14].

Celestial Mechanics and Chaos

For most of his career, Poincaré was as much a physicist as a mathematician. He taught courses on mechanics, optics, electromagnetism, thermodynamics, and elasticity, and contributed to the early development of relativity and quantum theory. He was even nominated for the Nobel Prize in physics and garnered a respectable number of votes. But it is probably his contribution to the 3-body problem that is of greatest interest today. Like some of his other groundbreaking work, his results on the 3-body problem were a triumph over initial mistakes.

The story was first uncovered by June Barrow-Green [1], and it is updated in Chapter 4 of Gray's book.

Poincaré's first attempt was an entry in a mathematical prize competition sponsored by King Oscar II of Sweden. The king had a more than amateur interest in mathematics, but the prize questions were probably devised by Poincaré's friend Gösta Mittag-Leffler to ensure Poincaré's participation. At any rate, Poincaré submitted an entry for question 1, on the stability of the solar system, and duly won. His entry was an essay on the 3-body problem, the simplest case where the stability is not obvious.

The judges—Mittag-Leffler, Weierstrass, and Hermite—were agreed that Poincaré should win because of the originality of his results, among them what later became known as the Poincaré return theorem. Nevertheless, they found his essay hard to follow, and Poincaré eventually added ninety-three pages to clarify it. This satisfied them enough to confirm their decision and to go ahead with publication of the essay. Meanwhile, Poincaré had continued thinking about the essay—and he discovered a mistake.

Fortunately, Poincaré was able to fix the error in a month, but not before Mittag-Leffler had frantically recalled all known published copies and asked Poincaré to pay for the printing of a corrected version. This Poincaré willingly did, even though the cost was 50 percent more than the value of the prize. The reason for the panic was that Mittag-Leffler for years had championed Poincaré's work, with its frequent intuitive leaps, over the objections of German mathematicians such as Kronecker. Kronecker was already miffed that he had not been chosen as one of the judges, so if he learned how much Poincaré had been "helped" to win the prize there would be hell to pay. Luckily, he never found out.

Perhaps with some discomfort over his first venture into celestial mechanics, Poincaré decided to write up his ideas properly in the 1890s. The result was the 3-volume *Les Méthodes Nouvelles de la Mécanique Céleste* (1892, 1893, 1899), a monumental work that launched the modern theory of dynamical systems. An English translation [15] was produced on the initiative of NASA. The third volume ends with a glimpse of chaos in the 3-body problem:

> One is struck by the complexity of this figure that I am not even attempting to draw. Nothing can give us a better idea of the complexity of the three-body problem and of all the problems of dynamics in general. . . .
>
> (*Méthodes Nouvelles*, Vol. 3, p. 389)

Before leaving the question of the stability of the solar system, I would like to mention a result that Gray hints at but seemingly forgets to explain. On p. 253 he remarks that the implications of Poincaré's later work

> . . .opened for serious investigation for the first time the idea that Newton's laws might permit a planet to exit the system altogether.

I am guessing that Gray here is referring to the result of Xia [21] that there is a 5-body system in which one of the bodies escapes to infinity in finite time. Xia's result is discussed in the book of Diacu and Holmes [5], a book that Gray does mention.

Algebraic Topology

To appreciate how much Poincaré did for algebraic topology, one needs to review the state of the subject before Poincaré burst onto the scene in 1892.

The topology of compact 2-dimensional manifolds ("closed surfaces") was quite well understood, partly because Riemann had seen the value of orientable surfaces in complex analysis and partly because of the lucky fact that topology is simple for orientable surfaces. It is captured by a single invariant number, the *genus* p (or, equivalently, by the *Euler characteristic*). Riemann [17] described p in terms of *connectivity*—the maximum number of closed cuts that can be made without separating the surface—and Möbius [7] introduced the normal form of a genus p surface, namely, the "sphere with p handles". Möbius also observed the existence of nonorientable surfaces, such as the Möbius band, but it turns out that they do not greatly complicate the topological classification of surfaces.

Thus by the 1880s the topology of surfaces was well understood and unexpectedly easy. Virtually the only progress in higher-dimensional topology was made by Betti [3] when he generalized Riemann's idea of connectivity to obtain a series of invariants that became known as the *Betti numbers*. A 3-dimensional manifold, for example, has a 2-dimensional connectivity number P_2, equal to the maximum number of disjoint surfaces in M that fail to separate it. M also has a 1-dimensional connectivity number P_1, equal to the maximum numbers of closed curves that can lie in M without forming the boundary of a surface.

The dream of topology, at this point, was to find a finite set of invariant numbers that characterize each compact n-dimensional manifold up to homeomorphism.

Poincaré [9] struck the first blow against this dream by showing that the Betti numbers do *not* suffice to characterize 3-dimensional manifolds M. He did so with the help of a new kind of invariant, the *fundamental group* $\pi_1(M)$, by showing that certain manifolds M with the same Betti numbers have different fundamental groups.

The fundamental group is an essentially algebraic invariant, because it does not readily reduce to a set of numbers (if at all). What one calculates from M is a set of *generators* and *defining relations* for $\pi_1(M)$, and it is generally hard to tell when two sets of defining relations define the same group. In the case of a surface S of genus ≥ 2, $\pi_1(S)$ is actually one of the Fuchsian groups introduced by Poincaré in the early 1880s. However, Poincaré at that time interpreted Fuchsian groups geometrically—as symmetry groups of tessellations—and it was Klein [6] who first saw the connection between the group and the topology of the surface obtained by identifying equivalent sides of a tile in the tessellation.

By the 1890s Poincaré had absorbed the topological viewpoint and was ready to extend it to three dimensions. In his first long paper on topology, entitled "Analysis situs" (the name then given to topology), Poincaré [10] constructed several 3-manifolds by identifying sides of polyhedra and calculated their Betti numbers and fundamental groups. They include, in more detail, his 1892 example showing that the fundamental group can distinguish certain 3-manifolds that the Betti numbers cannot. (He also remarks that the Betti numbers can be extracted from π_1 by allowing its generators to commute ("abelianization"), so π_1 is a strictly stronger invariant than the set of Betti numbers.)

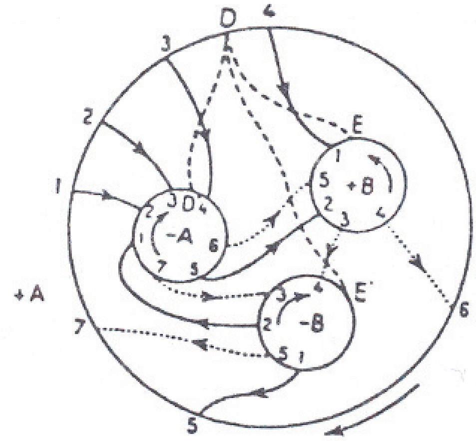

Figure 3. Poincaré's diagram for his homology sphere.

Nevertheless, Poincaré did not give up on the Betti numbers. He spent much of "Analysis situs" developing the algebra of homology, proving what we now call *Poincaré duality*, and concluding with a generalization of the Euler polyhedron formula to n dimensions.

He made some errors. In 1898 Heegaard pointed out that Poincaré duality was incorrect as it stood. Poincaré [11] responded with a supplement to "Analysis situs", revising his definition of the Betti numbers and formulating his homology theory in a more combinatorial way. He assumed that each manifold could be divided into cells and calculated its Betti numbers from incidence matrices. A second supplement [12] followed when he realized that he had overlooked the presence of *torsion* in the homology of manifolds. The word "torsion", which today appears as much in algebra as in topology, originates here. It reflects Poincaré's view that topological torsion is characteristic of manifolds that are "twisted upon themselves", such as the Möbius band.

Having now found all the invariant numbers that homology had to offer, Poincaré [12] dared to conjecture that *the three-dimensional sphere is the only closed three-dimensional manifold with trivial Betti and torsion numbers.* This was the first, and incorrect, version of the "Poincaré conjecture".

Eventually there were three more supplements, the most important of which is the fifth [13]. In it Poincaré makes two interesting excursions: the first is an analysis of 3-manifolds from the viewpoint of what we now call *Morse theory*; the second is an interesting application of *geometrization* (in this case, the hyperbolic structure of surfaces of genus ≥ 2) to simple curves on surfaces. These excursions lead, in a roundabout way, to one of

Poincaré's greatest discoveries: a 3-manifold H with trivial homology but nontrivial fundamental group—the *Poincaré homology sphere*.

The construction of H is based on the diagram in Figure 3, which I include mainly to show how unenlightening it is. Despite the apparent asymmetry of its construction, $\pi_1(H)$ falls out as a group with a 2-to-1 homomorphism onto the icosahedral group A_5. This shows that $\pi_1(H)$ is nontrivial, but, like A_5, it has a trivial abelianization. How much of this construction was good luck and how much good management is a mystery. Gray makes a plausible case that Poincaré picked the group first, then tinkered with manifolds until he found one that realized the group. (English translations of Poincaré's topology papers are given in [16].)

We now know that the Poincaré homology sphere can be arrived at by several different symmetric constructions. See, for example, [18]. So perhaps it is more "inevitable" than it seemed at first. We also know now that it is the *only* homology sphere with finite fundamental group.

The immediate effect of Poincaré's discovery was to refute the "Poincaré conjecture" of the second supplement and to amend it to its correct form: *the three-dimensional sphere is the only closed three-dimensional manifold with trivial fundamental group*. The latter is the conjecture that launched a thousand topology papers, from Tietze [20] and Dehn [4] to Perelman [8]. The early attacks on the conjecture assumed that Poincaré's combinatorial methods in topology and group theory would suffice to settle it. But that hope gradually faded, and in the 1970s Thurston upped the ante to a *geometrization conjecture*, greatly extending the geometrization long known for surfaces.

In the 1980s Hamilton proposed an approach to geometrization through differential geometry, hoping to approach the nice geometric structures conjectured by Thurston by letting nastier (but easily obtained) geometric structures flow towards nice ones. Hamilton's program was finally carried out by Perelman in 2003 by a tour de force of differential geometry and PDE. Among the mathematicians of Poincaré's era, perhaps only Poincaré himself would have felt at home with such high-powered geometric and analytic equipment.

General Remarks

The three topics above are thoroughly covered in Gray's book, together with their complex web of historical and mathematical connections. There is also much else to enjoy. The book is structured to lead the reader gently into Poincaré's work: first an introduction that could stand alone as a splendid essay on Poincaré, then a chapter on Poincaré's popular science essays, and then a chapter on Poincaré's career. All of this comes before the chapters on more specialized and difficult topics.

Of course, as a reviewer I read everything in the book, but I believe it would be easy to skip topics according to taste. It is a big book, and there is something for everyone.

I found the description of the French mathematical community and how they differed from the Germans particularly fascinating. Compared with most eminent mathematicians today and with the Germans then, Poincaré was unusually isolated. He had no graduate students and no immediate successors in France. The next generation of French mathematicians (Borel, Lebesgue, Hadamard) had different interests, and the generation after that was almost destroyed in World War I. Bourbaki emerged from the wreckage with a conscious effort to catch up with the Germans, who had forged ahead under the leadership of Klein and Hilbert.

Another interesting thread that runs through the book is Poincaré's interest in physics, particularly his near-discovery of special relativity. Gray shows how Poincaré took many of the right steps, starting from Maxwell's equations and getting as far as introducing the Lorentz group. But he lacked Einstein's physical insight, and the mathematical insight that could have made up for this, Minkowski's space-time, was not yet available. As Gray memorably puts it (p. 378):

> For Poincaré...to have grasped the full implications of special relativity he would have had to be not Einstein, but Minkowski.

Gray has obviously spent an enormous amount of time immersed in Poincaré's work and has become totally familiar with Poincaré's way of thinking. My only complaint is that occasionally he seems to channel Poincaré only too well, reliving some aspects of Poincaré that are hard for the modern reader to follow. Sometimes complicated geometric arguments are expressed in words when a picture would be clearer; sometimes he is too faithful to Poincaré's notation, as in the topology chapter, where relations in the fundamental group are written additively, even when they are not commutative.

But there is so much excellent exposition in this book that it is easy to skip the occasional difficult formula. I warmly recommend the book to anyone with an interest in the development of modern mathematics. It will surely be the definitive scientific biography of Poincaré for the foreseeable future.

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