WHAT IS... a Perfectoid Space?

Bhargav Bhatt

Perfectoid spaces are a class of algebro-geometric objects living in the realm of p-adic geometry that were introduced by Peter Scholze [Sch12] in his Ph.D. thesis. Their definition is heavily inspired by a classical result in Galois theory (see Theorem 1) due to Fontaine and Wintenberger, and the resulting theory has already had stunning applications.

Motivation

Fix a prime number p, and consider the field $L_0 := \mathbf{Q}_p$ of p-adic numbers, as well as the field $L_0^\flat := \mathbf{F}_p((t))$ of Laurent series over $\mathbf{F}_p$. These fields are formally quite similar: one can represent elements in L_0 as Laurent series in p with integer coefficients, and a similar description applies to $L_0^\flat$ with t replacing p. Of course there is no isomorphism $L_0 \simeq L_0^\flat$ of fields realizing this similarity: L_0 has characteristic 0, while $L_0^\flat$ has characteristic $p > 0$. Nevertheless, it is a fundamental insight of [FW79] that a robust relationship between the two *does* exist, at least after replacing L_0 with the larger field $L := \mathbf{Q}_p[p^{\frac{1}{p^\infty}}] := \cup_n \mathbf{Q}_p(p^{\frac{1}{p^n}})$, and $L_0^\flat$ with its perfection $L^\flat := \cup_n \mathbf{F}_p((t^{\frac{1}{p^n}}))$.

Theorem 1 (FW79). *The (absolute) Galois groups of L and $L^\flat$ are canonically isomorphic.*

Theorem 1 gives a correspondence between finite field extensions of L and $L^\flat$ which, heuristically, is established by replacing p with t. For example, the splitting field of $X^2 - t$ over $L^\flat$ corresponds to the splitting field of $X^2 - p$ over L. This mechanism can be somewhat demystified by noticing that the "integral" subrings $\mathbf{Z}_p[p^{\frac{1}{p^\infty}}] \subset L$ and $\mathbf{F}_p[t^{\frac{1}{p^\infty}}] \subset L^\flat$ are related by an isomorphism of rings

$$\mathbf{Z}_p[p^{\frac{1}{p^\infty}}]/(p) \simeq \mathbf{F}_p[t^{\frac{1}{p^\infty}}]/(t) \tag{1}$$

that carries $p^{\frac{1}{p^n}}$ to $t^{\frac{1}{p^n}}$. Besides its intrinsic beauty, this correspondence allows us to transport Galois-theoretic information between L and $L^\flat$.

Example 1. Certain invariants are very easy to compute for $L^\flat$ on account of the Frobenius automorphism; Theorem 1 can sometimes help transfer this computation to L. For example, using this strategy, one deduces that the $\mathbf{F}_p$-cohomological dimension of the absolute Galois group of L is ≤ 1 because the corresponding assertion for $L^\flat$ is classical (Hilbert).

Recall that fields are zero-dimensional varieties from an algebro-geometric perspective. The goal of the theory of perfectoid spaces is to extend Theorem 1 to higher dimensions, i.e., to relate (certain) algebras over L and $L^\flat$ in a relatively lossless manner.

Perfectoid Spaces

Fix L and $L^\flat$ as in the previous section. To introduce perfectoid spaces over these fields, it will be useful to recall some additional structures on L and $L^\flat$. Specifically, note that both L and $L^\flat$ are equipped with natural norms given by the p-adic and t-adic metrics respectively. As our subsequent constructions involve various limiting operations (such as the extraction of arbitrary p-power roots), it is convenient to cast all constructions in a slightly more analytic framework. We will thus pass from L to its p-adic completion K, and $L^\flat$ to its t-adic

Bhargav Bhatt is an associate professor of mathematics at the University of Michigan. His email address is bhattb@umich.edu.

DOI: http://dx.doi.org/10.1090/noti1166

completion $K^\flat$; the analogue of Theorem 1 holds for K and $K^\flat$ as completions do not change Galois groups. The basic definition is:

Definition 1. A *perfectoid K-algebra A* is a Banach K-algebra such that the subring A° of power-bounded elements is open and bounded, and the Frobenius endomorphism is surjective on A°/p; one similarly defines *perfectoid $K^\flat$-algebras*.

To unravel this definition, let us study some examples. The simplest example of such an algebra is K itself. Indeed, the norm on K endows K with a Banach algebra structure. The subring K° is the p-adic completion $\widehat{\mathbf{Z}_p[p^{\frac{1}{p^\infty}}]}$ and is thus open and bounded in the (p-adic) topology on K; the Frobenius on K°/p is surjective by construction. This example corresponds to a "point" in the world of perfectoid K-spaces. The next simplest example is that of a "line":

Example 2. Consider the p-adically complete K°-algebra $A' := \widehat{K^\circ[X^{\frac{1}{p^\infty}}]}$, and let $A := A'[\frac{1}{p}]$. Then one can endow A with a natural Banach K-algebra structure such that $A^\circ = A'$ is open and bounded. Moreover, as we have already extracted arbitrary p-power roots of X, the Frobenius on A°/p is surjective, so A is a perfectoid K-algebra; this algebra is often denoted $K\langle X^{\frac{1}{p^\infty}}\rangle$. Similarly, $A^\flat := K^\flat\langle X^{\frac{1}{p^\infty}}\rangle$ is a perfectoid $K^\flat$-algebra.

It is also easy to build examples in characteristic p:

Example 3. Let A_0 be any $K^{\flat,\circ}$ algebra. Extracting p-power roots of all elements in A (i.e., passing to the perfection) gives a new $K^{\flat,\circ}$-algebra $A_{0,\mathrm{perf}}$. This leads to the $K^\flat$-algebra $A := \widehat{A_{0,\mathrm{perf}}}[\frac{1}{t}]$ by inverting t in the t-adic completion. One may endow A with a natural Banach $K^\flat$-algebra structure to make it perfectoid. For example, applying this procedure to $A_0 := K^{\flat,\circ}[X]$ produces $A^\flat$ from Example 2.

Recall from algebraic geometry that affine varieties are completely described by their rings of functions, while varieties are built by glueing affine varieties together. The situation with perfectoid K-spaces is analogous: the "affine" objects correspond to perfectoid K-algebras, while perfectoid K-spaces are built by glueing these "affine" objects together. Actually, to retain the analytic flavor of Definition 1, this glueing is carried out in the world of rigid analytic geometry (incarnated through Huber's adic spaces [Hub96]). We will ignore this technical, but absolutely crucial, point here, and assume that the notion of a perfectoid K-space, built by glueing together "spectra" of perfectoid K-algebras, has been defined. The main theorem concerning these objects is:

Theorem 2 (Sch12, Theorems 1.9 and 1.11)**.** *The categories of perfectoid K-spaces and perfectoid $K^\flat$-spaces are canonically identified; this identification preserves the étale topology.*

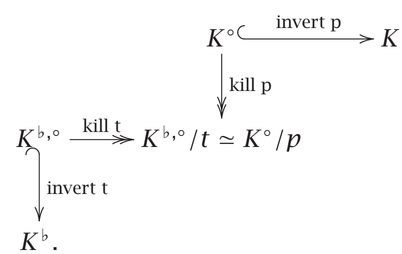
To describe this equivalence, observe that (1) gives a formula describing $K^\flat$ in terms of K:

$$K^\flat \simeq (\lim K^\circ/p)[\frac{1}{t}], \tag{2}$$

where the limit is along the Frobenius maps on K°/p, and $t \in \lim K^\circ/p$ is the p-power compatible system $(p^{\frac{1}{p^n}})$. The identification in Theorem 2 is given by exactly the same formula (for affines): one sends a perfectoid K-algebra A to the perfectoid $K^\flat$-algebra

$$A^\flat := (\lim A^\circ/p)[\frac{1}{t}].$$

The association $A \mapsto A^\flat$ is called *tilting*, while the inverse is called *untilting*; the nomenclature suggests viewing these operations as carrying us between the two ends of the following picture, resulting from (2):



We have already encountered some examples of tilting earlier in this note:

Example 4. The field $K^\flat$ is the tilt of K, as explained above, which clarifies the notation. Similarly, in Example 2, the ring $A^\flat$ is the tilt of A.

A more "global" example of tilting is given by:

Example 5. Fix an integer n. Globalizing Example 3 leads to a perfectoid $K^\flat$-space $\mathbf{P}^n_{K^\flat,\mathrm{perf}}$ obtained as the perfection of projective space $\mathbf{P}^n_{K^\flat}$ over $K^\flat$. Its untilt is (roughly) given by $\mathbf{P}^n_{K,\mathrm{perf}} := \lim \mathbf{P}^n_K$, where the transition maps raise all homogeneous coordinates to the p-th power.

The preservation of the étale topology under tilting is a deep result: it is a simultaneous generalization of Theorem 1 and of Faltings's "almost purity theorem," the key ingredient of his fundamental work in p-adic Hodge theory, which began in [Fal88] and allowed him to prove various conjectures of Fontaine.

Theorem 2 leads to the following picture summarizing the relationship of perfectoid spaces to classical algebraic geometry:

$$\begin{array}{ccc} \{\text{algebraic varieties}/K\} & & \{\text{algebraic varieties}/K^\flat\} \\ \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \text{perfection}} \\ \{\text{perfectoid spaces}/K\} & \underset{\text{untilt}}{\overset{\text{tilt}}{\rightleftarrows}} & \{\text{perfectoid spaces}/K^\flat\} \end{array}$$

Here the horizontal arrows come from Theorem 2, and the right vertical arrow is the globalization of Example 3. The mysterious dotted arrow Ψ is, in fact, nonexistent: there is no natural way to attach a perfectoid K-space to an algebraic K-variety. This apparent asymmetry can be explained by noticing that there is a *canonical* procedure for extracting all p-th roots in characteristic p (namely, taking the perfection), while there is no analogous construction in characteristic 0. Instead, given an algebraic K-variety X, each time one can *somehow* construct a related perfectoid K-space $\Psi(X)$, one learns a wealth of new information about X. We discuss some examples of this phenomenon in the section entitled "Examples and Applications."

Remark 1. In [Sch12], one finds a slightly more general version of the theory sketched here: the field K above is simply an example of a *perfectoid field*. For any such K, there is a tilt $K^\flat$ in characteristic p, and an analogous theory of perfectoid spaces over these fields (including, in particular, Theorem 2). An important example is $K = \mathbf{C}_p$ (the completed algebraic closure of $\mathbf{Q}_p$) whose tilt $K^\flat$ is the completed algebraic closure of $\mathbf{F}_p((t))$.

Examples and Applications

The theory of perfectoid spaces is rather young, but already extremely potent: each class of examples discovered so far has led to powerful and deep theorems in arithmetic geometry. We give a summary of some such examples next, with notation as in the previous section.

• Given a hypersurface $H \subset \mathbf{P}^n_K$, one can construct a perfectoid space U_ϵ which, essentially, is the tubular neighborhood of radius ϵ around the inverse image of H under $\mathbf{P}^n_{K,\mathrm{perf}} \to \mathbf{P}^n_K$, following the notation in Example 5. Using U_ϵ and Theorem 2, Scholze proved Deligne's weight-monodromy conjecture for smooth H in [Sch12] by reducing it to the analogous statement for a smooth hypersurface H' over the characteristic p field $K^\flat$ (as the latter was proven by Deligne en route to the Weil conjectures).

• Given a positive integer g, one may consider the moduli space $\mathcal{A}_g(p^\infty)$ parameterizing abelian varieties A over K equipped with a trivialization $\phi : \mathbf{Z}_p^{\oplus 2g} \simeq T_p(A)$ of their p-adic Tate modules. This space is rather large and pathological from the viewpoint of classical algebraic geometry. Nevertheless, in a recent preprint (titled "On torsion in the cohomology of locally symmetric varieties"), Scholze showed that $\mathcal{A}_g(p^\infty)$ is a well-behaved object: it is naturally a perfectoid K-space. In fact, he deduced a similar statement for any Shimura variety (of Hodge type) with full level structure at p. Using these spaces, he proved the following two results, which outwardly have nothing to do with perfectoid spaces (or even local fields): (a) a cohomological vanishing conjecture of Calegari and Emerton for Shimura varieties over $\mathbf{C}$ is true (much in the spirit of Example 1), and (b) one can attach Galois representations to *torsion* classes in the cohomology of locally symmetric spaces, which builds on recent work of Harris-Lan-Taylor-Thorne, and represents a significant step forward in the Langlands program.

• We end by touching on a theme that was largely skirted in the previous section. Namely, as perfectoid spaces live in the world of analytic geometry, they actually help study classical rigid-analytic spaces, not merely algebraic varieties (as in the previous two examples). In his "p-adic Hodge theory for rigid-analytic varieties" paper, Scholze pursues this idea to extend the foundational results in p-adic Hodge theory, such as Faltings's work mentioned above, to the setting of rigid-analytic spaces over $\mathbf{Q}_p$; such an extension was conjectured many decades ago by Tate in his epoch-making paper "p-divisible groups." The essential ingredient of Scholze's approach is the remarkable observation that *every* classical rigid-analytic space over $\mathbf{Q}_p$ is locally perfectoid, in a suitable sense.

The power of perfectoid spaces is only beginning to be exploited, and more applications will surely arise!

References

[Fal88] Gerd Faltings, p-adic Hodge theory, *J. Amer. Math. Soc.* **1**(1) (1988), 255–299.

[FW79] Jean-Marc Fontaine and Jean-Pierre Wintenberger, Extensions algébrique et corps des normes des extensions APF des corps locaux, *C. R. Acad. Sci. Paris Sér. A–B* **288**(8) (1979), A441–A444.

[Hub96] Roland Huber, Étale cohomology of rigid analytic varieties and adic spaces, *Aspects of Mathematics*, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Sch12] Peter Scholze, Perfectoid spaces, *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), 245–313.