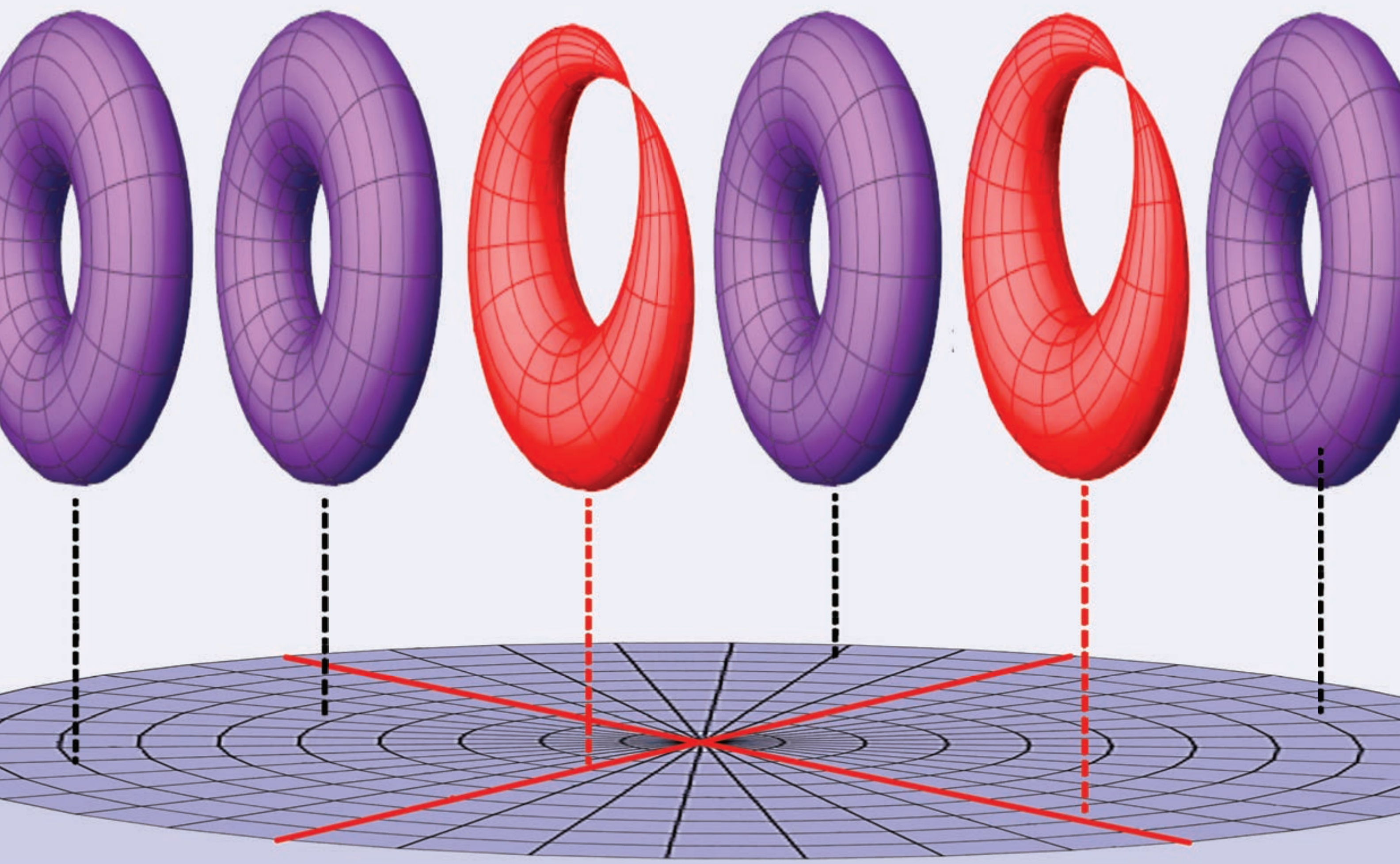
ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

May 2020

Volume 67, Number 5



The cover design is based on imagery from "Higgs Bundles—Recent Applications," *page 625.*

A WORD FROM...

Louise Jakobson, AMS Development Officer[1]

Courtesy of Billy Durvin.

The world is full of opportunities and problems big and small, whether at the planetary level, in far-flung regions of the world, or right on our doorstep. The array of environmental, social, educational, scientific, and medical issues that exist is endless. But, formidable as these issues are, human beings work collectively to make things better. They find solutions and implement them through government, industry, or business. At individual and group levels, they also make progress by giving and volunteering via nonprofits and other organizations.

Focusing on the United States, Americans have the reputation for being generous, and the data reflect that. The World Giving Index report for 2018 (`https://www.cafonline.org/about-us/publications/2018-publications/caf-world-giving-index-2018`) ranks the United States fourth worldwide (the top three countries are Indonesia, Australia, and New Zealand; Ireland, the UK, and Singapore are close behind the US). According to this study, 72% of Americans helped a stranger, 61% donated money, and 39% volunteered.

How much do people donate and what are they supporting? *Giving USA* reports that Americans gave $427.71 billion to nonprofits in 2018. Individuals account for 77% of that number (including bequests); foundations gave 18%, and corporations 5%. Where did donors direct their generosity? Religion (29%) received the largest segment, followed by Education (14%); Human Services (12%); Foundations (12%); Health (10%); Public-Society Benefit (7%); International Affairs (5%); Arts, Culture, and Humanities (5%); Environment/Animals (3%); and Individuals (2%).

There is a multitude of organizations to give to, and everyone has their own reasons for what they choose to support. At the AMS, donors regularly let us know why they are making their gift, large or small. In fact, one of the joys of working in fundraising is listening to donors share their stories! Often, they want to give back to their profession and support the younger generation, make opportunities available to others that they did not have themselves, spur important research in their field by supporting a prize, or honor a mentor or give in memory of an esteemed colleague or great mathematician. Each mathematics organization offers something different to the mathematics community. In many ways, the AMS is uniquely positioned to make an impact because of its longevity, financial stability, national and international reach, and active community of mathematicians in governance and committees.

There are many examples of mathematicians and people in other fields striving to advance a particular area to help solve some of the mathematical world's challenges. Paul Sally's passion for education drove him to establish the Arnold Ross Lectures endowed fund to bring top scholars to talented high school students; Don and Jill Knuth's contributions over recent years have helped Mathematical Reviews in many ways, including indexing, author disambiguation, and supporting native scripts; Joan and Joseph Birman created a Fellowship for Women Scholars to nurture outstanding research by mid-career women mathematicians; and many AMS donors have donated to the Epsilon Fund to benefit high school summer math camps.

Others prefer to give unrestricted funds, allowing the AMS flexibility to direct resources to the area of greatest need, including advocacy efforts for mathematics in Washington, DC; meetings and conferences; or additional funds to the MathSciNet for Developing Countries Program to ensure access to the mathematical literature. On average, approximately half of AMS annual donors choose to make their gift unrestricted.

Louise Jakobson is a development officer at the AMS. Her email address is `lxj@ams.org`.

[1]*The opinions expressed here are not necessarily those of the* Notices *or the AMS.*

As a donor, you may also want to think about *when* you want your gift to be used. If you want to effect change now, you can choose to have the AMS utilize your contribution immediately. If support for future needs and challenges is what is important to you, you can direct your gift to an existing endowment (meaning that the funds are invested and provide funding for a specified purpose for the years to come), or talk to development staff about creating a new one. The AMS has nineteen endowed funds supporting prizes; donors may also establish named endowed funds to support broad areas of need, including early career mathematicians, education, diversity, and advocacy. One such example is the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity; another is the Next Generation Fund, which was recently created to support the needs of current and future generations of early career mathematicians, an ongoing priority of the AMS.

I was interested to learn that endowments have a much longer history than one might imagine. The earliest known endowed chairs were established by the Roman emperor (and Stoic philosopher) Marcus Aurelius in Athens in AD 176 for the four major schools of philosophy, and the earliest known surviving endowed professorships were created by Lady Margaret Beaufort in Oxford and Cambridge in 1502 (these two funds are still in existence today!).

Finally, a number of donors also give through charitable estate planning, taking a longer-term view of their charitable giving. For example, Franklin Peterson, Cathleen Synge Morawetz, and mathematical couple Steven Schot and Joanna Wood Schot made unrestricted bequests to the AMS, while Edmund and Nancy Tomastik have declared their bequest intention to establish a prize in differential equations. All of these individuals demonstrate the wish to impact mathematics beyond their lifetimes. It can be very gratifying to know that you will be building mathematical research and scholarship for the future.

I know from experience that donors care very much what happens in the organizations they support. The annual AMS Contributors List for the year 2019 will be published in next month's issue of *Notices*. Approximately 1,400 donors a year support the AMS; as a development officer, I've had the privilege of talking with a number of these generous individuals. I can report that our donors care and make thoughtful and often deeply personal choices to invest in mathematics, its community, and its future.

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of the American Mathematical Society

May 2020

Cover Credit: The image used in the cover design appears in "Higgs Bundles—Recent Applications," p. 625, and is by Laura P. Schaposnik.

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[Notices of the American Mathematical Society (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA, GST No. 12189 2046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to Notices of the American Mathematical Society, PO Box 6248, Providence, RI 02904-6248 USA.] Publication here of the Society's street address and the other bracketed information is a technical requirement of the US Postal Service.

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LETTERS TO THE EDITOR

Letter to the Editor

Dear Colleagues,

Many thanks for a very interesting article, "How to Keep Your Secrets in a Post-Quantum World," published in the January 2020 issue of *Notices of the AMS*. This article describes ideas for "post-quantum cryptosystems that are not currently known to be breakable in polynomial time by a full-scale quantum computer." These are all great ideas, but readers who are not very familiar with this topic should be informed that already in the 1980s, researchers had developed quantum cryptography schemes—such as the 1984 Bennetts' and Brassard's Quantum Key Distribution scheme—which are not breakable even by a quantum computer. These are not just purely theoretical schemes: according to the Wikipedia page on quantum cryptography, several companies already manufacture such communication schemes, and they are actively used—in particular, for communications over hundreds of kilometers. Of course, this does not mean that the problem is fully solved: the existing quantum communication schemes have limitations, e.g., limitations on communication speed; from this viewpoint, it would be great to have faster alternative schemes, e.g., schemes described in the *Notices* article.

—Vladik Kreinovich and Luc Longpre
Department of Computer Science
University of Texas at El Paso

(Received December 20, 2019)

Where does "mathematical making" fit in our community?

At the end of a fantastic semester of Illustrating Mathematics at the Institute for Computational and Experimental Research in Mathematics (ICERM), many of the participants gathered to discuss the future of what we see as a growing movement. Where can we publish scholarly articles about mathematical visualization if the theorems alone might not justify publication? How does the mathematical community value the creation of new ways to see and communicate mathematics? The extraordinary creativity sparked by our being brought together makes us confident that more mathematicians will delight in taking up this enterprise. Those of us who have signed the Mathematical Makers' Manifesto below urge the mathematical community to support efforts in the same way ICERM so generously supported us this fall.

—Frank A. Farris
Santa Clara University

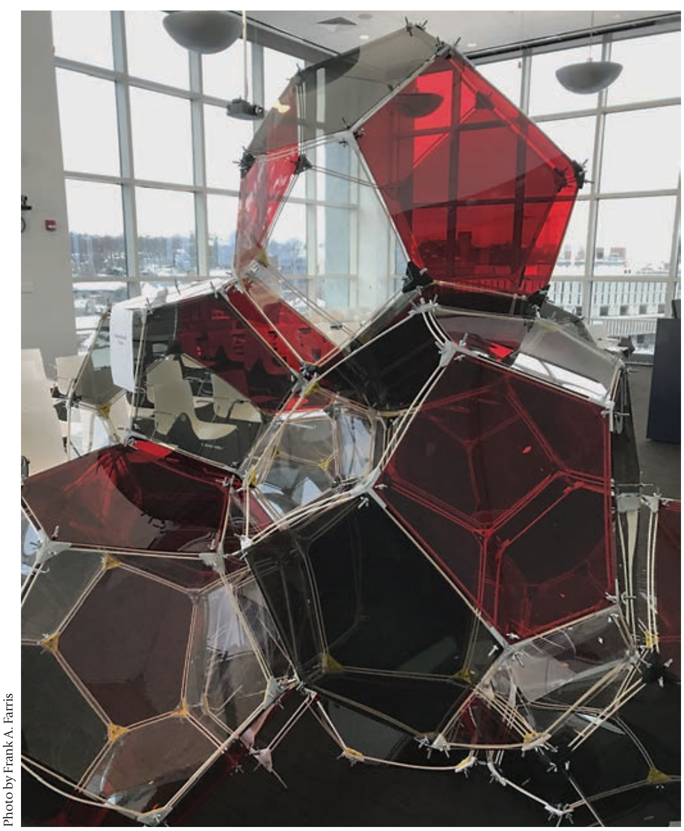

Photo by Frank A. Farris

A human-scale model of the Weaire-Phelan foam. Mathematical installation by Glen Whitney.

**We invite readers to submit letters to the editor at* `notices-letters@ams.org`.

Mathematical Makers' Manifesto

We are mathematical makers. We are makers because we make things, by which we might mean literal objects, such as sculptures, paintings, or fabrics, but our making includes creation of digital images, software, and even performance arts. We are mathematical makers because our creations require mathematical knowledge as a key ingredient. Why do we make these things? Our reasons are diverse, including education, outreach, and experimentation to investigate and create new mathematical understanding; we are also inspired to create works of art and useful crafts. We work to include mathematicians of many different backgrounds in our making, from beginning students to researchers in the farthest branches of mathematics. As the ultimate interdisciplinary subfield of mathematics, mathematical making deserves support from universities, museums, governments, and corporations around the world.

Signed by
Aaron Abrams, Washington and Lee University
Silviana Amethyst, University of Wisconsin–Eau Claire
Roger Antonsen, University of Oslo, Norway
Pierre Arnoux, Université d'Aix-Marseille
Jayadev Athreya, University of Washington, Seattle
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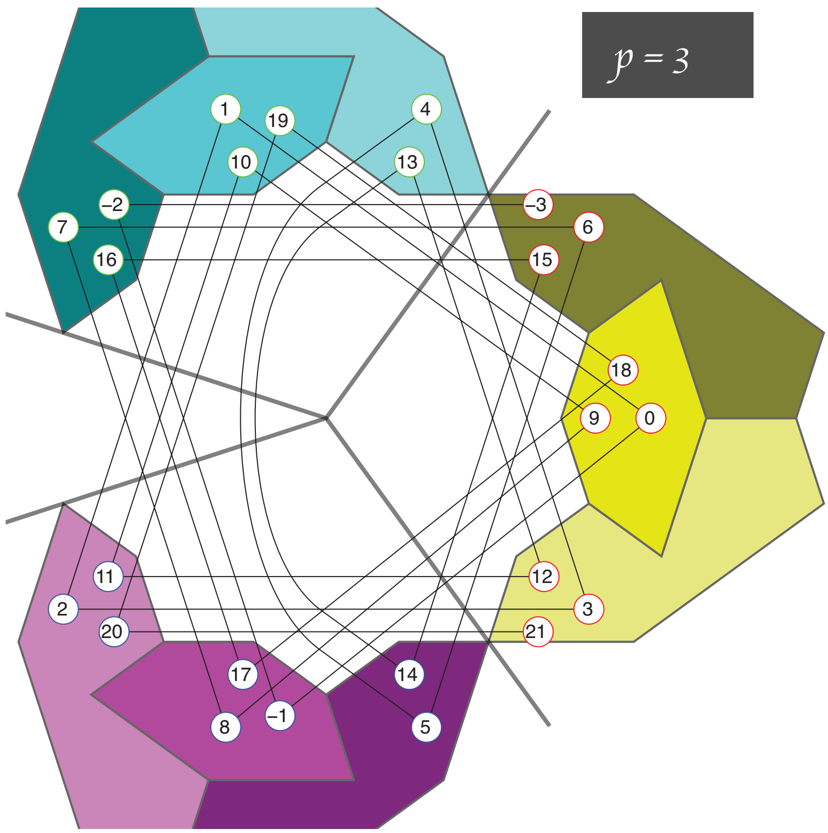
(Received January 23, 2020)

The Covering Method for Exponential Sums and Some Applications



Ivelisse M. Rubio

Introduction

Exponential sums over finite fields are an important tool for solving mathematical problems and have applications to many other areas. However, some of the methods and proofs of the results are nonelementary. The main purpose of this article is to present the covering method, an elementary and intuitive way to estimate or compute the p-divisibility of exponential sums, which is particularly convenient in the applications. The covering method allows us to determine solvability of systems of polynomial equations, improve the search for balanced Boolean functions, give better estimates for covering radius of codes, and has many other applications.

Solvability of systems of polynomial equations. One of the prominent problems in mathematics is to determine if a polynomial equation has solutions. In 1935 Artin conjectured that a homogeneous polynomial over a finite field has a nontrivial zero if the number of variables is larger than the degree. Chevalley obtained almost immediately a slightly better result changing the hypothesis of

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Communicated by Notices *Associate Editor Emilie Purvine.*

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DOI: https://doi.org/10.1090/noti2073

homogeneity to the weaker one of the polynomial having no constant term. Note that the homogeneous and the no-constant-term conditions imply that the polynomial has the trivial zero. The theorem guarantees additional zeros.

Warning improved Chevalley's result by proving that if the number of variables is larger than the sum of the degrees of a system of polynomials, then p, the characteristic of the field, divides the number of common zeros. This classical result is known as the Chevalley–Warning theorem and has an elementary proof [12, 20]. By elementary we mean that it uses only elementary results from number theory. Note that the number of zeros could be 0, but if the system has the trivial zero, Chevalley–Warning guarantees nontrivial solutions.

There are many results improving Chevalley–Warning's theorem. The results presented by Ax [2], Katz [11], Adolphson–Sperber [1], Moreno–Moreno [15], and Moreno et al. [17] have proofs that are nonelementary or semielementary. Other results presented by Moreno–Moreno [13], Wan [19], and Castro et al. [6] have entirely elementary proofs. As in the Chevalley–Warning theorem, solvability is not guaranteed; nontrivial solutions exist if the system has the trivial zero.

The covering method to study the p-divisibility of exponential sums is an elementary method introduced in [6] that lets us determine sufficient conditions to guarantee solvability and allows us to construct general families of solvable systems of polynomial equations [4, 5].

Applications to cryptography and coding theory. The divisibility of exponential sums has been used to characterize and prove properties in coding theory and cryptography [3, 7, 18]. The computation of bounds or the exact 2-divisibility of exponential sums of Boolean functions provides information on the Hamming weight of the function and can be used to obtain information on the covering radius and the weight distribution of certain codes. These properties are important for the analysis of decoding algorithms and are also related to cryptography, as they can be used to study nonlinearity and to search for balanced Boolean functions.

Exponential Sums Associated to Polynomials

We will restrict our exposition to exponential sums associated to polynomials in $\mathbb{F}_p[X_1, \dots, X_n]$, where p is a prime number and $\mathbb{F}_p$ is the finite field with p elements. The definition of these exponential sums depends on ζ, a pth root of unity over the p-adic field

$$\mathbb{Q}_p = \left\{a_r p^r + a_{r+1} p^{r+1} + \cdots \mid a_i \in \{0, \dots, p-1\}, r \in \mathbb{Z}\right\}.$$

For $a_r \neq 0$, define the **p-adic valuation of** $x = a_r p^r + a_{r+1} p^{r+1} + \cdots \in \mathbb{Q}_p$ as $v_p(x) = r$, the highest power of p dividing x, $v_p(0) = \infty$. We also call $v_p(x)$ the **p-divisibility of** x. If $v_p(x) \neq \infty$, we say that $v_p(x)$ is the **exact p-divisibility** of x.

Example 1. Consider $x = 36 = (3^2)(4) \in \mathbb{Z}$. Note that we can also represent x as $x = 3^2 + 3^3 \in \mathbb{Q}_3$. This implies that the exact 3-divisibility of 36 is 2. That is, $v_3(36) = 2$.

The set of p-adic integers is the local ring $\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p \mid v_p(x) \geq 0\right\}$ with maximal ideal $p\mathbb{Z}_p$ and residue field $\mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{F}_p$. One can use this valuation to define the **p-adic absolute value of** x by $|x|_p = p^{-v_p(x)}$ if $x \neq 0$ and $|0|_p = 0$.

The p-adic field $\mathbb{Q}_p$ is a completion of the rationals $\mathbb{Q}$, and its construction is similar to the construction of the real numbers $\mathbb{R}$ from $\mathbb{Q}$ but using the p-adic absolute value. So, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|.|_p$. The p-adic numbers offer a different perspective to study problems, and these methods can be helpful to understand concepts and prove properties that may be difficult without them. For an accessible introduction to the beautiful theory of p-adic numbers we refer the reader to [9].

For a polynomial $F \in \mathbb{F}_p[\mathbf{X}]$, where $\mathbf{X} = (X_1, X_2, \dots, X_n)$, and ζ a primitive pth root of unity over $\mathbb{Q}_p$, let $\zeta^a = \zeta^{a \bmod p}$, and define the **exponential sum associated to** F as

$$S(F) = \sum_{\mathbf{x} \in (\mathbb{F}_p)^n} \zeta^{F(\mathbf{x})} \in \mathbb{Z}_p.$$

The explicit evaluation of the exponential sum of a polynomial might be a difficult task, but for many applications it is enough to have estimates for $v_p(S(F))$. For simplicity, in some of the results in this article we consider only one polynomial, but the results can be extended to systems of polynomials $F_1, F_2, \dots, F_t \in \mathbb{F}_p[\mathbf{X}]$ by adding t extra variables $Y_1, \dots, Y_t$ (one per polynomial) and constructing a new polynomial $P = Y_1 F_1 + Y_2 F_2 + \cdots + Y_t F_t$. The **exponential sum associated to the system of polynomials** $F_1, F_2, \dots, F_t$ is the exponential sum associated to P. The relation between the number $\mathcal{N}$ of elements $(x_1, \dots, x_n) \in \left(\mathbb{F}_p\right)^n$ that are common zeros of the system and the exponential sum associated to the system is given by the following lemma:

Lemma 1. *Let $\mathcal{N}$ be the number of common zeros of $F_1, F_2, \dots, F_t \in \mathbb{F}_p[\mathbf{X}]$. Then*

$$\mathcal{N} = p^{-t} \sum_{\mathbf{x} \in (\mathbb{F}_p)^n, \mathbf{y} \in (\mathbb{F}_p)^t} \zeta^{y_1 F_1(\mathbf{x}) + y_2 F_2(\mathbf{x}) + \cdots + y_t F_t(\mathbf{x})}.$$

Since $\mathcal{N} = p^{-t} S(Y_1 F_1 + Y_2 F_2 + \cdots + Y_t F_t) = p^{-t} S(P)$, computing the exact value of $\mathcal{N}$ depends on the computation of $S(P)$, which is not easy. However, if we can get the exact p-divisibility of $S(P)$, $v_p(\mathcal{N}) < \infty$, we know that $p^{v_p(\mathcal{N})+1} \nmid \mathcal{N}$. This implies that $\mathcal{N} \neq 0$ and the system is solvable. Therefore, being able to compute the exact p-divisibility of an exponential sum of a system of polynomials gives a criterion for solvability of the system.

2-Divisibility of Exponential Sums of Boolean Functions

Most of the applications of exponential sums to coding theory and cryptography consider Boolean functions $f : (\mathbb{F}_2)^n \longrightarrow \mathbb{F}_2$. Any Boolean function f can be identified with a unique Boolean polynomial $F = \sum_{\mathbf{e}\in \mathrm{Supp}(F)} \mathbf{X}^{\mathbf{e}}$, where Supp$(F)$ is the set of exponents of the nonzero terms, $\mathbf{e} = (e_1, \dots, e_n) \in (\mathbb{F}_2)^n$, and $\mathbf{X}^{\mathbf{e}} = X_1^{e_1} X_2^{e_2} \cdots X_n^{e_n}$. This polynomial is known as the *algebraic normal form* of the Boolean function. The exponential sum of a Boolean polynomial $F \in \mathbb{F}_2[\mathbf{X}]$ is

$$S(F) = \sum_{\mathbf{x}\in(\mathbb{F}_2)^n} (-1)^{F(\mathbf{x})}.$$

The covering method for 2-divisibility. In [14], Moreno–Moreno introduced the covering method, which provides an elementary way to give a bound on the 2-divisibility of exponential sums of Boolean functions. Using this method, they gave an improvement to Ax's theorem in [2] for the binary case. However, the result does not give exact 2-divisibility and cannot be used to determine solvability or to find nonbalanced Boolean functions. Additional conditions have to be imposed to determine exact 2-divisibility. We now assume that any polynomial F is not a polynomial in some proper subset of the variables $X_1, \dots, X_n$.

Definition 1. A set C of monomials $F_{i_1}, \dots, F_{i_r}$ of a polynomial $F = F_1 + \cdots + F_m \in \mathbb{F}_2[\mathbf{X}]$ is called a **covering of** F if every variable X_i is in at least one monomial of C. The **size** of a covering C is its cardinality $|C|$. A set C is called a **minimal covering of** F if there is no other covering of F of smaller size.

Note that since, for $a \neq 0$, $X^a = X$ over $\mathbb{F}_2$, if we take the product of the monomials $F_{i_1} \cdots F_{i_r}$ in C we get $X_1X_2 \cdots X_n$, the monomial with all the variables. This fact will be useful in the generalization of the covering to any characteristic p.

Example 2. Let $F = X_1 + X_2 + \cdots + X_8 + X_1X_2X_3X_4 + X_3X_5X_6 + X_2X_7X_8 + X_4X_7X_8 \in \mathbb{F}_2[X_1, \dots, X_8]$. Then $C_1 = \{X_1, X_2, \dots, X_8\}$, $C_2 = \{X_1X_2X_3X_4, X_3X_5X_6, X_2X_7X_8\}$, and $C_3 = \{X_1X_2X_3X_4, X_3X_5X_6, X_4X_7X_8\}$ are coverings of F, but C_2, C_3 are the only minimal coverings of F.

Moreno–Moreno used minimal coverings of a Boolean function F to obtain a bound on the 2-divisibility of the exponential sum of F.

Theorem 1 ([14]). *Let C be a minimal covering of $F \in \mathbb{F}_2[\mathbf{X}]$. Then*

$$v_2(S(F)) \geq |C|.$$

One can use Theorem 1 and Lemma 1 to give a bound on the 2-divisibility of the number of solutions $\mathcal{N}$ of F. However, a bound does not guarantee that $\mathcal{N} \neq 0$. To determine solvability one needs to obtain exact 2-divisibility. Theorem 1 is general and tight in the sense that there are polynomials that attain the bound and have exact 2-divisibility $|C|$. This implies that to determine if a polynomial has exact 2-divisibility or to improve the bound, we need to impose additional conditions. The next theorem has simple conditions that are sufficient to obtain exact 2-divisibility.

Theorem 2 ([7]). *Let $F \in \mathbb{F}_2[\mathbf{X}]$, and let $C_1, \dots, C_c$ be all the minimal coverings of F. If, for each $1 \leq i \leq c$, each monomial in C_i has at least two variables that are not present in the other monomials of C_i, then $v_2(S(F)) = |C_i|$ if c is odd, and otherwise $v_2(S(F)) \geq |C_i| + 1$, where $|C_i|$ is the size of a minimal covering.*

With the given conditions, the above theorem refines Moreno–Moreno's Theorem 1.

Example 3. The polynomial in Example 2 has exactly two minimal coverings. Moreno–Moreno's Theorem 1 implies that $v_2(F) \geq 3$, but Theorem 2 guarantees that $v_2(F) \geq 4$. This might seem a small improvement, but in the applications even small improvements are important.

The next example shows that even though different Boolean functions might have the same unique minimal covering, and hence the same 2-divisibility, there is an ample spectrum for the exact value of $S(F)$.

Example 4. Consider $F = X_1X_2X_3X_4 + X_4X_5X_6X_7 + X_7X_8X_9$ and $F' = X_1X_2X_3X_4 + X_4X_5X_6X_7 + X_7X_8X_9 + X_1 + X_2 + \cdots + X_9$ in $\mathbb{F}_2[X_1, \dots, X_9]$. It can be verified that $S(F) = 8 \cdot 3 \cdot 13$ and $S(F') = 8$.

Although, in general, it is not an easy task to find all the minimal coverings of a given polynomial, one can easily construct polynomials for which one knows all the minimal coverings and hence knows the exact 2-divisibility. For example, to obtain unique minimal coverings it is enough to construct systems of polynomials with lead monomials of degree at least 2 and of disjoint support that cover all the variables.

Example 5. Consider the following system of polynomials in 13 variables, where $(\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{F}_2)^3$:

$$F_1 + G_1 = X_1X_2X_3X_4X_5 + \sum_i X_i - \alpha_1,$$

$$F_2 + G_2 = X_6X_7X_8X_9 + \sum_{i<j} X_iX_j - \alpha_2,$$

$$F_3 + G_3 = X_{10}X_{11}X_{12}X_{13} + \sum_{i<j<k} X_iX_jX_k - \alpha_3,$$

where $F_1 = X_1X_2X_3X_4X_5$, $F_2 = X_6X_7X_8X_9$, and $F_3 = X_{10}X_{11}X_{12}X_{13}$. Note that $C = \{Y_1F_1, Y_2F_2, Y_3F_3\}$ is the unique minimal covering of the associated polynomial

$P = Y_1(F_1+G_1) + Y_2(F_2+G_2) + Y_3(F_3+G_3)$. This implies that $S(P)$ has exact 2-divisibility $v_2(S(P)) = 3$.

Note that any system $F_1'+G_1', \dots, F_t'+G_t'$, where $F_1', \dots, F_t'$ have disjoint support and $\deg(G_i') < \min_i\{\deg(F_i')\}$, will also have an associated polynomial P with unique minimal covering, and hence $S(P)$ will have exact 2-divisibility $v_p(S(P)) = t$. One can determine other conditions so that families of "deformations" $F + G_i$ of a polynomial F have the same minimal coverings as F. This provides a way to obtain the 2-divisibility of exponential sums of polynomial deformations $F + G_i$ from the 2-divisibility of the exponential sum of the polynomial F.

Theorem 3 ([7]). *Let $F, G \in \mathbb{F}_2[\mathbf{X}]$. Suppose that the minimal coverings of F are the minimal coverings of $F+G$ and each monomial in each minimal covering C_F has at least two variables that are not present in the other monomials of C_F. Then $S(F+G) \equiv S(F) \pmod{2^{|C_F|+1}}$. Moreover, if the number of minimal coverings is odd, then $v_2(S(F+G)) = v_2(S(F)) = |C_F|$.*

Example 6. Consider $F = X_1X_2X_3 + X_4X_5X_6 \in \mathbb{F}_2[X_1,\dots,X_6]$ and let $F+G$ be any polynomial in $\mathbb{F}_2[X_1,\dots,X_6]$ with $\deg(G) \le 2$. Then $C = \{X_1X_2X_3, X_4X_5X_6\}$ is the unique minimal covering of F and $F+G$, and each monomial in C has three variables that are not present in the other monomial. This implies that $S(F)$ and $S(F+G)$ have exact 2-divisibility $v_2(S(F+G)) = v_2(S(F)) = |C| = 2$.

Example 7. Consider $F = X_1X_2X_3 + X_4X_5X_6 + X_1X_4X_5 + X_2X_4X_6 + X_3X_5X_6 \in \mathbb{F}_2[X_1,\dots,X_6]$ and let $F+G$ be any polynomial in $\mathbb{F}_2[X_1,\dots,X_6]$ where $\deg(G) \le 2$. Again, $C = \{X_1X_2X_3, X_4X_5X_6\}$ is the unique minimal covering of F and $F+G$ and $v_2(S(F+G)) = v_2(S(F)) = 2$.

Examples 6 and 7 provide families of polynomials whose exponential sums have exact 2-divisibility. The intuitive and simple condition of $F+G$ and F having the same minimal coverings allows us to easily construct families of deformations with exact 2-divisibility. We will see later that this has useful applications to the determination of nonbalanced Boolean functions.

Solvability. As mentioned above, one of the main applications of p-divisibility of exponential sums is to obtain information about the number of solutions of systems of equations. Lemma 1 gives the relation between exponential sums and the number of solutions $\mathcal{N}$ of a system of polynomial equations $F_1 = \cdots = F_t = 0$. Using Theorem 2 one could determine if $\mathcal{N}$ has exact 2-divisibility $v_2(\mathcal{N})$. If this happens, $2^{v_2(\mathcal{N})+1}$ does not divide $\mathcal{N}$, $\mathcal{N} \neq 0$, and the system is solvable.

Example 8. Consider the system

$$X_1X_2X_3X_4X_5 + \sum_i X_i = \alpha_1,$$

$$X_6X_7X_8X_9 + \sum_{i<j} X_iX_j = \alpha_2,$$

$$X_{10}X_{11}X_{12}X_{13} + \sum_{i<j<k} X_iX_jX_k = \alpha_3.$$

The solutions of this system are the zeros of the system of polynomials $F_1+G_1, F_2+G_2, F_3+G_3$ in Example 5. Since $v_2(S(P)) = 3$, $v_2(\mathcal{N}) = 0$ and $2 \nmid \mathcal{N}$. This implies that $\mathcal{N} \neq 0$ and the system is solvable for any $(\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{F}_2)^3$.

Other applications. Other important applications of exponential sums are to coding theory and cryptography. Error-correcting codes are used to protect digital information from accidental errors that might occur during transmission or storage; the aim is for the receiver to be able to detect and correct errors that were introduced accidentally and retrieve the original message that was sent. On the other hand, cryptography is used to hide information from intruders; the information transmitted should be understood only by its intended receiver. Coding theory and cryptography serve different purposes, but they both share some theoretical concepts and methods.

In the coding process an **encoder** adds redundancy to a block of symbols of length k that represents the **message m** to transform it into a **codeword c** of block length n so that when received, the **decoder** can detect and correct errors. The **code** $\mathcal{C}$ is the set of all codewords. One can identify the messages with k-tuples of symbols from a finite field $\mathbb{F}_q$ and give the code the structure of a vector space of dimension k over $\mathbb{F}_q$. The encoder is then a one-to-one linear map $\mathcal{E}_{\mathcal{C}} : (\mathbb{F}_q)^k \longrightarrow (\mathbb{F}_q)^n$, and the linear code $\mathcal{C} = \mathrm{Im}(\mathcal{E}_{\mathcal{C}})$.

At first one might think that the decoder could just be the inverse of the encoding function. But the problem is that after the codeword $\mathbf{c} = \mathcal{E}_{\mathcal{C}}(\mathbf{m})$ is transmitted, the received word is $\mathbf{r} = \mathbf{c} + \mathbf{e}$, where $\mathbf{e}$ is an error vector. So, $\mathcal{E}_{\mathcal{C}}^{-1}(\mathbf{r}) \neq \mathbf{c}$ if $\mathbf{e} \neq 0$. Hence we need "good" codes, coding and decoding algorithms that allow us to detect and correct errors. The main problem in coding theory is to find codes with large rate $\frac{k}{n}$ of information symbols k per total number of symbols n that can correct "enough" errors, where "large" and "enough" will depend on the transmission channel for which the code is designed.

From now on we will consider **binary linear codes**, that is, linear codes over $\mathbb{F}_2$. The **Hamming weight of a vector x**, $w_H(\mathbf{x})$, is the number of entries of $\mathbf{x}$ that are nonzero. The **Hamming distance between two vectors x, y**, $d_H(\mathbf{x}, \mathbf{y})$ defines a metric and is the number of places on which the vectors disagree; this is equivalent to the Hamming weight of $\mathbf{x} + \mathbf{y}$. The **minimum distance d of a code** $\mathcal{C}$ is the minimum (Hamming) distance between any two codewords, $d = \min\{d_H(\mathbf{x},\mathbf{y}) \mid \mathbf{x},\mathbf{y} \in \mathcal{C}\}$, and, since we are considering linear codes, this is equal to the minimum of the Hamming weight of the codewords $d = \min\{w_H(\mathbf{x}) \mid \mathbf{x} \in \mathcal{C}\}$.

Let $d \geq 2t+1$ be the minimum distance of a code $\mathcal{C}$ and suppose that a codeword $\mathbf{c}$ has been transmitted and $\mathbf{r} = \mathbf{c}+\mathbf{e}$ has been received, where $\mathbf{e}$ is an error vector with $w_H(\mathbf{e}) \leq t$. Then, $\mathbf{c}$ is the only codeword with distance from $\mathbf{r}$ less than or equal to t. Hence a possible decoding algorithm for a received word $\mathbf{r}$ is to look for the codeword that is closest in Hamming distance to $\mathbf{r}$. In practice this would be too inefficient but guarantees that if t or fewer errors occur, we can always correct them. This is why a linear block code with minimum distance $d \geq 2t+1$, block length n, and dimension k is called a t-**error-correcting code** with parameters (n,k,t). The goal of research in coding theory is to construct codes that have large d for the given rate $\frac{k}{n}$ and to design efficient algorithms to encode and decode them.

A **cryptographic system** is a set of transformations of the set of all messages into another space with certain properties. The message is **enciphered** into the **ciphertext** using a particular key that defines an injective mapping. Two important principles in the design of cryptographic systems are *confusion* and *diffusion*. The principle of diffusion makes different messages equally likely to occur; one way to measure diffusion is to determine if the function used to cipher is *balanced*. The principle of confusion measures the complexity of the decryption process; the *nonlinearity* of the functions used in the system gives a measure for confusion. Functions that are balanced and have large nonlinearity are desired.

Reed–Muller codes are some of the oldest and most-studied codes; nevertheless, there are still many open problems related to them that are important in both coding and cryptographic applications. To define a Reed–Muller code of size 2^n, given a fixed ordering of $(\mathbb{F}_2)^n$, one associates a Boolean polynomial $F \in \mathbb{F}_2[X_1,\ldots,X_n]$ with the vector of size 2^n consisting of all the values of $F(\mathbf{x})$ as $\mathbf{x}$ varies according to the ordering. This is also called the **truth table** of F. For example, if $(F_2)^3$ is ordered in lexicographic order with $X_1 > X_2 > X_3$, the truth table for $F(X_1,X_2,X_3) = X_1 + X_2X_3$ is $(0,0,0,1,1,1,1,0) \in (\mathbb{F}_2)^8$. When convenient, we will work with this representation of F in $(\mathbb{F}_2)^n$ instead of its polynomial representation; we will also alternate between calling F a function or a polynomial. The k**th order Reed–Muller code of length** 2^n, $R(k,n)$, is the set of truth tables of all the Boolean polynomials in n variables and degree less than or equal to k. That is, $R(k,n)$ can be identified with the set of Boolean polynomials in n variables and degree less than or equal to k.

The Reed–Muller code of order 1, $R(1,n)$, is the set of Boolean polynomials in n variables with degree less than or equal to 1. The *nonlinearity* of a Boolean function F is the Hamming distance from F to $R(1,n)$.

Exponential sums, Hamming weights, and nonlinearity. The **Hamming weight associated to a Boolean function** F, $w_H(F)$, is the Hamming weight of its truth table. This is the number of $\mathbf{x} \in (\mathbb{F}_2)^n$ such that $F(\mathbf{x}) = 1$. If $w_0(F)$ is the number of $\mathbf{x} \in (\mathbb{F}_2)^n$ such that $F(\mathbf{x}) = 0$, then $2^n = w_H(F) + w_0(F)$. Also, $S(F) = \sum_{\mathbf{x}\in(\mathbb{F}_2)^n}(-1)^{F(\mathbf{x})} = w_0(F)(-1)^0 + w_H(F)(-1)^1 = w_0(F) - w_H(F)$. This implies that $w_H(F) = 2^{n-1} - \frac{1}{2}S(F)$ and gives a correspondence between results on exponential sums and Hamming weights of Boolean functions. Hence, any result for exponential sums of a Boolean function also gives a corresponding result about the Hamming weight of the function.

Defining the Hamming distance of a Boolean function F to a vector $\mathbf{x}$ as the Hamming weight of the sum of $\mathbf{x}$ with the truth table of F, one can define the Hamming distance from F to a code $\mathcal{C}$ as $\min_{\mathbf{c}\in\mathcal{C}}\{w_H(F+\mathbf{c})\}$. This lets us define a measure for the principle of confusion in the cryptographic system, a sense of "how far is a Boolean function F from being linear." The **nonlinearity** of F is $Nl(F) = w_H(F + R(1,n))$, that is, the minimum Hamming distance between F and all the codewords in $R(1,n)$. This can be defined in terms of the exponential sums of cosets of $R(1,n)$,

$$Nl(F) = \min_{\mathbf{c}\in R(1,n)}\left\{2^{n-1} - \frac{1}{2}S(F+\mathbf{c})\right\},$$

and we can use results on exponential sums of deformations of Boolean functions to study the nonlinearity of a Boolean function F.

Covering radius of a code. The **covering radius** $\rho(\mathcal{C})$ is another important parameter of a code $\mathcal{C}$:

$$\rho(\mathcal{C}) = \max_{\mathbf{x}\in(\mathbb{F}_2)^n}\left\{\min_{\mathbf{c}\in\mathcal{C}}\{w_H(\mathbf{x}+\mathbf{c})\}\right\}.$$

This measure gives the maximum weight of a correctable error and can be used for the design of decoding algorithms. A code of minimum distance $2t+1$ is called **perfect** if $\rho(\mathcal{C}) = t$ and **quasi-perfect** if $\rho(\mathcal{C}) = t+1$.

The covering radius of the Reed–Muller code of order 1, $\rho(R(1,n))$, is the maximum Hamming distance of all n-variate Boolean polynomials to $R(1,n)$. We then have $Nl(F) \leq \rho(R(1,n))$. The covering radius of $R(1,n)$ gives a point of comparison for the nonlinearity of a Boolean polynomial and hence a sense of "how good" the function could be for cryptographic applications.

Results on the 2-divisibility of exponential sums have been used in several papers [16] to give elementary direct proofs of the covering radius of certain cyclic codes and to prove that families of cyclic codes are quasi-perfect.

Weight distribution. The weight distribution of a code $\mathcal{C}$ counts how many codewords of each weight there are. Much work has been done studying the weight distribution of Reed–Muller codes, but (as mentioned earlier) many problems remain open. Many of the properties of a Boolean function F that are important to cryptography can be related to the weight distribution of the coset $F+R(1,n)$ and can be studied using exponential sums. Canteaut [3] obtained a result that is a refinement of the Hamming weight version of Katz's theorem for the Boolean case and used it to study the weight distribution of cosets of first-order Reed–Muller codes. Her result is tight for the Boolean case and can be improved only by imposing additional conditions. The additional conditions to the covering method in Theorem 3 allowed us to obtain an improvement of her results [7].

Balanced functions. A Boolean function F is said to be **balanced** if the function is equal to 1 in half of the values of $\mathbf{x} \in (\mathbb{F}_2)^n$. Equivalently, an n-variate Boolean function F is balanced if the Hamming weight of its truth table, $w_H(F)$, is 2^{n-1}. This property is important in cryptographic applications because it follows the principle of diffusion: the function has no bias towards a value. The search for balanced Boolean functions and the development of new methods for constructing them are active areas of research.

It is easy to see that a Boolean function F is balanced if and only if $S(F) = 0$. If $S(F)$ has exact 2-divisibility, then $p^{v_p(S(F))+1} \nmid S(F)$, $S(F) \neq 0$, and F is not balanced. Hence, if one can describe families of Boolean functions with exact 2-divisibility, one is describing families of Boolean functions that are not balanced, and this can reduce the search for balanced Boolean functions.

In [10] Hou used the action of the group $\mathrm{GL}(n,2)$ on quotients of Reed–Muller codes $R(k,n)/R(k-1,n)$ to count the number of balanced polynomials in the cosets of $R(k-1,n)$. Note that the number of balanced polynomials in a coset of $R(k-1,n)$ is included in the weight distribution of the coset. Cosets of $R(k-1,n)$ belonging to the same orbit under this action have the same weight distribution and hence the same number of balanced polynomials. This implies that to know the number of balanced polynomials of all the cosets in an orbit, it is enough to study a coset representative for the orbit. Cosets of Reed–Muller codes $F+R(k-1,n)$ are sets of deformations of the polynomial F, and one can use Theorem 3 to determine nonbalanced polynomials a priori and improve the search for balanced functions.

Example 9. Consider the cosets of $R(3,6)/R(2,6)$, $X_1X_2X_3+X_4X_5X_6+R(2,6)$, and $X_1X_2X_3+X_4X_5X_6+X_1X_4X_5+X_2X_4X_6+X_3X_5X_6+R(2,6)$. The polynomials in these cosets satisfy the conditions in Examples 6 and 7 and hence have exact 2-divisibility. Therefore all the polynomials in these cosets are nonbalanced.

Hou presented representatives for each of the different orbits in $R(k,n)/R(k-1,n)$ for $k=3, n=6,7,8$. Example 9 shows two of the six cosets of $R(3,6)/R(2,6)$. Cusick and Cheon noticed in [8] the uneven distribution of the balanced functions in the table of balanced functions in the cosets of $R(3,6)/R(2,6)$. Two of the six cosets, the two cosets of Examples 6, 7, and 9, have zero balanced functions compared to more than 1.5 million in each of the other four cosets. The covering method gives a simple explanation for this phenomenon: as was seen in the examples, for any $G \in R(2,6)$, F and $F+G$ have the same unique minimal covering $\{X_1X_2X_3, X_4X_5X_6\}$, where each monomial has three variables not contained in the other monomial, and hence $F+G$ is not balanced.

It is not difficult to find sufficient conditions that can be used to determine a priori cosets of Reed–Muller codes that do not contain any balanced function, saving computational time. For example, we can use the covering method to identify by inspection 15 coset representatives (out of 32) in $R(3,8)/R(2,8)$ for which at least half of the functions in each coset are not balanced and provide constructions for these nonbalanced functions [7, 10]. This can be used to determine a priori types of polynomials to avoid in the search for balanced functions.

p-Divisibility of Exponential Sums

As mentioned above, the proof of most of the improvements and extensions of the Chevalley–Warning theorem are nonelementary. Ax and Katz used estimates on the p-divisibility of exponential sums to improve the Chevalley–Warning theorem. Katz [11] obtained that $\mathcal{N}$, the number of common zeros of polynomials $F_1,\dots,F_t$ in $\mathbb{F}_q[X_1,\dots,X_n]$ of degree $d_1,\dots,d_t$, is divisible by q^μ, where μ is the smallest nonnegative integer $\mu \geq \mu_0$:

$$\mu_0 = \frac{n-\sum_{i=1}^t d_i}{\max\{d_i\}}.$$

Note that the theorem gives information on the p-divisibility of $\mathcal{N}$ only if there are "enough variables" n; if $n < \sum_{i=1}^t d_i$, $\mu = 0$, the conclusion is that $1|\mathcal{N}$, and the theorem does not give any information. Adolphson–Sperber [1] improved Katz's result using a Newton polyhedra approach, and Moreno–Moreno gave an improvement by using the p-weight degree of the polynomials instead of their regular degree. The p-**weight degree** of a polynomial F, $w_p(F)$ is the maximal p-weight degree of its monomials. The p-weight degree of the monomial $\mathbf{X}^{\mathbf{e}} = X_1^{e_1}\cdots X_n^{e_n}$ is

$$w_p(\mathbf{X}^{\mathbf{e}}) = \sigma_p(e_1)+\cdots+\sigma_p(e_n),$$

where for $a = a_0 + a_1p + \cdots + a_rp^r$, $\sigma_p(a) = \sum_{i=0}^r a_i$. For $q = p^f$, Moreno–Moreno [13] found that $\mathcal{N}$ is divisible by

p^{μ}, where μ is the smallest nonnegative integer $\mu \geq \mu_0$,

$$\mu_0 = f\frac{n - \sum_{i=1}^{t} w_p(F_i)}{\max\{w_p(F_i)\}}.$$

Again, note that the theorem gives information on the p-divisibility of $\mathcal{N}$ only if there are "enough variables"; if the number of variables is less than or equal to the sum of the p-weight degree of the polynomials, $\mu = 0$ and the theorem does not give any information. A tight bound for the p-divisibility of exponential sums was given by Moreno et al. in [17]. That and Adolphson–Sperber's results use the exponents $\mathbf{e}_1, \dots, \mathbf{e}_m$ of all the monomials in the polynomial $F(\mathbf{X}) = \sum_{i=1}^{m} a_i X_1^{e_{1i}} \cdots X_n^{e_{ni}}$ in contrast to the Chevalley–Warning, Ax–Katz, and Moreno–Moreno results that use only the degree or the p-weight degree of the polynomial. In this sense the result in [17] resembles the covering method for Boolean polynomials. None of these results can be used to determine if the exponential sum has exact p-divisibility, and solvability cannot be determined.

The covering method for p-divisibility. In [6], Castro et al. introduced a generalization to any prime field of the covering method introduced in [14] for characteristic 2. With it they proved the prime field case of the theorem on the p-divisibility of exponential sums presented in [17]. The new proof was entirely elementary, and, as a consequence, elementary proofs and improvements of previous results on the p-divisibility of exponential sums were obtained.

For the case $q = p$, the tight bound in [17] relies on finding a minimal solution $(s_1, \dots, s_m)$ to a system of n modular equations $e_{j1}s_1 + e_{j2}s_2 + \cdots + e_{jm}s_m \equiv 0 \pmod{p-1}$, associated to the exponents of each variable in the polynomial $F(\mathbf{X}) = \sum_{i=1}^{m} a_i X_1^{e_{1i}} \cdots X_n^{e_{ni}}$. That is, one needs to find solutions to a system

$$\begin{cases} e_{11}s_1 + e_{12}s_2 + \cdots + e_{1m}s_m &= \lambda_1(p-1) \\ \qquad\vdots & \qquad\vdots \\ e_{n1}s_1 + e_{n2}s_2 + \cdots + e_{nm}s_m &= \lambda_n(p-1), \end{cases}$$

$\lambda_i \in \mathbb{N}$, where each column corresponds to a term and each row to a variable, that are minimal in terms of $L = \sum_{i=1}^{m} s_i$. In this case, $v_p(S(F)) \geq \frac{L}{p-1}$.

If the system is rewritten as

$$\begin{pmatrix} e_{11} \\ e_{21} \\ \vdots \\ e_{n1} \end{pmatrix} s_1 + \cdots + \begin{pmatrix} e_{1m} \\ e_{2m} \\ \vdots \\ e_{nm} \end{pmatrix} s_m = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} (p-1), \qquad (1)$$

one sees that the solutions that one is looking for are exponents s_i for the monomials $F_i = X_1^{e_{1i}} \cdots X_n^{e_{ni}}$ in F such that

$$F_1^{s_1} F_2^{s_2} \cdots F_m^{s_m} = X_1^{\lambda_1(p-1)} \cdots X_n^{\lambda_n(p-1)},$$

for $\lambda_1, \dots, \lambda_n \geq 1$, and such that $s_1 + \cdots + s_m$ is as small as possible [4]. If $p = 2$, then $C = \{F_1^{s_1}, F_2^{s_2}, \dots, F_m^{s_m}\}$ is a covering for F, as some of the s_i could be zero. This is the motivation for the definition of a minimal $(p-1)$-covering below. Note that the solutions do not depend on the coefficients of the polynomial F.

Definition 2. Let $F(\mathbf{X}) = a_1F_1 + a_2F_2 + \cdots + a_mF_m$. A set $C = \{F_1^{s_1}, \dots, F_m^{s_m}\}$ of powers of the monomials in F is a **minimal** $(p-1)$**-covering of** F if $F_1^{s_1} \cdots F_m^{s_m} = X_1^{\lambda_1(p-1)} \cdots X_n^{\lambda_n(p-1)}$ with $\lambda_i \geq 1$ and its **size**, $\sum_{i=1}^{m} s_i$, is minimal.

A $(p-1)$-covering need not use all the F_i's, and therefore some of the s_i's could be equal to 0.

Example 10. Let $F(\mathbf{X}) = X_1^2X_2^3 + X_1^2 + X_2^3 \in \mathbb{F}_7[X_1, X_2]$. Then $C_1 = \{(X_1^2X_2^3)^6\}$, $C_2 = \{(X_1^2)^3, (X_2^3)^2\}$, and $C_3 = \{(X_1^2X_2^3)^2, (X_1^2)^1\}$ are 6-coverings of F, and C_3 is the unique minimal 6-covering of F (of size 3).

A minimal $(p-1)$-covering of a polynomial F might not be unique, and the concept is independent of the coefficients of F. However, the exact p-divisibility of $S(F)$ and the solvability of equations involving F depend on both the minimal $(p-1)$-coverings and the relation among the coefficients. Also, if there are powers of the monomials in F that cover some (but not all) of the variables and are minimal in some sense, it is very hard to determine the exact p-divisibility. In Theorem 4 we avoid polynomials with this type of *minimal partial* $(p-1)$*-covering*.

Definition 3. Let $F(\mathbf{X}) = a_1F_1 + a_2F_2 + \cdots + a_mF_m$. A set $C = \{F_1^{s_1}, \dots, F_m^{s_m}\}$ of powers of the monomials in F is a **partial** $(p-1)$**-covering of** F if $F_1^{s_1} \cdots F_m^{s_m} = X_1^{\lambda_1(p-1)} \cdots X_n^{\lambda_n(p-1)}$ with $\lambda_i \geq 0$. The set C is a **minimal partial** $(p-1)$**-covering of** F if its size $\sum_{i=1}^{m} s_i + s(p-1)$, where s is the number of variables missing, is the size of a minimal $(p-1)$-covering of F.

Note that instead of requiring each exponent $\lambda_i(p-1)$ of X_i to be a positive multiple of $p-1$, in the definition of a partial $(p-1)$-covering, λ_i could be equal to 0, and therefore some variables could be missing. If $s = 0$, there are no variables missing, and we have the previous definition of the $(p-1)$-covering.

Example 11. Let $F(\mathbf{X}) = X_1^2X_2^3 + X_1^2 + X_2^3 \in \mathbb{F}_7[X_1, X_2]$ be the polynomial of Example 10. Then $C_4 = \{(X_2^3)^2\}$ is a partial 6-covering of F of size $2 + 6 = 8$. In Example 10 we saw that the minimal 6-coverings have size 3, and therefore C_4 is not a minimal partial 6-covering of F.

By avoiding minimal partial $(p-1)$-coverings we can improve previous results by computing exact p-divisibility of exponential sums or improving previous bounds. The next result [4] is a generalization of Theorem 2 to any characteristic.

Theorem 4. *Let $C_1, \dots, C_c$ be all the minimal $(p-1)$-coverings of size L of a polynomial $F = a_1F_1 + \cdots + a_mF_m$, $C_i = \{F_1^{s_{i1}}, \dots, F_m^{s_{im}}\}$, and suppose that any minimal partial $(p-1)$-covering is one of the C_i's. Then, $v_p(S(F)) = \frac{L}{p-1}$ if $\sum_{i=1}^{c} \frac{a_1^{s_{i1}} \cdots a_m^{s_{im}}}{s_{i1}! \cdots s_{im}!} \not\equiv 0 \pmod{p}$, and otherwise $v_p(S(F)) \geq \frac{L}{p-1} + 1$.*

The condition $\sum_{i=1}^{c} \frac{a_1^{s_{i1}} \cdots a_m^{s_{im}}}{s_{i1}! \cdots s_{im}!} \not\equiv 0 \pmod{p}$ is the generalization of the number c of minimal coverings being odd in the case of $p = 2$. The condition of the minimal partial $(p-1)$-coverings being one of the C_i's implies that any minimal partial $(p-1)$-covering does not have a missing variable. Similarly to the case where $p = 2$, a sufficient condition for not having minimal partial $(p-1)$-coverings with missing variables is to require that for each of the minimal $(p-1)$-coverings C_i, each monomial in C_i has at least two variables that are not present in the other monomials of C_i. This simple condition provides families of polynomials for which the "greater than or equal to" relation obtained in the classical results on p-divisibility is replaced by either equality or strict inequality.

Theorem 4, when applied to the number of solutions of systems of polynomial equations, gives refinements to the prime field case of many of the known results by giving precise conditions for when the system is solvable or the bound on the p-divisibility of the number of solutions $\mathcal{N}$ is improved. Moreover, it can also give information on the solvability or p-divisibility of $\mathcal{N}$ for cases that are not covered by previous theorems.

Example 12. Let $p \neq 2$ and consider the system

$$\begin{aligned} X_1^2X_2^4 + X_3^6X_4^2 + X_1 + X_2 + X_6 + X_7 &= \alpha, \\ X_5^2X_6^2 + X_7^6X_8^2 + X_3 + X_5 + X_8 &= \beta \end{aligned}$$

over $\mathbb{F}_p^*$. Note that the system has 8 variables and the sum of the degree of the polynomials is 16; hence Ax–Katz's theorem does not give any information on solvability nor p-divisibility of $\mathcal{N}$. For $p = 3, 5$ and $p > 5$ the sum of the p-weight degree of the polynomials is $8, 10$, and 16, respectively; hence Moreno–Moreno's theorem does not give any information either.

To use the covering method, we first compute the polynomial associated to this system:

$$P = Y_1\left(X_1^2X_2^4 + X_3^6X_4^2 + X_1 + X_2 + X_6 + X_7 - \alpha\right) + Y_2\left(X_5^2X_6^2 + X_7^6X_8^2 + X_3 + X_5 + X_8 - \beta\right).$$

It is easy to see that the unique minimal $(p-1)$-covering for P is

$$C = \left\{\left(Y_1X_1^2X_2^4\right)^{\frac{p-1}{2}}, \left(Y_1X_3^6X_4^2\right)^{\frac{p-1}{2}}, \left(Y_2X_5^2X_6^2\right)^{\frac{p-1}{2}}, \left(Y_2X_7^6X_8^2\right)^{\frac{p-1}{2}}\right\},$$

there are no minimal partial $(p-1)$-coverings with missing variables, and $\frac{a_1^{s_1} \cdots a_4^{s_4}}{s_1! \cdots s_4!} \not\equiv 0 \pmod{p}$. This implies that the exact p-divisibility of $\mathcal{N}$ is $v_p(\mathcal{N}) = 4\left(\frac{p-1}{2}\right)/(p-1) - 2 = 0$. Therefore, $p \nmid \mathcal{N}$, $\mathcal{N} \neq 0$, and the system is solvable for any $\alpha, \beta \in \mathbb{F}_p^*$.

By imposing conditions on polynomials G so that $F+G$ and F have the same minimal $(p-1)$-coverings, one can extend known results on the p-divisibility of $S(F)$ to results on $S(F+G)$ for deformations of F.

Example 13. Let $\mathcal{N}$ be the number of solutions of the system

$$\begin{aligned} aX_1^{p-1} + \cdots + aX_p^{p-1} + G &= 0, \\ b_1X_1 + \cdots + b_pX_p + \alpha &= 0, \end{aligned} \tag{2}$$

where $a, b_i \in {\mathbb{F}_p}^*, \alpha \in \mathbb{F}_p$, $G \in \mathbb{F}_p[\mathbf{X}]$, and $\deg G < p-1$.

This system has p variables, and sum of the degree and of the p-weight degree of the polynomials is also p. Hence Ax–Katz and Moreno–Moreno's theorems do not give any information on solvability or the p-divisibility of $\mathcal{N}$. There are p different minimal $(p-1)$-coverings with form

$$\left\{Y_1X_{i_1}^{p-1}, \dots, Y_1X_{i_{p-1}}^{p-1}, \left(Y_2X_{i_p}\right)^{p-1}\right\}$$

and size $L = 2(p-1)$. Since $\sum_{i=1}^{p} \frac{a^{p-1}b_i^{p-1}}{(p-1)!} = \frac{p}{(p-1)!}$, we have $v_p(\mathcal{N}) = -2 + v_p(S(P)) > -2 + \frac{2(p-1)}{p-1} = 0$. The result does not give information about solvability but gives some information about the p-divisibility of $\mathcal{N}$.

Conclusions

The covering method is an elementary method to obtain information about the p-divisibility of exponential sums. It provides an intuitive approach to the computation of exact p-divisibility that can be exploited in applications. It also gives a simple way to construct families of systems of polynomial equations that are solvable, determine p-divisibility of the number of solutions of systems for cases where previous results do not give information, and can be applied to answer questions in coding theory and cryptography.

ACKNOWLEDGMENTS. The author dedicates this work to the memory of her friend and collaborator, Francis Castro. The author appreciates the careful review, corrections, and comments made by the associate editor, the referees, Ricardo Cortez, H. F. Mattson Jr., Herbert Medina, and Luis Medina, which helped to improve the article.

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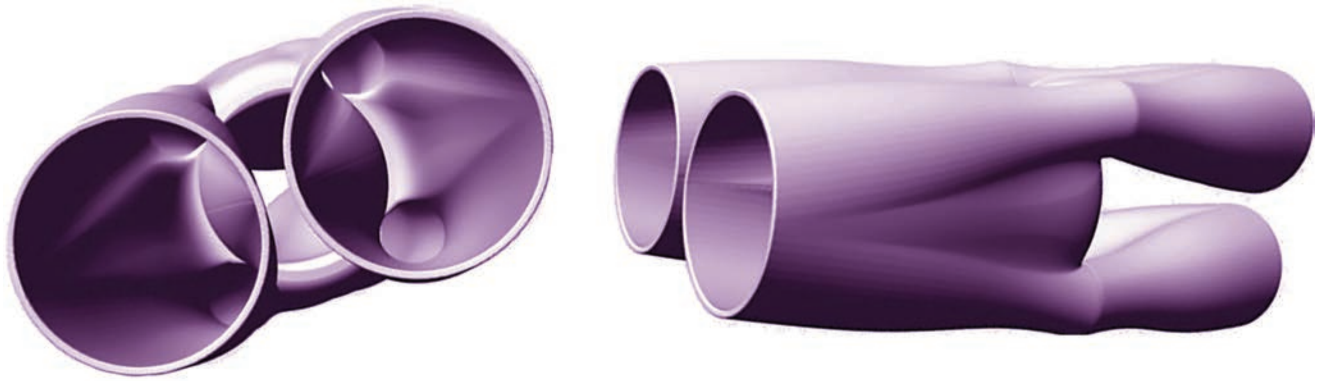
Higgs Bundles—Recent Applications



Laura P. Schaposnik

Introduction

This note is dedicated to introducing Higgs bundles and the Hitchin fibration, with a view towards their appearance within different branches of mathematics and physics, focusing in particular on the role played by the *integrable system* structure carried by their moduli spaces. On a compact Riemann surface Σ of genus $g \geq 2$, Higgs bundles are pairs (E, Φ) where

- E is a holomorphic vector bundle on Σ, and
- the Higgs field $\Phi : E \to E \otimes K$ is a holomorphic map for $K \coloneqq T^*\Sigma$.

Since their origin in the late 1980s in work of Hitchin and Simpson, Higgs bundles manifest as fundamental objects that are ubiquitous in contemporary mathematics and closely related to theoretical physics. For $G_{\mathbb{C}}$ a complex semisimple Lie group, the *Dolbeault moduli space* of $G_{\mathbb{C}}$-Higgs bundles $\mathcal{M}_{G_{\mathbb{C}}}$ has a hyperkähler structure, and via different complex structures it can be seen as different moduli spaces:

- Via the nonabelian Hodge correspondence developed by Corlette, Donaldson, Simpson, and Hitchin and in the spirit of Uhlenbeck–Yau's work for compact groups, the moduli space is analytically isomorphic as a real manifold to the *de Rahm moduli space* $\mathcal{M}_{dR}$ of flat connections on a smooth complex bundle.
- Via the Riemann–Hilbert correspondence there is a complex analytic isomorphism between the de Rham space and the *Betti moduli space* $\mathcal{M}_B$ of surface group representations $\pi_1(\Sigma) \to G_{\mathbb{C}}$.

Some prominent examples where these moduli spaces appear in mathematics and physics are:

- Through the Hitchin fibration, $\mathcal{M}_{G_{\mathbb{C}}}$ gives examples of hyperkähler manifolds that are *integrable systems*, leading to remarkable applications in physics, which we shall discuss later on.
- Building on the work of Hausel and Thaddeus relating Higgs bundles to *Langlands duality*, Donagi and Pantev presented $\mathcal{M}_{G_{\mathbb{C}}}$ as a fundamental example of mirror symmetry.
- Within the work of Kapustin and Witten, Higgs bundles were used to obtain a physical derivation of the *geometric Langlands correspondence* through mirror symmetry. Soon after, Ngô found Higgs bundles to be key ingredients when proving the fundamental lemma of the Langlands program, which led him to the Fields Medal a decade ago.

Higgs bundles and the corresponding Hitchin integrable systems have been an increasingly vibrant area, and

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The author was supported in part by the NSF grant DMS-1509693 and the NSF CAREER Award DMS-1749013. This material is based upon work supported by the NSF under Grant No. DMS1440140 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2019 semester.

Communicated by Notices *Associate Editor Chikako Mese.*

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DOI: https://doi.org/10.1090/noti2074

thus there are several expository articles, some of which we shall refer to: from the *Notices* article "What is... a Higgs bundle?" [3], to several graduate notes on Higgs bundles (e.g., the author's recent [18]), to more advanced reviews such as Ngô's 2010 ICM Proceedings article [4]. Hoping to avoid repeating material nicely covered in other reviews whilst still attempting to inspire the reader to learn more about the subject, we shall take this opportunity to focus on some of the recent work done by leading young members of the community.[1]

Higgs Bundles

Higgs bundles arise as solutions to self-dual Yang–Mills equations, a nonabelian generalization of Maxwell's equations that recurs through much of modern physics. Solutions to Yang–Mills self-duality equations in Euclidean 4D space are called instantons, and when these equations are reduced to Euclidean 3D space by imposing translational invariance in one dimension, one obtains monopoles as solutions. Higgs bundles were introduced by Hitchin in [10] as solutions of the so-called *Hitchin equations*, the 2-dimensional reduction of the Yang–Mills self-duality equations given by

$$F_A + [\Phi, \Phi^*] = 0, \qquad \overline{\partial}_A \Phi = 0, \tag{1}$$

where F_A is the curvature of a unitary connection $\nabla_A = \partial_A + \overline{\partial}_A$ associated to a Dolbeault operator $\overline{\partial}_A$ on a holomorphic principal $G_{\mathbb{C}}$ bundle P. The equations give a flat connection

$$\nabla_A + \Phi + \Phi^* \tag{2}$$

and express the harmonicity condition for a metric in the resulting flat bundle. Concretely, principal $G_{\mathbb{C}}$-Higgs bundles are pairs (P, Φ) where

- P is a principal $G_{\mathbb{C}}$-bundle, and
- Φ is a holomorphic section of $\mathrm{ad}(P) \otimes K$.

We shall refer to *classical Higgs bundles* as those described in the introduction and consider $G_{\mathbb{C}}$-Higgs bundles in their vector bundle representation: seen as classical Higgs bundles satisfying some extra conditions reflecting the nature of $G_{\mathbb{C}}$, dictated by the need for the (projectively) flat connection to have holonomy in $G_{\mathbb{C}}$. For instance, when $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$, a $G_{\mathbb{C}}$-Higgs bundle (E, Φ) is composed of a holomorphic rank n vector bundle E with trivial determinant $\Lambda^n E \cong \mathcal{O}$ and a Higgs field satisfying $\mathrm{Tr}(\Phi) = 0$, for which we shall write $\Phi \in H^0(\Sigma, \mathrm{End}_0(E) \otimes K)$.

Example 1. Choosing a square root of K, consider the vector bundle $E = K^{1/2} \oplus K^{-1/2}$. Then, a family of $\mathrm{SL}(2, \mathbb{C})$-Higgs bundles (E, Φ_a) parametrized by quadratic differentials $a \in H^0(\Sigma, K^2)$ is given by

$$\left(E = K^{1/2} \oplus K^{-1/2}, \Phi_a = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}\right). \tag{3}$$

One may also consider G-Higgs bundles for G a real form of $G_{\mathbb{C}}$, which in turn correspond to the Betti moduli space of representations $\pi_1(\Sigma) \to G$. For example, $\mathrm{SL}(2, \mathbb{R})$-Higgs bundles are pairs $(E = L \oplus L^*, \Phi)$ for L a line bundle and Φ off diagonal, a family of which is described in Example 1.

In order to define a Hausdorff moduli space of Higgs bundles, one needs to incorporate the notion of stability. For this, recall that holomorphic vector bundles E on Σ are topologically classified by their rank $rk(E)$ and their degree $\deg(E)$, through which one may define their *slope* as $\mu(E) \coloneqq \deg(E)/rk(E)$. Then, a vector bundle E is *stable* (*or semistable*) if for any proper subbundle $F \subset E$ one has that $\mu(F) < \mu(E)$ (or $\mu(F) \leq \mu(E)$). It is *polystable* if it is a direct sum of stable bundles whose slope is $\mu(E)$.

One can generalize the stability condition to Higgs bundles (E, Φ) by considering Φ*-invariant subbundles* F of E, vector subbundles $F \subset E$ for which $\Phi(F) \subset F \otimes K$. A Higgs bundle (E, Φ) is said to be *stable* (*semistable*) if for each proper Φ-invariant $F \subset E$ one has $\mu(F) < \mu(E)$ (equiv. $\leq$). Then, by imposing stability conditions, one can construct the moduli space $\mathcal{M}_{G_{\mathbb{C}}}$ of stable $G_{\mathbb{C}}$-Higgs bundles up to holomorphic automorphisms of the pairs (also denoted $\mathcal{M}_{Dol}$). Going back to Hitchin's equations, one of the most important characterizations of stable Higgs bundles is given in the work of Hitchin and Simpson and which carries through to more general settings: If a Higgs bundle (E, Φ) is stable and $\deg E = 0$, then there is a unique unitary connection ∇_A on E, compatible with the holomorphic structure, satisfying (1).

Finally, Hitchin showed that the underlying smooth manifold of solutions to (1) is a hyperkähler manifold, with a natural symplectic form ω defined on the infinitesimal deformations $(\dot{A}, \dot{\Phi})$ of a Higgs bundle (E, Φ) by

$$\omega((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_\Sigma \mathrm{tr}(\dot{A}_1 \dot{\Phi}_2 - \dot{A}_2 \dot{\Phi}_1), \tag{4}$$

where $\dot{A} \in \Omega^{0,1}(\mathrm{End}_0 E)$ and $\dot{\Phi} \in \Omega^{1,0}(\mathrm{End}_0 E)$. Moreover, he presented a natural way of studying the moduli spaces $\mathcal{M}_{G_{\mathbb{C}}}$ of $G_{\mathbb{C}}$-Higgs bundles through what is now called *the Hitchin fibration*, which we shall consider next.

Integrable Systems

Given a homogeneous basis $\{p_1, \dots, p_k\}$ for the ring of invariant polynomials on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$, we denote by d_i the degree of p_i. The *Hitchin fibration*, introduced in

[1] *As in other similar reviews, the number of references is limited to twenty, and thus we shall refer the reader mostly to survey articles where precise references can be found.*

[11], is then given by

$$h : \mathcal{M}_{G_{\mathbb{C}}} \longrightarrow \mathcal{A}_{G_{\mathbb{C}}} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}),$$
$$(E, \Phi) \mapsto (p_1(\Phi), \dots, p_k(\Phi)).$$

The map h is referred to as the *Hitchin map*: it is a proper map for any choice of basis and makes the moduli space into an integrable system whose base and fibres have dimension $\dim(\mathcal{M}_{G_{\mathbb{C}}})/2$. In what follows we shall restrict our attention to $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles, which are those Higgs bundles introduced in the first paragraph of these notes and whose Hitchin fibration in low dimension is depicted in Figure 1.

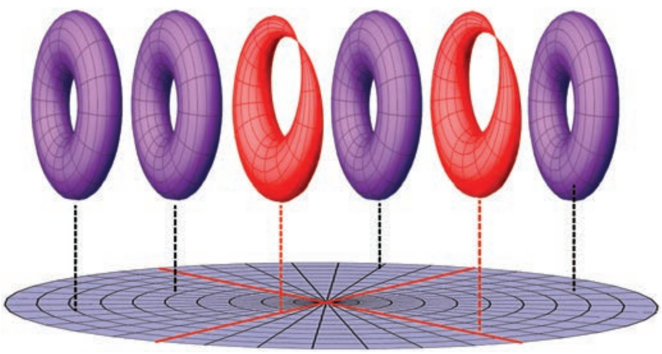

Figure 1. An example of a Hitchin fibration.

The generic or *regular* fibre of the Hitchin fibration—appearing in violet in Figure 1—is an abelian variety, leading to what is referred to as the *abelianization* of the moduli space of Higgs bundles and which can be seen geometrically by considering eigenvalues and eigenspaces of the Higgs field. Indeed, a Higgs bundle (E, Φ) defines a ramified cover $\pi : S \to \Sigma$ of the Riemann surface given by its eigenvalues and obtained through its characteristic equation,

$$S = \{\det(\Phi - \eta) = 0\} \subset \mathrm{Tot}K, \tag{5}$$

for η the tautological section of π^*K. This cover allows one to construct the *spectral data* associated to generic (E, Φ) given by:

- the spectral curve S from (5), generically smooth, defining a generic point in the Hitchin base, since the coefficients of $\{\det(\Phi - \eta) = 0\}$ give a basis of invariant polynomials, and
- a line bundle on S, defining a point in the Hitchin fibre and obtained as the eigenspace of Φ.

For classical Higgs bundles, the smooth fibres are Jacobian varieties $\mathrm{Jac}(S)$, and one recovers (E, Φ) up to isomorphism from the data $(S, L \in \mathrm{Jac}(S))$ by taking the direct images $E = \pi_* L$ and $\Phi = \pi_* \eta$.

When considering $G_{\mathbb{C}}$-Higgs bundles, one has to require appropriate conditions on the spectral curve and the line bundle reflecting the nature of $G_{\mathbb{C}}$. This approach originates in the work of Hitchin and of Beauville, Narasimhan, and Ramanan (see [18] for references), and we shall describe here an example to illustrate the setting. For $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles, the linear term in (5) vanishes since $\mathrm{Tr}(\Phi) = 0$, and the generic fibres are isomorphic to Prym varieties $\mathrm{Prym}(S, \Sigma)$ since $\Lambda^n E \cong \mathcal{O}$.

Example 2. For rank two Higgs bundles, we return to Example 1 in which the Hitchin fibration is over $H^0(\Sigma, K^2)$ and the Hitchin map is $h : (E, \Phi) \mapsto -\det(\Phi)$. The family (E, Φ_a) gives a section of the Hitchin fibration: a smooth map from the Hitchin base to the fibres, known as *the Hitchin section*. Moreover, this comprises a whole component of the moduli space of real $\mathrm{SL}(2, \mathbb{R})$-Higgs bundles inside $\mathcal{M}_{G_{\mathbb{C}}}$, which Hitchin identified with Teichmüller space and which is now referred to as a *Hitchin component* or *Teichmüller component*. Recall that the Teichmüller space $\mathcal{T}(S)$ of the underlying surface S of Σ is the space of marked conformal classes of Riemannian metrics on S.

In the early 1990s Hitchin showed that for any split group G, e.g., for the split form $\mathrm{SL}(n, \mathbb{R})$ of $\mathrm{SL}(n, \mathbb{C})$, the above components are homeomorphic to a vector space of dimension $\dim(G)(2g - 2)$ and conjectured that they should parametrize geometric structures. These spaces presented the first family of *higher Teichmüller spaces* within the Betti moduli space of reductive surface group representations $\mathcal{M}_B(G)$, which leads us to applications of Higgs bundles within *higher Teichmüller theory* for real forms G of $G_{\mathbb{C}}$.

Higher Teichmüller Theory

The moduli spaces of G-Higgs bundles have several connected components. For a split real form G of $G_{\mathbb{C}}$, the Hitchin component of G-Higgs bundles, or equivalently of surface group representations, can be defined as the connected component of the Betti moduli space $\mathcal{M}_B(G)$ containing Fuchsian representations in G, which are representations obtained by composing a discrete and faithful representation $\rho : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbb{R})$ (classically called Fuchsian) with the unique (up to conjugation) irreducible representation $\mathrm{SL}(2, \mathbb{R}) \to G$. Moreover, as mentioned before, these representations, called *Hitchin representations*, are considered the first example of higher Teichmüller space for surfaces: a component of the set of representations of discrete groups into Lie groups of higher ranks consisting entirely of discrete and faithful elements. In order to give a geometric description of Hitchin representations and motivated by dynamical properties, Labourie introduced the notion of *Anosov representations*, which can be thought of as a generalization of convex-cocompact representations to Lie groups G of higher real rank.[2]

[2] *For example, for representations in* $\mathrm{SL}(2, \mathbb{C})$, *these are quasi-Fuchsian representations.*

As beautifully described in Wienhard's ICM Proceedings article [20], building on Labourie's work, higher Teichmüller theory recently emerged as a new field in mathematics, closely related to Higgs bundles (see also [5, 13]). There are two known families of higher Teichmüller spaces, giving the only known examples of components that consist entirely of Anosov representations for surfaces:

- **(I)** the space of Hitchin representations into a real split simple Lie group G and
- **(II)** the space of maximal representations into a Hermitian Lie group G.

A representation $\rho : \pi_1(\Sigma) \to G$ is maximal if it maximizes the *Toledo invariant* $T(\rho)$, a topological invariant defined for any simple Lie group G of Hermitian type as

$$\frac{1}{2\pi}\int_\Sigma f^*\omega \tag{6}$$

for ω the invariant Kähler form on the Riemannian symmetric space, and $f : \tilde{\Sigma} \to X$ any ρ-equivariant smooth map.

Example 3. The Toledo invariant can be expressed in terms of Higgs bundles. For example, for $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles $(L \oplus L^*, \Phi)$, the Toledo invariant is $2\deg(L)$ and satisfies $0 \leq |2\deg(L)| \leq 2g-2$. Hence, the family (E, Φ_a) from Example 1 is maximal.

The existence of spaces other than those in **(I)** and **(II)** with similar properties to Teichmüller space is a topic of significant investigation. Expected candidates are spaces of θ-positive representations conjectured by Guichard–Wienhard, some of which were shown to exist via Higgs bundles [1].

Whilst Anosov representations give a clear link between discrete and faithful representations and geometric structures, there is no known Higgs bundle characterization of Anosov representations, and very little is known about which explicit geometric structures correspond to these spaces. For instance, work of Choi and Goldman shows that the holonomy representations of convex projective structures are the Hitchin representations when $G = \mathrm{PSL}(3,\mathbb{R})$.

Whilst there is no Higgs bundle characterization of Anosov representations,[3] Higgs bundles have been an effective tool for describing these structures. This brings us to one of the fundamental problems in modern geometry: the classification of geometric structures admitted by a manifold M. Recall that a model geometry is a pair (G, X) where X is a manifold (*model space*) and G is a Lie group acting transitively on X (*group of symmetries*). A (G, X)-structure on a manifold M is a maximal atlas of coordinate charts on M with values in X such that the transition maps are given by elements of G. Higgs bundles have played a key role in describing the closed manifold on which (G, X)-structures live. For example, Higgs bundles were used to show that maximal representations to $PO(2,3)$ give rise to (G, X)-manifolds when X is the space of null geodesics (photons) in particular Einstein manifolds and when $X = \mathbb{P}(\mathbb{R}^5)$ (e.g., see [5]). For an excellent review of geometric structures, see Kassel's ICM Proceedings [13].

[3]*Anosov representations are holonomy representations of geometric structures on certain closed manifolds.*

Harmonic Metrics

Equivariant harmonic maps play an important role in the nonabelian Hodge correspondence mentioned before (and beautifully reviewed in [3]), and thus we shall devote this section to some of the advances made in this direction. In our setting, from the work of Corlette and Donaldson, any reductive representation $\rho : \pi_1(\Sigma) \to G_\mathbb{C}$ has associated a ρ-equivariant harmonic map f from the universal cover $\tilde{\Sigma}$ of Σ to the corresponding symmetric space of $G_\mathbb{C}$, which in turn defines a Higgs bundle (E, Φ). Recall that a map $f : \tilde{\Sigma} \to M$ is called ρ-equivariant if $f(\gamma \cdot x) = \rho(\gamma) \cdot f(x)$ for all $x \in \tilde{\Sigma}$ and $\gamma \in \pi_1(\Sigma)$. Moreover, through a choice of metric on Σ, one may define the *energy density*

$$e(f) = \frac{1}{2}\langle df, df\rangle : \tilde{\Sigma} \to \mathbb{R}, \tag{7}$$

which is ρ-invariant and descends to Σ. Then, the *energy* of f is defined as

$$E(f) = \int_\Sigma e(f)d\mathrm{Vol}. \tag{8}$$

It depends only on the conformal class and is finite since Σ is compact. The map f is *harmonic* if it is a critical point of the energy functional $E(f)$ in (8).

Conversely, through the work of Hitchin and Simpson, a polystable Higgs bundle admits a Hermitian metric h on the bundle such that the associated Chern connection A solves the Hitchin equations (1), and such a metric is called harmonic. Moreover, the harmonic metric induces a completely reducible representation $\rho : \pi_1(\Sigma) \to G_\mathbb{C}$ and a ρ-equivariant harmonic map into the corresponding symmetric space. These two directions together give the celebrated nonabelian Hodge correspondence.

Understanding the geometric and analytic properties of the harmonic maps arising from Hitchin's equations (1) is of significant importance. For instance, one may ask how these metrics behave at the boundaries of the moduli space or how the energy densities of the corresponding harmonic maps at different points of the Hitchin fibration relate to each other (the reader may be interested in the reviews [15] and [7] and references therein).

From Hitchin's work, the moduli space of Higgs bundles has a natural $\mathbb{C}^*$-action $\lambda \cdot (E, \Phi) = (E, \lambda\Phi)$, whose fixed point sets allow one to study different aspects of the topology and the geometry of the space, as done in [11] (see also [5, 16]). Moreover, as shown by Simpson, the fixed points of this action are complex variations of Hodge structure (VHS). Recall that a VHS is a $\mathbb{C}^\infty$ vector bundle V with a Hodge decomposition $V = \bigoplus_{p+q} V^{p,q}$, a rational structure, and a flat connection satisfying the axioms of Griffiths transversality and existence of a polarization.

From the above, one may ask how the energy density of harmonic maps changes along the $\mathbb{C}^*$-flow on the moduli space of Higgs bundles. Whilst this remains a challenging open question in the area, a better understanding might come from the following conjectural picture of Dai–Li described in Figure 2 and through which the harmonic map of a fixed point set of the $\mathbb{C}^*$ action on $\mathcal{M}_{G_\mathbb{C}}$ gives rise to two other related harmonic maps.

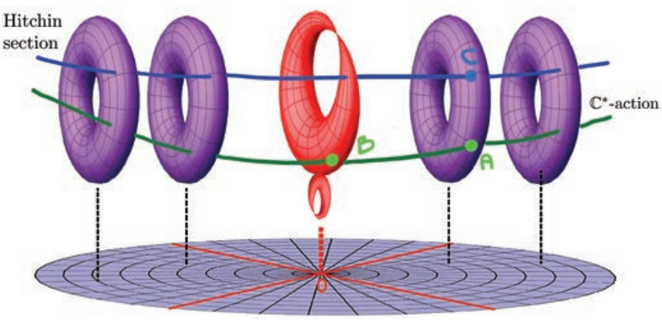

Figure 2. The nilpotent cone in red over the 0 and the points A, B, and C lying over the $\mathbb{C}^*$-flow and over the Hitchin section, respectively.

A point A within the Hitchin fibration naturally determines two other points: the point B, which is the limit of the $\mathbb{C}^*$-flow $\lambda \cdot A$ as $\lambda \to 0$ in the nilpotent cone, and the point C, which is the intersection point of the Hitchin fibre containing A and the Hitchin section. Then Dai–Li's conjecture states that the energy densities defined as in (7) of the corresponding harmonic maps f_A, f_B, f_C satisfy

$$e(f_B) < e(f_A) < e(f_C). \tag{9}$$

As evidence for the above conjecture, one can consider the integral version (through (8)) for which Hitchin showed that $E(f_B) < E(f_A)$, but where the other corresponding inequality in (9) remains open.

Limiting Structures

The study of ρ-equivariant harmonic metrics and higher Teichmüller theory through Higgs bundles has received much attention in recent years and brings us to one of the most important conjectures in the area. This conjecture, due to Labourie, states that for each Hitchin representation ρ there is a unique conformal structure X_ρ on the underlying surface S in which the ρ-equivariant harmonic metric is a minimal immersion. In particular, Labourie showed that for all Anosov representations such a conformal structure exists, but the difficulty lies in proving uniqueness; the conjecture has been established only for Lie groups of rank two ([5, 14]). To understand this problem, one may consider the study of deformations of conformal structures on surfaces and the corresponding harmonic metric.

Some of these deformations can be seen through the hyperkähler structure of the moduli space, by virtue of which it has a $\mathbb{CP}^1$-worth of complex structures labelled by a parameter ξ. Indeed, we can think of a hyperkähler manifold as a manifold whose tangent space admits an action of three complex structures I, J, and K satisfying the quaternionic equations and compatible with a single metric. In our case, I arises from the complex structure on the Riemann surface Σ, while J is from the complex structure on the group $G_\mathbb{C}$. In this setting, one has the following moduli spaces:

- for $\xi = 0$ the space of Higgs bundles,
- for $\xi \in \mathbb{C}^\times$ the space of flat connections[4]
$$\nabla_\xi = \xi^{-1}\Phi + \bar{\partial}_A + \partial_A + \xi\Phi^*, \tag{10}$$
- for $\xi = \infty$ the space of "anti-Higgs bundles."

The hyperkähler metric on Hitchin moduli space is expected to be of type "*quasi-ALG*," which is some expected generalization of ALG. A far-reaching open question is the understanding of the behavior of the metrics at the boundaries of the space, for instance along a path in the Hitchin base via the limit

$$\lim_{t\to\infty} (\bar{\partial}_A, t\Phi).$$

Almost a decade ago Gaiotto–Moore–Neitzke gave a conjectural description of the hyperkähler metric on $\mathcal{M}_{G_\mathbb{C}}$ near infinity, which surprisingly suggests that much of the asymptotic geometry of the moduli space can be derived from the abelian spectral data. Recent progress has been made by Mazzeo–Swoboda–Weiss–Witt, Dumas–Neitzke, and Fredrickson, but the global picture remains open (see [7]).

Finally, one further type of limiting structure we would like to mention is that of opers, appearing as certain limits of Higgs bundles in the Hitchin components. To see this, note that for a solution of (1) in the $\mathrm{SL}(n,\mathbb{C})$-Hitchin section, one can add a real parameter $R > 0$ to (10) to obtain a natural family of connections[5]

$$\nabla(\xi, R) := \xi^{-1}R\Phi + \bar{\partial}_A + \partial_A + \xi R\Phi^*. \tag{11}$$

Gaiotto conjectured that the space of opers (a generalization of projective structures that, like the Hitchin

[4] *In particular, for $\xi = 1$ we recover (2).*

[5] *Note that ∂_A and Φ^* depend on the scaling parameter R. Indeed, one needs to use the metric solving the Hitchin equation for $R\Phi$ to get a flat connection.*

section, is parametrized by the Hitchin base) should be obtained as the $\hbar$-conformal limit of the Hitchin section: taking $R \to 0$ and $\xi \to 0$ simultaneously while holding the ratio $\hbar = \xi/R$ fixed. The conjecture was recently established for general simple Lie groups by Dumistrescu–Fredrickson–Kydonakis–Mazzeo–Mulase–Neitzke, who also conjectured that this oper is the *quantum curve* in the sense of Dumitrescu–Mulase, a quantization of the spectral curve S of the corresponding Higgs bundle by *topological recursion* (see [6]). More recently, Collier–Wentworth showed that the above conformal limit exists in much more generality and gives a correspondence between (Lagrangian) strata for every stable VHS—and not only the Hitchin components. Specifically, they constructed a generalization of the Hitchin section by considering stable manifolds $\mathcal{W}^0(E_0, \Phi_0)$ arising from each VHS (E_0, Φ_0) given by

$$\{(E, \phi) \in \mathcal{M}_{G_\mathbb{C}} \mid \lim_{t\to 0} t \cdot (E, \Phi) = (E_0, \Phi_0)\}. \qquad (12)$$

The analog of the Hitchin section is then obtained by parameterizing $\mathcal{W}^0(E_0, \Phi_0)$ with a slice in the space of Higgs bundles through a global slice theorem, analogous to the definition of the Hitchin section.

Correspondences

The appearance of Higgs bundles as parameter spaces for geometric structures is an example of the study of correspondences between solutions to Hitchin's equations (1) and different mathematical objects. In what follows we shall restrict our attention to a few correspondences between Higgs bundles and two classes of mathematical objects: quiver varieties and hyperpolygons (e.g., see references in [12, 16]).

Recall that a quiver $Q = (V, A, h, t)$ is an oriented graph consisting of a finite vertex set V, a finite arrow set A, and head and tail maps $h, t : A \to V$. A Nakajima representation of a quiver Q can be written as families $W := ((W_v), \phi_a, \psi_a)$ for $a \in A$ and $v \in V$, where W_v is a finite-dimensional vector space, the map $\phi_a : W_{t(a)} \to W_{h(a)}$ is a linear map for all $a \in A$, and ψ_a is in the cotangent space to $\mathrm{Hom}(W_{t(a)}, W_{h(a)})$ at ϕ_a. In particular, a *hyperpolygon* is a representation of the *star-shaped* quiver, an example of which appears in Figure 3.

For the star-shaped quiver in Figure 3, for which the dimensions of W_v are indicated in each vertex, the cotangent space $T^*\mathrm{Rep}(Q)$ of representations of Q is

$$T^*\left(\bigoplus_{i=1}^{n} \mathrm{Hom}(\mathbb{C}, \mathbb{C}^r)\right) = T^*\left(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)\right).$$

Konno showed that hyperpolygon spaces are hyperkähler analogs of polygon spaces, which are representation spaces of the star-shaped quivers with simple arrows. Moreover, through the work of Fisher–Rayan, the space of

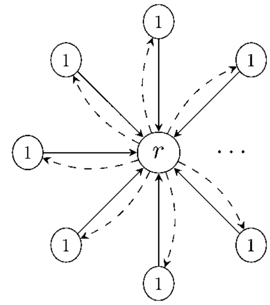

Figure 3. A star-shaped quiver.

hyperpolygons as in Figure 3 may be identified with a moduli space of certain rank r parabolic Higgs bundles on $\mathbb{P}^1$.

In this setting, one has to puncture $\mathbb{P}^1$ along a positive divisor D and then regard the Higgs field as being valued in $\mathcal{O}(q) = K \otimes \mathcal{O}(D)$, with poles along D and satisfying certain conditions on its residues at the poles. This takes us to a generalization of Higgs bundles on higher genus surfaces obtained by allowing the Higgs field to have poles, leading to the moduli spaces of tame or parabolic Higgs bundles (for logarithmic singularities) initiated by Simpson [19] or of wild Higgs bundles (for higher order poles) initiated by Boalch and Biquard; see references in [2] to learn more about these other settings. Understanding the more general appearance of parabolic (and wild) Higgs bundles on higher genus Riemann surfaces in terms of hyperpolygons remains an open question.

In a different direction, given a fixed Riemann surface Σ and a homomorphism between two Lie groups $\Psi : G_\mathbb{C} \to G'_\mathbb{C}$, there is a naturally induced map between the Betti moduli spaces of representations

$$\Psi : \mathcal{M}_B(G_\mathbb{C}) \to \mathcal{M}_B(G'_\mathbb{C}).$$

It follows that there must be a corresponding induced map between the Higgs bundle moduli spaces, but this does not transfer readily to the induced map on the Hitchin fibrations, in particular since the image might be over the singular locus of the base. When the maps arise through isogenies, together with Bradlow and Branco, the author obtained a description of the map for spectral data in terms of fibre products of spectral curves [18], but of much interest is the understanding of other maps arising in this manner.

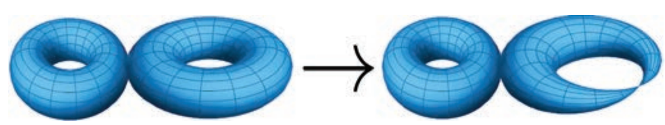
Figure 4. A degeneration of the Riemann surface.

Finally, when considering compactifications of the moduli space, one may ask how the moduli spaces transform when the base Riemann surface Σ changes (for instance, when degenerating the surface Σ as in Figure 4), a question closely related to the relation between Higgs bundles and singular geometry, which we shan't touch upon here; see [2] for a survey and open problems in this direction.

Mirror Symmetry and Branes

One of the most interesting correspondences of Hitchin systems arises through mirror symmetry. For ${}^LG_{\mathbb{C}}$ the Langlands dual group of $G_{\mathbb{C}}$, there is an identification of the Hitchin bases $\mathcal{A}_{G_{\mathbb{C}}} \simeq \mathcal{A}_{{}^LG_{\mathbb{C}}}$. The two moduli spaces $\mathcal{M}_{G_{\mathbb{C}}}$ and $\mathcal{M}_{{}^LG_{\mathbb{C}}}$ are then torus fibrations over a common base, and through the famous SYZ conjecture, mirror symmetry should manifest as a duality between the spaces of Higgs bundles for Langlands dual groups fibred over the same base via the Hitchin fibration. As first observed by Hausel–Thaddeus for $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{PGL}(n,\mathbb{C})$-Higgs bundles and shown by Donagi and Pantev for general pairs of Langlands dual reductive groups, the nonsingular fibres are indeed dual abelian varieties. Kapustin and Witten gave a physical interpretation of this in terms of S-duality, using it as the basis for their approach to the geometric Langlands program.

The appearance of Higgs bundles (and flat connections) within string theory and the geometric Langlands program has led researchers to study the *derived category of coherent sheaves* (B-model) and the *Fukaya category* (A-model) of these moduli spaces. It then becomes fundamental to understand Lagrangian submanifolds of $\mathcal{M}_{G_{\mathbb{C}}}$ (the support of A-branes), and their dual objects (the support of B-branes). By considering the support of branes, we shall refer to a submanifold of a hyperkähler manifold as being of type A or B with respect to each of the complex structures (I,J,K). Hence one may study branes of the four possible types: (B,B,B), (B,A,A), (A,B,A), and (A,A,B), whose dual partner is predicted by Kontsevich's homological mirror symmetry to be

$$(B,A,A) \longleftrightarrow (B,B,B), \tag{13}$$

$$(A,A,B) \longleftrightarrow (A,A,B), \tag{14}$$

$$(A,B,A) \longleftrightarrow (A,B,A). \tag{15}$$

In view of the SYZ conjecture, it is crucial to obtain the duality between branes within the Hitchin fibration. In particular, understanding the correspondence between branes contained completely within the irregular fibres has remained a very fruitful direction of research for decades. In 2006 Gukov, Kapustin, and Witten introduced the first studies of branes of Higgs bundles in relation to the geometric Langlands program and electromagnetic duality where the (B,A,A)-branes of real G-Higgs bundles were considered. These branes, which correspond to surface group representations into the real Lie group G, may intersect the regular fibres of the Hitchin fibration in very different ways (see [17, 18] for references):

- *Abelianization—zero-dimensional intersection.*
  When G is a split real form, the author showed that the (B,A,A)-brane intersects the Hitchin fibration in torsion two points.
- *Abelianization—positive dimensional intersection.*
  Moreover, we can also show that for other real groups such as $\mathrm{SU}(n,n)$, the intersection has positive dimension but may still be described solely in terms of abelian data.
- *Cayley/Langlands type correspondences.*
  Surprisingly, many spaces of Higgs bundles corresponding to nonabelian real gauge theories do admit abelian parametrizations via auxiliary spectral curves, as shown with Baraglia through Cayley/Langlands type correspondences for the groups $G = \mathrm{SO}(p+q,p)$ and $G = \mathrm{Sp}(2p+2q,2p)$.
- *Nonabelianization.*
  Together with Hitchin we initiated the study of branes that don't intersect the regular locus through the *nonabelianization of Higgs bundles*, which characterized the branes for $G = \mathrm{SL}(n,\mathbb{H})$, $\mathrm{SO}(n,\mathbb{H})$, and $\mathrm{Sp}(n,n)$ in terms of nonabelian data given by spaces of rank two vector bundles on the spectral curves.

Moreover, it has been conjectured (Baraglia–Schaposnik) that the Langlands dual in (13) to the above (B,A,A)-branes should correspond to the (B,B,B)-branes of Higgs bundles with structure group the *Nadler group* [17]. More generally, branes of Higgs bundles have proven notoriously difficult to compute in practice, and very few broad classes of examples are known; e.g., see [18] for a partial list of examples. We shall next describe a family of branes defined by the author and Baraglia, obtained by imposing symmetries to the solutions of (1); see [17] and references therein.

Higgs Bundles and 3-Manifolds

By considering actions on the Riemann surface Σ and on the Lie group $G_{\mathbb{C}}$, one can induce actions on the moduli space of Higgs bundles and on the Hitchin fibration and study their fixed point sets. Indeed, for ρ the compact structure of $G_{\mathbb{C}}$ and σ a real form of $G_{\mathbb{C}}$, together with Baraglia we defined the following:

(i) Through the Cartan involution θ of a real form G of $G_{\mathbb{C}}$ one obtains $i_1(\bar{\partial}_A,\Phi) = (\theta(\bar{\partial}_A), -\theta(\Phi))$.

(ii) A real structure $f : \Sigma \to \Sigma$ on Σ induces $i_2(\bar{\partial}_A,\Phi) = (f^*(\rho(\bar{\partial}_A)), -f^*(\rho(\Phi)))$.

(iii) Lastly, by looking at $i_3 = i_1 \circ i_2$, one obtains $i_3(\bar{\partial}_A,\Phi) = (f^*\sigma(\bar{\partial}_A), f^*\sigma(\Phi))$.

The fixed point sets of i_1, i_2, i_3 are branes of type (B,A,A), (A,B,A), and (A,A,B), respectively. The topological invariants can be described using KO, KR, and equivariant K-theory [17]. In particular, the fixed points of

i_1 give the (B, A, A)-brane of G-Higgs bundles mentioned in the previous section, an example of which appears in Figure 5 and which one can study via the monodromy action on the Hitchin fibration (e.g., see [18]).

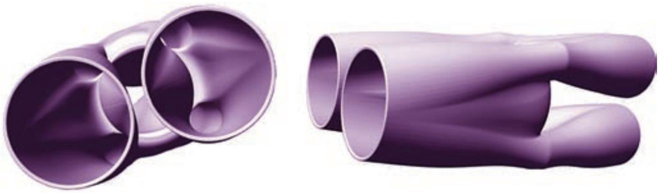

Figure 5. A real slice fixed by i_1 of the moduli space of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles, from two different angles, obtained through Hausel's 3D prints of slices of $\mathcal{M}_{G_{\mathbb{C}}}$.

The fixed points of i_3 are pseudo-real Higgs bundles. To describe the fixed points of the involution i_2, note that a real structure (or anticonformal map) on a compact connected Riemann surface Σ is an antiholomorphic involution $f : \Sigma \to \Sigma$. The classification of real structures on compact Riemann surfaces is a classical result of Klein, who showed that all such involutions on Σ may be characterized by two integer invariants (n, a): the number n of disjoint union of copies of the circle embedded in Σ fixed by f and $a \in \mathbb{Z}_2$ determining whether the complement of the fixed point set has one $(a = 1)$ or two $(a = 0)$ components; e.g., see Figure 6.

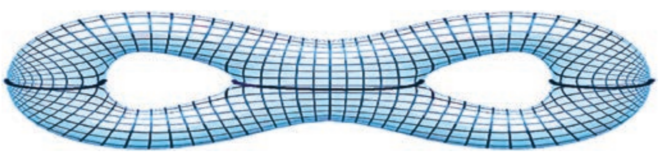

Figure 6. A genus 2 Riemann surface and its fixed point sets under an antiholomorphic involution with invariants $(n, a) = (3, 0)$.

A real structure f on the Riemann surface Σ induces involutions on the moduli space of representations $\pi_1(\Sigma) \to G_{\mathbb{C}}$ of flat connections and of $G_{\mathbb{C}}$-Higgs bundles on Σ, and the fixed point sets define the (A, B, A)-branes of Higgs bundles in (ii). These branes can be shown to be real integrable systems, given by (possibly singular) Lagrangian fibrations.

From a representation theoretic point of view, one may ask which interesting representations these branes correspond to, a question closely related to the understanding of *which representations of $\pi_1(\Sigma)$ extend to $\pi_1(M)$ for M a 3-manifold whose boundary is Σ.* Whilst this question in its full generality remains an important open problem, we can consider some particular cases in which the answer becomes clear from the perspective of Higgs bundles. For this, as seen in [17] and references therein, we considered the 3-manifolds

$$M = \frac{\Sigma \times [-1, 1]}{(x, t) \mapsto (f(x), -t)}, \tag{16}$$

for which $\partial M = \Sigma$ (e.g., a handle body). In this setting, together with Baraglia, we were able to show that a connection solving the Hitchin equations (1) on Σ extends over M given in (16) as a flat connection if and only if the Higgs bundle (E, Φ) is fixed by i_2 and the class $[E] \in \tilde{K}^0_{\mathbb{Z}_2}(\Sigma)$ in reduced equivariant K-theory is trivial. That is, the Higgs bundles that will extend are only those whose vector bundle is preserved by the lift of the involution i_2 and for which the action of i_2 over the fibres of E is trivial when restricted to each fixed circle.

Global Topology

The computation of topological invariants of Higgs bundle moduli spaces has received vast attention from researchers who have tackled this problem with a diverse set of mathematical tools (see references in [9, 16]). One of the central questions considered for Higgs bundles and their generalizations is what the Poincaré polynomial of the space is. A useful fact is that the total space of the Hitchin fibration deformation retracts onto the nilpotent cone $h^{-1}(0)$ via the gradient flow of the moment map of the $\mathbb{C}^*$-action introduced in the harmonic metrics section. The cohomology ring localizes to the fixed-point locus inside $h^{-1}(0)$: as first seen by Morse-theoretic methods in the work of Hitchin, the Poincaré series that generates the Betti numbers of the rational cohomology $H^\bullet(\mathcal{M}_{G_{\mathbb{C}}}, \mathbb{Q})$ is a weighted sum of the Poincaré series of the connected components of the fixed-point locus.

Example 4. As shown by Hitchin, for the family of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles in Example 1, when the genus of Σ is $g = 2$, the Poincaré series is

$$1 + t^2 + 4t^3 + 2t^4 + 34t^5 + 2t^6. \tag{17}$$

Using Morse theory, it has been possible to compute Poincaré polynomials only for low rank groups, and extending this to higher rank has been a challenging open problem for some time. More recently, interesting alternative techniques have been used to access the higher-rank Poincaré polynomials by Mozgovoy, Schiffmann, Mellit, and others. One may further ask about the structure of the ring $H^\bullet(\mathcal{M}_{G_{\mathbb{C}}}, \mathbb{Q})$ itself: for instance, Heinloth recently proved that the intersection pairing in the middle dimension for the smooth moduli space vanishes in all dimensions for $G_{\mathbb{C}} = \mathrm{PGL}(n,\mathbb{C})$, and Cliff–Nevins–Shen proved that the Kirwan map from the cohomology of the moduli stack of G-bundles to the moduli stack of semistable G-Higgs bundles fails to be surjective.

One of the most important cohomological conjectures in the area is de Cataldo–Hausel–Migliorini's *P=W conjecture*, which gives a correspondence between the weight filtration and the perverse filtration on the cohomology of $\mathcal{M}_B$ and $\mathcal{M}_{Dol}$, respectively, obtained via nonabelian Hodge theory. Only certain special cases are known, e.g., for rank two Higgs bundles, shown by de Cataldo–Hausel–Migliorini (see [9]), and for certain moduli spaces of wild Higgs bundles, proven recently by Shen–Zhang and Szabo.

Inspired by the SYZ conjecture mentioned before, Hausel–Thaddeus conjectured that mirror moduli spaces of Higgs bundles present an agreement of appropriately defined Hodge numbers:

$$h^{p,q}(\mathcal{M}_{G_{\mathbb{C}}}) = h^{p,q}(\mathcal{M}_{{}^L G_{\mathbb{C}}}). \tag{18}$$

Very recently, the first proof of this conjecture was established for the moduli spaces of $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{PGL}(n,\mathbb{C})$-Higgs bundles by Groechenig–Wyss–Ziegler in [8], where they established the equality of stringy Hodge numbers using p-adic integration relative to the fibres of the Hitchin fibration and interpreted canonical gerbes present on these moduli spaces as characters on the Hitchin fibres.

Further combinatorial properties of $\mathcal{M}_{G_{\mathbb{C}}}$ can be glimpsed through their twisted version, consisting of Higgs bundles (E,Φ) on Σ with $\Phi : E \to E \otimes \mathcal{L}$, where Σ now has any genus, L is a line bundle with $\deg(L) > \deg(K)$, but without any punctures or residues being fixed. The corresponding moduli spaces carry a natural $\mathbb{C}^*$-action but are not hyperkähler, and there is no immediate relationship to a character variety. Hence, there is no obvious reason for the Betti numbers to be invariant with regard to the choice of $\deg(E)$, which in the classical setting would follow from nonabelian Hodge theory. However, the independence holds in direct calculations of the Betti numbers in low rank and was recently shown for $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles by Groechenig–Wyss–Ziegler in [8]. This suggests that some topological properties of Hitchin systems are independent of the hyperkähler geometry (see references in [9, 16] for more details).

Finally, it should be mentioned that an alternative description of the Hitchin fibration can be given through cameral data, as introduced by Donagi and Gaitsgory, and this perspective presents many advantages, in particular when considering correspondences arising from mirror symmetry and Langlands duality, as those mentioned in previous sections studied by Donagi–Pantev.

ACKNOWLEDGMENTS. The author would like to thank Steve Bradlow, Laura Fredrickson, Tamas Hausel, Nigel Hitchin, Qiongling Li, Sara Maloni, Du Pei, James Unwin, Richard Wentworth, Anna Wienhard, the reviewers of the paper, and very especially Steve Rayan for useful comments on a draft.

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Laura P. Schaposnik

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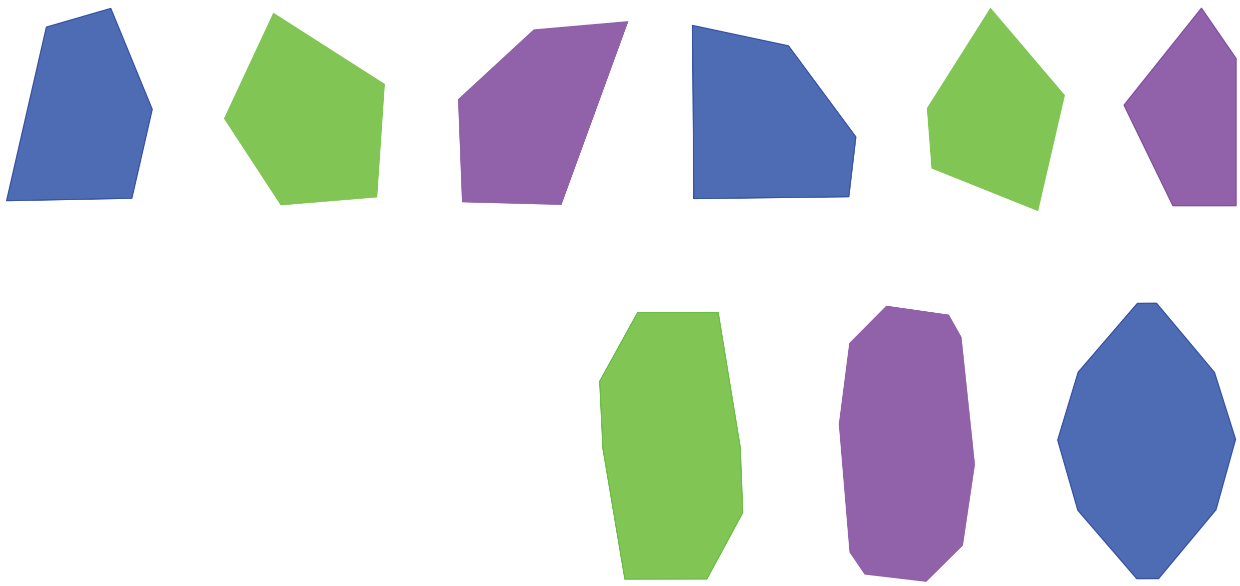
Can You Pave the Plane with Identical Tiles?



Chuanming Zong

Everybody knows that identical regular triangles or squares can tile the whole plane. Many people know that identical regular hexagons can tile the plane properly as well. In fact, even the bees know and use this fact! Is there any other convex domain that can tile the Euclidean plane? Of course, there is a long list of them! To find the list and to show the completeness of the list is a unique drama in mathematics which has lasted for more than one century, and the completeness of the list has been mistakenly announced more than once! Up to now, the list consists of triangles, quadrilaterals, fifteen types of pentagons, and three types of hexagons. In 2017, Michaël Rao announced a computer proof for the completeness of the list. Meanwhile, Qi Yang and Chuanming Zong made a series of unexpected discoveries in multiple tilings in the Euclidean plane. For example, besides parallelograms and centrally symmetric hexagons, there is no other convex domain that can form any two-, three-, or fourfold translative tiling in the plane. However, there are two types of octagons and one type of decagon that can form nontrivial fivefold translative tilings. In fact, parallelograms, centrally symmetric hexagons, and these three types of polygons are the only convex polygons that can form fivefold translative tilings. This paper reviews the dramatic progress.

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Communicated by Notices *Associate Editor Emilie Purvine.*

DOI: https://doi.org/10.1090/noti2075

Introduction

Tiling the plane is an ancient subject in our civilization. It has been considered in the arts by craftsmen since antiquity. According to Gardner [4], the ancient Greeks knew that, among the regular polygons, only the triangle, the square, and the hexagon can tile the plane. Aristotle apparently knew this fact, since he made a similar claim in the space: *Among the five Platonic solids, only the tetrahedron and the cube can tile the space*. Unfortunately, he was wrong: *Identical regular tetrahedra cannot tile the whole space*!

The first recorded scientific investigation into tilings was made by Kepler. Assume that $\mathcal{T}$ is a tiling of the Euclidean plane $\mathbb{E}^2$ by regular convex polygons. If the polygons are identical (congruent), the answer was already known to the ancient Greeks. When different polygons are allowed, the situation becomes more complicated and more interesting. In particular, an edge-to-edge tiling $\mathcal{T}$ by regular polygons is said to be of type $(n_1, n_2, \ldots, n_r)$ if each vertex $\mathbf{v}$ of $\mathcal{T}$ is surrounded by an n_1-gon, an n_2-gon, and so on in a cyclic order, where edge-to-edge means that every pair of neighbors shares an entire common edge. Usually, they are known as *Archimedean tilings*. In 1619, Kepler enumerated all such tilings as $(3,3,3,3,3,3)$, $(3,3,3,3,6)$, $(3,3,3,4,4)$, $(3,3,4,3,4)$, $(3,4,6,4)$, $(3,6,3,6)$, $(3,12,12)$, $(4,4,4,4)$, $(4,6,12)$, $(4,8,8)$, and $(6,6,6)$. Beautiful illustrations of the Archimedean tilings can be found in many references.

If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are n linearly independent vectors in the n-dimensional Euclidean space $\mathbb{E}^n$, then the set

$$\Lambda = \left\{ \sum z_i \mathbf{a}_i \;:\; z_i \in \mathbb{Z} \right\}$$

is an *n-dimensional lattice*. Clearly, lattices are the most natural periodic discrete sets in the plane and space. Therefore, many pioneering scientists like Kepler, Huygens, Haüy, and Seeber took lattice packings and lattice tilings as the scientific foundation for crystals. In 1885, the famous crystallographer Fedorov [3] discovered that *A convex domain can form a lattice tiling of $\mathbb{E}^2$ if and only if it is a parallelogram or a centrally symmetric hexagon; a convex body can form a lattice tiling in $\mathbb{E}^3$ if and only if it is a parallelotope, a hexagonal prism, a rhombic dodecahedron, an elongated octahedron, or a truncated octahedron.*

Usually, tilings allow very general settings without restriction on the shapes of the tiles and the number of the different shapes. However, to avoid complexity and confusion, in this paper we deal only with the tilings by identical convex polygon tiles. In other words, all the tiles are congruent to one convex polygon. In particular, we call it a *translative tiling* if all the tiles are translates of one another and call it a *lattice tiling* if it is a translative tiling and all the translative vectors together form a lattice.

In 1900, Hilbert [8] proposed a list of mathematical problems in his ICM lecture in Paris. As a generalized inverse of Fedorov's discovery, he wrote in the second part of his 18th problem that a *fundamental region of each group of motions, together with the congruent regions arising from the group, evidently fills up space completely. The question arises: whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible.* Here Hilbert did not restrict to convex ones.

Hilbert proposed his problem in the space; perhaps he believed that there is no such domain in the plane. When Reinhardt started his doctoral thesis at Frankfurt am Main in the 1910s, Bieberbach (see [14]) suggested that he *determine all the convex domains which can tile the whole plane* and to verify in this way that Hilbert's problem indeed has a positive answer in the plane. This is the origin of the following natural problem:

Bieberbach's Problem. To determine all the two-dimensional convex tiles.

In 1917 Reinhardt was an assistant of Hilbert's at Göttingen and likely discussed this problem with him. It is worth mentioning that in 1911, Bieberbach himself solved the first part of Hilbert's 18th problem: *Is there in n-dimensional Euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region*?

Reinhardt's List

In 1918, Reinhardt received his doctoral degree under the supervision of Bieberbach at Frankfurt am Main with a thesis "On Partitioning the Plane into Polygons" (Über die Zerlegung der Ebene in Polygone). This is the first approach to characterizing all the convex domains that can tile the whole plane. First, he studied the tiling networks (the vertices, edges, and faces of the tilings) and obtained an expression for the mean of the number of vertices over faces. As a corollary of the formula, he obtained the following result.

Theorem 1 (Reinhardt [14]). *A convex m-gon can tile the whole plane $\mathbb{E}^2$ by identical copies only if*

$$m \le 6.$$

In fact, as Reinhardt and several other authors pointed out (see [4, 10, 12, 14]), this theorem can be easily deduced by *Euler's formula*

$$v - e + f = 1, \tag{1}$$

where v, e, and f stand for the numbers of vertices, edges, and faces of a polygonal division of a finite polygon.

Let $\mathcal{T} = \{T_1, T_2, T_3, \ldots\}$ be a tiling of $\mathbb{E}^2$ such that all tiles T_i are congruent to a convex m-gon P_m, and let H_ℓ be a regular hexagon of edge length ℓ centered at the origin of $\mathbb{E}^2$. Assume that H_ℓ contains $f(\ell)$ tiles $T_1, T_2, \ldots, T_{f(\ell)}$ of $\mathcal{T}$ and the boundary of H_ℓ intersects $g(\ell)$ tiles $T_1', T_2', \ldots, T_{g(\ell)}'$ of $\mathcal{T}$, and let u_i denote the number of vertices of the tiling network on the boundary of T_i. Clearly we have $u_i \ge m$ and

$$\lim_{\ell \to \infty} \frac{g(\ell)}{f(\ell)} = 0. \tag{2}$$

Applying Euler's formula to $\mathcal{T} \cap H_\ell$, when ℓ is sufficiently large we get

$$f \le f(\ell) + g(\ell) \lesssim f(\ell), \tag{3}$$

$$e \ge \frac{1}{2} \sum_{i=1}^{f(\ell)} u_i \ge \frac{m}{2} \cdot f(\ell), \tag{4}$$

$$v \lesssim \frac{1}{3} \sum_{i=1}^{f(\ell)} u_i \lesssim \frac{m}{3} \cdot f(\ell), \tag{5}$$

$$v - e + f \lesssim \left(1 - \frac{m}{6}\right) f(\ell), \tag{6}$$

and therefore

$$m \le 6, \tag{7}$$

where $g(\ell) \lesssim cf(\ell)$ means

$$\lim_{\ell \to \infty} \frac{g(\ell)}{f(\ell)} \leq c.$$

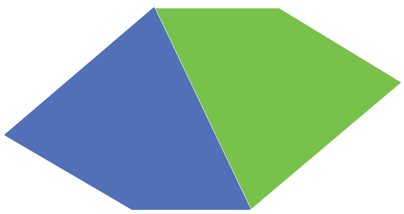

Figure 1. Two quadrilaterals make a centrally symmetric hexagon.

Apparently, two identical triangles can make a parallelogram and two identical quadrilaterals can make a centrally symmetric hexagon (see Figure 1). Thus, by Fedorov's theorem, identical triangles or quadrilaterals can always tile the plane nicely. However, it is easy to see that identical regular pentagons or some particular hexagons cannot tile the plane. Then, Bieberbach's problem can be reformulated as:

What kind of convex pentagons or hexagons can tile the plane?

Let P_n denote a convex n-gon with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in an anti-clock order, let G_i denote the edge with ends $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$, where $\mathbf{v}_0 = \mathbf{v}_n$, let ℓ_i denote the length of G_i, and let α_i denote the inner angle of P_n at $\mathbf{v}_i$.

Reinhardt's thesis obtained the following solution to the hexagon case of Bieberbach's problem.

Theorem 2 (Reinhardt [14]). *A convex hexagon P_6 can tile the whole plane $\mathbb{E}^2$ by identical copies if and only if it satisfies one of the three groups of conditions:*

(1) $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, *and* $\ell_1 = \ell_4$.
(2) $\alpha_1 + \alpha_2 + \alpha_4 = 2\pi$, $\ell_1 = \ell_4$, *and* $\ell_3 = \ell_5$.
(3) $\alpha_1 = \alpha_3 = \alpha_5 = \frac{2}{3}\pi$, $\ell_1 = \ell_2$, $\ell_3 = \ell_4$, *and* $\ell_5 = \ell_6$.

The "if" part of this theorem is relatively simple. It is illustrated by Figure 2. However, the "only if" part is much more complicated. Reinhardt deduced the only if part by considering six cases with respect to how many edges of the considered hexagon are equal. His proof was very sketchy and difficult to understand and check. It seems that he considered only the edge-to-edge tilings.

Fortunately, this theorem has been verified by several other authors. For example, without knowledge of Reinhardt's thesis, in 1963 Bollobás made the following surprising observation, which guarantees the sufficiency of Reinhardt's consideration.

Lemma 1 (Bollobás [2]). *If $\mathcal{T}$ is a tiling of the plane by identical convex hexagons and ℓ is any given positive number, there is a square of edge length ℓ in which the tiling is edge-to-edge and every vertex is surrounded by three hexagons.*

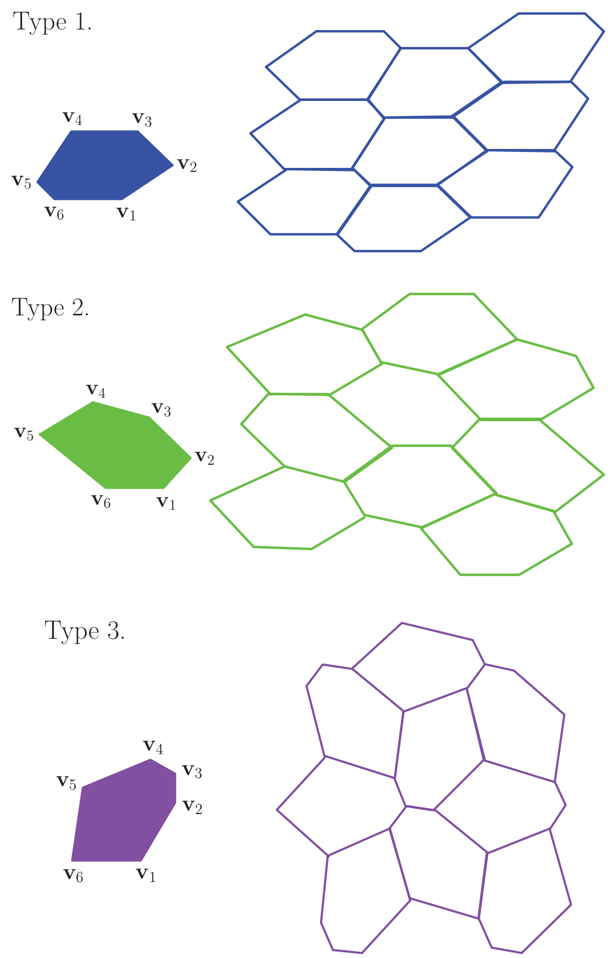

Figure 2. Reinhardt's hexagonal tiles and their local tilings.

Let S be a big square of edge length ℓ centered at the origin of the plane and consider the network of $N = \mathcal{T} \cap S$. Let n_1 denote the number of vertices in N that appear also in the relative interior of some edges, let n_2 denote the number of vertices in N at which three hexagons join properly at their vertices, and let n_3 denote the number of all other vertices of N. Lemma 1 can be proved by studying the quotients

$$\frac{n_2}{n_1 + n_3}$$

for S and its subsquares for sufficiently large ℓ. It is interesting to notice that there are hexagon tilings in which $n_3 \neq 0$.

For the pentagon tilings, by considering five cases with respect to how many edges are equal, Reinhardt obtained the following result.

Theorem 3 (Reinhardt [14]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies one of the five groups of conditions:*

(1) $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$.
(2) $\alpha_1 + \alpha_2 + \alpha_4 = 2\pi$, *and* $\ell_1 = \ell_4$.
(3) $\alpha_1 = \alpha_3 = \alpha_4 = \frac{2}{3}\pi$, $\ell_1 = \ell_2$, *and* $\ell_4 = \ell_3 + \ell_5$.
(4) $\alpha_1 = \alpha_3 = \frac{1}{2}\pi$, $\ell_1 = \ell_2$, *and* $\ell_3 = \ell_4$.
(5) $\alpha_1 = \frac{1}{3}\pi$, $\alpha_3 = \frac{2}{3}\pi$, $\ell_1 = \ell_2$, *and* $\ell_3 = \ell_4$.

Figuring out the list is nontrivial. However, as shown in Figure 3, it is easy to check that all the pentagons listed in Theorem 3 indeed can tile the plane. Reinhardt himself did not claim the completeness of the pentagon tile list. However, according to Gardner [4] *it is quite clear that Reinhardt and everyone else in the field thought that the Reinhardt pentagon list was probably complete.*

As observed by Reinhardt [14], all triangles, quadrilaterals, the three types of hexagons listed in Theorem 2, and the five classes of pentagons listed in Theorem 3 are indeed fundamental domains of some groups of motions. Hilbert and Bieberbach would have been happy to know this.

In 1928, Reinhardt discovered a (nonconvex) three-dimensional polytope that can form a tiling in the space but is not the fundamental domain of any group of motions! This is the first counterexample to the second part of Hilbert's 18th problem.

Inspired by Reinhardt's discovery, in 1935 Heesch [7] obtained a two-dimensional nonconvex counterexample to Hilbert's problem. In other words, there exists a nonconvex polygon that can tile the whole plane; however, it is not the fundamental region of any group of motions.

Thirty years later, Heesch and Kienzle presented a rather detailed treatment of plane tilings in a book entitled *Flächenschluß: System der Formen lückenlos aneinanderschliessender Flachteile*, including nonconvex tiles. No new convex tile was discovered. It was claimed that their treatment was complete.

An End, or a New Start

In 1968, fifty years after Reinhardt's pioneering thesis, Kershner surprisingly discovered three new classes of pentagons that can pave the whole plane without gaps or overlapping.

Theorem 4 (Kershner [10]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies one of the three groups of conditions:*

(6) $\alpha_1+\alpha_2+\alpha_4 = 2\pi$, $\alpha_1 = 2\alpha_3$, $\ell_1 = \ell_2 = \ell_5$, *and* $\ell_3 = \ell_4$.
(7) $2\alpha_2 + \alpha_3 = 2\alpha_4 + \alpha_1 = 2\pi$, *and* $\ell_1 = \ell_2 = \ell_3 = \ell_4$.
(8) $2\alpha_1 + \alpha_2 = 2\alpha_4 + \alpha_3 = 2\pi$, *and* $\ell_1 = \ell_2 = \ell_3 = \ell_4$.

According to Kershner, having been *intrigued by this problem for some 35 years, he finally discovered a method of classifying the possibilities for pentagons in a more convenient way than Reinhardt's to yield an approach that was humanly possible to carry to completion*. Unfortunately, Kershner's paper contains no hint of his method. Of course, the three classes of new pentagon tiles were indeed surprising, though verifications are simple (see Figure 5).

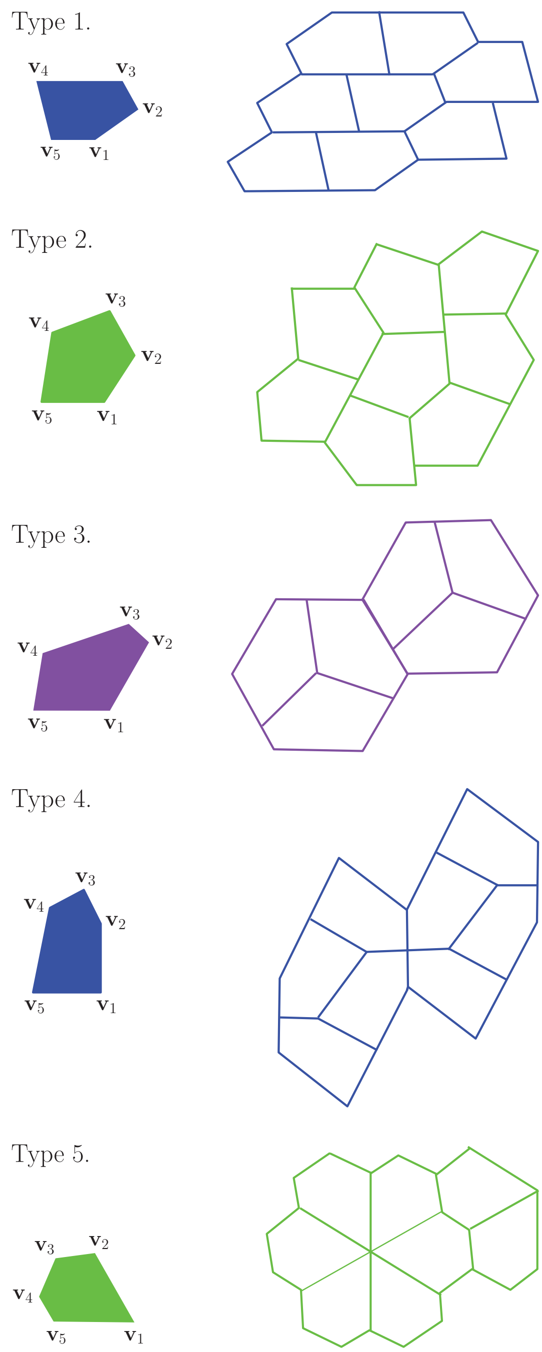

Figure 3. Reinhardt's pentagonal tiles and their local tilings.

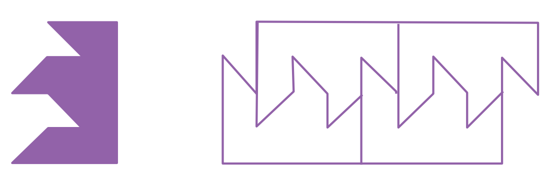
Figure 4. Heesch's counterexample to Hilbert's problem.

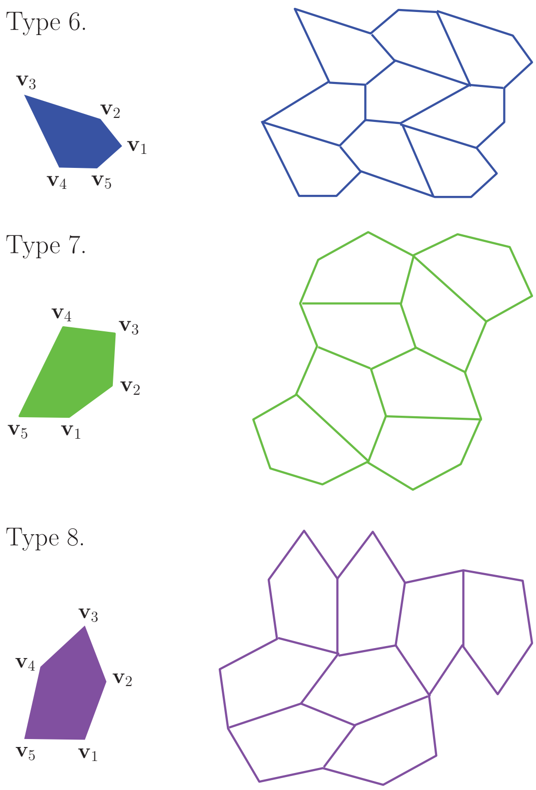

Figure 5. Kershner's pentagonal tiles and their local tilings.

Remark 1. Kershner's discovery was unexpected. Even more surprising was that all the pentagons of Types 6–8 are counterexamples to the second part of Hilbert's 18th problem! In other words, they can tile the whole plane; nevertheless they are not the fundamental regions of any group of motions. Hilbert, Bieberbach, Reinhardt, Heesch, and others would have been surprised by Kershner's elegant examples! Kershner himself did not mention this fact in his papers. Perhaps he overlooked it. This fact has been mentioned in many books and survey papers (see [6]). Inductively, n-dimensional counterexamples to Hilbert's problem can be constructed as cylinders over $(n-1)$-dimensional ones. For example, if D is a domain of Type 6 and H is the cylinder of height one over D, then H is a counterexample to the second part of Hilbert's 18th problem in $\mathbb{E}^3$.

In 1975, Reinhardt and Kershner's discoveries were introduced and popularized by Martin Gardner, a famous scientific writer, in the "Mathematical Games" column of the *Scientific American* magazine. Since then, the tiling problem has stimulated many amateurs who went on to make significant contributions to this problem.

Soon after Gardner's popular paper, based on the known Archimedean tiling $(4,8,8)$ by octagons and squares together, as shown in Figure 6, a computer scientist, Richard James III, discovered a class of new tiles.

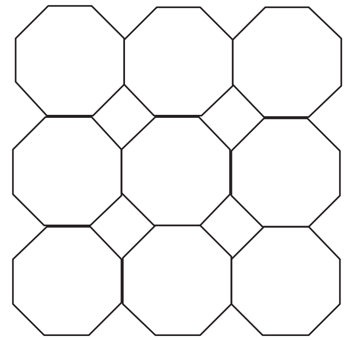

Figure 6. The Archimedean tiling of $(4,8,8)$ type.

Theorem 5 (James [9]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies the following group of conditions:*

(9) $\alpha_5 = \frac{\pi}{2}$, $\alpha_1 + \alpha_4 = \pi$, $2\alpha_2 - \alpha_4 = 2\alpha_3 + \alpha_4 = \pi$, *and* $\ell_1 = \ell_2 + \ell_4 = \ell_5$.

This result can be easily verified by argument based on Figure 7. In principle, Lemma 1 guarantees that every hexagon tiling is edge-to-edge. However, James's discovery shows that this is no longer true in some pentagon tilings. Theorem 5 also served to point out that Kershner had taken edge-to-edge as a hidden assumption in his consideration.

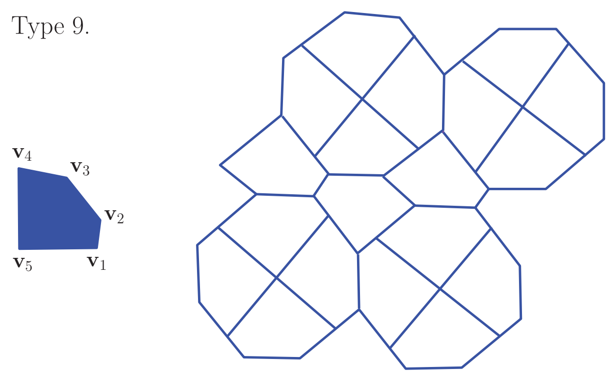

Figure 7. James's pentagonal tile and its local tiling.

Meanwhile, mathematical amateur Marjorie Rice made some truly astonishing discoveries that made the news. She was a true amateur. According to Schattschneider [15], Rice had no mathematical training beyond "the bare minimum they required...in high school over 35 years ago." Even so she was able to consider the problem with a systematic method based on the possible local structures of the pentagon tilings at a given vertex. By dealing with more than sixty cases, she discovered four types of new pentagon tiles!

Theorem 6 (Rice [15]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies one of the four groups of conditions:*

(10) $\alpha_2 + 2\alpha_5 = 2\pi$, $\alpha_3 + 2\alpha_4 = 2\pi$, *and* $\ell_1 = \ell_2 = \ell_3 = \ell_4$.

(11) $\alpha_1 = \frac{\pi}{2}$, $\alpha_3 + \alpha_5 = \pi$, $2\alpha_2 + \alpha_3 = 2\pi$, *and* $2\ell_1 + \ell_3 = \ell_4 = \ell_5$.

(12) $\alpha_1 = \frac{\pi}{2}$, $\alpha_3+\alpha_5 = \pi$, $2\alpha_2+\alpha_3 = 2\pi$, *and* $2\ell_1 = \ell_3+\ell_5 = \ell_4$.

(13) $\alpha_1 = \alpha_3 = \frac{\pi}{2}$, $2\alpha_2 + \alpha_4 = 2\alpha_5 + \alpha_4 = 2\pi$, $\ell_3 = \ell_4$, *and* $2\ell_3 = \ell_5$.

It is routine to verify this theorem based on Figure 8. Nevertheless, it is rather surprising to notice that the tilings produced by the pentagons of Type 10 are edge-to-edge, a fact that was missed by both Reinhardt and Kershner. It is even more surprising that all the pentagons of Types 9–13 are counterexamples to Hilbert's problem as well (see [6]). In other words, they can tile the whole plane; however, they are not the fundamental domains of any group of motions.

Marjorie Rice died on July 2, 2017, at the age of 94. A lobby floor of the Mathematical Association of America in Washington is paved with one of Rice's pentagon tiles in her honor. On July 11, 2017, *Quanta Magazine* published an article in her memory.

Rice's method was systematic, in the sense that it was based on a geometric principle. In any case, the method was not strong enough to guarantee the completeness of the list. In 1985, Rolf Stein reported another one.

Theorem 7 (Stein [16]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies the following conditions:*

(14) $\alpha_1 = \frac{\pi}{2}$, $2\alpha_2 + \alpha_3 = 2\pi$, $\alpha_3 + \alpha_5 = \pi$, *and* $2\ell_1 = 2\ell_3 = \ell_4 = \ell_5$.

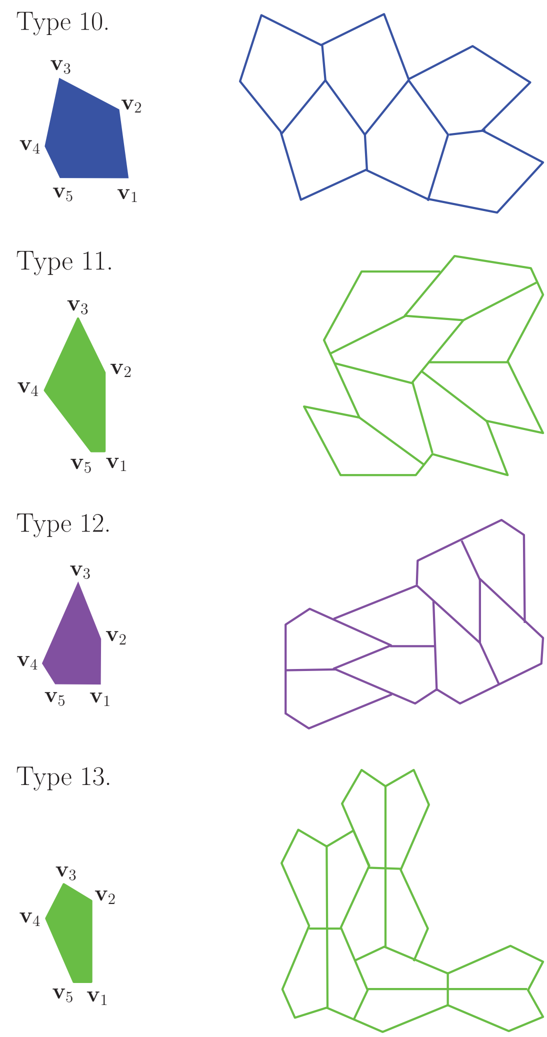

Figure 8. Rice's pentagonal tiles and their local tilings.

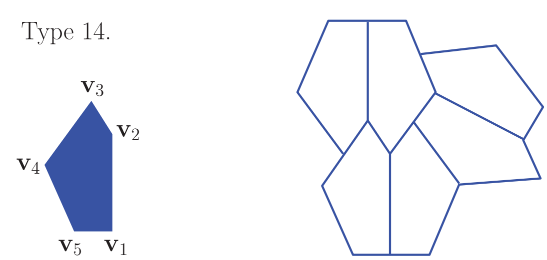

Figure 9. Stein's pentagonal tile and its local tiling.

Fifteen, and Only Fifteen

Let $\mathcal{T}$ denote a tiling of $\mathbb{E}^2$ with congruent tiles. A *symmetry* of $\mathcal{T}$ is an isometry of $\mathbb{E}^2$ that maps the tiles of $\mathcal{T}$ onto tiles of $\mathcal{T}$, and the *symmetry group* $\mathcal{G}$ of $\mathcal{T}$ is the collection of all such symmetries associated with isometry multiplications. Two tiles, T_1 and T_2 of $\mathcal{T}$, are said to be equivalent if there is a symmetry $\sigma \in \mathcal{G}$ such that $\sigma(T_1) = T_2$. If all the tiles of $\mathcal{T}$ are equivalent to one tile T, the tiling $\mathcal{T}$ is said to be *transitive* (or *isohedral*) and T is called a *transitive tile*. Then, the second part of Hilbert's 18th problem can be reformulated as:

Is every polytope that can tile the whole space a transitive tile?

A tiling $\mathcal{T}$ of $\mathbb{E}^2$ by identical convex pentagons is called an *n-block transitive tiling* if it has a block B consisting of n (minimum) connected tiles such that $\mathcal{T}$ is a transitive tiling of B. If a convex pentagon T can form an n-block transitive tiling but not an m-block transitive tiling for any $m < n$, then we call it an *n-block transitive tile*. Clearly, all the tiles of Types 1–5 are one-block transitive. In other words, they are transitive tiles. According to [12, 15], all the tiles of Types 5–14 except Type 9 are two-block transitive, and the tiles of Type 9 are three-block transitive.

From the intuitive point of view, it is reasonable to believe that periodic structure is inevitable in pentagon tilings and the period cannot be too large. Based on this belief, Mann, McLoud-Mann, and Von Derau [12] developed an algorithm for enumerating all the n-block transitive pentagon tiles. When they checked the three-block case, surprisingly, they discovered a new type of pentagon tiles.

Theorem 8 (Mann, McLoud-Mann, and Von Derau [12]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies the following conditions:*

$$(15)\quad \alpha_1 = \frac{\pi}{3},\ \alpha_2 = \frac{3\pi}{3},\ \alpha_3 = \frac{7\pi}{12},\ \alpha_4 = \frac{\pi}{2},\ \alpha_5 = \frac{5\pi}{6},\ \text{and}\ \ell_1 = 2\ell_2 = 2\ell_4 = 2\ell_5.$$

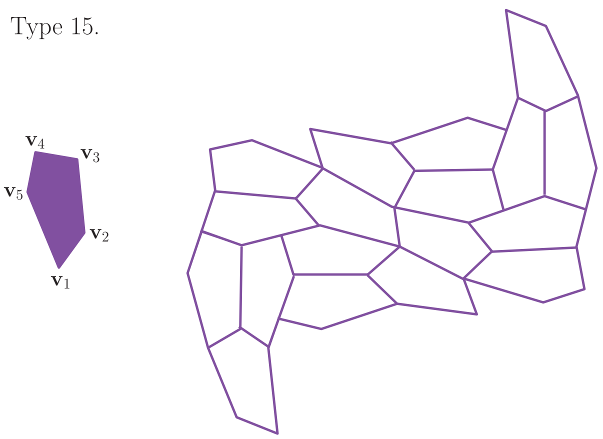

Figure 10. Mann, McLoud-Mann, and Von Derau's pentagonal tile and its local tiling.

Remark 2. It was shown by Mann, McLoud-Mann, and Von Derau [12] that there is no other n-block transitive pentagon tile with $n \le 4$. The completeness of the list emerges again.

Since Hales's computer proof for the Kepler conjecture, more and more geometers have turned to computers for help when their mathematical problems can be reduced into a large number of cases. Characterizing all the pentagon tiles seems to be a perfect candidate for such purpose.

In 2017, one century after Bieberbach proposed the characterization problem, Michaël Rao announced a computer proof for the completeness of the known pentagon tile list. Rao's approach is based on a graph expression. First he proved that if a pentagon tiles the plane, then it can form a tiling such that every vertex type has positive density. Clearly, this is a weak version of the periodic tiling. Second, it was shown that there are only a finite number of possible vertex types in the modified pentagon tiling. In fact, he reduced them to 371 types. Then, by testing the 371 cases, Rao announced the following theorem.

Theorem 9 (Rao [13]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if and only if it belongs to one of the fifteen types listed in Theorems* 3–8.

Computer proofs are still not as acceptable as transparent logical proofs within the mathematical community. However, we have to admit that the complexity of the mathematical problems ranges from zero to infinity, and there indeed exist problems that have no transparent logical proofs.

In 1980, Grünbaum and Shephard [6] made the following comment when they wrote about the tiling problems: *Current fashions in mathematics applaud abstraction for its own sake, regarding it as the highest intellectual activity—whether or not it is, in any sense, useful or related to other endeavors. Mathematicians frequently regard it as demeaning to work on problems related to "elementary geometry" in Euclidean space of two or three dimensions. In fact, we believe that many are unable, both by inclination and training, to make meaningful contributions to this more "concrete" type of mathematics; yet it is precisely these and similar considerations that include the results and techniques needed by workers in other disciplines.* Clearly, the proof history of Bieberbach's problem indeed confirms their comment.

Multiple Tilings

Intuitively speaking, tiling the plane is to pave the whole plane flat with identical tiles. As one can see from previous sections, only a few types of polygons are qualified for the job. However, if multiple layers are permitted, we will have many more choices for the shape of the tile.

Let K denote an n-dimensional convex body with interior int(K) and boundary $\partial(K)$. In particular, let D denote a two-dimensional convex domain.

Assume that $\mathcal{F} = \{K_1, K_2, K_3, \ldots\}$ is a family of convex bodies in $\mathbb{E}^n$ and k is a positive integer. We call $\mathcal{F}$ a *k-fold tiling* of $\mathbb{E}^n$ if every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at least k of these convex bodies and every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at most k of the int(K_i). In other words, a k-fold tiling of $\mathbb{E}^n$ is both a *k-fold packing* and a *k-fold covering* in $\mathbb{E}^n$. In particular, we call a k-fold tiling of $\mathbb{E}^n$ a *k-fold congruent tiling*, a *k-fold translative tiling*, or a *k-fold lattice tiling* if all K_i are congruent to K_1, all K_i are translates of K_1, or all K_i are translates of K_1 and the translative vectors form a lattice in $\mathbb{E}^n$, respectively. In these particular cases, we call K_1 a *k-fold congruent tile*, a *k-fold translative tile*, or a *k-fold lattice tile*, respectively. Clearly, a k-fold translative tiling of the plane $\mathbb{E}^2$ is a nice pavement with identical copies. In other words, it covers every point of the plane with the same multiplicity, excepting the boundary points of the tiles.

For a fixed convex body K, we define $\tau^{\bullet}(K)$ to be the smallest integer k such that K can form a k-fold congruent tiling in $\mathbb{E}^n$, $\tau(K)$ to be the smallest integer k such that K can form a k-fold translative tiling in $\mathbb{E}^n$, and $\tau^*(K)$ to be the smallest integer k such that K can form a k-fold lattice tiling in $\mathbb{E}^n$. For convenience, if K cannot form any multiple congruent tiling, translative tiling, or lattice tiling, we will define $\tau^{\bullet}(K) = \infty$, $\tau(K) = \infty$, or $\tau^*(K) = \infty$, respectively. Clearly, for every convex body K we have

$$\tau^{\bullet}(K) \le \tau(K) \le \tau^*(K). \tag{8}$$

By looking at the separating hyperplanes between tangent neighbors, it is obvious that a convex body can form a multiple tiling only if it is a polytope.

If σ is a nonsingular affine linear transformation from $\mathbb{E}^n$ to $\mathbb{E}^n$, then $\mathcal{F} = \{K_1, K_2, K_3, ...\}$ forms a k-fold tiling of $\mathbb{E}^n$ if and only if $\mathcal{F}' = \{\sigma(K_1), \sigma(K_2), \sigma(K_3), ...\}$ forms a k-fold tiling of $\mathbb{E}^n$. Consequently, for any n-dimensional convex body K and any nonsingular affine linear transformation σ we have both

$$\tau(\sigma(K)) = \tau(K) \tag{9}$$

and

$$\tau^*(\sigma(K)) = \tau^*(K). \tag{10}$$

Unfortunately, $\tau^{\bullet}(K)$ is not an invariant for the linear transformation group.

Clearly, onefold tilings are the usual tilings. In the plane, we have

$$\tau(D) = \tau^*(D) = 1 \tag{11}$$

if and only if D is a parallelogram or a centrally symmetric hexagon, and

$$\tau^{\bullet}(D) = 1 \tag{12}$$

if and only if D is a triangle, a quadrilateral, a pentagon belonging to one of the fifteen types listed in Theorems 3–8, or a hexagon belonging to one of the three types listed in Theorem 2.

Taking a usual tiling and stacking it on top of itself k times forms a k-fold tiling. Similarly, by stacking j copies of a k-fold tiling on top of each other, we get a jk-fold tiling. However, we are interested in the nontrivial multiple tilings.

Since 1936, multiple tilings have been studied by P. Furtwängler, G. Hajós, R. M. Robinson, U. Bolle, N. Gravin, M. N. Kolountzakis, S. Robins, D. Shiryaev, and many others. Nevertheless, many natural problems are still open. In the forthcoming sections we will introduce some fascinating new results about multiple tilings in the plane.

Multiple Lattice Tilings

In 1994, Bolle studied the two-dimensional lattice multiple tilings. Let D be a convex domain, let Λ be a lattice, and assume that $D + \Lambda$ is a k-fold lattice tiling of $\mathbb{E}^2$. It is easy to see that D must be a polygon. Let E be an edge of D, let L be the straight line containing E, let H_1 and H_2 denote the two closed half-planes with L as their boundary, and for a general point $\mathbf{p} \in \mathrm{int}(E)$ define

$$n_i(\mathbf{p}) = \sharp\{\mathbf{g} : \ \mathbf{g} \in \Lambda,\ D + \mathbf{g} \in H_i,\ \mathbf{p} \in \partial(D) + \mathbf{g}\}. \tag{13}$$

By studying $n_1(\mathbf{p})$ and $n_2(\mathbf{p})$ for general $\mathbf{p} \in \mathrm{int}(E)$, Bolle proved the following criterion:

Lemma 2 (Bolle [1]). *A convex polygon D is a k-fold lattice tile for a lattice Λ and some positive integer k if and only if the following conditions are satisfied:*

(1) *It is centrally symmetric.*
(2) *When D is centered at the origin, the relative interior of each edge G of D contains a point of $\frac{1}{2}\Lambda$.*
(3) *If the midpoint of G is not in $\frac{1}{2}\Lambda$, then G is a lattice vector of Λ.*

Based on Bolle's criterion, Gravin, Robins, and Shiryaev [5] discovered the following example.

Example 1. Let Λ denote the two-dimensional integer lattice $\mathbb{Z}^2$ and let P_8 denote the octagon with vertices $\mathbf{v}_1 = (\frac{1}{2}, -\frac{3}{2})$, $\mathbf{v}_2 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{v}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{v}_4 = (\frac{1}{2}, \frac{3}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, as shown in Figure 11. Then $P_8 + \Lambda$ is a sevenfold lattice tiling of $\mathbb{E}^2$.

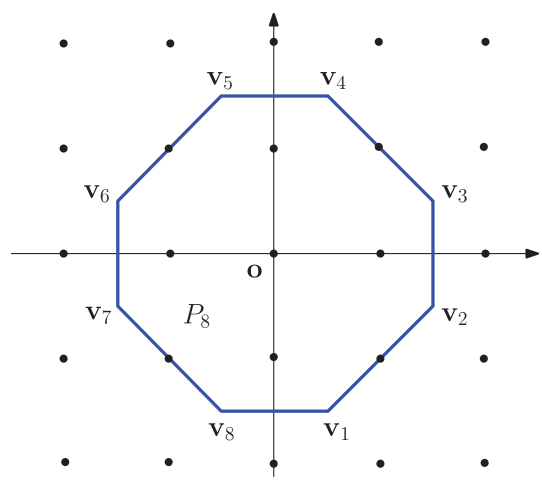

Figure 11. Gravin, Robins, and Shiryaev's octagonal sevenfold lattice tile.

Let $\mathcal{D}$ denote the family of all two-dimensional convex domains and let $\mathcal{P}_{2m}$ denote the family of all centrally symmetric convex $2m$-gons. Since the octagon of Example 1 is the simplest centrally symmetric polygon (except parallelograms and hexagons) satisfying the criterion of Lemma 2, one may conjecture that

$$\min_{D \in \mathcal{D}\setminus\{\mathcal{P}_4 \cup \mathcal{P}_6\}} \tau^*(D) \ge 7.$$

However, based on the known results on multiple lattice packings by V. C. Dumir, R. J. Hans-Gill, and G. Fejes Tóth (see Zong [19]), in 2017 Yang and Zong discovered the following unexpected result.

Theorem 10 (Yang and Zong [17]). *If D is a two-dimensional convex domain that is neither a parallelogram nor a centrally symmetric hexagon, then we have*

$$\tau^*(D) \ge 5,$$

where the equality holds at some particular decagons.

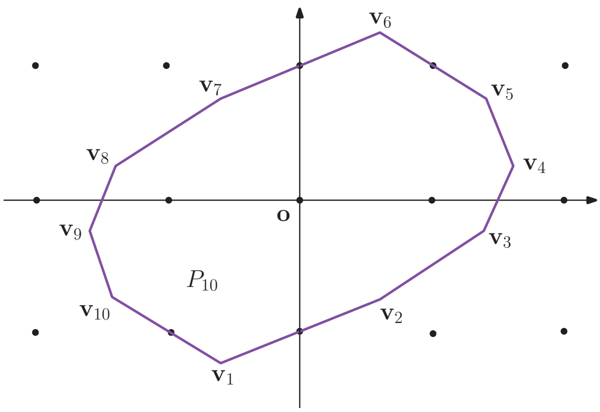

Figure 12. Yang and Zong's decagonal fivefold lattice tile.

Let Λ be the integer lattice $\mathbb{Z}^2$ and let P_{10} denote a decagon whose edge midpoints are $\mathbf{u}_1 = (0,-1)$, $\mathbf{u}_2 = (1,-\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2},0)$, $\mathbf{u}_4 = (\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_5 = (1,1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$, as shown in Figure 12. By Lemma 2, it can be easily verified that $P_{10} + \Lambda$ is indeed a fivefold lattice tiling of $\mathbb{E}^2$.

Even more unexpected, by studying lattice polygons, all the fivefold lattice tiles can be nicely characterized. There are two classes of octagons and one class of decagons besides the parallelograms and the centrally symmetric hexagons.

Theorem 11 (Zong [20]). *A convex domain D can form a fivefold lattice tiling of the Euclidean plane if and only if D is a parallelogram or centrally symmetric hexagon or, up to affine linear transformation, D is a centrally symmetric octagon with vertices $\mathbf{v}_1 = (-\alpha,-\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha,-\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha,-\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha,\frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{4}$, or with vertices $\mathbf{v}_1 = (\beta,-2)$, $\mathbf{v}_2 = (1+\beta,-2)$, $\mathbf{v}_3 = (1-\beta,0)$, $\mathbf{v}_4 = (\beta,1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, where $\frac{1}{4} < \beta < \frac{1}{3}$, or a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (0,-1)$, $\mathbf{u}_2 = (1,-\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2},0)$, $\mathbf{u}_4 = (\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_5 = (1,1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$.*

Let P_{2m} be a centrally symmetric convex $2m$-gon centered at the origin $\mathbf{o}$ of $\mathbb{E}^2$. It is reasonable to believe that $\tau^*(P_{2m})$ is big when m is sufficiently large. In fact, by studying the local structure of a multiple tiling (see next section), Yang and Zong [18] proved that

$$\tau^*(P_{2m}) \geq \begin{cases} m-1 & \text{if } m \text{ is even,} \\ m-2 & \text{if } m \text{ is odd.} \end{cases} \tag{14}$$

Furthermore, by detailed geometric analysis based on Lemma 2, Lemma 4, and Pick's theorem, Zong [20] proved that

$$\tau^*(P_{14}) \geq 6, \tag{15}$$

$$\tau^*(P_{12}) \geq 6, \tag{16}$$

$$\tau^*(P_{10}) \geq 5, \tag{17}$$

where equality in (17) holds if and only if (after a suitable affine linear transformation) P_{10} is a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (0,-1)$, $\mathbf{u}_2 = (1,-\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2},0)$, $\mathbf{u}_4 = (\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_5 = (1,1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ (as shown by Figure 12), and

$$\tau^*(P_8) \geq 5, \tag{18}$$

where the equality holds if and only if (after a suitable affine linear transformation) P_8 is either a centrally symmetric octagon $P_8(\alpha)$ (see Figure 13, top) with vertices $\mathbf{v}_1 = (-\alpha,-\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha,-\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha,-\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha,\frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{4}$, or a centrally symmetric octagon $P_8(\beta)$ (see Figure 13, bottom) with vertices $\mathbf{v}_1 = (\beta,-2)$, $\mathbf{v}_2 = (1+\beta,-2)$, $\mathbf{v}_3 = (1-\beta,0)$, $\mathbf{v}_4 = (\beta,1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, where $\frac{1}{4} < \beta < \frac{1}{3}$. Let Λ be the integer lattice $\mathbb{Z}^2$; it can be verified that both $P_8(\alpha) + \Lambda$ and $P_8(\beta) + \Lambda$ are indeed fivefold lattice tilings of $\mathbb{E}^2$.

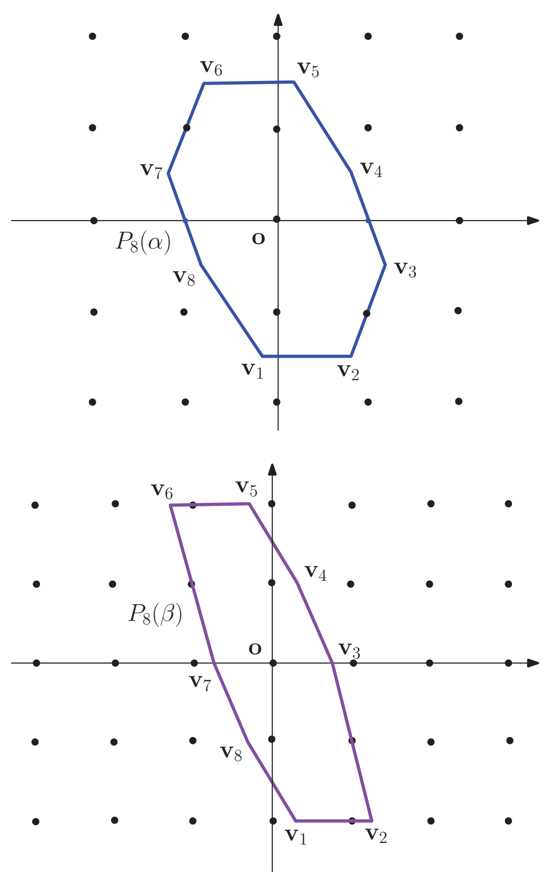

Figure 13. Zong's octagonal fivefold lattice tiles.

Clearly, Theorem 11 follows from (14)–(18). The proofs of these inequalities are complicated, in particular (17) and (18). Their proofs rely on carefully designed area estimations by dealing with many cases. Nevertheless, unlike Rao's proof for Theorem 9, computer checking is not necessary here.

To describe the structure of the decagon in Theorem 11 more explicitly we have the following theorem.

Theorem 12 (Zong [20]). *Let W denote the quadrilateral with vertices $\mathbf{w}_1 = (-\frac{1}{2}, 1)$, $\mathbf{w}_2 = (-\frac{1}{2}, \frac{3}{4})$, $\mathbf{w}_3 = (-\frac{2}{3}, \frac{2}{3})$, and $\mathbf{w}_4 = (-\frac{3}{4}, \frac{3}{4})$. A centrally symmetric convex decagon can take $\mathbf{u}_1 = (0, -1)$, $\mathbf{u}_2 = (1, -\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2}, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (1, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of its edges if and only if one of its vertices is an interior point of W.*

Similarly, the sixfold lattice tiles can be characterized as follows:

Theorem 13 (Zong [20]). *A convex domain D can form a sixfold lattice tiling of the Euclidean plane if and only if D is a parallelogram or centrally symmetric hexagon or, up to affine linear transformation, D is a centrally symmetric octagon with vertices $\mathbf{v}_1 = (\alpha - 1, 2)$, $\mathbf{v}_2 = (\alpha, 2)$, $\mathbf{v}_3 = (1 - \alpha, 0)$, $\mathbf{v}_4 = (1 + \alpha, -1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{6}$, a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (-1, \frac{1}{2})$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, 1)$, $\mathbf{u}_4 = (2, \frac{1}{2})$, $\mathbf{u}_5 = (2, 0)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$, or a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (-\frac{1}{2}, 1)$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (2, 0)$, $\mathbf{u}_5 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$.*

Theorem 14 (Zong [20]). *Let Q denote the quadrilateral with vertices $\mathbf{q}_1 = (0, 1)$, $\mathbf{q}_2 = (0, \frac{5}{6})$, $\mathbf{q}_3 = (-\frac{1}{4}, \frac{3}{4})$, and $\mathbf{q}_4 = (-\frac{1}{3}, \frac{5}{6})$. A centrally symmetric convex decagon P_{10} can take $\mathbf{u}_1 = (-1, \frac{1}{2})$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, 1)$, $\mathbf{u}_4 = (2, \frac{1}{2})$, $\mathbf{u}_5 = (2, 0)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of its edges if and only if one of its vertices is an interior point of Q.*

Let Q^* denote the quadrilateral with vertices $\mathbf{q}_1 = (0, \frac{5}{4})$, $\mathbf{q}_2 = (\frac{1}{6}, \frac{7}{6})$, $\mathbf{q}_3 = (0, 1)$, and $\mathbf{q}_4 = (-\frac{1}{6}, \frac{7}{6})$. A centrally symmetric convex decagon P_{10}^* can take $\mathbf{u}_1 = (\frac{1}{2}, -1)$, $\mathbf{u}_2 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{u}_3 = (2, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (\frac{1}{2}, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of their edges if and only if one of its vertices is an interior point of Q^*.

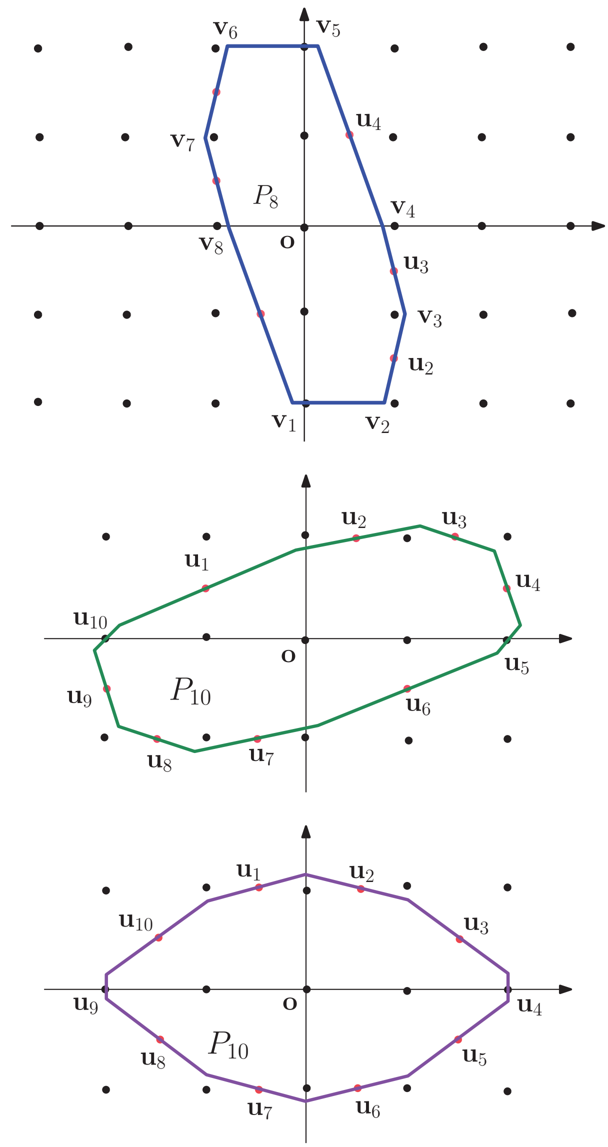

Figure 14. Zong's sixfold lattice tiles, one class of octagons and two classes of decagons.

Multiple Translative Tilings

In 2012, Gravin, Robins, and Shiryaev [5] proved that an n-dimensional convex body can form a multiple translative tiling of the space only if it is a centrally symmetric polytope with centrally symmetric facets. Therefore, to study multiple translative tilings in the plane, we need to deal only with the centrally symmetric polygons.

Let P_{2m} denote a centrally symmetric convex $2m$-gon centered at the origin, with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}$ enumerated in the clock-order, and write $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}\}$. Assume that $P_{2m} + X$ is a $\tau(P_{2m})$-fold translative tiling in $\mathbb{E}^2$, where $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$ is a discrete multiset with $\mathbf{x}_1 = \mathbf{o}$. By studying the local structure of $P_{2m} + X$ at the vertices $\mathbf{v} \in V + X$, Yang and Zong [18] discovered some fascinating results.

Theorem 15 (Yang and Zong [18]). *If D is a two-dimensional convex domain that is neither a parallelogram nor a centrally symmetric hexagon, then we have*

$$\tau(D) \geq 5,$$

where the equality holds if D is some particular centrally symmetric octagon or some particular centrally symmetric decagon.

Remark 3. It is known that

$$\tau(D) \leq \tau^*(D)$$

holds for every convex domain D. Therefore, Theorem 15 implies Theorem 10.

At this point, it is natural to ask for a characterization of all fivefold translative tiles and in particular to determine if these are just the known fivefold lattice tiles.

Theorem 16 (Yang and Zong [18]). *A convex domain D can form a fivefold translative tiling of the Euclidean plane if and only if D is a parallelogram or centrally symmetric hexagon or, up to affine linear transformation, D is a centrally symmetric octagon with vertices* $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$, $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{2}{3}$, *or with vertices* $\mathbf{v}_1 = (2-\beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \beta \leq 1$, *or a centrally symmetric decagon whose edge midpoints are* $\mathbf{u}_1 = (0, -1)$, $\mathbf{u}_2 = (1, -\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2}, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (1, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, *and* $\mathbf{u}_{10} = -\mathbf{u}_5$.

The proofs for Theorems 15 and 16 are extremely complicated. They consist of a series of lemmas showing that

$$\tau(P_{2m}) \geq \begin{cases} m-1 & \text{if } m \text{ is even,} \\ m-2 & \text{if } m \text{ is odd,} \end{cases} \tag{19}$$

$$\tau(P_{14}) \geq 6, \tag{20}$$

$$\tau(P_{12}) \geq 6, \tag{21}$$

$$\tau(P_{10}) \geq 5, \tag{22}$$

where in (22) equality holds if and only if P_{10} is a centrally symmetric decagon that can form a fivefold lattice tiling of $\mathbb{E}^2$, and

$$\tau(P_8) \geq 5, \tag{23}$$

where the equality holds if and only if (after a suitable affine linear transformation) P_8 is either a centrally symmetric octagon $P_8'(\alpha)$ (see Figure 15, top) with vertices $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$, $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{2}{3}$, or a centrally symmetric octagon $P_8'(\beta)$ (see Figure 15, bottom) with vertices $\mathbf{v}_1 = (2-\beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$,

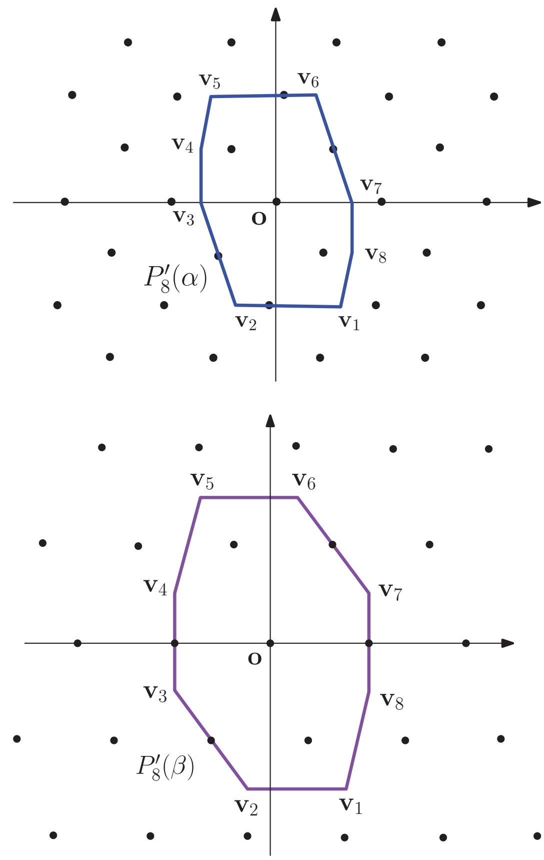

Figure 15. Yang and Zong's octagonal fivefold translative tiles.

$\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \beta \leq 1$.

Clearly, Theorems 15 and 16 follow by (19)–(23) and (17).

Though the statements (19)–(23) are more or less identical to (14)–(18), their proofs are very different. While the lattice case was based on lattice polygon checking, the translative case is based on combinatorial analysis.

In fact, the two classes $P_8(\alpha)$ and $P_8'(\beta)$ shown in Figures 13 and 15, respectively, are equivalent under suitable linear transformations, as well as the two classes $P_8(\beta)$ and $P_8'(\alpha)$. Therefore, we have the following theorem.

Theorem 17 (Yang and Zong [18]). *A convex domain can form a fivefold translative tiling of the Euclidean plane if and only if it can form a fivefold lattice tiling in $\mathbb{E}^2$.*

Open Problems

To end this paper, let us list three open problems about multiple tilings that are closely related to the known results.

Problem 1. Is there a two-dimensional convex domain D satisfying $\tau(D) \neq \tau^*(D)$?

In 2000, Kolountzakis [11] proved that if a convex polygon that is not a parallelogram can form a multiple translative tiling of the plane, then the translative set must be a finite union of translated lattices. To improve this result and to answer a question of Gravin, Robins, and Shiryaev [5], B. Liu and Q. Yang independently proved that if a convex domain D can form a multiple translative tiling of the plane, then it also can form a multiple lattice tiling of the plane. However, we do not know if $\tau(D) = \tau^*(D)$ holds for every convex domain D.

Problem 2. Is there an integer $k \geq 6$ such that $\tau(D) \neq k$ (or $\tau^*(D) \neq k$) holds for all the two-dimensional convex domains D?

As noticed by Yang and Zong [17], based on the two-dimensional examples, for any $n \geq 3$ one can construct n-dimensional centrally symmetric polytopes P satisfying

$$2 \leq \tau^*(P) \leq 5$$

and

$$2 \leq \tau(P) \leq 5.$$

Then, we have the following natural problem.

Problem 3. Assume that $k = 2$, 3, or 4, and $n \geq 3$. Is there an n-dimensional polytope P satisfying $\tau(P) = k$ (or $\tau^*(P) = k$)?

Besides these results and open problems for $\tau^*(P)$ and $\tau(P)$, analogous problems for $\tau^{\bullet}(P)$ are interesting and worth studying as well.

ACKNOWLEDGMENTS. The author is grateful to the referees and the handling editor for their helpful suggestions, which essentially improved the quality of the paper. This work was supported by the National Natural Science Foundation of China (NSFC11921001) and the National Key Research and Development Program of China (2018YFA0704701).

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Finding Solitons

Jorge Lauret

1. Introduction

The concept of soliton provides a useful way to find or discover elements in a given set that are somehow distinguished. Heuristically, only three ingredients are needed to define a soliton:

- A set Γ endowed with some kind of tangent space or space of directions $T_\gamma\Gamma$ at each $\gamma \in \Gamma$.
- An equivalence relation $\simeq$ on Γ collecting in each equivalence class $[\gamma]$ all the elements that cannot be distinguished from γ in relation to the question to be studied.
- An optimal or preferred direction at each point, $q(\gamma) \in T_\gamma\Gamma$, viewed as a "direction of improvement" in some sense.

In that case, $\gamma \in \Gamma$ is called a *soliton* if

$$q(\gamma) \in T_\gamma[\gamma]; \qquad (1)$$

that is, γ is in a way nice enough that it cannot be improved toward $q(\gamma)$ (see Figure 1). The relation $\simeq$ is typically defined by the action of a group H, and if q is H-equivariant, then γ is a soliton if and only if the whole class $[\gamma]$ consists of solitons.

By assuming enough differentiability in the situation, we may consider the evolution differential equation defined by q on Γ,

$$\frac{\partial}{\partial t}\gamma(t) = q(\gamma(t)), \qquad \gamma(0) = \gamma.$$

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DOI: https://doi.org/10.1090/noti2082

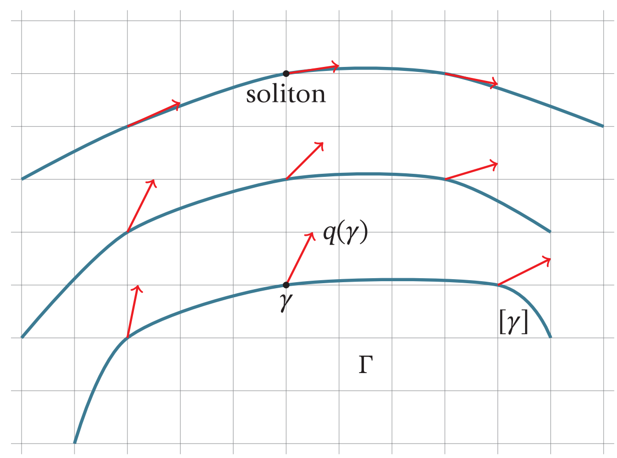

Figure 1. Solitons are not actually improved toward the "direction of improvement."

The existence of solutions is not guaranteed. Solitons are not in general fixed points of this evolution flow (i.e., zeroes of q). However, γ is a soliton if and only if $\gamma(t) \in [\gamma]$ for all t, called a *self-similar* solution (see Figure 1 and the opener image on the left). In other words, solitons are not improved by the flow, and so their existence is not that welcomed if one is hoping to use the flow to find a fixed point in Γ. Indeed, an element may be attracted or stopped in its way to a fixed point by a soliton.

On the other hand, the existence of solitons is great news for the search for canonical or distinguished elements in Γ beyond the zeroes of q.

We note that if $q(\gamma) = -\operatorname{grad}(F)|_\gamma$ for some functional $F : \Gamma \to \mathbb{R}$ that is constant on equivalence classes (or H-invariant), then γ is a soliton if and only if $q(\gamma) = 0$; that is, solitons are precisely the critical points of F or the fixed points of the corresponding flow. Most interesting phenomena occur when this is not the case.

The evocative word *soliton* first appeared in PDE theory in the context of the Korteweg–de Vries equation to name certain solutions resembling solitary water waves that evolve only by translation without losing their shape. More generally, in the study of geometric flows, solitons refer to geometric structures that evolve along symmetries of the flow (i.e., self-similar solutions). The use of the word *soliton* was initiated by Hamilton in the 1980s in the context of Ricci flow to name *Ricci solitons* (see [8]) and nowadays is spread over the fields of differential geometry and geometric analysis.

In this article we discuss and find solitons in many different contexts, including matrices, polynomials, plane curves, Lie group representations (moment maps), and the variety of Lie algebras, as well as in the context of geometric structures (Riemannian, Hermitian, almost-Kähler, and G_2) and their homogeneous versions on Lie groups.[1]

2. Matrices

Consider $\Gamma = \mathfrak{gl}_n$, the vector space of all real $n \times n$ matrices. As is well known, a matrix is semisimple (i.e., diagonalizable over $\mathbb{C}$) if and only if it is conjugate to a normal matrix (i.e., $[A, A^t] = 0$). The subset of normal matrices is invariant under scaling and the action of the orthogonal group $\mathrm{O}(n)$ by conjugation, and there is exactly one $\mathrm{O}(n)$-orbit of normal matrices in each semisimple conjugacy class. With the aim of finding distinguished matrices other than normal matrices, we consider orthogonal conjugation and scaling as the equivalence relation, that is,

$$[A] := H{\cdot}A = \{chAh^{-1} : c \in \mathbb{R}^*,\ h \in \mathrm{O}(n)\}, \quad H := \mathbb{R}^*\mathrm{O}(n).$$

It is easy to see that the tangent space at a matrix A of its conjugacy class $\mathrm{GL}_n \cdot A$ is given by $T_A \mathrm{GL}_n \cdot A = [A, \mathfrak{gl}_n]$, where GL_n is the group of all invertible real $n \times n$ matrices, so the simplest preferred direction that will make normal matrices solitons is

$$q(A) := [A, [A, A^t]].$$

According to (1), a matrix A is a soliton if and only if

$$[A, [A, A^t]] = cA + [A, B] \in T_A[A], \qquad c \in \mathbb{R}, \quad B \in \mathfrak{so}(n),$$

where $\mathfrak{so}(n)$ denotes the space of skew-symmetric matrices. Since $[A, B] \perp A, A^t$ (relative to the usual inner product $\operatorname{tr} XY^t$), we obtain that A is a soliton if and only if either A is normal ($c = 0$ implies $-|[A, A^t]|^2 = \langle [A, [A, A^t]], A \rangle = 0$) or A is nilpotent ($c \neq 0$ implies $\operatorname{tr} A^k = 0$ for any k) and satisfies the following matrix equation:

$$[A, [A, A^t]] = -\frac{|[A, A^t]|^2}{|A|^2} A.$$

[1] *In order to limit the number of references to twenty, most citations have been omitted, and mainly survey articles where the precise references can be found have been included.*

Remark 2.1. Arguing as above, one obtains that the even simpler possibility $q(A) = [A, A^t]$ gives that only normal matrices are solitons.

Besides normal matrices, it is straightforward to check that the following nilpotent matrices are also solitons:

$$\begin{bmatrix} 0 & \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & & \\ \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & & & & \\ 2 & 0 & & & \\ 0 & \sqrt{6} & 0 & & \\ 0 & 0 & \sqrt{6} & 0 & \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \quad \dots \quad, \begin{bmatrix} 0 & & & \\ a_1 & 0 & & \\ & \ddots & \ddots & \\ & & a_{k-1} & 0 \end{bmatrix}, \tag{2}$$

where $a_i := \sqrt{i(k-i)}$ (rather than the expected matrices with $a_i = 1$ for all i). Any nilpotent matrix is therefore conjugate to a soliton by using the Jordan canonical form.

An easy computation gives that actually $q(A) := -\frac{1}{4}\operatorname{grad}(\mathrm{E})|_A$, where $\mathrm{E} : \mathfrak{gl}_n \to \mathbb{R}$ is the functional $\mathrm{E}(B) := |[B, B^t]|^2$ measuring how far a matrix is from being normal. Note that E is not constant on $[A]$ due to scaling, but it is easy to see that A is a soliton if and only if it is a critical point of the normalized functional $\overline{\mathrm{E}}(B) := \mathrm{E}(B)/|B|^4$.

An interesting characterization of normal matrices, perhaps less known, is that they have minimal norm among their conjugacy classes. Also, it is not hard to show that if $A = S + N$, where S is semisimple, N nilpotent, and $[S, N] = 0$, then $S \in \overline{\mathrm{GL}_n \cdot A}$. In particular, if a conjugacy class $\mathrm{GL}_n \cdot A$ is closed, then A is necessarily semisimple (also note that $0 \in \overline{\mathrm{GL}_n \cdot N}$ for any nilpotent matrix N). The following nice properties of soliton matrices follow from well-known results in geometric invariant theory (GIT for short) and the fact that the moment map (to be defined later) for the GL_n-action on $\mathfrak{gl}_n$ by conjugation is precisely $\mathrm{m}(A) = [A, A^t]$:

- For any nilpotent matrix A, there is exactly one $\mathbb{R}^*\mathrm{O}(n)$-orbit of solitons in its conjugacy class $\mathrm{GL}_n \cdot A$.
- Any soliton A is a minimum of $\overline{\mathrm{E}}$ restricted to $\mathrm{GL}_n \cdot A$; that is, a nilpotent soliton is in a sense the matrix closest to being normal in its conjugacy class.
- A conjugacy class $\mathrm{GL}_n \cdot A$ is closed if and only if A is semisimple, if and only if $\mathrm{GL}_n \cdot A$ contains a matrix of minimal norm (or normal matrix).
- The negative gradient flow solution $A(t)$ of the functional $\overline{\mathrm{E}}$ starting at A stays in the conjugacy class $\mathrm{GL}_n \cdot A$, and if $A = S + N$ as above, then $A(t)$ converges as $t \to \infty$ to either a normal matrix in the conjugacy class of S or, in the case $S = 0$, to a soliton in the conjugacy class of N.

3. Polynomials

The space Γ is now given by $P_{n,d}$, the vector space of all homogeneous polynomials of degree d in n variables with coefficients in $\mathbb{R}$ (e.g., quadratic ($d = 2$) and binary ($n = 2$)

forms). There is a natural left GL_n-action on $P_{n,d}$ given by $h \cdot f := f \circ h^{-1}$ and the inner product for which the basis of monomials

$$\left\{x^D := x_1^{d_1} \cdots x_n^{d_n} : d_1 + \cdots + d_n = d\right\}, \quad D := (d_1, \dots, d_n),$$

is orthogonal and $|x^D|^2 := d_1! \cdots d_n!/d!$ is $\mathrm{O}(n)$-invariant. The role of normal matrices in the previous section is played here by the renowned *harmonic* polynomials, i.e., $\Delta f = 0$, where Δ is the *Laplace operator* defined by

$$\Delta : P_{n,d} \longrightarrow P_{n,d-2}, \qquad \Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Δ is $\mathrm{O}(n)$-equivariant, and so the subspace $\mathcal{H}_{n,d} \subset P_{n,d}$ of all harmonic polynomials is invariant under scalings and orthogonal maps. Moreover, $\mathcal{H}_{n,d}$ is known to be $\mathrm{O}(n)$-*irreducible* (i.e., the only $\mathrm{O}(n)$-invariant subspaces are $\{0\}$ and $\mathcal{H}_{n,d}$).

It is therefore natural to consider the equivalence defined by $H = \mathbb{R}^*\mathrm{O}(n) \subset \mathrm{GL}_n$ and as a preferred direction at $f \in P_{n,d}$,

$$q(f) := -r^2 \Delta f \in P_{n,d} = T_f P_{n,d},$$

where

$$r^2 := x_1^2 + \cdots + x_n^2 \in P_{n,2}.$$

Note that harmonic polynomials are the fixed points of the corresponding flow. A polynomial f is a soliton (see (1)) if and only if

$$r^2 \Delta f = cf + \theta(A) f \in T_f(H \cdot f), \qquad c \in \mathbb{R}, \quad A \in \mathfrak{so}(n),$$

where $\theta(A) f := \left.\frac{d}{dt}\right|_0 e^{tA} \cdot f$. Are there solitons other than harmonic polynomials?

We first note that

$$P_{n,d} = \mathcal{H}_{n,d} \oplus r^2 \mathcal{H}_{n,d-2} \oplus r^4 \mathcal{H}_{n,d-4} \oplus \cdots \qquad (3)$$

is a decomposition of $P_{n,d}$ in irreducible $\mathrm{O}(n)$-invariant subspaces[2] (note that r^k is fixed by $\mathrm{O}(n)$ for any $k \in \mathbb{N}$). This is suggesting candidates for solitons. For instance, a straightforward computation gives that if $f = r^{2k} g$ with $g \in \mathcal{H}_{n,d-2k}$, then

$$r^2 \Delta f = \lambda_k f, \qquad \text{where } \lambda_k := 2k(2d - 2k + n - 2), \qquad (4)$$

and thus f is a soliton.[3] Moreover, by writing any polynomial according to (3) and using that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and $\theta(\mathfrak{so}(n)) f \perp f$ for any $f \in P_{n,d}$, it is easy to see that any soliton is actually of this form. Thus the subset of solitons is precisely the union of the $\mathrm{O}(n)$-irreducible subspaces in decomposition (3).

[2] *Since $r^2 P_{n,d-2}$ is the orthogonal complement of $\mathcal{H}_{n,d}$ in $P_{n,d}$, which is straightforward to check, and hence $\Delta P_{n,d} = P_{n,d-2}$.*

[3] *This is strongly related to the fact that $d(d+n-2)$ is precisely the dth eigenvalue of the Laplace–Beltrami operator on the sphere S^{n-1} with eigenspace $\mathcal{H}_{n,d}|_{S^{n-1}}$.*

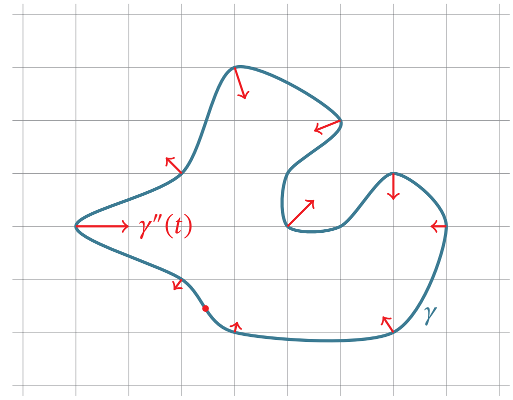

Figure 2. Curvature of a plane curve.

Concerning evolution, given $f = f_j + f_{j+1} + \cdots + f_k$, $f_j, f_k \neq 0$, $j < k$, relative to decomposition (3) (i.e., $f_i \in r^{2i}\mathcal{H}_{n,d-2i}$), the solution to the corresponding flow

$$\frac{d}{dt} f(t) = -r^2 \Delta f(t), \quad f(0) = f,$$

is given by $f(t) = e^{-t\lambda_j} f_j + \cdots + e^{-t\lambda_k} f_k$. This implies that

$$\lim_{t \to \infty} \frac{1}{|f(t)|} f(t) = \frac{1}{|f_j|} f_j,$$

and so each polynomial in the open and dense subset of $P_{n,d}$ defined by $f_0 \neq 0$ flows to some harmonic polynomial. On the contrary, any polynomial f with $f_0 = 0$ will be stopped in its way to $\mathcal{H}_{n,d}$ by a soliton.

4. Plane Curves

Plane curves naturally flow according to their curvature, and this may be considered as the gene of all geometric flows.

Let Γ be the space of all regular plane curves,

$$\Gamma := \{\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2 : \gamma \text{ is differentiable}\}.$$

Two curves are considered equivalent if their traces coincide up to rotations, translations, and scaling. We note that an element of $T_\gamma \Gamma$ consists of a vector field along the curve γ or, in other words, a smooth family of vectors, one at each point of the trace of the curve (see Figure 2). After assuming that γ is parametrized by arc length (i.e., $|\gamma'| \equiv 1$), the most natural preferred or optimal direction is its "curvature,"

$$q(\gamma) := \gamma''.$$

Being a measure of how sensitive your constant velocity car γ' is to passing through the point $\gamma(s)$, $\gamma''(s)$ certainly provides a good perception of how curved the trace of γ is at that point (see Figure 2).

The evolution equation defined by this preferred direction is called the *curve shortening flow* (CSF for short); the following are just a few of its several wonderful properties:

- Each of the following kinds of curves is invariant under the flow: embedded, closed, simple, and convex.

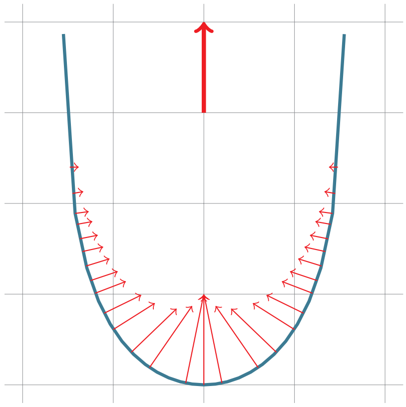

Figure 3. The unique translating soliton is the Grim Reaper found by Calabi, given by the graph of the function $y = -\log(\cos x)$. Its curvature produces the precise gentle breeze needed to move it up without losing its shape.

- For closed curves, the CSF is precisely the negative gradient flow of the length; that is, $q(\gamma) = \gamma''$ is the optimal direction to shorten a closed curve, explaining the name of the flow.
- Grayson proved that under CSF, any simple curve becomes convex (cf. Figure 2 and the opener image on the right), and Gage–Hamilton showed that once it is convex, it converges toward a round point (i.e., asymptotically becoming a circle), collapsing in finite time.

We note that according to the equivalence relation on Γ considered above, a curve is a soliton if and only if it evolves under CSF without losing its shape, i.e., by only a combination of rotations, translations, and scalings (possibly expanding or shrinking). It is therefore easy to convince ourselves that a circle γ is a soliton; indeed, $q(\gamma)$ is in the appropriate sense tangent to the subset of all circles with the same center as γ. It is not so easy, however, to figure out what would be another example of soliton, other than straight lines, which are the trivial solitons with $q = 0$.

The complete classification of CSF-solitons was obtained by Halldorsson in [7]. We give in Figures 3, 4, 5, and 6 examples of all the behaviors that appear. Note that in particular solitons that translate and are scaled at the same time do not exist.

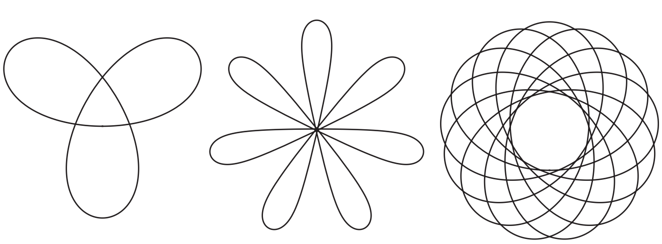

Figure 4. Three shrinking solitons from the infinite discrete family obtained in the classification by Abresch–Langer. Their evolution consists of the very same flower, just reducing its size.

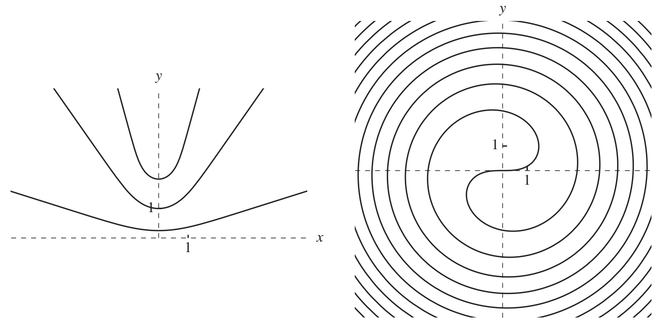

Figure 5. Three expanding solitons (left) and one rotating soliton (right).

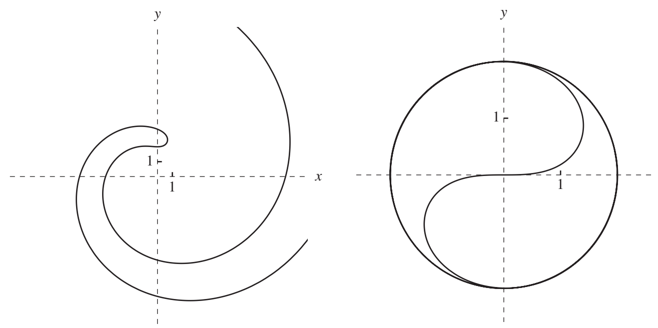

Figure 6. Rotating and expanding soliton (left) and rotating and shrinking soliton (right).

5. Lie Group Representations

As a massive generalization of the matrices example given above, we may consider any linear action of a *Lie group*[4] G on a real vector space V. Thus $\Gamma = V$, and a natural question arises: What would be a distinguished vector $v \in V$ analogous to a normal matrix? If we endow V with an inner product, then natural candidates are *minimal vectors*; i.e., $|v| \leq |h \cdot v|$ for any $h \in G$ (recall the characterization of normal matrices as minimal vectors in their conjugacy classes). Note that any closed G-orbit contains a minimal vector. The next question, more intriguing, is, What should the other solitons be, playing the role of nilpotent soliton matrices (see (2)) in this much more general context?

Motivated by the following equation satisfied in the case of matrices,

$$\langle [A, A^t], B\rangle = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \left|e^{tB}Ae^{-tB}\right|^2,$$

we fix inner products on the Lie algebra $\mathfrak{g}$ of G and on V and consider for each $v \in V$ the element $\mathrm{m}(v) \in \mathfrak{g}$ implicitly defined by

$$\langle \mathrm{m}(v), X\rangle = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \left|\exp tX \cdot v\right|^2 = \langle \theta(X)v, v\rangle \quad \forall X \in \mathfrak{g}, \tag{5}$$

[4] *A group that is also, compatibly, a differentiable manifold.*

where $\theta : \mathfrak{g} \to \mathfrak{gl}(V)$ is the corresponding Lie algebra *representation* (i.e., $\theta(X)v := \frac{d}{dt}\big|_0 \exp tX \cdot v \in T_v(G\cdot v)$). Thus $\mathrm{m}(v)$ encodes the behavior of the norm of vectors inside the orbit $G\cdot v$ in a neighborhood of v. Moreover, by (5), $-\theta(\mathrm{m}(v))v \in T_v(G\cdot v)$ is the direction of fastest norm decreasing tangent to the orbit $G\cdot v$ at v, in the sense that

$$\begin{aligned}\frac{d}{dt}\Big|_0 |\exp(-t\,\mathrm{m}(v))\cdot v|^2 &= -2\langle\theta(\mathrm{m}(v))v, v\rangle = -2|\,\mathrm{m}(v)|^2\\ &\le 2\langle \mathrm{m}(v), X\rangle = \langle 2\theta(X)v, v\rangle\\ &= \frac{d}{dt}\Big|_0 |\exp tX\cdot v|^2,\end{aligned}$$

for any $X\in\mathfrak{g}$ such that $|X| = |\,\mathrm{m}(v)|$, where equality holds if and only if $X = -\mathrm{m}(v)$ (note that any minimal vector v satisfies $\mathrm{m}(v)=0$).

We assume from now on that the following conditions on the G-action on V hold. If

$$\mathfrak{k} := \{X\in\mathfrak{g} : \theta(X)^t = -\theta(X)\},$$
$$\mathfrak{p} := \{X\in\mathfrak{g} : \theta(X)^t = \theta(X)\},$$

then $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ and $G = K\exp\mathfrak{p}$, where $K := \{h\in G : v\mapsto h\cdot v \text{ is orthogonal}\}$. It follows that $\mathfrak{g}$ is *reductive* (i.e., semisimple modulo an abelian factor), $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition (i.e., $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$), K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and the function $K\times\mathfrak{p}\to G$, $(h,X)\mapsto h\exp X$ is a diffeomorphism. By (5), $\mathrm{m}(v)\in\mathfrak{p}$ for any $v\in V$, and the function $\mathrm{m} : V\to\mathfrak{p}$, called in GIT the *moment map*[5] (or G-gradient map) for the action, is K-equivariant.

Similarly to matrices, we consider $\Gamma = V$, $[v] = K\cdot v$, and as the preferred direction,

$$q(v) := -\,\mathrm{grad}(\mathrm{E})|_v, \quad \text{where} \quad \mathrm{E} : V\setminus\{0\}\to\mathbb{R},$$
$$\mathrm{E}(v) := \frac{|\mathrm{m}(v)|^2}{|v|^2},$$

is the K-invariant functional measuring how far v is from being a minimal vector. Therefore, solitons are precisely the critical points of E (i.e., $q(v)=0$), and $\frac{d}{dt}v(t) = q(v(t))$ is the negative gradient flow of the functional E. A straightforward computation gives that

$$\mathrm{grad}(\mathrm{E})|_v = \tfrac{4}{|v|^2}\left(\theta(\mathrm{m}(v))v - |\,\mathrm{m}(v)|^2 v\right);$$

hence v is a soliton if and only if

$$\theta(\mathrm{m}(v))v\in\mathbb{R}v. \qquad (6)$$

This suggests that, as in the case of matrices, there may be solitons other than minimal vectors, probably having nonclosed G-orbits (see examples below). The following are nice and important results from real GIT (see [3, 9]):

- A G-orbit is closed if and only if it contains a minimal vector. The closure of any G-orbit contains exactly one K-orbit of minimal vectors.
- The subset of solitons of a given G-orbit is either empty or consists of exactly one K-orbit (up to scaling).
- Every soliton v is a minimum of the functional E restricted to $G\cdot v$. Solitons are therefore the vectors closest to being a minimal vector in their G-orbit in a sense.
- The negative gradient flow solution of E starting at any $v\in V$ stays in $G\cdot v$ and converges as $t\to\infty$ to a soliton $w\in\overline{G\cdot v}$. Moreover, there is exactly one K-orbit (up to scaling) of solitons $z\in\overline{G\cdot v}$ such that $\mathrm{m}(z)\in K\cdot\mathrm{m}(w)$, which is the limit set towards which the whole orbit $G\cdot v$ is flowing.

In what follows, we analyze the existence of solitons on some particular examples of representations.

Ternary cubics. Consider $G=\mathrm{SL}_3$ acting on $V = P_{3,3}$, the vector space of all homogeneous polynomials of degree 3 on 3 variables with real coefficients. It follows that $\mathfrak{g} = \mathfrak{sl}_3$, $K=\mathrm{SO}(3)$, $\mathfrak{k}=\mathfrak{so}(3)$, and $\mathfrak{p}=\mathrm{sym}_0(3)$, the space of traceless symmetric 3×3 matrices. It is easy to compute that the moment map $\mathrm{m} : P_{3,3}\to\mathrm{sym}_0(3)$ is given by

$$\mathrm{m}(f) = I - \tfrac{1}{|f|^2}\left[\left\langle x_j\frac{\partial f}{\partial x_i}, f\right\rangle\right].$$

Thus $\mathrm{m}(x^D) = \mathrm{Diag}(1-d_1, 1-d_2, 1-d_3)$ for any monomial x^D, $D=(d_1,d_2,d_3)$, and so any monomial is a soliton by (6) with critical value $\mathrm{E}(x^D) = -1+\sum d_i^2$ (this holds on any $P_{n,d}$). It also easily follows that $\mathrm{E}(p)=0$ for $p := x_1x_2x_3$; that is, p is a minimal vector and its SL_3-orbit is therefore closed. On the other hand, the polynomials

$$g = x_1^2x_3 + x_1x_2^2 \quad\text{and}\quad f = x_1^2x_3 + \left(\tfrac{5}{27}\right)^{\frac12} x_2^3$$

are both solitons with critical values $\mathrm{E}(g)=\frac12$ and $\mathrm{E}(f) = \frac{155}{49}-3 < \frac12$, respectively, which in particular implies that $f\notin\overline{\mathrm{SL}_3\cdot g}$.

Algebras. We consider the vector space

$$V = \mathcal{A} := \{\mu : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n : \mu \text{ is bilinear}\},$$

parametrizing the set of all n-dimensional algebras over $\mathbb{R}$. Note that isomorphism classes of algebras are precisely G-orbits, where $G=\mathrm{GL}_n$, relative to the standard GL_n-action on V given by $h\cdot\mu := h\mu(h^{-1}\cdot, h^{-1}\cdot)$. The corresponding representation,

$$\theta(A)\mu = A\mu - \mu(A\cdot,\cdot) - \mu(\cdot, A\cdot) \quad \forall A\in\mathfrak{gl}_n, \quad \mu\in\mathcal{A}, \quad (7)$$

measures how far A is from being a derivation of the algebra μ. Thus $\mathfrak{k}=\mathfrak{so}(n)$, $\mathfrak{p}=\mathrm{sym}(n)$, $K=\mathrm{O}(n)$, and it

[5] *The suggestive name comes from the fact that if we complexify everything, then* m *is precisely the moment map for the Hamiltonian action of K on the complex projective space $P(V_{\mathbb{C}})$.*

is straightforward to see that the moment map $\mathrm{m} : \mathcal{A} \to \mathrm{sym}(n)$ is given by

$$\begin{aligned}\langle \mathrm{m}(\mu)X, X\rangle = &-\tfrac{1}{2}\sum \langle \mu(e_i, e_j), X\rangle^2 \\ &+\tfrac{1}{8}\sum \langle \mu(X, e_i), e_j\rangle^2 \\ &+\tfrac{1}{8}\sum \langle \mu(e_i, X), e_j\rangle^2 \quad \forall X \in \mathbb{R}^n.\end{aligned} \tag{8}$$

Since $\operatorname{tr}\mathrm{m}(\mu) = -|\mu|^2$ by (5) and (7), the only minimal vector is the trivial algebra $\mu = 0$, which is also the only closed GL_n-orbit (note that actually $0 \in \overline{\mathrm{GL}_n \cdot \mu}$ for any $\mu \in \mathcal{A}$). However, there are closed SL_n-orbits and SL_n-minimal vectors.

According to (6), an algebra product $\mu \in \mathcal{A}$ is a soliton if and only if the following nice compatibility condition between μ and the fixed inner product $\langle\cdot,\cdot\rangle$ holds:

$$\mathrm{m}(\mu) = cI + D, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mu). \tag{9}$$

Which algebras are isomorphic to a soliton? How special are they?

Lie algebras. We now list only a few of several known results on solitons in the case of Lie algebras (see [12]). Note that the set of all n-dimensional Lie algebras is parametrized by the GL_n-invariant algebraic subset

$$\mathcal{L} \subset \mathcal{A} \tag{10}$$

of all algebras that are in addition skew-symmetric and satisfy the Jacobi condition,[6] called the *variety of Lie algebras*.

- $\mathrm{SL}_n \cdot \mu$ is closed if and only if μ is semisimple. Moreover, μ is an SL_n-minimal vector if and only if the Killing form B_μ is either a negative multiple of $\langle\cdot,\cdot\rangle$ and μ is compact semisimple, or B_μ has exactly two opposite eigenvalues (relative to $\langle\cdot,\cdot\rangle$) and the eigenspace decomposition is a Cartan decomposition.
- There is a soliton in the isomorphism class of each of the fifty nilpotent Lie algebras of dimension ≤ 6 (see [20]). In dimension 7, there are infinitely many nilpotent Lie algebras that are not isomorphic to a soliton, and a complete classification was obtained in [6].
- The only known general obstruction in the nilpotent case is that any soliton μ has to admit an $\mathbb{N}$-gradation. Everything seems to indicate that a full structural characterization of nilpotent solitons may be hopeless.
- A Lie algebra μ is a soliton if and only if its nilradical $\mathfrak{n}$ is a soliton and the orthogonal complement $\mathfrak{r}$ of $\mathfrak{n}$ is a reductive Lie algebra such that $\mathrm{ad}_\mu X|_\mathfrak{n}^t \in \mathrm{Der}(\mathfrak{n})$ for any $X \in \mathfrak{r}$. In that case, $\mathrm{ad}_\mu X|_\mathfrak{n}$ is a normal operator for any X in the center of $\mathfrak{r}$ and the subspaces $\mathfrak{k} := \{X : \mathrm{ad}_\mu X|_\mathfrak{n}^t = -\mathrm{ad}_\mu X|_\mathfrak{n}\}$ and $\mathfrak{p} := \{X : \mathrm{ad}_\mu X|_\mathfrak{n}^t = \mathrm{ad}_\mu X|_\mathfrak{n}\}$ give rise to a Cartan decomposition $[\mathfrak{r},\mathfrak{r}] = \mathfrak{k} \oplus \mathfrak{p}$ of the semisimple Lie subalgebra $[\mathfrak{r},\mathfrak{r}]$.

[6] *$\mu(\mu(e_i, e_j), e_k) + \mu(\mu(e_j, e_k), e_i) + \mu(\mu(e_k, e_i), e_j) = 0$ for all i, j, k.*

The study of soliton Lie algebras was strongly motivated by their relationship with left-invariant Ricci solitons and Einstein metrics on Lie groups (see [12]). The author is not aware of any study of solitons in other classes of algebras, such as associative or Jordan algebras.

6. Geometric Structures

Let M be a differentiable manifold. We consider the space Γ of all geometric structures on M of a given type, e.g., Riemannian metrics, almost-Hermitian structures, G_2-structures, etc. As usual, Γ is identified with a subset of the vector space $\mathcal{T}^{r,s}M$ of all tensor fields of some type (r,s), or tuples of tensors, and the equivalence relation is scaling and pulling back by diffeomorphisms. Thus the equivalence class of $\gamma \in \Gamma$ is determined by the natural action of the group $H := \mathrm{Diff}(M) \times \mathbb{R}^*$ on tensor fields:

$$[\gamma] := H \cdot \gamma = \{ch^*\gamma : c \in \mathbb{R}^*,\ h \in \mathrm{Diff}(M)\}.$$

The preferred direction

$$\gamma \mapsto q(\gamma) \in T_\gamma\Gamma \subset \mathcal{T}^{r,s}M$$

is typically given by a curvature tensor associated to some affine connection associated with γ, or the gradient of a natural geometric functional, or the Hodge–Laplacian on differential forms, etc. Thus q is in most cases diffeomorphism equivariant, i.e., $q(h^*\gamma) = h^*q(\gamma)$ for any $h \in \mathrm{Diff}(M)$, which implies that if γ is a soliton, then any $h^*\gamma$ is also a soliton.

Γ is many times open in a vector subspace $\mathcal{T} \subset \mathcal{T}^{r,s}M$, in which case one has that $T_\gamma\Gamma = \mathcal{T}$, and so any tensor field in $\mathcal{T}$ can be the "direction of improvement" $q(\gamma)$ at the structure $\gamma \in \Gamma$.

Once the space Γ and the preferred direction q have been specified, it follows from (1) that $\gamma \in \Gamma$ is a soliton if and only if

$$q(\gamma) \in T_\gamma(H \cdot \gamma), \qquad H = \mathrm{Diff}(M) \times \mathbb{R}^*,$$

which is equivalent to

$$q(\gamma) = c\gamma + \mathcal{L}_X\gamma, \qquad c \in \mathbb{R}, \quad X \in \mathfrak{X}(M), \tag{11}$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X of M. Indeed, recall that if X is defined by a one-parameter family $f(t) \in \mathrm{Diff}(M)$ with $f(0) = \mathrm{id}$, then $\frac{d}{dt}\big|_0 f(t)^*\gamma = \mathcal{L}_X\gamma$.

Example 6.1 (Ricci solitons). Consider $\Gamma = \mathcal{M}$, the space of all Riemannian metrics on M. Thus $\mathcal{M}$ is open in $\mathcal{T} = \mathcal{S}^2M \subset \mathcal{T}^{2,0}M$, the vector space of all symmetric 2-tensors on M. A natural preferred direction is $q(g) := -2\,\mathrm{Ric}_g$, where Ric_g is the Ricci tensor of the metric $g \in \mathcal{M}$, giving rise to the well-known Ricci solitons (see [5, Chapter 1]). Note that the corresponding evolution equation is precisely the famous Ricci flow $\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)}$ introduced in the 1980s by Hamilton and used as a primary tool by Perelman to prove the Poincaré and geometrization conjectures.

In the case we want to consider a space Γ of geometric structures satisfying some extra properties, e.g., an integrability-like condition such as Hermitian, almost-Kähler, or closedness/coclosedness for G_2-structures, we have to reduce accordingly the group $H \subset \mathrm{Diff}(M) \times \mathbb{R}^*$ determining the equivalence between structures. The possibilities for preferred directions also decrease, and the vector field X in the definition of soliton (11) must be tangent to H as $q(\gamma) \in T_\gamma(H \cdot \gamma)$. Note that q is assumed to be only H-equivariant in this situation rather than diffeomorphism equivariant. On the other hand, Γ is no longer open in $\mathcal{T}$.

Example 6.2. If a complex manifold (M,J) is fixed and Hermitian metrics or any other kind of geometric structures on (M,J) are to be considered, then $H = \mathrm{Aut}(M,J) \times \mathbb{R}^*$, where $\mathrm{Aut}(M,J)$ is the group of bi-holomorphic diffeomorphisms, q is assumed to be only $\mathrm{Aut}(M,J)$-equivariant, and X has to be a holomorphic field. Analogously, in the symplectic case, $H = \mathrm{Aut}(M,\omega)$, the group of symplectomorphisms of a fixed symplectic manifold (M,ω).

Concerning the associated evolution equation,

$$\frac{\partial}{\partial t}\gamma(t) = q(\gamma(t)), \qquad \gamma(0) = \gamma, \tag{12}$$

one easily obtains that γ is a soliton if and only if

$$\gamma(t) = c(t)f(t)^*\gamma, \qquad c(t) \in \mathbb{R}, \quad f(t) \in \mathrm{Diff}(M); \tag{13}$$

that is, $\gamma(t)$ is a self-similar solution. It is worth pointing out at this point that the natural preferred direction q chosen may or may not produce a flow, as the existence of solutions to the PDE (12) is not always guaranteed. So possibly, a study of solitons can be worked out without any reference to a flow. An example of this situation is Ricci solitons in pseudo-Riemannian geometry (see [4]). On the other hand, even though the flow is not defined on the whole space Γ, there may be special subclasses $\Gamma' \subset \Gamma$ on which solutions to (12) do exist, e.g., homogeneous structures (see below).

Assuming that the scaling behavior of the preferred direction q is given by $q(c\gamma) = c^\alpha \gamma$ for any $c \in \mathbb{R}^*$, $\gamma \in \Gamma$, for some fixed $\alpha < 1$, it is easy to check that the scaling in (13) is given by

$$c(t) = ((1-\alpha)ct + 1)^{\frac{1}{1-\alpha}},$$

where c is the constant appearing in the soliton equation (11). The soliton γ is therefore called *expanding*, *steady*, or *shrinking* depending on whether $c > 0$, $c = 0$, or $c < 0$. The maximal time intervals of the self-similar solutions are respectively given by

$$(-T_\alpha, \infty), \quad (-\infty, \infty), \quad (-\infty, T_\alpha), \qquad \text{where} \quad T_\alpha := \frac{1}{(1-\alpha)|c|} > 0,$$

often called *immortal*, *eternal*, and *ancient* solutions, respectively. For instance, $\alpha = 0$ if q is the Ricci tensor or form of any connection associated to a metric or to an almost-Hermitian structure, and $\alpha = \frac{1}{3}$ for most of the known flows for G_2-structures.

In what follows, we give an overview of different kinds of solitons in complex, symplectic, and G_2 geometries.

Chern–Ricci solitons. For a given complex manifold (M,J), consider the space Γ of all Hermitian metrics on M (or J-invariant, i.e., $g(J\cdot,J\cdot) = g$). Thus $\mathcal{T} \subset \mathcal{T}^{2,0}M$ is the vector space of holomorphic (or J-invariant) symmetric 2-tensors, and the group providing the equivalence relation is the subgroup $H \subset \mathrm{Diff}(M)$ of bi-holomorphic diffeomorphisms (i.e., $h^*J = J$). The Chern connection ∇^C, being the only *Hermitian connection* (i.e., $\nabla^C g = 0$, $\nabla^C J = 0$) with an anti-J-invariant torsion, provides us with the natural preferred direction $q(g) := -2\,\mathrm{Ric}_g^C \in \mathcal{T}$, where Ric_g^C is the corresponding Chern–Ricci tensor of the Hermitian metric g. Note that g is a soliton, called the *Chern–Ricci soliton*, if and only if $\mathrm{Ric}_g^C = cg + \mathcal{L}_X g$, for some $c \in \mathbb{R}$ and holomorphic vector field X on M. The corresponding Chern–Ricci flow was introduced by Gill and has been studied by Tosatti–Weinkove among others. Examples of homogeneous Chern–Ricci solitons were given in [16].

Remark 6.3. Another possible preferred direction for Hermitian metrics is to just take the J-invariant part of the Ricci tensor, given by

$$q(g) := -\mathrm{Ric}_g^{1,1} = -\tfrac{1}{2}\left(\mathrm{Ric}_g + \mathrm{Ric}_g(J\cdot,J\cdot)\right).$$

This choice does not give rise to any geometric flow on the set of all Hermitian metrics. However, as recently shown by Lafuente–Pujia–Vezzoni, it coincides with the *Hermitian curvature flow* (HCF for short) introduced by Streets–Tian among left-invariant Hermitian metrics on complex unimodular Lie groups.

Pluriclosed solitons. Consider now on (M,J) the space Γ of all Hermitian metrics on M that satisfy the *pluriclosed* condition $\partial\bar{\partial}\omega = 0$, where $\omega = g(J\cdot,\cdot) \in \Omega^2 M$ (also called SKT metrics). A natural preferred direction here is given by $q(g) := -\left(\mathrm{Ric}_g^B\right)^{1,1}$, where Ric_g^B denotes the Bismut–Ricci tensor of g associated to the Bismut connection.[7] Pluriclosed solitons (i.e., $\mathrm{Ric}_g^B = cg + \mathcal{L}_X g$) and the corresponding pluriclosed flow, which coincides with HCF

[7] *The only Hermitian connection whose torsion satisfies that* $(X,Y,Z) \mapsto g(T^B(X,Y),Z)$ *is a 3-form.*

on SKT metrics, have been studied by several authors (see [1]).

Anticomplexified Ricci solitons. Let (M,ω) be a symplectic manifold and let Γ denote the space of all compatible metrics (i.e., $J^2 = -I$ if $\omega = g(J\cdot,\cdot)$). For each $g \in \Gamma$, the pair (ω, g) is called an *almost-Kähler structure* (in other words, an almost-Hermitian structure (ω, g, J) such that $d\omega = 0$). It follows that $\mathcal{T}$ is the vector space of anti-J-invariant symmetric 2-tensors (i.e., $q(J\cdot,J\cdot) = -q$) and $H \subset \mathrm{Diff}(M)$ is the subgroup of symplectomorphisms (i.e., $h^*\omega = \omega$). As a preferred direction, the simplest option is

$$q(g) := \mathrm{Ric}_g^{2,0+0,2} = \tfrac{1}{2}\left(\mathrm{Ric}_g - \mathrm{Ric}_g(J\cdot,J\cdot)\right),$$

the anti-J-invariant part of the Ricci tensor Ric_g. This flow was introduced by Le–Wang, and examples of homogeneous solitons were given by Fernández–Culma (see [13]).

Symplectic curvature flow solitons. We now consider the larger and trickier space Γ of all almost-Hermitian structures (i.e., a 3-tuple (ω, g, J) such that $g = \omega(J\cdot,\cdot)$) on a manifold M. It is easy to see that in this case

$$\mathcal{T} = \left\{(\overline{\omega}, \overline{g}) \in \Omega^2 M \times \mathcal{S}^2 M \,:\, \overline{g}^{1,1} = \overline{\omega}^{1,1}(\cdot, J\cdot)\right\}, \qquad (14)$$

and we take the full $H = \mathrm{Diff}(M)\times\mathbb{R}_{>0}$, so the equivalence class of an almost-Hermitian structure $\gamma = (\omega, g)$ is given by $[(\omega,g)] = \{(ch^*\omega, ch^*g) : c \in \mathbb{R}^*,\ h \in \mathrm{Diff}(M)\}$.

As above, we consider the Chern–Ricci tensor $\mathrm{Ric}^C_{(\omega,g)}$ and the corresponding Chern–Ricci form $p(\omega, g) := \mathrm{Ric}^C_{(\omega,g)}(J\cdot,\cdot)$, which is a very natural preferred direction at ω to choose (indeed, $dp = 0$ if $d\omega = 0$, so p is tangent to the set of almost-Kähler structures). According to (14), the J-invariant part of the preferred direction at g must be given by $p(\omega, g)^{1,1}(\cdot, J\cdot)$; hence it only remains to set the anti-J-invariant part, for which we can just choose the simplest one considered above. In this way, we arrive at the following natural preferred direction:

$$q(\omega, g) := \left(p(\omega, g),\ p(\omega, g)^{1,1}(\cdot, J\cdot) + \mathrm{Ric}_g^{2,0+0,2}\right) \in \mathcal{T}.$$

Thus (ω, g) is a soliton if and only if there exist $c \in \mathbb{R}$ and $X \in \mathfrak{X}(M)$ such that

$$\begin{cases} p(\omega, g) = c\omega + \mathcal{L}_X\omega, \\ p^{1,1}(\cdot, J\cdot) + \mathrm{Ric}^{2,0+0,2} = cg + \mathcal{L}_X g. \end{cases}$$

The flow is called *symplectic curvature flow* (SCF for short) and was introduced by Streets–Tian; see [17] for a study of homogeneous SCF-solitons. The fact that the three structures ω, g, and J are actually evolving is particularly challenging.

Remark 6.4. The classification of complex surfaces is a major problem motivating the study of all the above flows among different subclasses of structures. They all coincide with the Kähler–Ricci flow among *Kähler* stuctures (i.e., $\nabla J = 0$ for the Levi–Civita connection ∇ of g).

Laplacian solitons. A G_2-*structure* on a 7-dimensional differentiable manifold M is a differential 3-form φ that is positive (or definite), in the sense that φ (uniquely) determines a Riemannian metric g on M together with an orientation. The space Γ of all G_2-structures on M is an open subset of $\mathcal{T} = \Omega^3 M \subset \mathcal{T}^{3,0}$, and the equivalence is determined by $H = \mathrm{Diff}(M)\times\mathbb{R}^*$. A very natural preferred direction is $q(\varphi) = \Delta_\varphi\varphi$, where $\Delta_\varphi = *d*d - d*d*$ denotes the Hodge Laplace operator on forms and $*$ the Hodge star operator attached to g and the orientation. Indeed, if M is compact and $\Delta_\varphi\varphi = 0$, then g is Ricci flat and has holonomy group contained in the exceptional compact simple Lie group G_2.

The corresponding *Laplacian flow* $\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t)$ was introduced back in 1992 by Bryant as a tool to try to deform closed G_2-structures toward holonomy G_2 and has recently been deeply studied by Lotay–Wei (see [18]). We refer to [14] for an account of the existence and structure of Laplacian solitons. Interestingly, the shrinking Laplacian solitons found on certain solvable Lie groups are the only Laplacian flow solutions with a finite-time singularity known so far.

7. Geometric Structures on Lie Groups

The role of (locally) homogeneous manifolds in Ricci flow theory has been very important (see [2]). More recently, Lie groups have also played an even stronger role in the study of geometric flows in complex, symplectic, and exceptional holonomy geometries, due mainly to the lack of explicit examples (see [13]).

We continue in this section the study of solitons in differential geometry initiated above. However, our fixed manifold is here a *Lie group*[8] G, and we assume that Γ consists of *left-invariant* geometric structures, i.e., $L_a^*\gamma = \gamma$ for any $a \in G$, where L_a is the diffeomorphism of G defined by $L_a(b) = ab$ for all $\gamma \in \Gamma$. Thus each γ is determined by its value at the identity $e \in G$ and so is identified with the tensor γ_e (or a tuple of tensors) on the Lie algebra $\mathfrak{g} = T_eG$ of G. In this way,

$$\Gamma \subset T^{r,s}\mathfrak{g},$$

the finite-dimensional vector space of all left-invariant tensor fields of type (r,s) on G. Note that Γ is usually contained in a single $\mathrm{GL}(\mathfrak{g})$-orbit, which is many times open in some suitable vector subspace $T \subset T^{r,s}\mathfrak{g}$ (e.g., inner products, nondegenerate 2-forms, almost-Hermitian structures, SU(3)-structures, positive 3-forms, etc.).

Accordingly, we consider the equivalence between left-invariant structures to be defined by scalings and the particular diffeomorphisms of G that are also group morphisms, i.e., by the group

$$H := \mathrm{Aut}(G)\times\mathbb{R}^*,$$

[8] *A differentiable manifold that is also, compatibly, a group.*

which is identified with $\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*$, where $\mathrm{Aut}(\mathfrak{g})$ is the automorphism group of $\mathfrak{g}$, as G is assumed to be simply connected from now on.

Any preferred direction q from the general case that is diffeomorphism equivariant produces a left-invariant one. Indeed, one obtains a preferred direction determined by a function

$$\gamma \mapsto q(\gamma) \in T_\gamma\Gamma \subset T^{r,s}\mathfrak{g}.$$

We therefore have that $\gamma \in \Gamma$ is a soliton, called a *semi-algebraic soliton* in the literature, if and only if

$$q(\gamma) = c\gamma + \mathcal{L}_{X_D}\gamma, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mathfrak{g}), \tag{15}$$

where $X_D \in \mathfrak{X}(G)$ is defined at each point $a \in G$ by $X_D(a) = \frac{d}{dt}\big|_{t=0} f(t)(a)$ and $f(t) \in \mathrm{Aut}(G)$ is determined by $df(t)|_e = e^{tD} \in \mathrm{Aut}(\mathfrak{g})$ (since $X_D(e) = 0$, these fields are never left invariant). Note that (G,γ) is also a soliton from the general point of view considered in (11). It is easy to check that, algebraically, the Lie derivative is simply given by

$$\mathcal{L}_{X_D}\gamma = -\theta(D)\gamma, \tag{16}$$

where θ denotes the usual $\mathfrak{gl}(\mathfrak{g})$-representation on tensors. As in the general case, additional conditions on the derivation D may apply if Γ satisfies extra properties (see Example 6.2).

Remark 7.1. These vector fields X_D on Lie groups attached to a derivation D may be viewed as a generalization of *linear vector fields* on $\mathbb{R}^n$ (i.e., $X_v = Av$, $A \in \mathfrak{gl}_n$) and have been strongly used in control theory since a pioneering article by Ayala–Tirao.

It is important to point out that in the Lie group case considered in this section, the corresponding geometric flow

$$\tfrac{d}{dt}\gamma(t) = q(\gamma(t)), \qquad \gamma(0) = \gamma, \tag{17}$$

is actually an ODE rather than a PDE. In particular, short time existence and uniqueness of solutions are always guaranteed. It is also known that $|q(\gamma(t))|$ must blow up at any finite-time singularity (see [13]). Note that γ is a semialgebraic soliton if and only if the solution $\gamma(t)$ to (17) is given by $\gamma(t) = c(t)f(t)^*\gamma$ for some $c(t) \in \mathbb{R}$ and $f(t) \in \mathrm{Aut}(G)$.

Due perhaps to its neat definition as a combination of geometric and algebraic aspects of (G,γ) (cf. (15) and (16)), the concept of semialgebraic soliton has a long and fruitful history in the Ricci flow case (see [10–12]) and has also been a quite useful tool to address the existence problem of soliton structures for all the geometric flows in complex, symplectic, and G_2 geometries given above.

Remark 7.2. Given a semialgebraic soliton (G,γ), if G has a cocompact discrete subgroup Λ, then the solution $\gamma(t)$ also solves (17) on the compact manifold G/Λ. However, in general, the locally homogeneous manifold $(G/\Lambda,\gamma)$ is no longer a soliton, since the field X_D does not descend to G/Λ. The solution $(G/\Lambda,\gamma(t))$ is very peculiar though: it is "locally self-similar" in the sense that $\gamma(t)$ is locally equivalent to γ up to scaling for all t.

The moving-bracket approach. The following viewpoint is suggested by the fact that all the geometric information on a Lie group endowed with a left-invariant geometric structure, say (G,γ), is encoded in just the tensor $\gamma \in T^{r,s}\mathfrak{g}$ and the Lie bracket μ of $\mathfrak{g}$. We consider the variety of Lie algebras $\mathcal{L} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ as in (10) (i.e., the algebraic subset of all Lie brackets on the vector space $\mathfrak{g}$) and fix a suitable tensor γ on $\mathfrak{g}$. Each $\mu \in \mathcal{L}$ is therefore identified with (G_μ,γ), the simply connected Lie group G_μ with Lie algebra $(\mathfrak{g},\mu)$ endowed with the left-invariant geometric structure on G_μ defined by the fixed γ:

$$\mu \longleftrightarrow (G_\mu,\gamma). \tag{18}$$

The natural $\mathrm{GL}(\mathfrak{g})$-actions on tensors provide the following key equivalence between geometric structures:

$$(G_{h\cdot\mu},\gamma) \underset{\simeq}{\longrightarrow} (G_\mu, h^*\gamma) \quad \forall h \in \mathrm{GL}(\mathfrak{g}), \tag{19}$$

given by the Lie group isomorphism $G_{h\cdot\mu} \to G_\mu$ with derivative h^{-1}. Since $\Gamma \subset \mathrm{GL}(\mathfrak{g})\cdot\gamma$, it follows from (18) and (19) that the isomorphism class $\mathrm{GL}(\mathfrak{g})\cdot\mu$ contains all geometric structures of the same type of γ (up to equivalence) on the Lie group G_μ for each $\mu \in \mathcal{L}$. Thus one has inside $\mathcal{L}$, all together, all Lie groups of a given dimension endowed with left-invariant geometric structures of a given type.

Remark 7.3. The usual convergence of a sequence of brackets produces convergence of the corresponding geometric structures in well-known senses such as pointed (or Cheeger–Gromov) and smooth up to pull-back by diffeomorphisms, under suitable conditions (see [13]). In particular, a degeneration (i.e., $\lambda \in \overline{\mathrm{GL}(\mathfrak{g})\cdot\mu} \setminus \mathrm{GL}(\mathfrak{g})\cdot\mu$) gives rise to the convergence of a sequence of geometric structures on a given Lie group toward a structure on a different Lie group, which may be nonhomeomorphic.

The moving-bracket approach has actually been used for decades in homogeneous geometry (see [13, Section 5] and [15]). In most applications, concepts and results from GIT, including moment maps and their convexity properties, closed orbits, stability, categorical quotients, and Kirwan stratification, have been exploited in one way or another.

A particularly fruitful interplay occurs in the Riemannian case, which relies on the fact that if μ is nilpotent, then the Ricci operator of $(G_\mu,\langle\cdot,\cdot\rangle)$ is precisely the moment map $\mathrm{m}(\mu)$ defined in (8) (up to scaling). This implies that, remarkably, soliton nilpotent Lie algebras (see (9)) and semialgebraic Ricci solitons (see (15) and (16)) on nilpotent Lie groups (called *nilsolitons*) are the same thing. In particular, the uniqueness up to isometry and scaling of nilsolitons on a given nilpotent Lie group follow from the uniqueness of critical points of $E(\mu) = |\,m(\mu)|^2$ on a given nilpotent $GL(\mathfrak{g})$-orbit up to the action of $O(\mathfrak{g})$ and scaling.

The bracket flow. Provided by equivalence (19), a main tool to study the geometric flow (17) is a dynamical system defined on the variety of Lie algebras $\mathcal{L}$ called the *bracket flow*, which is equivalent in a precise sense to the geometric flow (17). It is defined by

$$\frac{d}{dt}\mu(t) = \theta(Q_{\mu(t)})\mu(t), \mu(0) = \mu, \qquad (20)$$

where $Q_\mu \in \mathfrak{gl}(\mathfrak{g})$ is a suitable (unique) operator such that $\theta(Q_\mu)\gamma = q(G_\mu, \gamma)$.[9] Since $\frac{d}{dt}\mu(t) \in T_{\mu(t)}(GL(\mathfrak{g}) \cdot \mu(t))$, the solution $\mu(t) \in GL(\mathfrak{g}) \cdot \mu$ for all t, and so each $\mu(t)$ represents a structure on G_μ. However, $\mu(t)$ may converge to a Lie bracket $\lambda \in \overline{GL(\mathfrak{g}) \cdot \mu}$, i.e., toward a structure on a different Lie group G_λ (cf. Remark 7.3). For instance, this occurs already in dimension 3 for the Ricci flow.

The bracket flow is useful to better visualize the possible pointed limits of solutions under diverse rescalings, as well as to address regularity issues. Immortal, ancient, and self-similar solutions naturally arise from the qualitative analysis of the bracket flow (see [1, 2, 13, 17]).

Algebraic solitons. In light of the equivalence between the flows (17) and (20), a natural question arises: How do solitons evolve according to the bracket flow? It is natural to expect an evolution of a very special kind.

If μ is a fixed point (up to scaling) of the bracket flow (20), i.e., $\mu(t) = c(t)\mu$ for $c(t) \in \mathbb{R}$, then $\theta(Q_\mu)\mu = -c\mu$ for some $c \in \mathbb{R}$ by evaluating (20) at $t = 0$, and one obtains from (7) that

$$Q_\mu = cI + D, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mu). \qquad (21)$$

In that case, (G_μ, γ) is called an *algebraic soliton*. Note that these are semialgebraic solitons, since by (16), $q(G_\mu, \gamma) = \bar{c}\gamma - \mathcal{L}_{X_D}\gamma$ for some $\bar{c} \in \mathbb{R}$. They are distinguished though; indeed, it is proved in [15] that in terms of the operator Q_μ, (G_μ, γ) is a semialgebraic soliton if and only if $Q_\mu = cI + p(D)$ for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mu)$, where $p : \mathfrak{gl}(\mathfrak{g}) \to \mathfrak{q}$ is the projection with respect to the decomposition[10]

$$\mathfrak{gl}(\mathfrak{g}) = \mathfrak{k} \oplus \mathfrak{q}, \qquad \mathfrak{k} := \{A \in \mathfrak{gl}(\mathfrak{g}) : \theta(A)\gamma = 0\}.$$

Thus algebraic solitons are the solitons for which $Q_\mu = cI + p(D)$ holds for a special derivation D such that $p(D) = D$ (see (21)).

Any Ricci soliton on a Lie group is isometric to an algebraic soliton (see [10]). On the other hand, examples of Laplacian and pluriclosed semialgebraic solitons that are not isometric to any algebraic soliton were found in [19] and [1], respectively.

[9] *The existence of such an operator relies on the fact that* $GL(\mathfrak{g}) \cdot \gamma$ *is open in* T.

[10] *For instance,* $p(D) = \frac{1}{2}(D + D^t)$ *if* γ *is a metric or a closed* G_2*-structure.*

As expected, algebraic solitons are distinguished from many other points of view. Some of the results supporting this follow:

- Consider the Ricci pinching functional

$$F(g) := \frac{\mathrm{scal}_g^2}{|\,\mathrm{Ric}_g\,|^2},$$

  measuring in a sense how far a homogeneous metric g is from being Einstein (indeed, $F(g) \le \dim M$, and equality holds if and only if (M, g) is Einstein). As shown by Lauret–Will, algebraic Ricci solitons are precisely the global maxima for F restricted to the set of all left-invariant metrics on any unimodular Lie group, as well as on any solvable Lie group with codimension one nilradical.
- Böhm–Lafuente proved that the dimension of the isometry group of an algebraic Ricci soliton on a solvable Lie group S (called *solvsolitons*) is maximal among all left-invariant metrics on S. A stronger symmetry maximality condition was shown to hold in the case when S is in addition unimodular by Jablonski: the isometry group of a solvsoliton contains all possible isometry groups of left-invariant metrics on S up to conjugation by a diffeomorphism.
- A closed G_2-structure φ is called *extremally Ricci-pinched* (ERP for short) if

$$d\tau = \tfrac{1}{6}|\tau|^2\varphi + \tfrac{1}{6} * (\tau \wedge \tau),$$

  where $\tau = - * d * \varphi$ is the torsion 2-form of φ. They are characterized in the compact case as the structures at which equality holds in the following Ricci curvature estimate for closed G_2-structures discovered by Bryant:

$$\int_M \mathrm{scal}^2 *1 \le 3 \int_M |\,\mathrm{Ric}\,|^2 * 1. \qquad (22)$$

  It was proved by Lauret–Nicolini that any left-invariant ERP G_2-structure on a Lie group is necessarily a steady algebraic Laplacian soliton and its attached metric is an expanding algebraic Ricci soliton.

Concerning bracket flow evolution of a semialgebraic soliton that is not algebraic, we know that $\mu(t)/|\mu(t)|$ is either periodic or not periodic and the following chaotic behavior occurs: for each t_0 there exists a sequence $t_k \to \pm\infty$ such that $\mu(t_k)/|\mu(t_k)|$ converges to $\mu(t_0)/|\mu(t_0)|$ (see [15]). The existence of a soliton of this last kind is an open problem.

Remark 7.4. More generally, the whole picture developed in this section essentially works for G-invariant geometric structures on a homogeneous space G/K, though a more technical exposition would be necessary (see [13, 15]).

8. Concluding Remarks

As discussed in the Introduction, solitons play the role of "best" elements in a given set in the case when the most natural ones are not available. A main aim of this article was to show how fruitful this has been in the study of geometric structures on manifolds, with particular strength on Lie groups.

On each solvable Lie group, there is at most one solvsoliton up to isometry and scaling. This allows us to endow several Lie groups that do not admit Einstein metrics (e.g., nilpotent or unimodular solvable Lie groups) with a canonical Riemannian metric. Analogously, Chern–Ricci, pluriclosed, and HCF (resp., SCF) algebraic solitons provide distinguished Hermitian (resp., almost-Kähler) structures for Lie groups on which Kähler metrics do not exist. Laplacian algebraic solitons play the same role in the homogeneous case, where holonomy G_2 is out of reach since Ricci flat implies flat.

The moving-bracket approach allows the rich interplay between soliton geometric structures on Lie groups and soliton Lie algebras, paving the way to many beautiful applications of GIT to differential geometry.

ACKNOWLEDGMENTS. The author is very grateful to Ramiro Lafuente, Emilio Lauret, and Cynthia Will for invaluable conversations during the writing of this paper. Current and past support from CONICET, FONCyT (ANPCyT), and SECyT (UNC) is acknowledged.

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Jorge Lauret

Credits

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The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section. Next month's theme will be variations in academic math jobs.

Research

How to Balance Research with Everything Else We Have to Do

David Zureick-Brown

My institution's tenure and promotion guidelines describe my position as "40% research, 40% teaching, 20% service." Teaching includes three courses each year, but also advising PhD or honors thesis students, and service includes committees, organizing conferences, refereeing, recommendation letters, etc.

I interpret this breakdown literally and typically set aside two full days each week for undistracted research time, two days for teaching (prep, lectures, office hours, meeting with students, grading, reading courses, student learning seminars), and one day for service. In practice the teaching and service days blur together, and I also set aside a day for (mostly) nonacademic work (errands, shopping, food prep, coordinating travel, processing email (usually in batch mode), and planning out my week).

My research time is sacred. One would never skip teaching a lecture or a committee meeting for reasons related to research, and I encourage the converse. Work off campus if necessary, or at least with a closed door, headphones on, immune to distraction.

Create (and protect) research time. I used to "joke" with contemporaneous early career faculty that our workload doubled each year. The joke quickly aged.

I recommend reading some book like *Getting Things Done* by David Allen or *Deep Work* by Cal Newport, or the archives of an academic blog with a heavy advice subtheme (e.g., by Terry Tao[1] or Matt Might[2]). Here are a few ways that I implement their ideas.

Become organized. A main message of books like *Getting Things Done* is: if you can do something in 1–2 minutes, just do it; otherwise, schedule when you are planning to do it and put it out of sight and (especially) out of mind until then.

I do this via a calendar (synced with my phone, tablet, etc.). If I have a recommendation letter to write (or slides to make, PhD applications to read, etc.), I schedule a block of time (not just a reminder) to work on the task.

When I get a request to (for example) upload an already written recommendation to yet another UC school, I do it immediately (and organize so that it takes literally under a minute).

Email. Process in batch mode. Usually I'll set aside a few hours over the weekend to handle as many menial tasks as I can all at once. Throughout the week, I'll usually set aside half an hour for email in the mornings and another fifteen minutes at the end of the day, and otherwise I mostly don't check email (especially not on my phone, which might reveal some task that I can't attend to right away).

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DOI: https://dx.doi.org/10.1090/noti2080

[1] https://terrytao.wordpress.com/career-advice/

[2] http://matt.might.net/articles/productivity-tips-hints-hacks-tricks-for-grad-students-academics/

Focus. I turn off most mobile notifications and set my phone to "do not disturb." I stay off social media when working, and (worth emphasizing again) I don't check email as a distraction (new work will manifest, and if I can't dedicate 10–20 minutes to process, then this lingers and distracts). Any emails related to a joint project are printed to pdf and put in a folder related to that project as they arrive so that I don't need to check email when working.

Automate. I coordinate our weekly Algebra and Number Theory research seminar. For each outside speaker, I send 5–6 emails (announcing the seminar, requesting that the staff make and post flyers, processing reimbursement, etc.). The emails are basically the same every week, with a few variables; I have a bash script that writes the emails for me.

Saying no. Initially I accepted every reasonable referee request. Eventually, several senior colleagues explained that they turn down many requests. Good citizens seem to referee roughly three papers for each paper they submit, and I try to stick to that (even if the paper looks interesting and I'm an appropriate referee).

It's difficult to say no, and psychologically, I needed "permission" to start declining requests. It was helpful that my department's service expectations were clear and direct. Similarly, I found writing for *Mathematical Reviews* stressful and difficult but organizing conferences enjoyable and natural; these days I do more of the latter.

Efficiency. I could fill another article with habits and routines that create and protect time.

I bike to work: it takes about 25 minutes, compared to 35–40 to drive and park. Our department has a shower on my floor. Cycling doubles as cardio and boosts my mood and mental health.

Each semester I poll students to find a time that accommodates everyone who might possibly attend office hours. I reserve a classroom for office hours. Students can show up even if they don't have focused questions, and I can leave immediately when finished. (In any case I can't fit more than 3–4 people in my office.)

I scan my lecture notes after each class; I include the date, lecture number, and course number in the filename, and keep a terse outline of what I covered each lecture. Teaching a course a second (or ninth…) time is a breeze, and this type of organization creates time and space to focus on improving the course. And if I'm stuck somewhere (e.g., delayed flight), I can access the notes via Dropbox and use that time for review and preparation.

In fact, I scan everything; my office is across from the printer/scanner, and Dropbox has a useful mobile app (if I have a meal receipt that needs to be reimbursed, I scan it immediately). If I give a chalkboard talk, I scan the notes (and put the scan somewhere easy to find again).

Discuss! I've had countless conversations about these topics (at tea, conference dinners, in hallways) and have benefited greatly. Our profession tends to pile on extra work with little guidance, and reaching out to experienced colleagues can be productive and therapeutic.

David Zureick-Brown

Credits

Author photo is courtesy of Sarah Zureick-Brown.

How to Read a Research Paper

Matt Baker

Before attempting to read a research paper, I recommend first deciding *why* you want to read it, what you hope to get out of the paper, and how much time you're willing to commit. Then place the paper into one of the following three categories:

- **Speed Read:** A paper whose introduction you plan to read in order to get an overview of the results and then possibly skim further.
- **Substantial Skim:** A paper that you plan to skim all the way through, perhaps reading certain parts in detail.
- **Deep Dive:** A paper that you wish to thoroughly study and understand.

It's useful to have different categories because there's *so* much interesting math research produced every day, and it's impossible to keep up with everything. I go through the arXiv preprint listings in three different categories almost religiously every single morning. I get email notifications for all new postings and revisions in these categories, and I make an effort most days to "speed read" at least one new paper while drinking my morning coffee. I also bookmark papers to come back to later (although to be honest, I end up not having time to come back to many of these). Skimming through the arXiv abstracts in the daily digest and then speed reading at least one paper a day keeps me feeling in touch with what's happening in the fields I'm most interested in.

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DOI: https://dx.doi.org/10.1090/noti2077

With a Speed Read, I don't spend a lot of time trying to come to terms with complex definitions or understand precisely why the hypotheses in various theorems are what they are; I just want to get an overall impression of the paper and learn something new, even if it's mostly superficial knowledge. This kind of reading will take me about 15 minutes for an average paper.

If the paper seems really interesting and I'm wanting to read and learn more, I'll upgrade it to the "Substantial Skim" category and come back to it later in more detail. I've found that over time, the collection of quick impressions I obtain through my Speed Reads actually provides a reasonably deep foundation for understanding various bits of the mathematical landscape at large.

For a smaller number of papers (roughly one a week, in my case, though the variance is high), I will do a more Substantial Skim. This involves reading through all of the definitions and statements, and at least some of the proofs, in the whole paper and making an attempt to actually understand what's going on. My goal here is to embed the paper firmly enough into my mind that I'll be able to incorporate some of the ideas into my thinking later on. When I read a paper in this way, I will sometimes jot down key definitions or statements (either in a physical notebook or Evernote file), and I might go through certain arguments carefully in order to understand particular points in detail. However, I don't make an attempt to check the paper for correctness or try to fully understand all of the technical points. This is actually my favorite category of reading, because I learn a tremendous amount without having to put in *too* many hours of work. Depending on the complexity of the paper, this kind of reading might take me anywhere between one and three hours per 10 pages.

Finally, there's the Deep Dive. These days, I probably do this for only about one paper every month or two (though when I was younger and less overwhelmed with responsibilities, it was quite a bit more frequent). Here, I spend as much time as it takes to understand the definitions, the conditions of the theorems, and the logic behind the arguments. I will sometimes write out detailed notes in a notebook, and as often as possible I'll stop reading and see if I can figure out the next step myself. If I'm really interested in understanding the paper and retaining the knowledge, I'll try to explain the results to someone else; there's no better way than teaching to internalize complex information! I'll also look up background facts in the references and, as long as it doesn't lead to an exponential recursion, try to understand that material as well. And I'll keep a list of typos or other errors I find and send it to the author as appropriate. This kind of reading can take anywhere from 15 minutes to a couple of hours per page, depending on the subject matter and how familiar I am with the underlying concepts.

To be perfectly honest, these days I find the time for a Deep Dive only when a paper is either directly relevant to my current research or I've agreed to referee it. But that's an artifact of choices I've made and priorities I've set in my own life, and of course if you really want to build up a formidable knowledge base, you should take Deep Dives as often as possible. If, like me, you're constantly feeling busy and overwhelmed, another way to "read" a paper is to assign it to a student to explain to you! Or put together a study group and parcel out the task to various students and colleagues. I did this recently with the paper "Lorentzian Polynomials" by Petter Brändén and June Huh over the course of a whole semester. (For an example of how to organize such a seminar, see `https://sites.google.com/view/gtlorentzian`.)

One other thing I think is important is to read widely and try to push the boundaries of your understanding. Especially with Speed Reads and Substantial Skims, don't just read a bunch of papers about the same topic all the time. By reading about different topics within a short time frame, your brain will automatically start to explore connections and begin thinking "outside the box."

This is my own personal approach to reading math papers, and I don't claim that the same techniques will work for everyone. But I do think that every working mathematician needs to develop a system for reading research papers in order to attain, over time, both the breadth and depth of knowledge required to keep up with the relentless but thrilling march of progress in modern mathematics.

Matt Baker

Credits

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Changing Focus

Mark Andrea de Cataldo

Alla memoria di mamma e papà, con amore.

I write down here a few thoughts about changing the direction of one's research, mostly by referring to my personal experience. It is not my intention to have the reader interpret my writing as a description of a move towards "better" math (whatever that means). Very plainly, I have tried to share selected memories of what happened when I met new math that I found to be interesting and beautiful. Math that I could not resist. It would be silly to try to pose that what follows contains any kind of universal truth. I hope the young reader glances at these recollections and draws some hopefully useful conclusions.

In My Youth

In my youth, I had lots of fun playing rugby as a left-wing for the University of Milano. In this role, change of pace and direction are very important, and they occur as if in a dizzying ride, the result of the split-second decisions you make running with the ball. The speed at which they occur notwithstanding, these changes are planned. On the other hand, I think that, more often than not, the changing of research direction does not occur because of a choice made at one point in time. Oftentimes, it is the result of a series of (mostly) fortunate events. This new evolution is not a sudden occurrence but the result of a mental disposition towards the beautiful math that comes to us by reading, thinking, working out problems, writing, and, most importantly, talking to people.

Grad School: Midwest

As a graduate student, I had been working on a problem I did not like. Ah! What a unique situation I found myself in. At that time, higher-dimensional algebraic geometry was in a boom (still is), and, darn it, I wanted to be part of it. So I set myself to work on it, drop my problem, change advisor, and do great new things. Bust! It came to nothing. The field itself was very exciting, but I was unprepared to enter it for what I now think was the wrong reason. I backtracked. However, I did not go back to the old line of work. With the help of my (very patient) advisor, I found myself a new problem, involving the special properties of codimension two submanifolds of complex hyperquadrics, solved it for the most part, and got a few papers out of it. And a degree.

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DOI: https://dx.doi.org/10.1090/noti2079

Postdoc Numero Uno: Midwest

Postdoc! First job! Two years. Upon arrival, in the early fall of so many years ago, I submitted my first NSF grant research proposal. In writing it, I realized, to my horror, that I was not interested in the problems I came up with and was proposing. The subsequent decline letter from the foundation only confirmed my suspicion that many others were in total agreement with me. Meantime, I had stumbled upon some new beautiful math, estimates for solutions to $\bar\partial$ and applications to algebraic geometry. I started a learning seminar with faculty and graduate students. I asked myself a question, answered it, wrote two papers and my second NSF proposal, which was funded just in time for my second postdoc to begin.

Postdoc Numero Due: Germany

Postdoc! Second job! One year. Upon my arrival in Germany, a math physicist said, "Ah! You are an algebraic geometer. Why don't you explain to us, in a series of talks (not one, not two…a series of them, as many as you need), the action of the Heisenberg algebra on the cohomology of the Hilbert scheme of points on a surface?" I said, "Uh?" The truth is, I did not want to do it: I did not know that field; it would have been too much work; I wanted to focus on $\bar\partial$; I could not care less. I needed to get out of this predicament. But the physicist was…my office mate. I tried avoiding him. I failed. He was gently persistent. Hey, he was a fun guy with a great disposition—towards math, and towards life. I started reading up. Wow! Love at fifteenth sight. It took a lot of work to start realizing that I liked very much what I was not fully understanding. By the end of the year, my second postdoc was over; I was working a little bit on the $\bar\partial$ project, and more on Hilbert schemes.

Postdoc Numero Tre: East Coast

Postdoc! Third job! One year. Fancy US institution. Fellowship from the AMS (thank you). Stellar mentor. But (there is always a "but") I felt something was a bit off: I had been awarded the NSF grant, the fellowship, and the postdoc because of $\bar\partial$, but I could not stop binging on Hilbert schemes. Was I even on the right track? Upon arrival, a newly minted postdoc, a fellow algebraic geometer, explained in very, very lengthy detail his achievements and his goals. He asked me about my work, and ten seconds into my pitch, he dismissed me quickly: "That's not algebraic geometry." What should you do when something like this happens? I smiled. During that year, I ended up devoting a lot of my attention to finding a tenure-track job, giving over two dozen talks, and straddling the two research topics. I know now that nothing was off. Math, as a whole, must invest in its own future: encourage young people to try new things.

Tenure-Track: Long Island

Tenure-track! Move to Long Island. Benefits, tenure-clock, dean, sub-dean, uber-dean, mini-dean, provost, president, chair, departmental politics, grant proposals, teaching, setting up a webpage (1999! Still have the same one; looks like it's from 1993). More importantly, what to do mathwise? My good old friend $\bar{\partial}$ was no longer on my horizon. Hilbert schemes? Yes, but it was not the Hilbert schemes anymore, really. Somehow, the topology of complex algebraic varieties, with its difficulties, was occupying my thoughts. It had started dwelling in my mind without much fanfare as a result of thinking about Hilbert schemes. It was there, and it did not budge. What was going to happen? I was on tenure-track, needed to write papers, get grants, and this math was so new to me (and so weirdly exciting). What if it did not pan out? What about the grant renewal? Tenure (tick-tock)? In the end, there was really no choice. It was too interesting, difficult, and beautiful. The light was too blinding. I went towards it, and that was it. Luckily, as it turned out, it was not a truck.

Today

Undergrad years, compulsory military service, grad school, postdocs, tenure-track, promotions… It's all a vivid, awesome blur. There have been more changes in direction since then, but, at least mathwise, none more pronounced than the ones of my younger years. When I look back (when do I do that? There's no time!), it seems to me that the real changes of direction in my research were not recognized by me as such when they started happening: I was simply drawn to work on something that fascinated me, and to do that, I needed to learn and discover new math. And it was fun, pure and simple. At some point, suddenly, I found myself already moving in a new direction. It seems that there was no moment in time when I made a conscious decision to change. Quite simply, it just happened. And I am very, very happy it did.

Mentoring Undergraduate Research: Advanced Planning Tools and Tips

Courtney R. Gibbons

My first experience with mentoring undergraduate researchers was as the undergraduate mentee, and the critical importance of that summer features prominently in my mathematical coming-of-age story. That summer was a success because my mentor had a pretty clear idea of what we would do: the problem, the schedule, the follow-up. With this in mind, I'm sharing five homework exercises that I assign to myself (now that I'm the mentor!) to make sure that my students and I have a fun and productive experience.

But first, a reminder: Whom you mentor matters. Students form their mathematician identity through their experiences with us, the mathematicians in their lives. Our outside recognition of them as "the kind of people who do math" is a component of this development, especially for students who don't see themselves represented in mathematics often or at all. For an entry into the literature on this topic, see the recent paper [RCJ19] and its thorough bibliography. The upshot? Inviting students from underrepresented groups in mathematics to work on our research projects is one way of committing to diversity and inclusion in our profession.

Exercises

I recommend completing these exercises several months in advance of your anticipated research project with undergraduates. For example, if you are thinking of working with students over a summer, consider working on them between the fall and spring semesters.

Exercise 1. Answer two questions: What do you hope to get from the collaboration? What do you hope your students get from it?

Your answers here (and below) don't need to be deep, philosophical reflections; keep it workable. For example, when I was a pretenure REU mentor, my goal for me was to produce a paper with my team. My goal for my team was to give them each ownership over specific parts of the project.

It's a good time to state that every piece of advice comes with exceptions! While I usually ask students to work on a project I designed, I've had gung-ho students pursue their own problems, and they've come up with surprising (to me) approaches and results. In those cases, I revised my goals for myself and made sure I felt like I had enough background to jump in and help.

Exercise 2. Write up a project proposal, including the problem(s) and a potential pathway to a solution.

The first time I did this, I didn't have a choice. I had agreed to be a visiting REU mentor, and I had to write this as part of the grant proposal. I think it's the most useful exercise on the list. From this, you will figure out what kind of background your future research students will need and what supplemental material they'll need to learn. You will start to see how to divide up the project into parts.

You can guide students along a path and stay a few steps ahead of them. You can even assign students some preliminary homework if they seem interested in working on research with you!

Exercise 3. Decide on the length of the collaboration and structure your time.

1. Break the project into phases: ramp-up (literature review, "classes" to cover background material, and time for students to build up stamina), active research, and

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DOI: https://dx.doi.org/10.1090/noti2078

ramp-down (writing up results in various formats). Fill in some of the details for each phase. What topics will you cover? What software will your students learn to use? What will students be responsible for presenting?

2. Draft a sample weekly and daily schedule. Big blocks of unstructured time are great for research if you already know how to support your research process. Those doing research for the first time benefit from having finite chunks of time planned for specific purposes.

Your weekly schedule can include a lunch outing, times for group meetings, or whatever you like to break the time up. For first-time researchers, I schedule 2–3 hours twice a week to sit side by side and work on math so that I can help them develop research practices for what to do when they're stuck. If there are other faculty working with students along a similar timeline, I coordinate with them to have joint "show-and-tell" meetings.

Exercise 4. Write a paragraph dedicated to your future research student to share your expectations for them and yourself. How much time do you expect them to work on the project each day/week, and how will you keep them accountable?

I have found that remote mentoring is far less effective than mentoring in person. For me, this means when I decide to work with undergraduates, I'm committing to being in the same room (at the beginning) and in the same building (later on) for most of the collaboration.

Exercise 5. Describe how you will commit to your students' professional development after the time is officially up. Will you send them info about conferences where they can present? Will you write down some preliminary notes for letters of recommendation?

I like to keep a file with notes about what my students accomplished during our project and any other tidbits I think might be useful for future formal or informal recommendations. It can be as simple as a text file.

Parting Advice

Working with undergraduates was a highlight of my early career. Not every project turned into a bullet point in my research portfolio for tenure and promotion, but the energy that each young researcher brought to a collaboration gave me a booster shot of enthusiasm for all of my projects. Without other commutative algebraists around to talk to in my department, I found it was talking to my research students that did the most to keep me motivated and productive. I'll finish with the advice I received that I keep in mind when I start a new project with undergraduates: be ambitious for your students; they can learn a lot and do excellent work. They may just be your best local collaborators.

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Courtney R. Gibbons

Credits

Author photo is by Victor Lou.

Ola Bratteli and His Diagrams

Tone Bratteli, Trond Digernes, George A. Elliott, David E. Evans, Palle E. T. Jorgensen, Aki Kishimoto, Magnus B. Landstad, Derek W. Robinson, and Erling Størmer

Introduction

Magnus B. Landstad and George A. Elliott

Ola Bratteli is known first and foremost for what are now called Bratteli diagrams, a kind of infinite, bifurcating, graded graph. He showed how these diagrams (cousins of Coxeter–Dynkin diagrams) can be used to study algebras that are infinite increasing unions of direct sums of matrix algebras. They turned out to be very useful tools, giving a large class of examples, and later led to a *K*-theoretical classification both of the algebras just mentioned and, more recently, of an enormously larger class (all "well-behaved" simple amenable C*-algebras). According to MathSciNet, Ola Bratteli has 113 publications with 21 coauthors and he received various awards. He was a member of the Norwegian Academy of Science and Letters and a member of the AMS for forty-three years.

As to his biography, Ola Bratteli graduated with distinction from the University of Oslo in 1971 and took his doctorate there in May 1974. He was a research fellow at New York University 1971–73, had various postdoc positions 1973–77, was an associate professor at the University of Oslo 1978–79, a full professor at the University of Trondheim (now NTNU) 1980–91, and since 1991 at the University of Oslo.

Figure 1. Ola Bratteli (1946–2015).

Ola's father, Trygve Bratteli, was a Norwegian politician from the Labour Party and prime minister of Norway in 1971–72 and 1973–76. During the Nazi invasion of Norway, he was arrested in 1942, and was a *Nacht und Nebel* prisoner in various German concentration camps from 1943 to 1945 but miraculously survived. Ola's mother, Randi Bratteli, was a respected journalist and author of several books. Ola was born October 24, 1946, and died February 8, 2015. He is survived by his wife, Rungnapa (Wasana), and their son, Kitidet.

For a detailed biography, see the MacTutor History of Mathematics Archive: `www-history.mcs.st-and.ac.uk/Biographies/Bratteli.html`.

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DOI: https://dx.doi.org/10.1090/noti2085

Bratteli Diagrams

George A. Elliott

One of Bratteli's most important discoveries, I think, was what is now called a Bratteli diagram, which he found as a way of codifying the data of an approximately finite-dimensional (AF) C*-algebra (the completion of an increasing sequence of finite-dimensional C*-algebras, i.e., finite direct sums of full matrix algebras), in a far-reaching extension of the thesis of Glimm. (Glimm considered the case of simple finite-dimensional C*-algebras, with unital embeddings. The nonunital case was later studied by Dixmier.)

In one sense, Bratteli diagrams had perhaps already been invented, as, for one thing, the idea is so simple—a vertically arranged sequence of horizontal rows of points, with numbered lines connecting the points of each row to the points of the row below, these numbers recording the multiplicities of the partial embeddings of the simple direct summands at one stage of the sequence of algebras into those at the next stage.

In the unital case, with unital embeddings, the orders of the simple direct summands at each stage, which can be written as numbers accompanying the corresponding points in the diagram, are determined in a simple way by the multiplicities, assuming that the first stage is just the complex numbers (which clearly is no loss of generality).

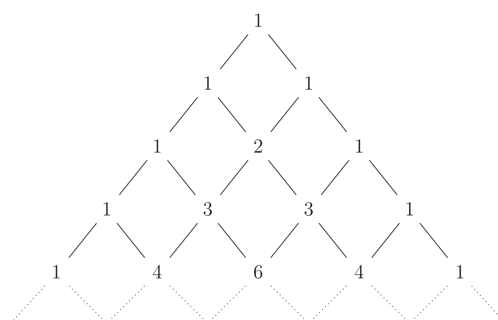
Pascal's triangle is a Bratteli diagram—with the multiplicities equal to one, and the numbers appearing in the rows being of course the successive degrees of binomial coefficients. The unital AF C*-algebra this diagram encodes arises in physics as the gauge-invariant subalgebra of the so-called CAR algebra, the C*-algebra of the canonical anticommutation relations (also AF). (This well-known subalgebra is referred to as the GICAR algebra.)

Bratteli was not content just to look at the diagrams—he isolated the equivalence relation between them that is determined by isomorphism of the associated C*-algebras. This was prophetic, as he in fact was noticing that the diagrams formed a category, in which his equivalence is just isomorphism. It was later noticed that this category is equivalent to the category of ordered groups arising from the algebras in question via the *K*-functor. This led eventually to a *K*-theoretical classification of an enormous—still evolving!—class of amenable C*-algebras, analogous to the classification by Connes and Haagerup and others of amenable von Neumann algebras.

Bratteli diagrams arose in a fundamental way in Jones's theory of subfactors. Given a subfactor of Jones index less than four, the increasing sequence of relative commutants in the Jones tower are finite-dimensional and so give rise to an AF algebra. Its Bratteli diagram is periodic with period two, with the step from the second row to the third being just the reflection of the step from the first row to the second. Both these one-step Bratteli diagrams are obtained from a single Coxeter–Dynkin diagram by pleating it in the two different possible ways.

Bratteli diagrams with an order structure were introduced by Vershik to describe a measurable transformation, and were adapted by Herman, Putnam, and Skau to describe a minimal transformation of the Cantor set. Using this description, Giordano, Putnam, and Skau classified the orbit structures of such transformations, the invariant being (in the generic case) the ordered *K*-group of the associated C*-algebra, also classified by this ordered group.

Below is a Bratteli diagram representing the GICAR algebra (see above). Note the resemblance to Pascal's triangle.



The corresponding inductive chain system (depicted vertically) is

$$\mathbb{C} \xrightarrow{\phi_0} \mathbb{C} \oplus \mathbb{C} \xrightarrow{\phi_1} \mathbb{C} \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \xrightarrow{\phi_2} \mathbb{C} \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \xrightarrow{\phi_3} \mathbb{C} \oplus M_4(\mathbb{C}) \oplus M_6(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus \mathbb{C} \xrightarrow{\phi_4} \cdots$$

where the (injective) connecting homomorphisms are given by

$$\begin{aligned}
\phi_0(a) &= a \oplus a,\\
\phi_1(a,b) &= a \oplus \begin{pmatrix} a & \\ & b \end{pmatrix} \oplus b,\\
\phi_2(a,B,c) &= a \oplus \begin{pmatrix} a & \\ & B \end{pmatrix} \oplus \begin{pmatrix} B & \\ & c \end{pmatrix} \oplus c,\\
\phi_3(a,B,C,d) &= a \oplus \begin{pmatrix} a & \\ & B \end{pmatrix} \oplus \begin{pmatrix} B & \\ & C \end{pmatrix} \oplus \begin{pmatrix} C & \\ & d \end{pmatrix} \oplus d,\\
&\ \vdots
\end{aligned}$$

where blank means one or more zeros; a, b, c, d are complex numbers; and B, C are matrices.

Ola was always a fountain, or mountain, of good sense. Once, when he was visiting Toronto, we went to the gym to go swimming. I was a member, but it was a little murky what guest privileges I had. I thought I had them as a faculty member and proceeded to explain that. This met with resistance, which gradually became more protracted. In the meantime, Ola slipped past the desk, picked up a towel, and half an hour later had finished his swim—perhaps even had a sauna too. At that point I gave up and we left.

Ola—A Child of Peace, a Man of the Outdoors, and a Family Man

Tone Bratteli

On United Nations Day, October 24, 1946, a child of peace was born in a crowded maternity department in Oslo. He was one of many in the baby boom that followed World War II.

This event was no matter of course. My father came home in 1945 after years in extermination camps in Germany. He survived by a hair's breadth. Soon after his return he met my mother. Her father had also come home from concentration camps. The two found each other quickly, despite my father's shyness. My mother's sociable nature made up for that. And in October 1946 he could lift his son up in the air in joy. He had not been sure whether he would be able to have children after the appalling treatment in the camps.

A year and a half later, May 8, 1948, I was born. And after another three years, May 20, 1951, our little sister, Marianne, arrived. Ola's sometimes insistent little sister.

My father was on his way into politics, with the result that when Ola was five years old we moved to a so-called official residence in which our parents could also carry out social duties. On the day we moved, Ola and I scrambled around searching for our old home. The little we had with us was swallowed up by the huge rooms. Many leading politicians from other countries came to this apartment. We children hid away in our rooms and were not especially eager to introduce ourselves. Ola was a quiet boy, but one evening he came in to join the houseguests and started tugging on the men's ties. This was a big surprise. What had happened to Ola? He had been in the kitchen, where

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Figure 2. The Bratteli family, 1952.

some half-empty wine glasses had caught his eye. After tasting that juice, the shy, self-conscious boy was briefly transformed into a party animal.

But Ola was soon ready for Bolteløkka School and a meeting with the subject that became his passion—mathematics. My mother was rather taken aback the day she discovered that Ola had wallpapered his room with equations. He did not play football or hang out with friends. Instead he solved equations and went for very long walks in the forests and the mountains. He also did my maths homework for me, because to me that subject was a struggle. Ola achieved the best grade in mathematics; mine was the worst.

As a young man, Ola spent many hours skiing, and the trips could easily reach 50 to 60 kilometres. He also tried to find detours to make them even longer. When he came home, a slice of bread or two was not enough. He ate the whole loaf.

Early on, Ola showed an interest in music and visual arts as well. He made an attempt to teach himself to play the cello. It was not a success. But he spent a lot of time listening to music and he went to exhibitions. Our sister, Marianne, is an artist, so my brother and sister had something in common there. Later in life, when I travelled to several continents in my job as a journalist and in other connections, Ola always knew which gallery I should head for if there was only time for one. The composition of a picture and the solution to a mathematical puzzle must have something in common.

Ola graduated from the University of Oslo in 1971. He was awarded the best possible grade and was what we call "reported to the King." That meant that the entire government was informed about his academic accomplishments. It was a big day for everyone in the family, but perhaps most of all for my father. He was the prime minister who came from such a poor background that he never completed his education himself, but was self-taught. Now he was to inform the king and government about his son's academic triumph. As usual, Ola was unassuming and self-conscious, but he was no doubt satisfied.

It was difficult to be completely anonymous and work undisturbed with his own research in Oslo. Being the child of a prime minister has many sides. You have to take a variety of comments and media coverage in your stride. When Ola went to New York in 1971, it was a kind of escape. From his apartment in Greenwich Village, it was easy to get to theatres and exhibitions, and Ola soaked up everything he could get to. One day when it snowed in New York—that does happen, after all—he skied up and down Fifth Avenue. Finally, skiing weather in the Big Apple....

Ola returned home in 1973 and took his doctorate in 1974. This was during Dad's second term as prime minister and also led to coverage in the media.

In his personal life, Ola was simply a very kind, generous, caring and family-loving man. With a twinkle in his eye, it was easy for him to establish rapport with children.

During the years he lived abroad, he kept in touch with our mother by phone and wanted to know how we were doing.

Before Ola moved back to Norway, he was fortunate enough to meet Rungnapa (Wasana). He was to share almost half his life with her. Kitidet—his son—was his pride and joy.

Figure 3. Wedding in Trondheim, 1986.

Rungnapa and Ola travelled widely all over the world. There were holidays, but she also accompanied him to conferences and on visits to universities. After a while, they settled down in Norway. They enjoyed good years together. In the last years of his life, Ola's health deteriorated. Rungnapa was enormously supportive and did all she could to make it possible for him to live at home as long as possible. He was a lucky man.

Now Rungnapa and Kitidet have moved back to Phitsanulok in Thailand. Ola and Rungnapa built a house in her home city several years ago and had no doubt planned to spend the winters there eventually. It did not turn out that way; Ola died so early. But outside the house there is a small temple for Ola. So in a way he is there too.

Tone Bratteli

Reminiscences of Ola

Trond Digernes

I first met Ola around 1970 when we were both Master's students at the University of Oslo. At that time Ola was a slender young man with a passion for the outdoors, especially mountain hiking.

During the year 1970–71 there were three of us who spent much time together: Ola, John Erik Fornæss, and myself. We played bridge, took a skiing vacation in the Norwegian mountains, and went mountain hiking in the summer. At the end of summer 1971 our roads parted. We all went to the US for PhD studies, but to different institutions: Ola to the Courant Institute, John Erik to the University of Washington, and I to UCLA. Other adventures with Ola in the early seventies included a multiday hike in the Sierra Nevadas in the summer of 1972, and a trip by jeep through the roadless interior of Iceland in the summer of 1973. The latter involved getting stuck in rivers and sleeping out in the open. After Iceland, Ola returned to New York, whereas I was headed for a year's stay at CPT/CNRS, Marseille.

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Figure 4. Ola and Trond in Trondheim, 1983.

At CPT/CNRS, Marseille, the year 1973–74 was organized as a special year dedicated to operator algebras and mathematical physics. It attracted several high-powered researchers, among them Alain Connes and Masamichi Takesaki, and Derek Robinson and Daniel Kastler were already there. Ola joined the Marseille group in January 1974, and this was also when he started his long-time collaboration with Derek Robinson. During the decade 1980–90 I joined Ola and Derek on several occasions for discussions at ANU, Canberra, and Derek also visited Trondheim. This resulted in a few joint publications, sometimes also with other coauthors.

I was involved in only a fraction of Ola's mathematical work, but since we spent much time together, we had many interesting discussions. Given Ola's incisive mind and deep understanding of everything he was involved in, it was always a rewarding experience to exchange ideas with him. He is dearly missed, both as friend and colleague.

Trond Digernes

Ola Bratteli, Friend and Mathematician

Erling Størmer

I met Ola for the first time when he was about to start on his Master's thesis. Then he was a dark-haired lad with a beard, radiating health and fitness, who often went for very long skiing trips. It soon struck me that he was a highly effective person who could pick up new theory extremely quickly. By then he had taught himself a great deal about the field of operator algebras, which he wanted to work on. In the late 1960s the physicists became interested, and operator algebras became a popular field. So Ola's timing was excellent when he passed the Master's examination in 1971 with top grades. When it became clear to me how good the thesis was, I said to Ola that we had made a big mistake; this thesis should have been used for a doctoral degree. And it was precisely the results here ([1]) that made Ola well known as a mathematician from an early stage.

In 1959, James Glimm studied operator algebras that were achieved by taking an infinite union of an increasing family of $n \times n$ matrices. This became a famous piece of work, and Ola quickly discovered that he could generalize Glimm's work by studying infinitely increasing unions of direct sums of matrix algebras. He then ended up with an infinitely large diagram that described all the inclusions, which thus also described how the operator algebra was constructed. His main finding was that this diagram fully described the operator algebra, enabling a classification of all such operator algebras, now known as AF algebras. This result proved far more important than Ola and I had anticipated, and the figures are now called Bratteli diagrams.

After graduating, Ola studied for two years at New York University, where Glimm was. There was here an excellent environment in mathematical physics but also for cultural life and food. His appearance changed in some ways; much of his hair disappeared, his beard was gone, and he put on so much weight that when I met him later, I did not recognize him, so I introduced myself to him.

Ola returned home and took his doctorate in Oslo in 1974. After that, he went to Marseilles, where there was a very active and high-quality environment led by physicists who developed the theory of quantum physics in operator algebras. There he met Derek Robinson, and they started a cooperation that lasted for the rest of Ola's career. Robinson moved to Australia, and on one of Ola's journeys to Australia he stopped in Thailand. Here he was fortunate

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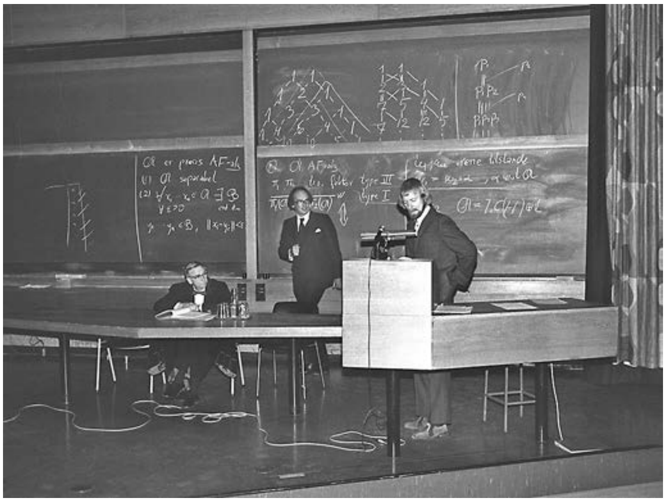

Figure 5. Ola's PhD defense, May 1974. From left: the dean, Ola, Gert K. Pedersen.

enough to meet Wasana, whom he married. Together, they had a good life.

Ola had many more coauthors and was altogether a very popular person to work with. Everyone liked Ola. When there were several of us together, he was not a man of many words, but in private he would talk. Ola radiated a good spirit, and it was easy to become fond of him. He was a person with a warm heart and a subtle humor that will stay with us for the rest of our lives.

Ola Bratteli passed away at the age of sixty-eight after several years of steadily declining health. It was an extremely sad experience to see how he had become weaker each time I saw him in recent years. His strength began to fail at a fairly early stage. His last research articles were published in 2008, and after that he had little energy to do more. So his brilliant career as a mathematician came to an end far too early. Ola will be remembered and missed for a long time, both as a mathematician and as the wonderful person he was.

Erling Størmer

Figure 6. Derek and Ola, 1988.

Life with Ola

Derek W. Robinson

I met Ola at the beginning of 1974 in Marseille. He was introduced to me by Trond Digernes. It was to be a pivotal moment in each of our lives, although we had no premonition of this at the time. There were many unpredictable consequences: a few years later Trond was to meet his wife, Hallie, at an open air opera in Sydney; Ola became the owner of a mushroom farm in northern Thailand; and Ola and I were to write a book that is still bought, read, and regularly cited thirty-five years later. At that time I was professor of physics at the University of Marseille, where, under the influence of the late Daniel Kastler, there was a strong visitors program in mathematical physics. This explained Ola's and Trond's presence.

My collaboration with Ola began the day we met. I explained to him some of the ideas I had about quantum dynamics and derivations on C*-algebras, and shortly after we wrote our first paper on these topics. By 1976 we had coauthored three other papers. When we met we had different interests, different backgrounds, and quite different personalities. Ola was quiet, well organized, and introspective, characteristics I did not share. Ola rapidly assimilated the Mediterranean lifestyle—the sun, the sand, and the seafood, especially the seafood. It was a time of calm cooperation amid the chaos of French academic life. In 1975 I began to think about writing a book on operator algebras and their applications in physics. I discussed the idea in spring 1976 with Ola, thinking that with our

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disparate backgrounds and successful working relationship we would be able to do justice to the subject and its recent developments. I was pleased he did not dismiss the idea immediately.

Initially the book was intended as a relatively short-term project: a monograph of 300–400 pages with the early chapters on mathematical background and the later chapters on applications to quantum statistical mechanics. We quickly realized that we would exceed the estimated length, so the short-term project turned out to be a long-term project, and the book changed from one volume to two.

We began each chapter with a tentative sketch of the intended sections. After discussing the general presentation of the material in each section, we began to draft alternate sections. We then exchanged drafts and edited each other's work. This process would be repeated until we were each satisfied with the outcome. We often had different notions of the relative significance of the material and the emphasis to be given to various statements and results. At times my first draft would be completely changed by Ola and vice versa. Somehow the process always reached equilibrium after a reasonably short time, with one exception, the section on modular theory. This took seven exchanges before we were both satisfied. This procedure had various advantages. It naturally introduced a uniformity of style. It also gave a fairly foolproof method of avoiding error, although we were not completely successful in that respect.

The first volume of the book was completed by September 1977, which left three months to complete the second volume of the book before I left France to take up a position in Australia. In that time we managed to write about 40 percent of Volume 2, aided by discussions with Akitaka Kishimoto, who had just arrived in Marseille as a postdoc. I returned to Marseille in June 1978, and work began again. I then returned to Sydney and Ola moved to Trondheim, but we returned to Marseille in June 1979 to tackle the final work. We completed the book by working nonstop for three and a half weeks. The second volume finally appeared in 1981, so the total operation of writing and publishing the

Figure 7. Lysebu, Oslo, 2017. From left: Aki Kishimoto, Palle Jorgensen, Derek Robinson, Tone Bratteli, George Elliott, Reiko Kishimoto, and David Evans.

Figure 8. Sjusjøen, Lillehammer, Easter 2004.

1000-page, two-volume book took about three years. That was not the end, however; we returned to it again twice, preparing the second edition, but that is a different story.

After the book was finished, our collaboration continued, with Ola visiting Australia almost every year. He was lucky to survive his first visit. Whilst touring the almost deserted roads of a national park, he reverted, European style, to driving on the right side, the wrong side for Australia. This led to a head-on collision that wrecked both cars, fortunately without any personal injuries.

Ola realized that by timing his Australian visits correctly he could ensure that it was almost always summer. It also had the advantage that we could spend a maximum of time working at the family beach house. Ola would then take his daily swim before heading to the local oyster shop. He bought freshly harvested oysters by the bag to be opened, seasoned with a lemon from our garden, and savoured as the sun went down.

"Those were the days my friend, I thought they'd never end."

Derek W. Robinson

Figure 9. Ola preparing bouillabaisse in Trondheim.

A Tribute to Ola Bratteli

Aki Kishimoto

Before I first met him in September of 1977, I must have read his early paper ([1]) of 1972 and his more recent series of papers on unbounded derivations, mostly written with D. W. Robinson, because his image had been firmly established in my mind as a formidable mathematician with whom I could hardly be compared. And he was, and I think I was quite lucky to come to know him in the earliest possible days, which brought me chances to collaborate with him for three decades.

Ola and Derek were writing the book in Marseille, a kind of book I love and could get familiar with, when I visited in 1977. Derek soon left for Sydney, but Ola and I spent almost a year together there. Though he was just one year my senior, he was a kind of mentor in mathematics and everything else during the stay. He also arranged for me to attend two conferences, sort of encouraging me against my timidity. In all these activities when transportation was required, Ola was in charge, which eliminated a practical worry for me.

In one outing I remember we hiked on rocky coastal paths, where I learned to mumble "Bonjour" to strangers we encountered. (Later I found this practice ubiquitous, and Ola would say "Konnichiwa" awkwardly when we hiked in Japan.) We were not sports types, but he was better than me in everything. So when we had to descend a steep slope made of gravel at one point, I was surprised to find myself rather enjoying skidding down while Ola tried to walk down steadily, as if reflecting his meticulous style of doing mathematics.

Ola's paper ([1]) on AF algebras became an inspirational source for me. This class might look rather special, but now we might say *if a C*-algebra is not obviously not AF, then it would be AF*. As a touchstone of this credo Ola and I examined the fixed point algebra F_θ of a C*-algebra $C_\theta(u,v)$ generated by two unitaries u,v with $uv=e^{2\pi i\theta}vu$ with θ irrational, under the period two automorphism σ: $\sigma(u)=u^{-1}$, $\sigma(v)=v^{-1}$. ($C_0(u,v)$ with $\theta=0$ still can be defined and comprises the continuous functions on the torus, while F_0 is on the sphere or rather a pillow with four corner points.) We managed to show F_θ, a noncommutative pillow, is AF (when θ is irrational). Another notable result in [1] was that any two irreducible representations of a simple AF algebra are bridged by an automorphism, which we established in a different setting, published as "Homogeneity of the pure state space of the Cuntz algebra" in *J. Funct. Anal.* **171** (2000). This gave me hope of successfully attacking a more general case, and I did it with some help from others.

Ola had many works; I cannot touch on all of them. Among them the books [3,4] are a good reference for those in the field of mathematical physics, me included. Our last work "Approximately inner derivations," *Math. Scand.* **103** (2008) with Derek is an attempt to shed light on a topic dealt with there, which yet haunts me to this day.

Aki Kishimoto

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Mathematics Collaborations on Three Continents

Palle E. T. Jorgensen

Ola Bratteli had a profound influence on modern analysis, especially themes connected with operator algebras, classification, noncommutative harmonic analysis, and

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representation theory. While Ola's first paper was solo, almost all that followed were joint.

In January of 2017, a weeklong conference was organized in Oslo, with the aim of presenting some of the many collaborative advances in mathematics and its applications involving Ola.

My own collaboration with Ola started by chance, dates back to the mid-1970s, and lasted for four decades. Our early work was in noncommutative geometry, and our later research moved in a diverse number of directions.

My own collaborations involved my visiting Ola in Oslo. In addition, we both made research visits to Derek Robinson, Dai Evans, and George Elliott. In all, I have thirty joint research publications (including two AMS Memoirs) with Ola, and a book.

Early joint research includes the themes Lie algebras of operators, smooth Lie group actions on noncommutative tori, and a study of decomposition of unbounded derivations into invariant and approximately inner parts. These topics are part of a systematic analysis of unbounded ∗-derivations as infinitesimal generators in operator algebras, with direct connections to quantum statistical mechanics. Other applications include noncommutative geometry, as envisioned by Alain Connes. Our later joint research moved more in the direction of representations and certain applications.

My most recent, and substantial, joint work with Ola was wavelets. That part includes a book *Wavelets through a Looking Glass: The World of the Spectrum*, which presents the subject from a representation theoretic viewpoint.

A common theme in my joint research with Ola is my insistence on the central role to be played by representations and decomposition theory. Some of our early work dealt with representations of Lie groups, of C∗-algebras, and of multiscale systems.

A quite different representation theoretic theme was the theory of numerical AF invariants, representations and centralizers of certain states on the Cuntz algebras, and a related but different study of combinatorial notions we called iterated function systems and permutation representations of Cuntz algebras. They play a key role in our understanding of such multilevel systems as wavelet multiresolution scales, in addition to multiband filters in signal processing.

Later joint work between Ola, me, K.-H. Kim, and F. Roush was inspired by Bratteli diagrams. It is known that the nonstationary case defies classification (order-isomorphism is undecidable), but we discovered that the stationary case could be decided by explicit classification numbers and associated finite algorithms. In this work, for the stationary dimension groups, we obtained explicit computation of numerical isomorphism invariants. We proved decidability of the isomorphism problem for stationary AF algebras and the associated ordered simple dimension groups.

Sadly, Ola's health declined towards the end, but I am happy to have had the benefit of many intense research experiences from the early part of our careers.

Palle E.T. Jorgensen

Ola and Orbifolds

David E. Evans

I first briefly met Ola in the new year of 1977 when I was a postdoc in Oslo. Later I got to know him very well, not only through our work and joint papers (fifteen altogether) but also through holidays taken together, particularly skiing ones in Rondane and Sjusjøen. He spent six months with me in Warwick in 1982, which was the start of our collaboration. Our first conference together was at Arco Felice, Naples, in March 1978. Ola and Aki drove there from Marseille in his Citroën deux chevaux. This was a meeting organized by Vittorio Gorini on Mathematical Problems in Quantum Theory of Irreversible Processes that brought together our mutual interests in derivations and generators of one parameter semigroups of positive maps, on which we would later collaborate. At a meeting in Chennai organized by Sunder, Ola took me on an expedition through the local markets to find saffron, which was later put to good use in his bouillabaisse, the finest I have ever had, with fresh fish from the harbor back in Norway.

In his PhD thesis ([1]), Ola classified AF C∗-algebras in terms of what are now known as Bratteli diagrams. This work was not only pivotal in Elliott's classification of AF algebras through *K*-theory; it is ubiquitous in operator algebras, dynamical systems, and in Jones's theory of subfactors.

My own work with Ola started with dynamical systems of one-parameter semigroups of positive maps in 1982. Later visits to Swansea led to the collaboration with George Elliott and Akitaka Kishimoto on noncommutative spheres, the irrational rotation algebra, and the *K*-theoretic obstructions to classifying such amenable C∗-algebras and their dynamical systems. This was particularly motivated by the remarkable construction by Blackadar (*Annals of*

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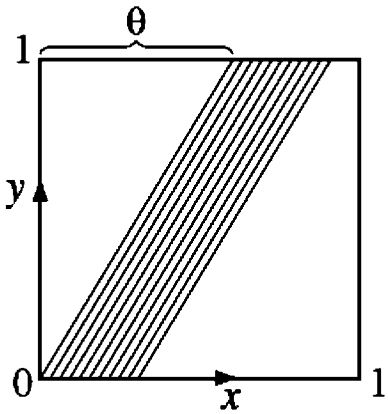

Figure 10. Kronecker flow on a torus. The flip on the irrational or Kronecker flow on the torus yields a nonsingular flow on the sphere which is zero-dimensional in a strong sense. The corresponding C*-algebra is approximately finite-dimensional.

Math, 1990) of a $\mathbb{Z}_2$-symmetry on the Fermion algebra with non-AF fixed point algebra, and the subsequent work of Kumjian in showing that a $\mathbb{Z}_2$-symmetry of a Bunce–Deddens algebra and the corresponding crossed product $C(\mathbb{T})\rtimes$ dyadic rotations $\rtimes\mathbb{Z}_2$ yielded an AF algebra. The issue of existence of such symmetries had been a well-known open problem.

This led us to consider the $\mathbb{Z}_2$-symmetry on the rotation algebra with the *flip* σ: $\sigma(u)=u^{-1}$, $\sigma(v)=v^{-1}$ on the generators. On the classical two torus, $\mathbb{T}^2$, this yields a singular orbifold $\mathbb{T}^2/\mathbb{Z}_2$ which can be thought of as a tetrahedron, topologically a sphere, but with four singular vertices. This led us to refer to the fixed point algebras and crossed products of rotation algebras as noncommutative spheres, but as Alain Connes pointed out, they are better described as noncommutative toroidal orbifolds, as they do not have the *K*-theory of a sphere.

Taking matrix-valued functions constrained at the singular points to have dimension drops and their inductive limits led us on a path towards studying group actions on approximately finite-dimensional AF algebras with non-AF fixed point algebras and crossed product algebras.

This way we also showed the existence of non-AF C*-algebras that when tensored with a certain UHF algebra (i.e., an infinite tensor product of matrix algebras) become UHF as well, that the fixed point algebra of an irrational rotation algebra by the flip is AF, and that the irrational rotation algebras are inductive limits of sums of matrix algebras over the continuous functions on a circle.

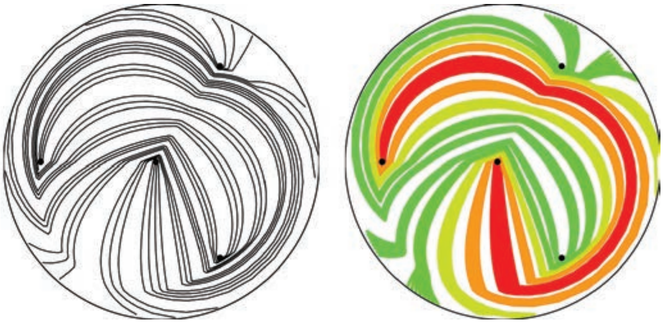

Figure 11. Non-commutative orbifold from folding the Kronecker flow on a torus.

The work described above laid the foundations for subsequent work over the last twenty-five years on the classification of amenable C*-algebras by *K*-theoretic data, before which the classification was out of sight and did not appear feasible. Our work on orbifolds also directly led to the study of orbifold subfactors (Evans and Kawahigashi, *Comm. Math. Phys.*, 1994), as reported in the first Danish-Norwegian Workshop on Operator Algebras at Røros in 1991.

Ola had a long connection and affection for Thailand, making regular visits; his wife for more than thirty years, Wasana, was from Phitsanulok. In 2016, Paulo Bertozzini initiated at Thammasat University in Bangkok the Ola Bratteli Mathematical Physics and Mathematics in Thailand Colloquium, where I was honored to give the first talk.

Ola had a generous spirit and integrity. We will miss his presence and friendship.

David E. Evans

ACKNOWLEDGMENTS. Editing has been done by Magnus B. Landstad, translations from Norwegian by Margaret Forbes, Bratteli diagrams by Petter Nyland, and John Erik Fornæss assisted with the submission. A preliminary (and longer version) of this article has been posted on https://arxiv.org/abs/1903.01524.

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Credits

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Memories of Goro Shimura

Don Blasius, Toni Bluher, Haruzo Hida, Kamal Khuri-Makdisi, Kenneth Ribet, Alice Silverberg, and Hiroyuki Yoshida

Goro Shimura, a mathematician who greatly influenced number theory in the second half of the twentieth century, was born in Japan on February 23, 1930. Over a career spanning six decades, he repeatedly made transformational discoveries that stimulated new lines of investigation and played a central role in the development of the field. Shimura earned his degrees at the University of Tokyo and held appointments at the University of Tokyo and Osaka University. He was a professor at Princeton University from 1964 until he retired in 1999. He authored numerous influential books and papers and was awarded a Guggenheim Fellowship in 1970, the Cole Prize in Number Theory in 1977, the Asahi Prize in 1991, and the Steele Prize for Lifetime Achievement in 1996. Goro Shimura passed away in Princeton on May 3, 2019, at the age of eighty-nine.

Figure 1. Goro, Chikako, Haru, and Tomoko returning to the US from Haneda Airport in 1971.

Don Blasius

Goro Shimura advised my 1981 Princeton thesis and he was, through his research and guidance, the central figure of my intellectual life in graduate school and for many years afterwards. His ideas about how to do mathematics have influenced me throughout my career.

Arriving at Princeton in fall 1977, I had no plan to study number theory, had never read a book about it, and had never heard of Shimura, Iwasawa, Dwork, Langlands, or even Weil. Most graduate courses had incomprehensible course descriptions. In this context Professor Shimura's offering of an introductory course in algebraic number theory stood out as a beacon of light. Algebra already had great appeal for me, and I was hooked right away by this perfect course. Later I took his courses on families of abelian varieties, special values of L-functions, period relations, the arithmetic theory of automorphic forms, Eisenstein series, theta functions, etc., all topics essential to his current research and, as it turned out, my initial research. He really taught for his students, not just to have a context to explore a topic of interest to himself. Each course started

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Communicated by Notices *Associate Editor William McCallum.*

DOI: https://doi.org/10.1090/noti2076

with minimal assumptions of background and went far into its subject with both clarity and an economy of means. He always had his lecture fully written out in a notebook, which would be on the table in front of him. He would lecture without it except for complex formulas. For them, he would pick it up to refresh his memory, always explaining clearly as he went. Shimura was always perfectly prepared and engaging.

By early in my second year, it was clear to me that I wanted Shimura to supervise my dissertation. When I queried him about it, he asked me to read his famous 1971 text *Introduction to the Arithmetic Theory of Automorphic Functions*. This book is a masterpiece of exposition, starting from scratch and giving, mostly with proofs, the key themes of his research up to that point. For me it was the perfect preparation for reading the articles on which it was based. It was a transformative undertaking. As an anecdote, I mention that when I told him I had finished Chapter 3, the one on Hecke algebras and the relation with L-functions, he asked me how long I had spent on it. I told him I had spent about a month, and he said that was "very fast" (meaning too fast). He was correct! Later that year, when I asked to be his student formally, he agreed but imposed the condition that I promise not to "fire" him. He explained that a student with whom he had been working for some time had just done that, and he was visibly hurt when he spoke about it. I told him that there was no chance of that.

Shimura did not tend, at least with me, to engage in lengthy, detailed discussion of mathematics. Usually, after a brief discussion of math, he would shift the conversation to something else, ranging from departmental gossip to Chinese stories. He did not give me a research problem right away. Instead, he asked me to read articles. The first was [3], mentioned below, and the second was [17]. Sometime while reading the second paper, he gave me the simple-sounding research problem concerning its main theme of relations between the periods of abelian varieties: "Find more precise results. Find natural fields of definition." This was a great problem because it connected with so many topics, including values of L-functions and the extension of the canonical models formalism from automorphic functions to automorphic forms, both new areas.

Before giving me this problem, Shimura had supported my desire to come up with my own. Several times I made suggestions to him, but he found compelling issues with them, such as being too hard or already done. When I finally found some results on his problem, he was happy and exclaimed "Isn't it nice to have some success?!" Later, I asked him about how he worked. He drew a really messy self-intersecting path on the blackboard and declared the end as his destination after the fact. In words, he told me he starts with a general idea but no fixed goal or conjecture.

Many people who did not know him have some imagination of Shimura as exclusively serious or severe. This is simply not true, and I mention two among many moments of wit that we both enjoyed. After I started a conversation with a remark about Hecke operators, he said, "The first thing you say to me is always interesting." And after I complained to him that I found his paper on confluent hypergeometric functions hard, he said, "You need to go see an analyst!"

About other mathematicians, he freely acknowledged his debts to Eichler, Hecke, and Weil, who was a friend for over four decades. He also had the highest respect for Siegel. Once he told me, "His proofs are correct and you can just use his theorems." He did not think that about many mathematicians. One day we discussed Weil's 1967 paper on the converse theorem. After a while he said, with intensity, "Weil is a genius." I never heard him say that about anyone else, even when discussing major works. About ideas, although he knew a great deal, he was something of a minimalist in his preferred way of writing. For example, when I referred to automorphic forms as sections of vector bundles, he queried me sharply as to whether this language added anything. I had to admit that, in the context, except for curb appeal to some, it did not. As a consequence, I avoided such usage in my dissertation.

I was asked to make a brief summary concerning Shimura's theory of canonical models and its antecedents. In 1953, at the start of his career, Shimura created ([1]) the first theory of reduction mod p of varieties in arbitrary dimensions. In a December 1953 letter to Shimura, Weil called it "a very important step forward" and emphasized its promise for the further development of complex multiplication. He also wrote that it was "just what is needed [to study] modular functions of several variables." The first direction became the famous collaboration of Shimura and Yutaka Taniyama, which was well underway by 1955. They defined and studied abelian (group) varieties of CM type and proved the Shimura–Taniyama reciprocity law, which describes explicitly the action of a Galois group on the points of finite order of the variety. The main underlying result here is a celebrated formula for the prime ideal decomposition of the endomorphism that reduces to the Frobenius morphism attached to a given prime of the field of definition. As a key application of this formula, they computed the Hasse–Weil zeta function at almost all places, thereby verifying Hasse's conjecture for such functions. Their research was summarized in the well-known 1961 monograph *Complex Multiplication of Abelian Varieties and Its Applications to Number Theory*, which Shimura wrote after Taniyama's death. In fact, Shimura wrote research articles about complex multiplication and abelian varieties over his career, even publishing in 1998 an expository monograph *Abelian Varieties with Complex Multiplication and Modular Functions*. This text includes more recent fundamental topics of his research, such as reciprocity laws for values of modular functions at CM points and the theory of period relations for abelian varieties of CM type.

From the late 1950s until the late 1960s Shimura made a continuing study, mostly published in the *Annals of Mathematics*, of the fields of definition for certain varieties defined by arithmetic quotients of bounded symmetric domains. For me, the start was the 1963 article "On analytic families of polarized abelian varieties and automorphic functions." In its first part, this highly readable paper showed that arithmetic quotients of three of the four classical domains arise, via normalizing period matrices, as analytic parameter spaces of the varieties of a given type. In 1966 he followed up with [7], which introduced the well-known notion of *PEL type*. He constructed a moduli space for each type as a model of an arithmetic quotient, thus providing a large supply of varieties whose points had definite meaning, and remarking of their fields of definition k_Ω: "In many cases we have verified that k_Ω is an abelian extension of K'." Here K' is a number field, frequently called the reflex field, which is central to the subject and which first arose in the Shimura–Taniyama theory. In 1964 in [4], he studied the varieties (quotients of products of upper half-planes) attached to quaternion algebras over a totally real field of arbitrary ramification behavior at infinite places. This paper introduced the cases where the fields of definition are abelian extensions of totally real fields. The algebraic varieties themselves had already been studied in [5] as moduli spaces associated to quaternion algebras over CM fields, in which case the canonical fields of definition are abelian extensions of the reflex field, a CM field. Thus in [4] it was a question of a further descent (Shimura used the term "bottom field"). In 1967, in [9] he considered further the cases of [5] where the dimension is one and computed, via a congruence relation analogous to Eichler's, the Hasse–Weil zeta function at almost all places, thereby proving Hasse's conjecture for the curves. These are the famous "Shimura curves." This article also introduces the notion of a canonical model as one uniquely characterized by an explicit description, obtained by virtue of the uniformization, of the Galois action over the reflex field on the images of CM points (see Main Theorem 1 of the article). In 1967 as well, a further paper [8] extended the canonical model theory of [9] to arithmetic quotients of higher-dimensional Siegel spaces. All these papers were written in the language of ideals. Finally, in 1970 this long development culminated with two articles, now famous: "On canonical models of arithmetic quotients of bounded symmetric domains I, II," both published in the *Annals*. In them, Shimura gave an adelic version of [8]. This viewpoint enabled him to define an action of the finite adeles (a way of introducing Hecke operators) of the associated reductive group on the system of models. He conjectured that such a theory would exist for any reductive group giving rise to a product of classical domains. Indeed he wrote, "The completion of this task does not seem so difficult."

Figure 2. Chikako and Goro in 1993 in Nagano prefecture, where they had a summer house.

Shimura's students Shih and Miyake each proved cases of Shimura's conjecture, and Deligne made major progress, as well as reformulating the theory in a general axiomatic way. In 1983 Borovoi and Milne constructed canonical models for all reductive groups of Hermitian symmetric type. They did this by proving a conjecture of Langlands that extended the reciprocity law at the fixed points to arbitrary automorphisms of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ instead of $\mathrm{Gal}(\overline{\mathbf{Q}}/K')$. This conjecture was based itself on Langlands's remarkable extension, later proven by Deligne, of the reciprocity law of Shimura and Taniyama for abelian varieties of CM type. Shimura himself did not really return to the theory after 1970, except, as mentioned, for extending it to automorphic forms in some special cases. Instead, the 1970s were for him a period of great and diverse achievements in new fields such as the theory of critical values of L-functions. Yoshida and Khuri-Makdisi mention these in their contributions, so I will stop here.

I miss Shimura deeply. Before beginning to write, I looked again at many of his articles that were so important to me. I fell again under the spell of my teacher and I found a problem to work on.

Toni Bluher

One of the reasons that Professor Shimura had so many graduate students was his extraordinary commitment to

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teaching. He gave masterful lectures that were very popular among the graduate students. We circulated among ourselves mimeographed copies of notes from prior years. Most influential to me was his course on Siegel modular forms, given during my first year of graduate school in 1984. Based on my experience in his course, I asked to be his graduate student and told him that I particularly enjoyed the material on theta functions. He remembered that comment and designed a thesis topic for me that included theta functions of half-integral weight. He prepared a series of readings that introduced all the concepts that I would need for my thesis topic, including some material on bounded symmetric domains, several of Shimura's articles, and Andre Weil's *Sur certain groups d'operateurs unitaires* (Acta Math **111**, 1964, 143–211). Everything fit together perfectly, and the thesis was progressing well. I got married in my second year of graduate school and had a son in September of my fourth year. Professor Shimura was supportive and nominated me for a Sloan Scholarship, which relieved me of teaching duties and made it possible to focus on writing my thesis. Being Shimura's student was akin to hiking Mount Everest with a skilled guide—he cleared the path so that we could reach the summit. I remember him saying that he would have preferred an approach that would give us more time to read and gain perspective, but he adopted his style because at that time it was expected that graduate students should finish in four years, something that is hard to do in such a technical field.

I have fond memories of when Professor Shimura invited me, my husband, and other graduate students to his house on a few occasions and also a dinner party where we met Andre Weil. At one of those occasions, I learned about his sense of humor. He said that puns are not part of Japanese culture, and he could not see how they were funny. "So what is funny to you?" I asked. He told the following joke. Some mosquitos were annoying the guests, so the host said he would take care of it. He put out a bowl of sake and many small pieces of tissue paper. "How will this help get rid of the mosquitos?" the guests asked. The host replied that the mosquitos would drink the sake and then fall asleep. "Ah, very clever," said the guests, "and what is the tissue for?" "It's so that the mosquitos will have a place to rest their heads after they drink the sake!"

Haruzo Hida

In the mid-1970s, I was a senior undergraduate at Kyoto University and had just started learning mathematics.

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Somehow I had met Professor Doi in Kyoto slightly earlier, and because of a friend of mine (an ardent mathematics addict) I had started reading mathematics books at college level and above. This day, following a suggestion of Doi, I planned to attend a lecture by Professor Shimura at Tokyo University of Education. I took a bullet train in the morning and reached Tokyo about an hour before the lecture. In the lecture, he talked about CM abelian varieties and their fields of moduli. I understood the content well, as I had already read his red book and the English version of the book he coauthored with Taniyama. Because of my late start, I was obsessively reading, as quickly as possible, many mathematics books. I did well, giving myself good background knowledge of analysis, algebraic and analytic number theory, and algebraic geometry, including the viewpoints of both Weil and Grothendieck. Prior to this point in my life, I had read a great deal but mostly to have fun. Reading books, including Chinese classics that Shimura loved, had been my way of life.

After the lecture, senior PhD students were invited into a smaller room to pose questions to the speaker. Out of curiosity, I walked into the room also. As Shimura had a rare charisma, at first nobody dared to ask him questions. This seemed impolite to me, so I started asking some well-posed questions about CM abelian varieties. This went well, and he answered me, looking directly into my eyes, treating me correctly as a novice, and emphasizing the importance of studying periods of CM automorphic forms. Then there was another awkward silence, so I decided to ask a somewhat imprecise question about a minimal field of definition of a given CM abelian variety, as it seemed to be related to some results on the field of moduli that I had seen in a preprint of his that Doi had received ([15]). Once I had described the question (saying that Doi allowed me to see his new results), he got excited and gave me a terse reply, saying that you could ask questions about facts, but trying to get some "guess" or some "way" towards a new result from somebody else is not morally sound. You should think about them on your own and should find a way out. His last words were, "Do your own mathematics."

I am from a family belonging to a traditional commercial class of people in Osaka-Kyoto. My family had been successful in banking from the late shogunate era through the Meiji Restoration. In Japan, we had a "rice exchange," which was analogous to a European stock exchange, as early as the late seventeenth century. There were "rice stocks" (*kome-tegata*) that were traded at the exchange by banking officials, and the price of rice, including that collected as tax by each feudal province, was determined nationally at the exchange. Banking (called *Ryogae*, literally "money-exchange") was a prosperous business for three or four centuries. Even though Japan was governed by a combination of the shogun's samurai (principled and well literate in Chinese classics) and the emperor's aristocrats (often indulging in subtle poems with hidden meanings), the economy at the time was essentially an unofficial capitalism. This is one of the reasons Japan was able to modernize itself so quickly in the colonial period of other Asian nations. For people in banking, a deep understanding of the governing samurai class is fundamental for their business. So most children of either gender in such a family had a sort of private tutor/nurse/nanny to train them how to divine the undercurrent thinking of people. (I mean, training so that your front personality can make friendly contact while at the same time your rear personality can delve into the counterpart's intents, often by posing questions that appear innocuous.) I had learned this type of slightly schizophrenic approach to people. Thus, when Shimura made his reply to my question, I was calm, but inside I was quite amused by his excitement (as I was looking for a way to cope well with him). I decided to avoid henceforth indulging myself too much in unfounded thoughts with him and instead to focus on asking him well-posed, maybe conjectural, questions and to present him with new ideas, not necessarily in mathematics. I made a firm note in my mind that this should be my way to cope with his principled personality.

Immediately after his lecture, I stopped my indulgence of reading mathematics broadly and focused on books whose content I felt I really needed. This freed my time, and I started writing a research article that became my first paper, published in 1978. In March 1976, there was a Takagi anniversary conference at RIMS, and at it I had a short conversation with Shimura without much content (though he remembered me well). I felt shame that I could not produce something entertaining to him by this time, although I had the seed of an idea of creating complex multiplication on the complex torus spanned by CM theta series in the middle degree Jacobian of the Hilbert modular variety. I finished this project a year after the conference. Doi had left for the Max Planck Institute for Mathematics just after the conference, and in April 1976, I entered the graduate school of Kyoto University. I was next to meet Doi again in Sapporo only two years later, so I was alone. Fortunately, Hiroyuki Yoshida returned to Kyoto at this time after his PhD study with Shimura in Princeton, and he had a good understanding of Hilbert modular varieties. In fact, while working on this problem, I talked only to Yoshida. When I was ready, I wrote about my results to Shimura at Princeton and, surprisingly, this attracted him. Indeed, the work suggested that higher-dimensional periods of Hilbert modular CM theta series are somehow related to periods of CM elliptic curves, at least if the base totally real field has odd degree. Here I should mention that CM period relations were a main topic of Shimura's research at the time. I knew this conjecture, but I did not explicitly write it in the letter or in the published paper. In any case, I got a job at Hokkaido University with the help of Doi, who moved there after his trip to Germany. For my next project, at Hokkaido, I classified CM factors of Jacobians of Shimura curves, extending Shimura's work for the case of modular curves. I thus, with Shimura's support, was offered a one-year visit to the Institute for Advanced Study.

I arrived at Princeton in August 1979, and right away I called Shimura on the phone. He invited me and my wife to a dinner at his home. At the dinner, he asked us a funny question: Why do Osaka people use the plural form *oko-tachi* in referring to an only child? The part "tachi" is a plural indication, although the Japanese language does not have a systematic plural form. My answer was that for a family in commerce, having several children is more desirable than having one, and hence the talker is apparently showing friendship by way of courtesy. He was not at all convinced, giving me a couple of counterexamples from the usage found in Kyoto aristocracy. This was typical in his conversation. He would come up with totally unexpected questions, and if one's answer was off the mark, he used it as a seed-topic of often poignant stories he loved to talk about. My answer could be wrong but not bad either (at least not provable in either way). After this conversation, he started calling me *Haruzo-san* and told me to call him *Goro-san*, which I never did in our conversation. (I called him, to his dismay, always *Sensei*.) I was told at the dinner to come to see him in Fine Hall at tea time, that is, every Thursday at 3 pm.

I kept busy every week to concoct something new to tell Sensei on Thursday. If I had not done that, I would have had a hard time listening to his short stories. To cope with them I needed all my skill of conversation. Perhaps he was doing this intentionally to pull out the most from me. I was fairly successful in the first year, and I wrote three papers, later published in *Inventiones*. By these, I think, I earned an extension to stay at the IAS for a second year. This year was difficult, although I had a conjectural idea about p-adic deformation of modular forms and the use of the Hecke algebra to make something like a GL(2)-version of Iwasawa theory. They were fuzzy thoughts at the time and only once or twice useful in our conversation. Repetition of topics, without much progress, was not a useful strategy for conversation with Sensei. But if you are an entertainer hired by somebody, you need to produce an attraction every time you perform! Thus, I had to skip the meeting several times. Fortunately, I eventually came up with a use of the partial Fourier transform to compute q-expansions of orthogonal and unitary Eisenstein series, as well as some series that Shimura had invented (I call them Shimura series). Then I went to the tea. The first words he threw at me were "I thought you are dead." I replied to him, "Like the characters 'Yosaburo and Otomi' of a Kabuki play (Japanese opera), somebody's survival could not be known even to Buddha." This amused him. I recovered, and I was to find a place twenty-five years later for this computation, in the case of Siegel's theta series, in my article in the Coates volume of *Documenta Math*. dealing with the anti-cyclotomic main conjecture.

Those were demanding but happy days for me at IAS. Shimura was able to squeeze out of me every bit of mathematics I potentially had. I still have a good stock of usable results from the notes at the time. He did not teach me much mathematics, but he guided me how to pull out something useful from my own mind, not from books or articles somebody else wrote. I am grateful for his unusual effort for my development. Farewell to his existence, which was so richly difficult, rewarding, and fun for me. *Sayou-nara*! (literally, "if things have gone that way, we part").

Kamal Khuri-Makdisi

Personal memories. Goro Shimura was my thesis advisor in the early 1990s, and the relationship developed into friendship over the subsequent years. While I was his graduate student, he gave me full support and mentorship, patiently guiding me through first reading a number of his articles in preparation for my thesis, then the actual thesis work. Our weekly meetings would usually last one to one and a half hours, during which he was unstintingly generous with his attention and advice. When, as a result of my youthful inexperience, I made a naive mathematical speculation, or was mistaken about a certain point, he would diplomatically correct me with the phrase, "That is completely correct, but...." He also regularly exhorted me during my thesis and subsequent career choices to be "practical," especially in terms of finding research subjects to work on that were both attainable and interesting. He was so aware of my progress that when I stalled for a while on my thesis, he was able to diagnose the problem without my having told him precisely what I had been stuck on. He simply presented me one day with a few pages of notes where he explained what I had most probably overlooked (an issue where Maass-type Hilbert modular forms could be either odd or even at each archimedean place, which led to different constructions). His advice was of course right on the mark, and he included in his notes suggestions on

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Figure 3. Chikako and Goro with Kamal Khuri-Makdisi in Beirut in May 2000.

how to overcome the blockage.

During my time at Princeton, Shimura's graduate courses alternated between lectures at the introductory or more advanced level and seminars where students had to present material out of his 1971 book *Introduction to the Arithmetic Theory...* or from terse notes of his on L-functions of modular forms and Artin L-functions. I have kept these notes preciously over the years. At the end of the semester, particularly with a seminar, he would invite the students over to his house for a "dinner in lieu of final exam," an occasion to be more informal than in class.

Shimura had a real interest in art, with an impressive collection of both prints and porcelain, the latter not only from East Asia but also the Middle East; his nonmathematical writings include a book on Imari porcelain. He and his wife Chikako traveled several times to the Middle East, visiting Turkey a few times (K. Ilhan Ikeda and I did our PhDs with him at the same time), Iran for a conference, and Lebanon on two occasions, when it was a real pleasure to be able to host Goro and Chikako. He managed to combine the mathematical aspect of his trips with visits to museums and archaeological sites, plus the inevitable antique shops.

Some aspects of his mathematical legacy. Shimura's mathematical contributions are so fundamental and wide-ranging that no one person can write about them all. I will go over many important topics too quickly and will skip others altogether. I hope that this discussion can at least do justice to some fraction of his work.

Shimura knew thoroughly the earlier work on modular forms by Hecke, Siegel, Maass, Petersson, Fricke, and Weber, among others. He had also carefully studied Lie (and algebraic) groups from Chevalley's book, as well as algebraic geometry in the language of Weil's *Foundations*. By the late 1970s, though, his articles tended to contain less algebraic geometry and more analysis. He never worked explicitly in the language of automorphic representations but was very happy to move between the adelic language and explicit computations with real or Hermitian symmetric spaces and arithmetic groups. In general, his articles are complete, thorough, and very demanding to read in terms of the intricacy of the calculation. However, all the material is there, with very few errors (which usually get corrected in errata at the end of a subsequent paper). So a patient and determined reader can make it to the end but must just be prepared for a slow rate of progress per page.

Readers in number theory will be familiar with Shimura's foundational contributions to the arithmetic of modular curves and abelian varieties, such as the decomposition of the Jacobian of a modular curve. The section by Blasius in this memorial article summarizes his introduction of what are now known as Shimura varieties and their canonical models. I will only mention Shimura's important insight that in the non-PEL case one can use the CM-points to pin down the rationality and produce a canonical model over the "correct" number field.

Another famous contribution by Shimura is in the area of modular forms of half-integral weight, beginning with [11]. Shimura studied many aspects, not just over $\mathrm{SL}(2, \mathbf{Q})$ as in the first paper above, but also over symplectic groups over totally real fields, so in the Siegel–Hilbert case. Over $\mathrm{SL}(2, \mathbf{Q})$, as is well known, Shimura showed that to a Hecke eigenform f of half-integral weight $k = m + 1/2$ there corresponds a Hecke eigenform g of (even) integral weight $2m$, so on $P\mathrm{GL}(2, \mathbf{Q})$, with matching Hecke eigenvalues. Shimura's original proof of this went via constructing the L-functions of twists of g by Dirichlet characters and then invoking Weil's converse theorem. Later, after work of Shintani and Niwa and with further hindsight, this "Shimura correspondence" was recognized as an early example of a theta-correspondence, here between the double cover of SL(2) and $O(2, 1)$, which is essentially the same as $P\mathrm{GL}(2)$. Shimura revisited his correspondence from this point of view in [21] and subsequent articles for the Hilbert modular case, and describes the theta-correspondence viewpoint over $\mathbf{Q}$ in a readable account for students in his last book, *Modular Forms: Basics and Beyond*, published in 2012. As another result using half-integral weight on SL(2), Shimura was the first to prove the remarkable result [13] that the symmetric square L-function of a classical modular form has an analytic continuation to $\mathbf{C}$. Prior to that, one had only a meromorphic continuation with possible poles at all the zeros of the Riemann zeta function or a Dirichlet L-function. This proof used a careful analysis of Eisenstein series of half-integral weight. Shimura of course studied many other aspects of half-integral weight on larger groups, including but also going well beyond questions about the behavior of Eisenstein series, as part of his large program on arithmeticity, which was a large focus of his work from the mid-1970s through the late 1990s.

Before I mention Shimura's work on arithmeticity, however, I will briefly mention his significant production of books and articles from the mid-1990s until 2012, during his retirement ("only from teaching," he once told me). In a 1997 monograph, *Euler Products and Eisenstein Series*, he broke new ground in the explicit construction of L-functions using essentially the "doubling method" of Gelbart, Piatetski-Shapiro, and Shalika, obtaining all Euler factors and gamma factors. Also, in two monographs, *Arithmetic and Analytic Theories of Quadratic Forms and Clifford Groups* (2004) and *Arithmetic of Quadratic Forms* (2010), Shimura obtained new results in the theory of quadratic forms and new explicit forms of the celebrated Siegel mass formula. He further summed up and refined in monograph form many strands of his earlier work that had previously appeared in articles: his work with Taniyama from the 1950s on complex multiplication in *Abelian Varieties with Complex Multiplication and Modular Functions* (1998); elementary and less elementary topics in modular forms, including the Shimura correspondence and the simplest cases of arithmeticity in the books *Elementary Dirichlet Series and Modular Forms* (2007) and *Modular Forms: Basics and Beyond* (2012); and a more comprehensive treatment of his program of arithmeticity in *Arithmeticity in the Theory of Automorphic Forms* (2000).

Shimura's program on arithmeticity, a large focus of his work from the mid-1970s onwards, can be viewed as a very large and elaborate outgrowth of the two seminal articles [12], [14]. I shall single out two themes: from the first article, the arithmeticity of the values of nearly holomorphic modular forms at CM points, and from the second, the arithmeticity of special values of L-functions of modular forms, and the relation of these special values to more fundamental periods attached to the forms.

The first theme above generalizes Shimura's reciprocity law for holomorphic modular functions at CM points to certain nonholomorphic functions, where the relation to algebraic geometry is less direct. In the classical context, a nearly holomorphic form is a function $f : \mathcal{H} \to \mathbf{C}$ on the usual upper half-plane which transforms as expected under a congruence subgroup of $\mathrm{SL}(2, \mathbf{Z})$. Instead of requiring f to be holomorphic, we require $f = \sum_{n=0}^{N} f_n(z) y^{-n}$, where $y = \mathrm{Im}(z)$ and the f_n are holomorphic. (Actually, for arithmeticity reasons, it is better to use $(\pi y)^{-n}$.) A typical example is the Eisenstein series $E_2 = (8\pi y)^{-1} - (1/24) + \sum_{n \geq 1} \sigma_1(n) q^n$. One can also obtain nearly holomorphic forms by applying certain differential operators to holomorphic forms. Shimura introduced an ingenious way to evaluate such an f at a CM-point $z_0 = (a + b\sqrt{-D})/c$, by comparing f and $f|\alpha$ for an element α that stabilizes z_0, and combining this with taking various derivatives. The generalization of this to larger groups is of course more involved.

The second theme, in the setting of the arithmeticity of the special values of standard L-functions of classical modular forms, can be studied in terms of the Eichler–Shimura cohomology groups or (as formulated by Manin) in terms of modular symbols. The new approach in [14] is quite different and allows for a generalization to many other groups and L-functions. In the classical setting, let f be a newform (in Shimura's terminology, a primitive form). Instead of considering a single (twisted) special L-value $L(k, \chi, f)$ for a Dirichlet character χ, Shimura considers products of two such special values, which he obtains via an integral of Rankin–Selberg type as $L(k_1, \chi_1, f)L(k_2, \chi_2, f) = \langle f, G\rangle$ for an explicit modular form G. Here G is a product of two Eisenstein series and can be expanded as $G = c_E E + \sum_i c_i g_i$, where E itself is an Eisenstein series, and the g_i are cuspidal Hecke eigenforms (not necessarily newforms; one can have $g_i(z) = h_i(N_i z)$ for a newform h_i). Since G, E, and the h_i have algebraic Fourier coefficients, the c_i and c_E are themselves algebraic, and then one obtains (after some more work) an algebraic expression for $L(k_1, \chi_1, f)L(k_2, \chi_2, f)$ in terms of those c_i where $h_i = f$. This is the heart of the idea in the classical case, and it generalizes somewhat directly to Hilbert modular forms. However, for the generalization to larger groups and other L-functions, one requires two significant inputs: first, a good understanding of the analytic (not just meromorphic) continuation of Eisenstein series on larger groups, with a precise proof of arithmeticity of their Fourier expansions, and, second, once again a thorough understanding for higher groups of the differential operators that already appeared for evaluation at CM-points. (The differential operators are needed even in the classical case but are more tractable there.) Tackling both of these questions in more general settings involved a large body of work by Shimura (and his students, in their theses) over some twenty-five years, and the computations are quite intricate. Besides the books mentioned above, *Euler Products and Eisenstein Series* (1997) and *Arithmeticity in the Theory of Automorphic Forms* (2000) and their references, I will single out the articles [19], [20] as a memorable illustration of the careful study that Shimura was able to undertake of the analytic continuation and Fourier expansions of Eisenstein series in integral and half-integral weight. The difficulty resides largely in the number theory, but there are also genuine analytic challenges in terms of special (confluent hypergeometric) functions on the symmetric spaces of these higher-rank groups.

Figure 4. Goro Shimura and Kenneth Ribet in the summer of 1973.

Kenneth Ribet

I first saw Goro Shimura at the Antwerp conference Modular Functions of One Variable in 1972. I was a graduate student at the time and was shy around senior mathematicians. Some of the faculty members at that conference encouraged informal contact with graduate students, but Shimura's body language did not convey any encouragement. I was further intimidated by a comment that was made by one of the Princeton graduate students: "Many mathematicians have knowledge. Shimura has wisdom."

Twelve months later, I had finished my dissertation work and was about to begin a lecturer position at Princeton. Shimura welcomed me when I arrived in the Princeton math department and gave me tips for dealing with students. In particular, he recommended that I respond to students' questions by writing down answers in their notebooks so that they would have easy access to my comments even after their memories of my remarks had faded.

Shimura then asked me a mathematical question that turned out to be extremely fruitful. Specifically, let N be a prime number and let J be the Jacobian of the modular curve $X_0(N)$ that classifies degree N isogenies between elliptic curves. Then J is an abelian variety over the field $\mathbf{Q}$ of rational numbers. Moreover, J has endomorphisms T_n $(n \geq 1)$ that are geometric versions of the Hecke operators on modular forms that had been studied by Hecke and Shimura. These endomorphisms generate a subring $\mathbf{T}$ of the ring of endomorphisms of J over $\mathbf{Q}$. Shimura asked me whether $\mathbf{T} \otimes \mathbf{Q}$ is the full endomorphism algebra of J.

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Figure 5. Goro, Chikako, and Haru at the Ukiyo-e Museum in Matsumoto, Japan.

I found first that $\mathbf{T} \otimes \mathbf{Q}$ is the algebra of all endomorphisms of J that are defined over $\mathbf{Q}$, but this was not news to Shimura. He asked me whether or not there were endomorphisms of J over the algebraic closure of $\mathbf{Q}$ beyond the ones that are already defined over $\mathbf{Q}$. Using results that had been obtained by Deligne and Rapoport a year or two before, I proved that all endomorphisms of J are defined over $\mathbf{Q}$. Technically, I used the theorem of Deligne–Rapoport to the effect that J has semistable reduction at the prime N (and thus at all primes, because J has good reduction outside N); my result was really about abelian varieties with semistable reduction.

Shimura was delighted by my result and asked me to explain my proof to him in detail. In response, he wrote down a polynomial identity that simplified the main computation that I had presented to him. Shimura then suggested that we write a joint article with the result. Perhaps selfishly, I told him that I was reluctant to write a joint paper because I had never yet published any mathematical article. Shimura accepted my answer and encouraged me to publish the result on my own. I did so—and credited Shimura for posing the original problem and for the simplification that he made to my argument. I was grateful to him for this act of generosity, but now worry that I was wrong to decline his offer.

By the way, my theorem shows that $(\operatorname{End} J)/\mathbf{T}$ is a torsion abelian group. Here, $\operatorname{End} J$ is the full ring of endomorphisms of J, and $\mathbf{T}$ again is the subring of those endomorphisms that come from Hecke operators. In 1977, Barry Mazur proved that the quotient $(\operatorname{End} J)/\mathbf{T}$ is *torsion free* in his "Eisenstein ideal" article. Our results together imply that the quotient is *trivial*, i.e., that $\mathbf{T}$ is the full ring of endomorphisms of J.

Alice Silverberg

The way I became a PhD student of Goro Shimura was a bit unusual. Sometime in my first year, I asked Nick Katz to be my thesis advisor. He told me to talk to him after I passed my General Exam in the spring. After the exam, Katz went away for the summer. At a conference early that summer, I ran into John Coates, who asked me whom I planned to work with. When I said Katz, he exclaimed, "But Alice, you can't possibly work with Nick Katz! He won't be at Princeton. He's accepted a job at Berkeley."

I had no way to contact Katz to check this (and to find out that it wasn't correct). So I decided that I had better have a back-up plan. Former students of Professor Shimura had told me that the first thing he tells prospective students is to read the red book, *Introduction to the Arithmetic Theory of Automorphic Functions*. That summer, I started to read the book and do the exercises.

In the fall, I needed to talk to the director of graduate studies about an administrative matter. That happened to be Katz. One day after tea I followed him to the elevator and asked to talk with him. As we waited for the elevator he turned to me and said, "So are you my student or aren't you?" Based on his tone of voice, I reflexively responded, "No, I'm not." Then he asked me who my advisor was. Without thinking I blurted out "Shimura." Then, horrified at the thought that he might ask Shimura and find out it wasn't true, I hurriedly added, "But he doesn't know it yet!"

Luckily, Shimura agreed to be my advisor.

A former student told me that his experience was that he would bring a notebook to his meetings with Shimura. Shimura would write a problem in the notebook and ask the student to solve it for their next meeting. If the student didn't solve it, Shimura wrote the solution in the student's notebook. My experience was very different; I worked very independently.

Shimura suggested a nice thesis problem, and I went away and worked on it. He then took the problem away from me. I heard through the grapevine that Shimura had given the problem to a former student to do for his thesis, but the student hadn't solved the problem then and now wanted it back.

Next, perhaps to make up for the time I had spent on the first problem, Shimura gave me a choice of three problems. When I chose the one furthest from his interests at the time, he was pleased with my choice.

Shimura gave me a first step to solve and told me to come back in about two weeks to report on my progress.

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By the end of those two weeks I managed to understand the question, but I hadn't made progress towards a solution. I didn't feel that was good enough, so I didn't arrange to see him and kept working. After another two weeks I had answered the question, but I didn't feel I could show up after four weeks having only accomplished what I should have done in two. So I kept working and made more progress, but it never seemed like enough, given the time I had spent on it. After a few months, I realized that I needed to let my advisor know I was still alive, so I met with Shimura and told him what I had accomplished. He could have been angry that I hadn't kept him informed. Instead, and luckily for me, he was pleased with both the work I had accomplished and my independence.

If I were restricted to one word to describe Shimura as a thesis advisor, I would say that he was "responsible." That's higher praise than it sounds; conscientiousness seemed like a rare and unusual trait among thesis advisors when I was a Princeton graduate student.

As my interests moved away from automorphic forms and into cryptography, I found that I continued to use Goro's work, especially the theory of complex multiplication, which is useful for modern-day cryptography. I also used Shimura reciprocity to help construct an algorithm related to point counting on elliptic curves over finite fields.

Goro Shimura had very high standards. I do best when the standards for me are high, so I am very grateful to Goro for having high standards for me, for telling me that a mathematician *must* be an optimist, and for believing in me as a mathematician. While he didn't often communicate that he thought highly of me, he did it enough (to both me and mathematicians who made decisions about me) to have a positive effect on my life and career. I will cherish the memories of our mathematical father-daughter relationship.

Hiroyuki Yoshida

The impact of student protests in Paris in May 1968 spread to the world, and Kyoto University in Japan was swallowed up in big waves from the beginning of 1969. No courses were offered to students for one year. In the autumn of 1969, I visited, with classmates, Professor Hiroaki Hijikata, who was then a young associate professor, in his office to ask him to give us a seminar on number theory. I was a senior mathematics major at Kyoto University. Hijikata asked me what I was interested in. I responded, "Complex multiplication." I knew at that time, without precise understanding, the legend of Shimura–Taniyama–Weil on complex multiplication and Shimura's work on Shimura curves. Hijikata selected some suitable literature for a seminar, and I was able to proceed rather quickly and learned that Shimura was developing the theory of higher-dimensional arithmetic quotients of bounded symmetric domains. Hiroshi Saito also attended Hijikata's seminar.

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Figure 6. Group photograph taken at the NSF-CBMS conference at Texas Christian University in honor of Shimura in May 1996.

Next summer, Hijikata, through his friend Professor Yasutaka Ihara, introduced me to Shimura. In the spring of 1971, I was admitted to graduate school in mathematics at Princeton University with a scholarship, to start in the autumn. That spring, Professors Koji Doi and Hidehisa Naganuma came back to Kyoto from IAS in Princeton. I first met Professor Goro Shimura on July 14, 1971, on the Ishibashi campus of Osaka University. Shimura was invited by Professor Taira Honda to give a lecture in Osaka. My first impression was that he resembled the famous philosopher Kitaro Nishida of the Kyoto school.

Talks with Shimura in person. I arrived at Princeton on September 13, 1971, and met Shimura the next day in his office, and he kindly took me to his home. Shimura had just published his now standard textbook *Introduction to the Arithmetic Theory of Automorphic Functions*. I asked him, "What is your present interest?" He replied that he was studying modular forms of half integral weight. He hinted that he had discovered a relation between modular forms of half integral weight and modular forms of integral weight. A preprint became available only the next spring, and now this relation between modular forms of half integral weight $(2k+1)/2$ and modular forms of integral weight $2k$ is called the *Shimura correspondence* ([11]). A technical core of the proof is an ingenious application of the Rankin–Selberg convolution and Weil's converse theorem.

We also talked about complex multiplication of abelian varieties and construction of class fields. Though an abelian variety A has (sufficiently many) complex multiplications by an algebraic number field K, the class field is obtained over the reflex field K', which is different from K in general. Shimura explained this curious phenomenon, first discovered by Hecke in a simple case, as follows. Suppose that A is defined over a subfield k of $\mathbf{C}$. Then the field generated by the division points of A is determined as a subfield of $\mathbf{C}$. But K is not determined as a subfield of $\mathbf{C}$; only its isomorphism class has a definitive meaning. In contrast to this, the reflex field is determined as a subfield of $\mathbf{C}$ from K and the CM-type of A. I felt a revelation and got deeply interested in this "thought experiment."

In the summer of 1972, Shimura took me to the Antwerp conference on modular functions of one variable. One afternoon, we went out for sightseeing in the city. I was impressed that Shimura very efficiently found a route to visit museums and places of interest with a handy map. This experience turned out to be very useful for my travels in later years. He went back to his hotel after buying some paper of good quality for his calculations.

I received a PhD in 1973 under Shimura's guidance and stayed in Princeton as a postdoc until August of 1975. In 1975, when I was preparing to leave Princeton for Kyoto, Shimura was studying critical values of L-functions associated with modular forms. This time the Rankin–Selberg method was employed again. The basic formula is

$$(4\pi)^{-s}\Gamma(s)D(s,f,g) = \int_{\Gamma_0(N)\backslash H} \overline{f(z)}g(z)E_{k-l}(z,s+1-k)y^{s-1}dxdy. \quad (1)$$

Here s is a complex variable, H is the complex upper half-plane, and $z \in H$ is written as $z = x+iy$, $y>0$, $x \in \mathbf{R}$; $f(z) = \sum_{n=1}^{\infty} a_n q^n$ and $g(z) = \sum_{n=0}^{\infty} b_n q^n$, with $q = e^{2\pi i z}$, are holomorphic modular forms with respect to $\Gamma_0(N)$ of weights k and l, respectively, with $k>l$. For simplicity, we assume that the characters of f and g are trivial (Haupttypus). For $0 \le \lambda \in \mathbf{Z}$, $E_\lambda(z,s)$ is an Eisenstein series defined by

$$E_\lambda(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (cz+d)^{-\lambda}|cz+d|^{-2s},$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \middle| \, m \in \mathbf{Z} \right\}.$$

Our first objective is to study special values of $L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$. Suppose that f is a Hecke eigen cusp form. Then there exist periods $u^{\pm}(f)$ such that $L(m,f)/\pi^m u^{\pm}(f)$ is algebraic for $1 \le m \le k-1$, $(-1)^m = \pm 1$, $m \in \mathbf{Z}$. (A cohomological approach was discovered by Shimura fifteen years earlier ([2]).)

Moreover, $u^+(f)u^-(f) = i^{1-k}\pi\langle f,f\rangle$, where $\langle f,f\rangle$ is the normalized Petersson norm of f. Shimura stressed the importance of $D(s,f,g)$ and not only $L(s,f)$. Suppose that g is also a Hecke eigen cusp form. Then $\zeta(2s+2-k-l)D(s,f,g)$ has an Euler product of degree 4 and may be denoted as $L(s, f\otimes g)$. Then $(2\pi)^{l-1-2m}L(m,f\otimes g)/u^+(f)u^-(f)$ is algebraic for $l \le m < k$, $m \in \mathbf{Z}$ ([14], [16]). It is very interesting that the form of higher weight gives the dominant contribution. Shimura discovered an ingenious but now standard way to deduce these results from (1).

Shimura was a great master of using the Rankin–Selberg convolution to squeeze arithmetical information from it. In the half integral weight case mentioned above, f is a form of half integral weight, g is a classical theta function, and E is an Eisenstein series of half integral weight in (1). In the later years, Shimura generalized the method to higher-dimensional cases.

In the summer of 1985, Shimura suddenly called me. (In the meantime, I was communicating with Shimura by occasional letters.) He told me he had come to Kyoto with his wife but with no business. His wife was visiting her friend. So we went to Kurama temple for sightseeing. In the bus from his hotel to the local train station, I talked about the then-popular movie *Amadeus*. I talked about the sad destiny of Salieri's music. Then he said, "I am Mozart" immediately. Shimura also declared at this time that the modularity conjecture for elliptic curves over $\mathbf{Q}$ was his. Shimura explained to me some interesting work of his recent PhD students. The name of Don Blasius was among them. I think Shimura wished to give me some stimulation. We parted this time after he promised to visit Kyoto University in the summer two years later.

Shimura had a summer villa in Tateshina, Nagano prefecture, Japan. After 2011, Shimura invited me three times to visit his villa. The villa was acquired around the time when Shimura moved to Princeton from Osaka. An old, rather small building was standing in a large site. Tateshina is very nice to stay in during the summertime. Nearby are villas of celebrities. Shimura spent summers here with his family every couple of years during his professorship in Princeton. His study was small, less than 5m^2. It is amazing that monumental papers were written in this place. As I was accompanying a young mathematician, I asked Shimura what would you do in mathematics if you were young. It was around 2013. He responded that he would study Siegel. In fact, volume V of his collected papers contains several important papers on quadratic forms, and the influence of Siegel and Eichler was manifest.

Shimura sometimes told me that he loved "tricks" in mathematics. The trace formula and the Rankin–Selberg convolution are examples to him. A long winding road of reasoning leads to a simple theorem, illuminated by concrete (sometimes numerical) examples that can be seen by everybody. This is a tradition among number theorists from ancient times. For a layman, this may look like magic. I mention [2], [6], [11], [14], and [25].

On a few occasions, we talked about big problems such as the Hodge conjecture and the Riemann hypothesis. Shimura was negative about attacking a big problem without having good ideas. But he had the opinion that for the Riemann hypothesis, differential operators will play a crucial role. An interested reader may consult [23], [24].

In Shimura's class. Shimura's students know that the master has excellent skills of exposition and is also an entertainer.

In a lecture around 1972 at Princeton University, he explained the theory of canonical models, now called Shimura varieties. To study the model further, he said, "To desingularize or not to desingularize: that is the question." Everybody, knowing *Hamlet*, enjoyed the performance and laughed. (This incident is recorded in his book written in Japanese *How to Teach Mathematics*, p. 20. This book is the last item of the bibliography of volume V of his collected papers.) Also in another lecture around the same time, he talked about exotic ℓ-adic representations constructed using the theory of canonical models ([10], and §8 of "On canonical models of arithmetic quotients of bounded symmetric domains. I"). He said that the eigenvalues of the Frobenius automorphism have the property of "Riemann–Ramanujan–Weil type." Everybody enjoyed it and laughed.

In a lecture around 1989 at Kyoto University, he talked about the critical values of Dirichlet series and periods of automorphic forms ([22]). Automorphic forms are of Hilbert modular type, and L-functions are of standard type or of Rankin–Selberg type. But he considers all configurations of weights including the half integral weight case. Shimura's exposition was clear, but the situation is complicated and divided into several cases. For the half integral case, the period to be considered is the minus period of the integral weight form that corresponds to the half integral weight form by the Shimura correspondence, while it is the product of plus and minus periods when the form is integral weight. He explained this by saying, "The period becomes the half because the weight is half." Everybody laughed but this time with some feeling of relief. (To catch the point quickly, the reader is advised to see the introduction of [18].)

In Paris in 2000, after finishing a talk at a conference, Shimura was asked to give advice for a young audience by the chairman. He replied, "Don't prove anybody's conjecture," and everybody laughed. This episode is recorded in *How to Teach Mathematics*, p. 33. In this book, Shimura explains this advice in some depth.

I stayed at IAS in Princeton for 1990–91. In the spring of 1991, Shimura showed me reports of senior students evaluating Shimura's course. Most students evaluated the course highly; a few said that they never attended such splendid lectures during their student time in Princeton. Shimura told me that they were not mathematics majors; he lectured on number theory with an emphasis on historical perspectives. He was very proud of the students' evaluations and saw some of them in his Princeton home for decades.

Shimura published two autobiographies; one in English (*The Map of My Life*), one in Japanese. The contents are basically the same, of course, but they are independently written. In the book, he wrote that when he was young (1952–56), he taught at the University of Tokyo and also in a preparatory school because the salary was so low. Hijikata told me that he was then in preparatory school and impressed by Shimura's lecture; that experience led him to study mathematics.

Perhaps I should comment briefly on the historical facts concerning the Shimura–Taniyama conjecture, i.e., the modularity conjecture of elliptic curves over $\mathbf{Q}$. The issue is analyzed in detail in his autobiographies. The English version gives a more detailed account. He understood well that the issue was quite social. I quote one paragraph from Shimura's book: "The reader may ask why there were so many people who called the conjecture in various strange ways. I cannot answer that question except to say that many of them had no moral sense and most were incapable of having their own opinions."

As a man of culture. We had several interests in common, so it was not difficult for me to start a conversation with Shimura. One interest is Japanese chess (*Shogi*). The chess board is 9×9, and we can use captured pieces again. The other rules are basically the same as the western chess game. When I visited him in his Princeton home, he showed me a Japanese chess problem (*Tsume-Shogi*) composed by him. When I showed the correct answer of 71 moves written on a paper in my next visit, Shimura was very pleased. I still keep this problem and another problem of 169 moves. Shimura said he composed chess problems when he was very young, spending considerable time.

As my family served as Buddhist priests for centuries, I had some training to read Buddhist scriptures in Chinese translation from my childhood. Shimura liked to read Chinese classics and published two books. He also had some original perspectives on Buddhism.

Shimura loved music very much. His writings about music are scattered in many places in his books. In my first visit to his home, he played the well-tempered clavier on a record player. In later years, he regularly visited the Metropolitan Opera with his wife.

Shimura had an interest in antiques. He published a book on Imari porcelains. He had this interest since he was young; he wrote so in his book written in Japanese. But it was enhanced by a famous professor of oriental art at Princeton University, I think. Shimura's wife told me that her husband had a very strong memory of shapes, perhaps stronger than that professor.

Final recollections. Writing this article, I felt strongly the coherency of Shimura's character. Time has passed, and I am now much older than Shimura was when we first met. Nobody can stop aging and change their ultimate destiny. What I can be proud of, if anything, is that, after 1987, I invited Shimura about ten times to Kyoto University to give intensive courses, which must benefit students and young researchers. I also helped Shimura to edit his collected papers I–V with Alice Silverberg and Doi. But compared to what Shimura gave me to live as a mathematician, they are very small contributions. I thank Professor Goro Shimura again, this time with the deepest feeling of loss. Shimura is not among us anymore and will not respond to our emails. But his mathematics will live like the music of Mozart.

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Credits

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The Legacy of Józef Marcinkiewicz: Four Hallmarks of Genius

In Memoriam of an Extraordinary Analyst

Nikolay Kuznetsov

This article is a tribute to one of the most prominent Polish mathematicians, Józef Marcinkiewicz, who perished eighty years ago in the Katyń massacre. He was one of nearly 22,000 Polish officers interned by the Red Army in September 1939 and executed in April–May 1940 in the Katyń forest near Smolensk and at several locations elsewhere. One of these places was Kharkov (Ukraine), where more than 3,800 Polish prisoners of war from the Starobelsk camp were executed. One of them was Marcinkiewicz; the plaque with his name (see Figure 1) is on the Memorial Wall at the Polish War Cemetery in Kharkov.[1] This industrial execution was authorized by Stalin's secret order dated 5 March 1940 and organized by Beria, who headed the People's Commissariat for Internal Affairs (the interior ministry of the Soviet Union) known as NKVD.

Turning to the personality and mathematical achievements of Marcinkiewicz, it is appropriate to cite the article [24] of his supervisor Antoni Zygmund (it is published in the *Collected Papers* [13] of Marcinkiewicz; see p. 1):

> Considering what he did during his short life and what he might have done in normal circumstances one may view his early death as a great blow to Polish Mathematics, and probably its heaviest individual loss during the second world war.

From the Marcinkiewicz Biography [9]

On the occasion of the centenary of Marcinkiewicz's birth, a conference was held on 28 June–2 July 2010 in Poznań. In its proceedings, L. Maligranda published the detailed article [9] about Marcinkiewicz's life and mathematical results; sixteen pages of this paper are devoted to his biography, where one finds the following about his education and scientific career.

Education. Klemens Marcinkiewicz, Józef's father, was a farmer well-to-do enough to afford private lessons for him at home (the reason was Józef's poor health) before sending him to elementary school and then to gymnasium in Białystok. After graduating in 1930, Józef enrolled in the Department of Mathematics and Natural Science of the Stefan Batory University (USB) in Wilno (then in Poland, now Vilnius in Lithuania).

From the beginning of his university studies, Józef demonstrated exceptional mathematical talent that attracted the attention of his professors, in particular, of A. Zygmund. Being just a second-year student, Marcinkiewicz attended his lectures on orthogonal series, requiring some erudition, in particular, knowledge of the Lebesgue integral; this was the point where their collaboration began. The first paper of Marcinkiewicz (see [13, p. 35])

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[1] https://www.tracesofwar.com/sights/10355/Polish-War-Cemetery-Kharkiv.htm.

Communicated by Notices *Associate Editor Daniela De Silva.*

DOI: https://doi.org/10.1090/noti2084

Figure 1. Plaque for Marcinkiewicz on the Memorial Wall at the Polish War Cemetery in Kharkov.

was published when he was still an undergraduate student. It provides a half-page proof of Kolmogorov's theorem (1924) guaranteeing the convergence almost everywhere for partial sums of lacunary Fourier series. Marcinkiewicz completed his MSc and PhD theses (both supervised by Zygmund) in 1933 and 1935, respectively; to obtain his PhD degree he also passed a rather stiff examination. The second dissertation was the fourth of his almost five dozen publications; it concerns interpolation by means of trigonometric polynomials and contains interesting results (see [24, p. 17] for a discussion), but a long publication history awaited this work. Part of it was published in the *Studia Mathematica* the next year after the thesis defense (these two papers in French are reproduced in [13, pp. 171–185 and 186–199]). The full, original text in Polish appeared in the *Wiadomości Matematyczne* (the *Mathematical News*) in 1939. Finally, its English translation was included in [13, pp. 45–70].

Scientific career. During the two years between defending his MSc and PhD theses, Marcinkiewicz did the one year of mandatory military service and then was Zygmund's assistant at USB. The academic year 1935–1936 Marcinkiewicz spent as an assistant at the Jan Kazimierz University in Lwów. Despite twelve hours of teaching weekly, he was an active participant in mathematical discussions at the famous Scottish Café (see [3, ch. 10], where this unique form of doing mathematics is described), and his contribution to the *Scottish Book* compiled in this café was substantial, taking into account that his stay in Lwów lasted only nine months. One finds the history of this book in [14, ch. I], whereas problems and their solutions, where applicable, are presented in ch. II. Marcinkiewicz posed his own problem; it concerns the uniqueness of the solution for the integral equation

$$\int_0^1 y(t) f(x-t)\, \mathrm{d}t = 0, \qquad x \in [0,1].$$

He conjectured that if $f(0) \neq 0$ and f is continuous, then this equation has only the trivial solution $y \equiv 0$ (see problem no. 124 in [14, pp. 211 and 212]). He also solved three problems; his negative answers to problems 83 and 106 posed by H. Auerbach and S. Banach, respectively, involve ingenious counterexamples. His positive solution of problem 131 (it was formulated by Zygmund in a lecture given in Lwów in the early 1930s) was published in 1938; see [13, pp. 413–417].

During the next two academic years, Marcinkiewicz was a senior assistant at USB and after completing his habilitation in June 1937 became the youngest docent at USB. The same year, he was awarded the Józef Piłsudski Scientific Prize (the highest Polish distinction for achievements in science at that time). His last academic year 1938–1939, Marcinkiewicz was on leave from USB; a scholarship from the Polish Fund for National Culture afforded him opportunity to travel. He spent October 1938–March 1939 in Paris and moved to the University College London for April–August 1939, also visiting Cambridge and Oxford.

This period was very successful for Marcinkiewicz; he published several brief notes in the *Comptes rendus de l'Académie des Sciences Paris*. One of these, namely [12], became widely cited because the celebrated theorem concerning interpolation of operators was announced in it. Now this theorem is referred to as the Marcinkiewicz or Marcinkiewicz–Zygmund interpolation theorem (see below). Moreover, an important notion was introduced in the same note: the so-called weak-L^p spaces, known as Marcinkiewicz spaces now, are essential for the general form of this theorem.

Meanwhile, Marcinkiewicz was appointed to the position of Extraordinary Professor at the University of Poznań in June 1939. On his way to Paris, he delivered a lecture there and this, probably, was related to this impending appointment. Also, this was the reason to decline an offer of professorship in the USA during his stay in Paris.

Marcinkiewicz still was in England when the general mobilization was announced in Poland in the second half of August 1939; the outbreak of war became imminent. His colleagues advised him to stay in England, but his ill-fated decision was to go back to Poland. He regarded

Figure 2. Józef Marcinkiewicz.

himself as a patriot of his homeland, which is easily explainable by the fact that he was just eight years old (very sensitive age in forming a personality) when the independence of Poland was restored.

Contribution of Marcinkiewicz to Mathematics

Marcinkiewicz was a prolific author, as demonstrated by the almost five dozen papers he wrote in just seven years (1933–1939); see *Collected Papers* [13, pp. 31–33]. He was open to collaboration; indeed, more than one third of his papers (nineteen, to be exact) were written with five coauthors, of which the lion's share belongs to his supervisor Zygmund.

Marcinkiewicz is known, primarily, as an outstanding analyst, whose best results deal with various aspects of real analysis, in particular, theory of series (trigonometric and others), inequalities, and approximation theory. He also published several papers concerning complex and functional analysis and probability theory. In the extensive paper [9] dedicated to the centenary of Marcinkiewicz's birth, one finds a detailed survey of all his results.

This survey begins with the description of five topics concerning functional analysis ([9, pp. 153–175]). No doubt, the first two of them—the Marcinkiewicz interpolation theorem and Marcinkiewicz spaces—are hallmarks of genius. One indication of the ingenuity of the idea behind these results is that the note [11], in which they first appeared, is the most cited work of Marcinkiewicz.

Another important point about his work is that he skillfully applied methods of real analysis to questions bordering with complex analysis. A brilliant example of this mastery—one more hallmark of genius—is the Marcinkiewicz function μ introduced as an analogue of the Littlewood–Paley function g. It is worth mentioning that the short paper [10], in which μ first appeared, contains other fruitful ideas developed by many mathematicians subsequently.

One more hallmark of genius one finds in the paper [11] entitled "Sur les multiplicateurs des séries de Fourier." There are many generalizations of its results because of their important applications. This work was the last of eight papers that Marcinkiewicz published in the *Studia Mathematica*; the first three he submitted during his stay in Lwów, and they appeared in 1936.

Below, the above-mentioned results of Marcinkiewicz are outlined in their historical context together with some further developments. One can find a detailed presentation of all these results in the excellent textbook [18] based on lectures of the eminent analyst Elias Stein, who made a considerable contribution to further development of ideas proposed by Marcinkiewicz.

Marcinkiewicz Interpolation Theorem and Marcinkiewicz Spaces

There are two pillars of the interpolation theory: the classical Riesz–Thorin and Marcinkiewicz theorems. Each of these serves as the basis for two essentially different approaches to interpolation of operators known as the complex and real methods. The term "interpolation of operators" was, presumably, coined by Marcinkiewicz in 1939, because Riesz and Thorin, who published their results in 1926 and 1938, respectively, referred to their assertions as "convexity theorems."

It is worth emphasizing again that a characteristic feature of Marcinkiewicz's work was applying real methods to problems that other authors treated with the help of complex analysis. It was mentioned above that in his paper [10] published in 1938, Marcinkiewicz introduced the function μ without using complex variables but so that it is analogous to the Littlewood–Paley function g, whose definition involves these variables. In the same year, 1938, Thorin published his extension of the Riesz convexity theorem, which exemplifies the approach based on complex variables. Possibly this stimulated Marcinkiewicz to seek an analogous result with proof relying on real analysis. Anyway, Marcinkiewicz found his interpolation theorem and announced it in [12]; concurrently, a letter was sent to Zygmund that contained the proof concerning a particular case. Ten years after World War II, Zygmund reconstructed the general proof and published it in 1956; for this reason the theorem is sometimes referred to as the Marcinkiewicz–Zygmund interpolation theorem.

An excellent introduction to the interpolation theory one finds in the book [1] based on the works of Jaak Peetre (he passed away on 1 April 2019 at age eighty-three), whose contribution to this theory cannot be overestimated. In collaboration with Jacques-Louis Lions, he introduced the "real method interpolation spaces" (see their fundamental article [8]), which can be considered as "descendants" of the Marcinkiewicz interpolation theorem.

An important fact of Peetre's biography is that his life was severely changed during World War II (another reminder about that terrible time). With his parents, Jaak escaped from Estonia in September 1944 just two days before his home town of Pärnu was destroyed in an air raid of the Red Army. He was only ten years old when his family settled in Lund (Sweden), where he spent most of his life. But let us turn to mathematics again.

The Marcinkiewicz interpolation theorem for operators in $L^p(\mathbb{R}^n)$. We begin with this simple result because it has numerous applications, being valid for subadditive operators mapping the Lebesgue spaces $L^p(\mathbb{R}^n)$ with $p \geq 1$ into themselves (see, e.g., [18, ch. 1, sect. 4]). We recall that an operator $T : L^p \to L^p$ is subadditive if

$$|T(f_1 + f_2)(x)| = |T(f_1)(x)| + |T(f_2)(x)| \quad \text{for every } f_1, f_2 .$$

Furthermore, T is of *weak type* (r, r) if the inequality

$$\alpha^r \text{mes}\{x : |T(f)(x)| > \alpha\} \leq A_r \|f\|_r^r$$

holds for all $\alpha > 0$ and all $f \in L^r$ with A_r independent of α and f. Here, mes$\{\dots\}$ denotes the Lebesgue measure of the corresponding set, and

$$\|f\|_p = \Big[\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \Big]^{1/p}$$

is the norm in $L^p(\mathbb{R}^n)$. Now, we are in a position to formulate the following.

Theorem 1. *Let $1 \leq r_1 < r_2 < \infty$, and let T be a subadditive operator acting simultaneously in $L^{r_i}(\mathbb{R}^n)$, $i = 1, 2$. If it is of weak type (r_i, r_i) for $i = 1, 2$, then for every $p \in (r_1, r_2)$ the inequality $\|T(f)\|_p \leq B\|f\|_p$ holds for all $f \in L^p(\mathbb{R}^n)$ with B depending only on A_{r_1}, A_{r_2}, r_1, r_2, and p.*

When B is independent of f in the last inequality, the operator T is of *strong type* (p, p); it is clear that T is also of weak type (p, p) in this case.

In the letter to Zygmund mentioned above, Marcinkiewicz included a proof of this theorem for the case $r_1 = 1$ and $r_2 = 2$. Presumably, it was rather simple; indeed, even when $r_2 < \infty$ is arbitrary, the proof is less than two pages long in [18, ch. 1, sect. 4].

Marcinkiewicz spaces. Another crucial step, made by Marcinkiewicz in [12], was the introduction of the *weak L^p* spaces playing the essential role in his general interpolation theorem. They are now called the *Marcinkiewicz spaces* and usually denoted $L^{p,\infty}$.

To give an idea of these spaces, let us consider a measure space (U, Σ, m) over real scalars with a nonnegative measure m (just to be specific). For a real-valued f, which is finite almost everywhere and m-measurable, we introduce its distribution function

$$m(\{x : |f(x)| > \lambda\}), \qquad \lambda \in (0, \infty)$$

and put

$$|f|_{p,\infty} = \sup_{\lambda>0} \lambda [m(\{x : |f(x)| > \lambda\})]^{1/p} \quad \text{for } p \in [1, \infty).$$

Then $L^{p,\infty} = \{f : |f|_{p,\infty} < \infty\}$, and it is clear that $L^p \subset L^{p,\infty}$ for $p \in [1, \infty)$, because $|f|_{p,\infty} \leq \|f\|_p$ in view of Chebyshev's inequality. The Marcinkiewicz space for $p = \infty$ is L^∞ by definition.

It occurs that $|f|_{p,\infty}$ is not a norm for $p \in [1, \infty)$, but a quasi-norm because

$$|f + g|_{p,\infty} \leq 2(|f|_{p,\infty} + |g|_{p,\infty})$$

(see, e.g., [1, p. 7]). However, it is possible to endow $L^{p,\infty}$, $p \in (1, \infty)$, with a norm $\|\cdot\|_{p,\infty}$, converting it into a Banach space. Moreover, the inequality

$$|f|_{p,\infty} \leq \|f\|_{p,\infty} \leq p(p-1)^{-1} |f|_{p,\infty}$$

holds for all $f \in L^{p,\infty}$. It is worth mentioning that $L^{p,\infty}$ belongs (as a limiting case) to the class of *Lorentz spaces* $L^{p,q}$, $q \in [1, \infty]$ (see, e.g., [1, sect. 1.6] and references cited in this book).

Another generalization of $L^{p,\infty}$, known as the Marcinkiewicz space M_φ, is defined with the help of a nonnegative, concave function $\varphi \in C[0, \infty)$. This Banach space consists of all (equivalence classes of) measurable functions for which the norm

$$\|f\|_\varphi = \sup_{t>0} \frac{1}{\varphi(t)} \int_0^t f^*(s) \, \mathrm{d}s$$

is finite. Here f^* denotes the nonincreasing rearrangement of f, i.e.,

$$f^*(s) = \inf_{\lambda>0} \{\lambda : m(\{x : |f(x)| > \lambda\}) \leq s\} \quad \text{for } s \geq 0,$$

and so is nonnegative and right-continuous. Moreover, its distribution function $m(\{x : |f^*(x)| > \lambda\})$ coincides with that of f. If $\varphi(t) = t^{1-1/p}$, then the corresponding Marcinkiewicz space is $L^{p,\infty}$, whereas $\varphi(t) \equiv 1$ and $\varphi(t) = t$ give L^1 and L^∞, respectively.

The Marcinkiewicz interpolation theorem for bounded linear operators. This kind of continuous operator is usually considered as mapping one normed space to another one, in which case the operator's norm is an important characteristic. However, the latter can be readily generalized for a mapping of L^p to $L^{p,\infty}$. Indeed, if $|Tf|_{p,\infty} \leq C\|f\|_p$, then it is natural to introduce the norm (or quasi-norm) of T as the infimum over all possible values of C. Now we are in a position to formulate the following.

Theorem 2. *Let p_0, q_0, p_1, $q_1 \in [1,\infty]$ satisfy the inequalities $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$, and let p, $q \in [1,\infty]$ be such that $p \leq q$ and the equalities*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{1}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{1}{q_1}$$

hold for some $\theta \in (0,1)$. If T is a linear operator that maps L^{p_0} into $L^{q_0,\infty}$ and its norm is N_0 and simultaneously $T : L^{p_1} \to L^{q_1,\infty}$ has N_1 as its norm, then T maps L^p into L^q and its norm N satisfies the estimate

$$N \leq C N_0^{1-\theta} N_1^{\theta}, \tag{1}$$

with C depending on p_0, q_0, p_1, q_1, and θ.

The convexity inequality (1) is a characteristic feature of the interpolation theory. The general form of this theorem (it is valid for quasi-additive operators, whose special case are subadditive ones described prior to Theorem 1) is proved in [23, ch. XII, sect. 4]. In particular, it is shown that one can take

$$C = 2\left(\frac{q}{|q-q_0|} + \frac{q}{|q-q_1|}\right)^{1/q} \frac{p_0^{(1-\theta)/p_0}\, p_1^{\theta/p_1}}{p^{1/p}};$$

see [23, Vol. II, p. 114, formula (4.18)], where, unfortunately, the notation differs from that adopted here. Special cases of Theorem 2 and diagrams illustrating them can be found in [9, pp. 155–156]. It should be emphasized that the restriction $p \leq q$ is essential; indeed, as early as 1964, R. A. Hunt [6] constructed an example demonstrating that Theorem 2 is not true without it. For a description of this example see, e.g., [1, pp. 16–17].

It was Marcinkiewicz himself who proposed an extension of his interpolation theorem to other function spaces; namely, the so-called diagonal case (when $p_0 = q_0$ and $p_1 = q_1$) of his theorem is formulated for Orlicz spaces in [12]. References to papers containing further results on interpolation in these and other spaces (e.g., Lorentz and M_φ) can be found in [1, pp. 128–129] and [9, pp. 163–166].

Applications of the interpolation theorems. (1) In his monograph [23], Zygmund gave a detailed study of the one-dimensional Fourier transform

$$F(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, \exp\{-i\,\xi x\}\, dx, \qquad \xi \in \mathbb{R}.$$

See Vol. II, ch. XVI, sects. 2 and 3, where, in particular, it is demonstrated that F, originally defined on a dense set in L^p, $p \in [1,2]$, is extensible to the whole space as a bounded operator $F : L^p \to L^{p'}$, $p' = p/(p-1)$, and so the integral converges in $L^{p'}$. To prove this assertion and its n-dimensional analogue one can use Theorem 2. Indeed, $F : L^1 \to L^\infty$ is bounded (this is straightforward to see), and by Plancherel's theorem F is bounded on L^2, and so this theorem is applicable. On the other hand, the Riesz–Thorin theorem, which has no restriction $p \leq q$, yields a more complete result valid for the inverse transform F^{-1} as well. The latter operator acting from $L^{p'}$ to L^p is bounded; here $p' \in [2,\infty)$, and so $p = p'/(p'-1) \in (1,2]$.

(2) In studies of conjugate Fourier series, the singular integral operator (the periodic Hilbert transform)

$$H(f)(s) = \frac{1}{2\pi} \lim_{\epsilon\to 0} \int_{\epsilon \leq |t| \leq \pi} f(s-t) \cot\frac{t}{2}\, dt$$

plays an important role. Indeed, by linearity it is sufficient to define H on a basis in $L^2(-\pi,\pi)$, and the relations

$$H(\cos nt) = \sin ns \text{ for } n \geq 0, \quad H(\sin nt) = -\cos ns \text{ for } n \geq 1$$

show that it expresses passing from a trigonometric series to its conjugate. Moreover, these formulae show that H is bounded on $L^2(-\pi,\pi)$ and its norm is equal to one.

In the mid-1920s, Marcel Riesz obtained his celebrated result about this operator; first, he announced it in a brief note in the *Comptes rendus de l'Académie des Sciences Paris*, and three years later published his rather long proof that H is bounded on $L^p(-\pi,\pi)$ for $p \in (1,\infty)$; i.e., for every finite $p > 1$ there exists $A_p > 0$ such that

$$\|H(f)\|_p \leq A_p \|f\|_p \quad \text{for all } f \in L_p(-\pi,\pi). \tag{2}$$

However, (2) does not hold for $p = 1$ and ∞; see [23, Vol. I, ch. VII, sect. 2] for the corresponding examples and a proof of this inequality.

There are several different proofs of this theorem; the original proof of M. Riesz was reproduced in the first edition of Zygmund's monograph [23], which appeared in 1935. In the second edition published in 1959, this proof was replaced by that of Calderón obtained in 1950. Let us outline another proof based on the Marcinkiewicz interpolation theorem analogous to Theorem 1 but involving L^p-spaces on $(-\pi,\pi)$ instead of the spaces on $\mathbb{R}$.

First we notice that it is sufficient to prove (2) only for $p \in (1,2]$. Indeed, assuming that this is established, then for $f \in L_p$ and $g \in L_{p'}$ we have

$$\int_{-\pi}^{\pi} [H(f)(s)]\, g(-s)\, ds \leq A_p \|f\|_p\, \|g\|_{p'}$$

by the Hölder inequality (as above $p' = p/(p-1)$, and so

$p' \geq 2$ when $p \leq 2$). Since

$$\int_{-\pi}^{\pi} [H(f)(s)]\, g(-s)\, ds = \int_{-\pi}^{\pi} f(-s)\, [H(g)(s)]\, ds\,,$$

the inequality $\|H(g)\|_{p'} \leq A_p^{-1} \|g\|_{p'}$ is a consequence of the assertion converse to the Hölder inequality.

It was mentioned above that H is bounded in L^2. Hence, in order to apply Theorem 1 for $p \in (1, 2]$, it is sufficient to show that this operator is of weak type $(1, 1)$, and this is an essential part of Calderón's proof; see [23, Vol. I, ch. IV, sect. 3]. Moreover, an improvement of the latter proof allowed S. K. Pichorides [16] to obtain the least value of the constant A_p in (2). It occurs that $A_p = \tan \pi/(2p)$ and $\cot \pi/(2p)$ is this value for $p \in (1, 2]$ and $p \geq 2$, respectively.

There are many other applications of interpolation theorems in analysis; see, e.g., [1, ch. 1], [23, ch. XII], and references cited in these books.

Further development of interpolation theorems. Results constituting the interpolation space theory were obtained in the early 1960s and are classical now. This theory was created in the works of Nachman Aronszajn, Alberto Calderón, Mischa Cotlar, Emilio Gagliardo, Selim Grigorievich Krein, Jacques-Louis Lions, and Jaak Peetre, to list a few. We leave aside several versions of complex interpolation spaces developed from the Riesz–Thorin theorem (see, e.g., [1, ch. 4]) and concentrate on "espaces de moyennes" introduced by Lions and Peetre in their celebrated article [8]. These "real method interpolation spaces," usually denoted $(A_0, A_1)_{\theta,p}$, are often considered as "descendants" of the Marcinkiewicz interpolation theorem.

Prior to describing these spaces, it is worth mentioning another germ of interpolation theory originating from Lwów. Problem 87 in the *Scottish Book* [14] posed by Banach demonstrates his interest in nonlinear interpolation. Presumably, it was formulated during Marcinkiewicz's stay in Lwów. Indeed, he solved problems 83 and 106 in [14], which were posed before and after, respectively, Banach's problem on interpolation. A positive solution of the latter problem (due to L. Maligranda) is presented in [14, pp. 163–170].

Let us turn to defining the family of spaces $\{(A_0, A_1)_{\theta,p}\}$ involved in the real interpolation method; here $\theta \in (0, 1)$ and $p \in [1, \infty]$. In what follows, we write $A_{\theta,p}$ instead of $(A_0, A_1)_{\theta,p}$ for the sake of brevity. Let A_0 and A_1 be two Banach spaces, both continuously embedded in some (larger) Hausdorff topological vector space. Then for a pair (θ, p) the space $A_{\theta,p}$ with $p < \infty$ consists of all $a \in A_0 + A_1$ for which the norm

$$\|a\|_{\theta,p} = \left\{ \int_0^\infty \left[t^{-\theta} K(t, a) \right]^p \frac{dt}{t} \right\}^{1/p}$$

is finite. Here $K(t, a)$ is defined on $A_0 + A_1$ for $t \in (0, \infty)$ by

$$\inf_{a_0, a_1} \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} \,:\, a_0 \in A_0,\ a_1 \in A_1 \text{ and } a_0 + a_1 = a \}.$$

This K-functional was introduced by Peetre. If $p = \infty$, then the expression $\sup_{t>0} \{t^{-\theta} K(t, a)\}$ gives the norm $\|a\|_{\theta,\infty}$ when finite.

Every $A_{\theta,p}$ is an intermediate space with respect to the pair (A_0, A_1), i.e.,

$$A_0 \cap A_1 \subset A_{\theta,p} \subset A_0 + A_1.$$

Moreover, if $A_0 \subset A_1$, then

$$A_0 \subset A_{\theta_0, p_0} \subset A_{\theta_1, p_1} \subset A_1,$$

provided either $\theta_0 > \theta_1$ or $\theta_0 = \theta_1$ and $p_0 \leq p_1$. For any p, it is convenient to put $A_{0,p} = A_0$ and $A_{1,p} = A_1$. Now we are in a position to explain what the interpolation of an operator is in terms of the family $\{A_{\theta,p}\}$ and another family of spaces $\{B_{\theta,p}\}$ constructed by using some Banach spaces B_0 and B_1 in the same way as A_0 and A_1.

Let $T : A_0 + A_1 \to B_0 + B_1$ be a linear operator such that its norm as the operator mapping $A_0\,(A_1)$ to $B_0\,(B_1)$ is equal to $M_0\,(M_1)$. Then the operator $T : A_{\theta,p} \to B_{\theta,p}$ is also bounded, and its norm is less than or equal to $M_0^{1-\theta} M_1^{\theta}$. Along with the method based on the K-functional, there is an equivalent method (also developed by Peetre) involving the so-called J-functional. Further details concerning this approach to interpolation theory can be found in [1, chs. 3 and 4].

The Marcinkiewicz Function

In the *Annales de la Société Polonaise de Mathématique*, volume 17 (1938), Marcinkiewicz published two short papers. Two remarkable integral operators were considered in the first of these notes (see [10] and [13, pp. 444–451]); they and their numerous generalizations became indispensable tools in analysis. One of these operators is always called the "Marcinkiewicz integral"; see [23, ch. IV, sect. 2] for its definition and properties. In particular, it is used for investigation of the structure of a measurable set near an "almost arbitrary" point; see [18, sects. 2.3 and 2.4], whereas further references to papers describing some of its generalizations can be found in the monographs [18] and [23]. The second operator is usually referred to as the "Marcinkiewicz function" (see, e.g., [9, pp. 192–194]), but it also appears as the "Marcinkiewicz integral." Presumably, the mess with names began as early as 1944, when Zygmund published the extensive article [22], section 2 of which was titled "On an Integral of Marcinkiewicz." In fact, this 14-page section is devoted to a detailed study of the Marcinkiewicz function μ, whose properties were just outlined by Marcinkiewicz himself in [10]. It is not clear whether Zygmund had already received information about Marcinkiewicz's death when he decided to present in detail the results from [10] (the discovery of mass graves in the Katyń forest was announced by the Nazi government in April 1943).

Zygmund begins his presentation with a definition of the Littlewood–Paley function $g(\theta; f)$, which is a nonlinear operator applied to an integrable, 2π-periodic f. The purpose of introducing $g(\theta; f)$ was to provide a characterization of the L^p-norm $\|f\|_p$ in terms of the Poisson integral of f. After describing some properties of $g(\theta)$, Zygmund notes:

> It is natural to look for functions analogous to $g(\theta)$ but defined without entering the interior of the unit circle.

After a reference to [10], Zygmund continues:

> Marcinkiewicz had the right idea of introducing the function
>
> $$\mu(\theta) = \mu(\theta; f) = \left\{ \int_0^\pi \frac{[F(\theta+t) + F(\theta-t) - 2F(\theta)]^2}{t^3} \mathrm{d}t \right\}^{1/2} = \left\{ \int_0^\pi t \left[\frac{F(\theta+t) + F(\theta-t) - 2F(\theta)}{t^2} \right]^2 \mathrm{d}t \right\}^{1/2}$$
>
> where $F(\theta)$ is the integral of f,
>
> $$F(\theta) = C + \int_0^\theta f(u)\, \mathrm{d}u\,.$$
>
> More generally, he considers the functions
>
> $$\mu_r(\theta) = \left\{ \int_0^\pi \frac{|F(\theta+t) + F(\theta-t) - 2F(\theta)|^r}{t^{r+1}} \mathrm{d}t \right\}^{1/r} = \left\{ \int_0^\pi t^{r-1} \left| \frac{F(\theta+t) + F(\theta-t) - 2F(\theta)}{t^2} \right|^r \mathrm{d}t \right\}^{1/r},$$
>
> so that $\mu_2(\theta) = \mu(\theta)$. He proves the following facts which are clearly analogues of the corresponding properties of $g(\theta)$.

These facts are the estimates

$$\|\mu_q\|_q \le A_q \|f\|_q \quad \text{and} \quad \|f\|_p \le A_p \|\mu_p\|_p$$

valid for $q \ge 2$ and $1 < p \le 2$, respectively, where f has the zero mean value in the second inequality and the assertion: *For every $p \in (1, 2]$ there exists a continuous, 2π-periodic function f such that $\mu_p(\theta; f) = \infty$ for almost every θ.*

Furthermore, Marcinkiewicz conjectured that for $p > 1$ the inequalities

$$A_p \|f\|_p \le \|\mu\|_p \le B_p \|f\|_p \tag{3}$$

hold, where again f must have the zero mean value in the second inequality. Moreover, he foresaw that it would not be easy to prove these inequalities; indeed, the proof given by Zygmund in his article [22] is more than 11 pages long.

The first step towards generalization of the Marcinkiewicz function was made by Daniel Waterman; his paper [21] was published seven (!) years after presentation of the work to the AMS. However, its abstract appeared in the *Proceedings of the International Congress of Mathematicians* held in 1954 in Amsterdam. Waterman considered the μ-function

$$\mu(\tau; f) = \left\{ \int_0^\infty \frac{[F(\tau+t) + F(\tau-t) - 2F(\tau)]^2}{t^3} \mathrm{d}t \right\}^{1/2},$$

where $\tau \in (-\infty, \infty)$ and F is a primitive of $f \in L^p(-\infty, \infty)$, $p > 1$. His proof of inequalities (3) for $\mu(\tau; f)$ heavily relies on the M. Riesz theorem about conjugate functions on $\mathbb{R}^1$ (see [21, p. 130] for the formulation), and its proof involves the Marcinkiewicz interpolation theorem described above.

Another consequence of inequalities (3) for $\mu(\tau; f)$ is a characterization of the Sobolev space $W^{1,p}(\mathbb{R})$, $p \in (1, \infty)$. Indeed, putting

$$\mathrm{M}(\tau; f) = \left\{ \int_0^\infty \frac{[f(\tau+t) + f(\tau-t) - 2f(\tau)]^2}{t^3} \mathrm{d}t \right\}^{1/2}$$

for $f \in W^{1,p}(\mathbb{R})$, we have that $\mathrm{M}(\tau; f) = \mu(\tau; f')$. Then (3) can be written as

$$A_p \|f'\|_p \le \|\mathrm{M}(\cdot; f)\|_p \le B_p \|f'\|_p\,,$$

which implies the following assertion. *Let $p \in (1, \infty)$. Then $f \in W^{1,p}(\mathbb{R})$ if and only if $f \in L^p(\mathbb{R})$ and $\mathrm{M}(\cdot; f) \in L^p(\mathbb{R})$.*

Stein extended these results to higher dimensions in the late 1950s and early 1960s (it is worth mentioning that μ is referred to as the Marcinkiewicz integral in his paper [17]). For this purpose he applied the real-variable technique used in the generalization of the Hilbert transform

$$\text{P.V.} \int_0^\infty \frac{f(x+t) - f(x-t)}{t}\, \mathrm{d}t$$

to higher dimensions. Indeed, this can be written as

$$\int_0^\infty \frac{F(x+t) + F(x-t) - 2F(x)}{t^2}\, \mathrm{d}t,$$

which resembles the expression for $\mu(\tau; f)$, and so Stein, in his own words, was

> guided by the techniques used by A. P. Calderón and A. Zygmund [2] in their study of the n-dimensional generalizations of the Hilbert transform; connected with this are some earlier ideas of Marcinkiewicz.

The definition of singular integral given in [2], to which Stein refers, involves a function $\Omega(x)$ defined for $x \in \mathbb{R}^n$ and assumed (i) to be homogeneous of degree zero, i.e., to depend only on $x' = x/|x|$; (ii) to satisfy the Hölder condition with exponent $\alpha \in (0,1]$; and (iii) to have the zero mean value over the unit sphere in $\mathbb{R}^n$. Then

$$S(f)(x) = \lim_{\epsilon\to 0} \int_{|y|>\epsilon} \frac{\Omega(y')}{|y|^n} f(x-y)\,\mathrm{d}y$$

exists almost everywhere provided $f \in L^p(\mathbb{R}^n)$, $p \in [1,\infty)$. Furthermore, this singular integral operator is bounded in $L^p(\mathbb{R}^n)$ for $p > 1$; i.e., the inequality $\|S(f)\|_p \le A_p\|f\|_p$ holds with A_p independent of f.

Moreover, in the section dealing with background facts, Stein notes that μ is a nonlinear operator and writes (see [17, p. 433]):

> An "interpolation" theorem of Marcinkiewicz is very useful in this connection.
>
> In quoting the result of Marcinkiewicz, [...] we shall not aim at generality. For the sake of simplicity we shall limit ourselves to the special case that is needed.

After that the required form of the interpolation theorem (see Theorem 1 above) is formulated and used later in the paper, thus adding one of the first items in the now long list of its applications. Since the term interpolation was novel, quotation marks are used by Stein in the quoted piece. Indeed, Zygmund's proof of the Marcinkiewicz theorem had appeared in 1956, just two years earlier than Stein's article.

Stein begins his generalization of the Marcinkiewicz function $\mu(\tau; f)$ with the case when $f \in L^p(\mathbb{R}^n)$, $p \in [1,2]$. Realizing the analogy described above, he puts

$$F_t(x) = \int_{|y|\le t} \frac{\Omega(y')}{|y|^{n-1}} f(x-y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^n, \qquad (4)$$

where Ω satisfies conditions (i)–(iii), and notes that if $n = 1$ and $\Omega(y) = \operatorname{sign} y$, then

$$F_t(x) = F(x+t)+F(x-t)-2F(x) \quad \text{with } F(x) = \int_0^x f(s)\,\mathrm{d}s.$$

Therefore, it is natural to define the n-dimensional Marcinkiewicz function as follows:

$$\mu(x; f) = \left\{\int_0^\infty \frac{[F_t(x)]^2}{t^3}\,\mathrm{d}t\right\}^{1/2}. \qquad (5)$$

Stein begins his investigation of properties of this function by proving that $\|\mu(\cdot; f)\|_2 \le A\|f\|_2$, where A is independent of f, and his proof involving Plancherel's theorem is not elementary at all. Even less elementary is his proof that $\mu(\cdot; f)$ is of weak type $(1,1)$. Then the Marcinkiewicz interpolation theorem (see Theorem 1 above) implies that $\|\mu(\cdot; f)\|_p \le A\|f\|_p$ for $p \in (1,2]$ provided $f \in L^p(\mathbb{R}^n)$. For all $p \in (1,\infty)$ this inequality is proved in [17] with assumptions (i)–(iii) changed to the following ones: $\Omega(x')$ is absolutely integrable on the unit sphere and is odd there, i.e., $\Omega(-x') = -\Omega(x')$. A few years later, A. Benedek, A. P. Calderón, and R. Panzone demonstrated that for a C^1-function Ω, condition (iii) implies the last inequality for all $p \in (1,\infty)$.

In another note, Stein obtained the following generalization of the one-dimensional result.

Let $p \in (2n/(n+2),\infty)$ and $n \ge 2$. Then f belongs to the Sobolev space $W^{1,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and

$$\left\{\int_{\mathbb{R}^n} \frac{[f(\cdot+y)+f(\cdot-y)-2f(\cdot)]^2}{|y|^{n+2}}\mathrm{d}y\right\}^{1/2} \in L^p(\mathbb{R}^n).$$

For $n > 2$ this does not cover $p \in (1, 2n/(n+2)]$ and so is weaker than the assertion formulated above for $n = 1$.

In the survey article [9, pp. 193–194], one finds a list of papers concerning the Marcinkiewicz function. In particular, further properties of μ were considered by A. Torchinsky and S. Wang [19] in 1990, whereas T. Walsh [20] proposed a modification of the definition (4), (5) in 1972.

Multipliers of Fourier Series and Integrals

During his stay in Lwów, Marcinkiewicz collaborated with Stefan Kaczmarz and Juliusz Schauder,[2] who had awakened his interest in *multipliers* of orthogonal series. Studies in this area of analysis were initiated by Hugo Steinhaus in the 1920s; in its general form, the problem of multipliers is as follows. Let B_1 be a Banach space with a Schauder basis $\{g_n\}_{n=1}^\infty$. The (linear) operator T is called a *multiplier* when there is a sequence $\{m_n\}_{n=1}^\infty$ of scalars of this space and T acts as follows:

$$B_1 \ni f = \sum_{n=1}^\infty c_n g_n \;\;\to\;\; Tf \sim \sum_{n=1}^\infty m_n c_n g_n\,.$$

Here $\sim$ means that the second sum assigned as Tf can belong to the same space B_1 or be an element of another Banach space B_2; this depends on properties of the sequence. Multipliers of Fourier series are of paramount interest, and this was the topic of the remarkable paper [11] published by Marcinkiewicz in 1939.

[2] *Both perished in World War II. Being in the reserve, Kaczmarz was drafted and killed during the first week of war; the circumstances of his death are unclear. Schauder was in hiding in occupied Lwów, and the Gestapo killed him in 1943 while he was trying to escape arrest.*

Not long before Marcinkiewicz's visit to Lwów started, Kaczmarz investigated some properties of multipliers in the function spaces (mainly $L^p(0,1)$ and $C[0,1]$) under rather general assumptions about the system $\{g_n\}_{n=1}^{\infty}$. Further results about multiplier operators were obtained in the joint paper [7] of Kaczmarz and Marcinkiewicz. It was submitted to the *Studia Mathematica* in June 1937; i.e., their collaboration lasted for another year after Marcinkiewicz left Lwów. This paper has the same title as that of Kaczmarz and concerns the case when $L^p(0,1)$ with $p \neq \infty$ is mapped to $L^q(0,1)$, $q \in [1,\infty]$; it occurs that the case $q = \infty$ is the simplest one. In this paper, it is assumed that every function g_n is bounded, whereas the sequence $\{g_n\}_{n=1}^{\infty}$ is closed in $L^1(0,1)$. In each of four theorems that differ by the ranges of p and q involved, certain conditions are imposed on $\{m_n\}_{n=1}^{\infty}$, and these conditions are necessary and sufficient for the sequence to define a multiplier operator $T : L^p \to L^q$.

After returning to Wilno, Marcinkiewicz kept on his studies of multipliers initiated in Lwów, and in May 1938, he submitted (again to the *Studia Mathematica*) the seminal paper [11], in which the main results are presented in a curious way. Namely, Theorems 1 and 2, concerning multipliers of Fourier series and double Fourier series, are formulated in the reverse order. Presumably, the reason for this is the importance of multiple Fourier series for applications and generalizations. Let us formulate Theorem 1 in a slightly updated form.

Let $f \in L^p(0,2\pi)$, $p \in (1,\infty)$, be a real-valued function and let its Fourier series be

$$a_0/2 + \sum_{n=1}^{\infty} A_n(x), \ \textit{where}\ A_n(x) = a_n \cos nx + b_n \sin nx.$$

If a bounded sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is such that

$$\sum_{n=2^k}^{2^{k+1}} |\lambda_n - \lambda_{n+1}| \leq M \ \textit{for all}\ k = 0,1,2,\dots, \tag{6}$$

where M is a constant independent of k, then the mapping $f \mapsto \sum_{n=1}^{\infty} \lambda_n A_n$ is a bounded operator in $L^p(0,2\pi)$.

It is well known that for $p = 2$ this theorem is true with condition (6) omitted, but this is not mentioned in [11]. The assumptions that f is real-valued and $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ were not stated in [11] explicitly but used in the proof. This was noted by Solomon Grigorievich Mikhlin [15], who extended this theorem to complex-valued multipliers and functions. Also, he used the exponential form of the Fourier expansion:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp inx\,.$$

The trigonometric form was used by Marcinkiewicz for double Fourier series as well, and his sufficient conditions on bounded real multipliers $\{\lambda_{mn}\}$ look rather awkward. Now, the restrictions on $\{\lambda_{mn}\} \subset \mathbb{C}$ are usually expressed in a rather condensed form by using the so-called dyadic intervals; see, e.g., [18, sect. 5.1]. Applying these conditions to multipliers acting on the expansion

$$\sum_{m,n=-\infty}^{\infty} c_{mn} \exp i\{mx + ny\}$$

of $f \in L^p((0,2\pi)^2)$, $p \in (1,\infty)$, one obtains an updated formulation of the multiplier theorem; see, e.g., [9, p. 201].

A simple corollary derived by Marcinkiewicz from this theorem is as follows (see [11, p. 86]). The fractions

$$\frac{m^2}{m^2+n^2}, \quad \frac{n^2}{m^2+n^2}, \quad \frac{|mn|}{m^2+n^2} \tag{7}$$

provide examples of multipliers in L^p for double Fourier series. The reason to include these examples was to answer a question posed by Schauder, and this is specially mentioned in a footnote. Moreover, after remarking that his Theorem 2 admits an extension to multiple Fourier series, Marcinkiewicz added a straightforward generalization of formulae (7) to higher dimensions, again referring to Schauder's question. This is evidence that the question was an important stimulus for Marcinkiewicz in his work.

A natural way to generalize Marcinkiewicz's theorems is to consider multipliers of Fourier integrals. Study of these operators was initiated by Mikhlin in 1956; see note [15], in which the first result of that kind was announced. Several years later, Mikhlin's theorem was improved by Lars Hörmander [5], and since then it has been widely used for various purposes. To formulate this theorem we need the n-dimensional Fourier transform

$$F(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \exp\{-i\,\xi \cdot x\}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n,$$

defined for $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $p \in (1,\infty)$. It is clear that any bounded measurable function Λ on $\mathbb{R}^n$ defines the mapping

$$T_\Lambda(f)(x) = F^{-1}[\Lambda(\xi)F(f)(\xi)](x)\,, \qquad x \in \mathbb{R}^n,$$

such that $T_\Lambda(f) \in L^2(\mathbb{R}^n)$. If $T_\Lambda(f)$ is also in $L^p(\mathbb{R}^n)$ and T_Λ is a bounded operator, i.e.,

$$\|T_\Lambda(f)\|_p \leq B_{p,n}\|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n) \tag{8}$$

with B_p independent of f, then Λ is called a multiplier for L^p.

The description of all multipliers for L^2 is known as well for L^1 and L^∞ (it is the same for these two spaces); see [18, pp. 94–95]. However, the question about characterization of the whole class of multipliers for other values of p is far from resolved. The following assertion gives widely used sufficient conditions.

Theorem (Mikhlin, Hörmander). *Let Λ be a function of the C^k-class in the complement of the origin of $\mathbb{R}^n$. Here k is the least integer greater than $n/2$. If there exists $B > 0$ such that*

$$|\xi|^\ell \left| \frac{\partial^\ell \Lambda(\xi)}{\partial \xi_{j_1} \partial \xi_{j_2} \cdots \partial \xi_{j_\ell}} \right| \leq B\,, \qquad 1 \leq j_1 < j_2 < \cdots < j_\ell \leq n\,,$$

for all $\xi \in \mathbb{R}^n$, $\ell = 0, \ldots, k$, and all possible ℓ-tuples, then inequality (8) *holds; i.e., Λ is a multiplier for L^p.*

In various versions of this theorem, different assumptions are imposed on the differentiability of Λ. In particular, Hörmander [5, pp. 120–121] replaced the pointwise inequality for weighted derivatives of Λ by a weaker one involving certain integrals (see also [18, p. 96]). Recently, Loukas Grafakos and Lenka Slavíková [4] obtained new sufficient conditions for Λ in the multiplier theorem, thus improving Hörmander's result. Their conditions are optimal in a certain sense explicitly described in [4].

Corollary. *Every function that is smooth everywhere except at the origin and is homogeneous of degree zero is a Fourier multiplier for L^p.*

Its immediate consequence is the Schauder estimate

$$\left\| \frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}} \right\|_p \leq C_{p,n} \|\Delta u\|_p\,, \qquad 1 \leq j_1, j_2 \leq n\,,$$

valid for u belonging to the Schwartz space of rapidly decaying infinitely differentiable functions. For this purpose one has to use the equality

$$F\left(\frac{\partial^2 u}{\partial x_{j_1} \partial x_{j_2}} \right)(\xi) = \frac{\xi_{j_1} \xi_{j_2}}{|\xi|^2} F(\Delta u)(\xi)\,, \qquad 1 \leq j_1, j_2 \leq n\,,$$

and the fact that the function $\xi_{j_1} \xi_{j_2} / |\xi|^2$ is homogeneous of degree zero.

ACKNOWLEDGMENTS. The author thanks Irina Egorova for the photo of Marcinkiewicz's plaque and Alex Eremenko, whose comments helped to improve the original manuscript.

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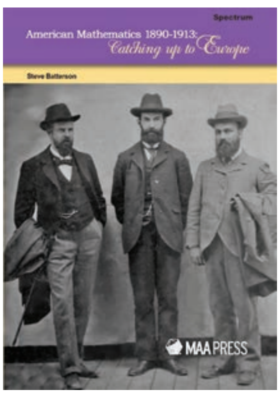
BOOK REVIEW

American Mathematics 1890–1913: Catching Up to Europe

Reviewed by Deborah Kent



In 1994, the AMS published *The Emergence of the American Mathematical Research Community, 1876–1900: J. J. Sylvester, Felix Klein, and E. H. Moore*, written by Karen Parshall and David Rowe. This now classic work laid carefully documented groundwork for subsequent study and articulated a periodization for American mathematics. By targeting the years 1876–1900, Parshall and Rowe explicitly structure two other periods in the history of American mathematics—one from 1776 to 1876 and at least one after 1900. Such periodization is frequently utilized as a helpful analytical framework in historical scholarship, and justifications for the selected start and end dates become part of a work's historical argument.[1]

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Communicated by Notices *Book Review Editor Stephan Ramon Garcia.*

[1]*Historical periodization sometimes presents challenges, particularly in cross-cultural contexts. For an example, see work of Bernard Lewis,* https://www.gatestoneinstitute.org/323/the-periodization-of-history---excerpts *or* https://www.youtube.com/watch?v=OS7-ZRFwVxM. *For an overview of the postmodernist position on periodization, see especially Chapter 1 in Lawrence Besserman, ed.,* The Challenge of Periodization: Old Paradigms and New Perspectives, *1996. A sampling of other perspectives is available in J. H. Bentley, "Cross-Cultural Interaction and Periodization in World History," American Historical Review (June 1996), 749–770; and L. Grinin, "Periodization of History: A theoretic-mathematical analysis." In:* History & Mathematics, *Moscow: KomKniga/URSS, pp. 10–38, 2007.*

DOI: https://dx.doi.org/10.1090/noti2072

The Emergence of the American Mathematical Research Community begins with an overview of American mathematics during its first one hundred years. During this period, American mathematics evolved in the context of general scientific structure building. Mathematical practitioners in the United States worked alongside geologists, physicists, botanists, and other self-identified scientists to organize societies, develop employment opportunities, and provide outlets for scientific publication. The notion of research in academic disciplines emerged as practitioners looked to Europe as a model for scientific achievement. Specifically, German educational institutions introduced seminar-style instruction to train graduate students as future researchers.

The foundation of The Johns Hopkins University in 1876 marked a major change in American higher education. Before that, American colleges were designed primarily for undergraduate instruction and did little to encourage faculty research. Johns Hopkins, though, adopted the German vision for a university faculty charged to educate undergraduate students, conduct independent research, and train graduate students to be future researchers. This influential German model likewise informed Clark University when it started in 1889 and the University of Chicago, which opened for classes in 1892. The quarter century of central consideration in Parshall and Rowe thus begins in the year shared by the US centennial and the start of a new kind of academic institution in America. In the detailed discussion in the meaty middle eight chapters, Parshall and Rowe focus on James Joseph Sylvester at Johns Hopkins; Felix Klein at Leipzig, then Göttingen; and Eliakim Hastings Moore (see Figure 1) at the University of Chicago. The authors articulate how their focus on students and research from these schools illustrates "the process of maturation of an American mathematical research community which had fully emerged by 1900" [Parshall and Rowe, p. xv].

In an epilogue chapter, Parshall and Rowe sketch developments in the American mathematical research community during a period of consolidation and growth from 1900 to 1933. The bookend year of 1933 marks an influx of European mathematicians escaping the Nazi regime. Parshall and Rowe describe how, when scholars like Hermann Weyl and Albert Einstein joined the original faculty at the Institute for Advanced Study, they entered a mathematical community of similar quality to the one they'd left behind in Europe.

In title and content alike, *The Emergence of the American Mathematical Research Community, 1876-1900: J. J. Sylvester, Felix Klein, and E. H. Moore* invited subsequent studies concerning the development of American mathematics influenced by people beyond these "big three," in places outside of Hopkins, Göttingen, and Chicago, as well as in times before and after the delineated quarter century.

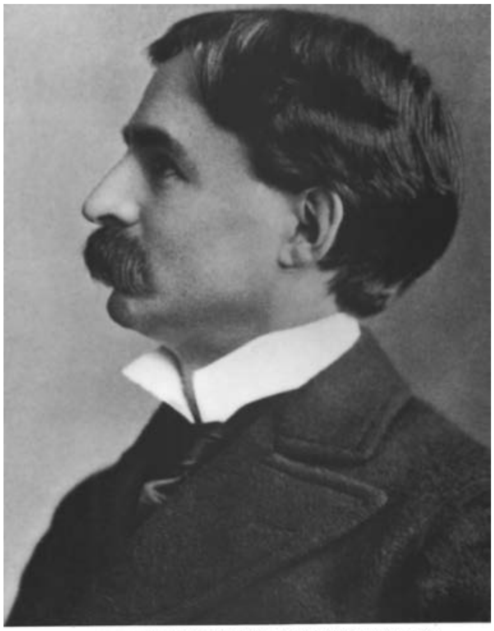

Figure 1. Eliakim Hastings Moore, 1902.

American Mathematics 1890–1913: Catching Up to Europe aligns itself as exactly such scholarship. From the outset, Batterson states very accurately that his book "has a nontrivial overlap" with the one by Parshall and Rowe [Batterson, p. xii]. Batterson's closing acknowledgments further note that "[r]eaders familiar with their work will undoubtedly recognize its impact" [Batterson, p. xii]. Such familiar readers might also wonder why so much similar material is revisited in this work. Batterson explains that his "1890–1913 periodization provides a different window for distinguishing the heroes" involved in "the coming of age of American mathematics" [Batterson, p. xiii]. Batterson especially wants to frame William Osgood and Maxime Bôcher as "intellectual pioneers" who "overcame the obstructions to change that were imposed by institutional culture and economic conditions at American universities" [Batterson, xii]. Batterson starts his periodization when he does because Osgood began an instructorship at Harvard in 1890, and Bôcher joined him the following year. Then E. H. Moore moved to the University of Chicago in 1892 and, says Batterson, "American mathematics began an exponential rise" [Batterson, p. 89].

Building a historical argument is not unlike writing a mathematical paper. One frames a research question in the context of existing scholarship, then provides substantial evidence for a sustained and focused argument that builds to the stated conclusion, a claim that will advance the boundary of knowledge in the field. To add a contribution to historical scholarship, one first must be conversant with a range of existing literature, especially relevant recent work, and then either provide contextualized, carefully documented critical analysis of previously unstudied primary source material or articulate new, conclusively argued insights or perspective on known sources. Reviewing a work of historical scholarship, then, involves considering to what extent the work adds to the existing scholarship and how effectively the author builds a historical argument in support of stated claims.

With the title *American Mathematics 1890–1913: Catching Up to Europe*, Batterson implies two claims. First, that his book will justify this particular periodization in the history of American mathematics. Why choose *these* particular dates? What makes 1890 and 1913 uniquely significant within the larger picture of American mathematics? Second, the title indicates that the text will demonstrate that American mathematics reaches what Batterson calls "parity with European nations" within the delineated period [Batterson, p. 197]. The preface confirms these expectations by stating that the book "examines the 1890–1913 transformation in American mathematics" [Batterson, p. xii] and claiming that American mathematics achieved "international standing" between the years 1900 and 1913 [Batterson, p. xiii]. If a *transformation* occurred, then there must be a *before* circumstance that is somehow demonstrably different from the situation *after* said transformation. So any thoughtful review of *Catching Up to Europe* must take into consideration the degree to which the author situates his claims in the context of related literature and how effectively evidence is marshaled in support of these claims.

Some precedent exists for using Batterson's chosen dates as significant timestamps in the development of American mathematics. At one end, AMS president Thomas Fiske gave a periodization of the development of pure mathematics in America in an address at the AMS meeting in 1904. Like Parshall and Rowe, Fiske drew a line at the establishment of Johns Hopkins University in 1876 to demarcate the first attempt made to "stimulate in a systematic manner research in the field of pure mathematics" [Fiske, p. 3].[2] He ends the second, and thus begins the third, period in 1891, when the New York Mathematical Society (this would become the American Mathematical Society in 1894) assumed a broader scope and began to publish the *Bulletin*. On the

[2] *Smith and Ginsberg also drew a line at the foundation of Johns Hopkins* [*Smith and Ginsberg, 1934*].

other end, Garret Birkhoff described the year 1912 as "a milestone marking the transition from primary emphasis on mathematical *education* at Harvard to primary emphasis on *research*" [G.D. Birkhoff in Duren, p. 27]. It is too bad that Batterson seems unaware of both Fiske's and Birkhoff's perspectives, when an appeal to either could have helped his cause and connected his work to foundational scholarship in the history of American mathematics.

The opening chapter, "An American Colony in Göttingen," revisits narratives told elsewhere about the group of American graduate students who studied in Europe (many with Felix Klein in Göttingen) in the 1880s. The second chapter is a somewhat uneven discussion labeled "19th-Century American Notions of Scholarship." Presumably, the purpose of this chapter is to articulate a picture of American mathematics *before* the transformation Batterson intends to highlight. A tighter narrative rooted in existing scholarship could have sketched the contours more efficiently to allow more space and focus on the primary stated argument of the book. For one example, in describing the well-known Neptune controversy, Batterson ignores work that focuses on how American mathematicians seized on this event as an opportunity to assert their competence at a time of perceived European scientific superiority [Kent, 2011]. The long digression instead becomes unnecessarily detailed and loses the point about American involvement in the story [Batterson, pp. 30–36]. Batterson's passages on Harvard and Yale similarly exclude most of the voluminous literature on nineteenth-century higher education in America. For one example, Batterson seems not to have consulted Roger Geiger's masterful treatise on the metamorphosis of American research universities into world leaders in scientific research between the years 1900 and 1940 [Geiger, 2004]. Any relevant subsequent work, such as *Catching Up to Europe*, would stand firmer on the solid scholarly foundation provided by Geiger's work.

This trend of omitted sources continues in the third chapter, which focuses on the academic institution building of Charles Eliot, president of Harvard from 1869 to 1909, and Daniel Coit Gilman, president of Johns Hopkins from the founding in 1876 until 1901. Again, for minimal examples, Batterson references neither a directly relevant 2013 book on the history of mathematics at Harvard [Nadis and Yau, 2013] nor a standard source with a critical perspective on Gilman's contributions to American higher education [Cordasco, 1960]. Much like mathematical knowledge, historical scholarship is built up by researchers who tackle open questions—they find and fill gaps in existing work or illuminate entirely new directions, perhaps with new sources or new methodologies. Overall, Batterson adds little to the existing coverage of the efforts and impacts of both Eliot and Gilman. It is clear that he has conducted extensive archival work, although unfortunate that such effort and expense was not applied more uniformly to new material. There is no shortage of open questions or unmined archival material relevant to the history of American mathematics.

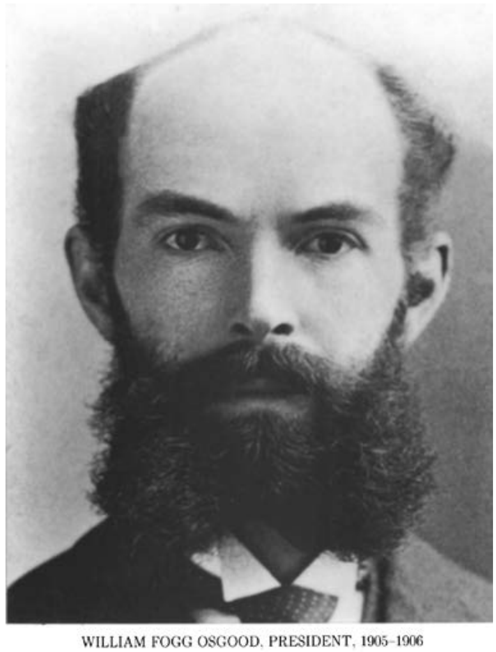

Figure 2. William Fogg Osgood, 1896.

Figure 3. George David Birkhoff, 1913.

Not until Chapter 4 does *American Mathematics 1890–1913: Catching Up to Europe* turn to the events on which Batterson based the start date for his periodization. This "Harvard and Chicago Hire Osgood, Bôcher, and Moore" chapter offers a few new insights into the Harvard careers of Osgood (see Figure 2) and Bôcher, expanding a bit on Batterson's own previous work [Batterson, 2009]. It is unfortunate that in his efforts to champion Osgood, he includes no reference to Diann Porter's recent biography, *William Fogg Osgood at Harvard. Agent of a Transformation of Mathematics in the United States*, even the title of which indicates this could be a rich source of evidence untapped by Batterson [Porter, 2013]. Whatever Batterson's analysis of Porter's work, current standards of historical scholarship dictate at least *some* mention of such closely aligned contemporary work, if for no other reason than to demonstrate the author is aware of it.

The most original part of the book comes in Chapter 5, where Batterson delves into the archives of astronomer

Simon Newcomb and mathematician Henry White to explore discussions about the possibility that the AMS might take over the *American Journal of Mathematics*. J. J. Sylvester, the first professor of mathematics at Johns Hopkins, had founded the *American Journal of Mathematics* in 1878 in keeping with Gilman's research university mandate. After almost seventy-five years of sporadic private publication of mathematical periodicals in the US, the *American Journal of Mathematics* was the first mathematical journal designed to circulate original research and backed by institutional support. It rapidly became the top publication outlet for increasingly research-minded American mathematicians. After Sylvester left America in 1883, "stories of sloppy editing and poor refereeing" demoralized a young cohort keen to publish original work [Batterson, p. 126]. In this well-written and carefully documented chapter, Batterson details how the politics and personalities in this situation led to a new research journal, the *Transactions of the American Mathematical Society*. A revised version of this chapter also appeared as an article in the May 2019 issue of the *Notices*.

The sixth chapter focuses on the preceptor program at Princeton. Woodrow Wilson introduced this program in 1905 to improve undergraduate education by introducing some research-oriented instructors to the Princeton faculty. Oswald Veblen, Luther Eisenhart, and Joseph Wedderburn all started as Princeton preceptors. The former two later were particularly influential in establishing mathematics as a strength at Princeton, especially after National Research Council Fellowships in mathematics started in 1923. In the seventh chapter, Batterson examines events that moved George Birkhoff (see Figure 3) from his Princeton preceptorship to his longtime Harvard faculty position in 1913. This final chapter title, "The *Verge* of Parity with Europe," seems like a bit of a disclaimer. Those words belong to Veblen, who did not give a time frame or definition for parity with Europe [Batterson, p. xi].

Indeed, it is also not clear what Batterson means by "reaching international standing" [Batterson, p. xiii]. The evidence he presents for the claim that this happened for American mathematics by 1913 likewise invites closer examination. The main claims of having "caught up" to Europe seem to be Birkhoff's publication of his proof of Poincare's Geometric Theorem,[3] James Alexander's proof that different constructions of the same manifold must have the same Betti numbers and torsion coefficients [Batterson, p. 187], the 1912 ICM plenary speaking invitations for Osgood and Bôcher, and a memory from Courant that in 1913 mathematicians in Göttingen looked admirably across the Atlantic *for the first time*. These claims invite a more nuanced and critical analysis than what is provided.

There is no question that both the American-educated Birkhoff and Alexander achieved significant mathematical results. Even so, are two big results enough to "catch up" with Europe? It seems not, based on Veblen's correspondence during the period that depicts Hilbert alone at the pinnacle of mathematics [Barrow-Green, 2011]. It must be noted, too, that the 1912 ICM was hosted in Cambridge, UK. Perhaps invitations to speak at a meeting in English-speaking Britain might not say much about the perception of Osgood and Bôcher and American mathematics more generally within continental Europe. Prior to those 1912 invitations, Simon Newcomb gave an invited plenary lecture in applied mathematics and had been a vice president at the ICM in Rome in 1908.[4] Batterson's statement about how American mathematics appeared "to European eyes" heavily relies on a reference to a biography [Reid, 1976] in which the author provides no actual attribution to Courant himself [Batterson, pp. 184–185]. Hence a key piece of Batterson's claim is not substantiated by documented evidence. More, and more reliable, data would likely have presented a more accurate picture of the European perception of American mathematics. For example, it may have been helpful to include reports from Poincaré or Darboux during a 1904 visit [Zitarelli, 2011] or some other documented firsthand accounts.

There is a striking lack of data in the book overall. Some quantification related to numbers and types of publications, PhDs granted, and academic trajectories might have helped Batterson fortify his argument that a new era in American mathematics indeed began in 1913. Rich material along these lines is available in recent articles publishing data about participants in the American mathematical community, their educational attainments, and scholarly output [Fenster and Parshall, 1994; Zitarelli, 2001]. Modern search engines make it fairly straightforward to get a sense of data for years that may be missing in these studies. Considered as a whole, work on the history of American mathematics known to this reviewer still suggests that, in fact, American mathematics had "earned standing" by the time of the *Second* World War rather than the First as Batterson claims to prove [Batterson, p. xi].

The *Catching Up to Europe* book jacket claims that by the First World War, not only had the gap between mathematical research in European and American universities "largely closed" but also that "[i]t would remain so." Reality is not so tidy. The twenty years from 1913 to 1933 brought internal tensions within the American mathematical community, significant institutional transitions, and a major world war, all of which resulted in a trajectory that is not strictly monotone increasing. To demonstrate that parity had, in fact, been sustained from 1913 onwards would

[3]*Poincaré's Geometric Theorem can be stated as follows: If* T *is an area-preserving and boundary-component-preserving homeomorphism of an annulus that rotates the inner boundary and outer boundary in opposite directions, then* T *has at least two fixed points* [*Batterson, p. 182*].

[4]*Other invited American speakers at the Rome ICM included David Eugene Smith and L. E. Dickson. Dickson had also been invited to speak at the Paris ICM in 1900. Americans Artemas Martin, Harris Hancock, and Irving Stringham were also invited to speak in Paris.*

involve acknowledging, addressing, and minimizing known evidence to the contrary. Significant work by Reinhard Siegmund-Schultze includes many sources that contradict the claim of parity by 1913 [Siegmund-Schultze, 2001 and 2009]. Batterson neither addresses these counterexamples nor much discusses the impact of World War I on either American mathematical research or Europe's view of it. The 2014 book *The War of Guns and Mathematics: Mathematical Practices and Communities in France and Its Western Allies around World War I* could have shed some light on these matters [Aubin and Goldstein, 2014]. Work completed since the publication of *Catching Up to Europe* likewise reveals rich archives that also support the much more complex picture suggested by earlier scholarship on American mathematics during the first third of the twentieth century.

As a contribution to historical scholarship, *Catching Up to Europe* leaves some things to be desired. A broader familiarity with relevant scholarship, both classic and current, could have both enriched and focused the significant archival endeavors of this project. Connecting with descendants of historical actors of interest can be a thrill for many historians—Batterson notably accomplished contact with a grandson of Osgood and a great-grandson of Bôcher. Beyond these efforts, more careful attention to building a historical argument may have altered the outcome of insufficient evidence necessary to justify the claim that American mathematics had indeed caught up to Europe by 1913. Or it might have altered the claim. As it stands, readers unfamiliar with much of the history of mathematics in the United States may nonetheless find in this well-written book some items of interest about "how life has evolved for research mathematicians in the United States" [Batterson, p. xiii].

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Deborah Kent

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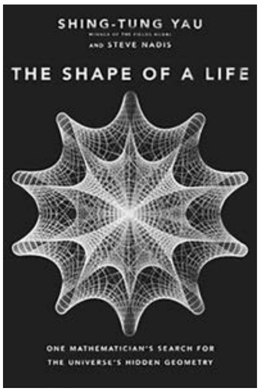

Yale, 2019, 328 pages

The Shape of a Life
One Mathematician's Search for the Universe's Hidden Geometry
by Shing-Tung Yau and Steve Nadis

The Shape of a Life is an autobiography of Shing-Tung Yau, winner of the 1982 Fields Medal and many other prestigious awards. The book is coauthored with science writer and *Discover* contributing editor Steve Nadis, who previously collaborated with Yau in 2010 on the popular science book *The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions*. Although there is no explicit mathematics in the book, the writing style is clear enough that the mathematically literate reader will be able to understand, at an intuitive level, the stunning breakthroughs and remarkable discoveries that Yau and his collaborators contributed to.

The Shape of a Life is an interesting and engaging read, written in a detailed yet lively style. The book vividly documents Yau's trials as a desperately poor child in Hong Kong; his unlikely path to the UC Berkeley graduate program; his deteriorating relationship with his doctoral advisor, Shiing-Shen Chern; the development of geometric analysis; the discovery of Calabi-Yau manifolds; and more.

The Shape of a Life maintains an appropriate pace, never dragging its feet nor skipping important details. Yau is always frank and forthright. He has strong opinions and does not hesitate to share them. Whether one agrees with him or not, one must conclude that *The Shape of a Life* is a deeply personal reflection that provides a keen insight into the life and mind of one of the world's top mathematicians.

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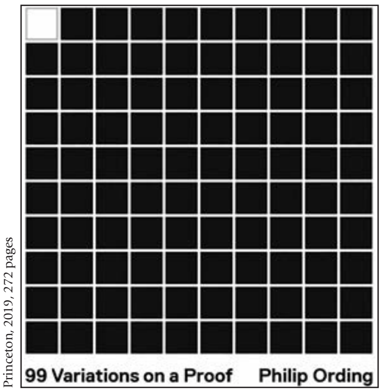

Princeton, 2019, 272 pages

99 Variations on a Proof
by Philip Ording

This entire book is devoted to 99 proofs, liberally interpreted, of the following statement: "If x^3-$6x^2$+$11x$-6 = $2x$-2, then x=1 or x=4." It seems improbable that one could write more than a few pages on such a modest topic, but Ording has managed to parlay this simple question into a perceptive reflection on mathematics and its culture. Students and professors alike will enjoy this unusual book.

The inspiration for *99 Variations on a Proof* comes from Raymond Queneau's *Exercises in Style*, a 1947 work that retells the same story in 99 strikingly different ways. The book is divided into 99 short chapters, each of which explores a different "proof" of the main result. The word "proof" here appears in quotes since most would not pass muster in an undergraduate course. Each comes with a short parenthetical description. For example, Proof 36 (Social Media) appears in the form of a fictitious tweet by Girolamo Cardano: "Cube & 9 times first power equals 6 times square & 4 solved by reduction to @delferro's equation `arxiv.org/abs/4307.1160` #cubic #tartaglia". Some approaches seem uncomfortably familiar, such as Proof 44 (Omitted with Condescension): "There is a simply beautiful theorem which provides all solutions of the equation x^3-$6x^2$+$11x$-6 = $2x$-2. Alas, any further explanation would deny you the satisfaction of discovering it on your own…" and Proof 94 (Authority): "Of course, if x^3-$6x^2$+$11x$-6 = $2x$-2, then it follows from Euler that the real number in question must be 1 or 4." There are also some serious mathematical proofs, graphical proofs, proofs by experiment, and several unique expositions that demonstrate a great deal of artistry. For example, there are proofs in dialogue form and doggerel, along with screenplays and blog entries. Fittingly, the book ends with Proof 99 (Prescribed): "The proof is left to the reader."

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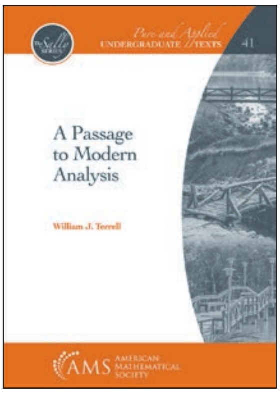

A Passage to Modern Analysis

by William J. Terrell

The author's title reveals his purpose—he wishes to give readers a thorough grounding in the rudiments of analysis so they can pursue further study in those areas of pure and applied mathematics that build upon it. The first third of the book is a careful development of the analysis of real-valued functions of a single real variable. The middle third generalizes to $\mathbf{R}^n$ and introduces metric spaces and normed vector spaces. The final third presents some tools of advanced analysis: Fourier series, ordinary and partial differential equations, the Lebesgue integral, and function spaces.

The exposition is careful and clear throughout. The defining feature is the author's clear-eyed vision of where he wants his readers to go and his foresight in planning the path along this passage. Everything feels completely natural and inevitable because the ground in $\mathbf{R}$ and $\mathbf{R}^n$ has been so thoroughly prepared. Instructors will observe Terrell carefully laying this groundwork, students will just feel a natural progression to higher levels of abstraction. The nearly 600 exercises illuminate the exposition and give instructors plenty of opportunities to push and stretch their students. There is more than enough material here for three semesters of coursework, or instructors can pick and choose topics to construct a year-long course. The prerequisites are low—a few semesters of calculus, some linear algebra, and exposure to the notion of proof—but the end goal is lofty. Readers who study the entire book will be exceedingly well prepared for graduate study leading to research in pure or applied topics in analysis.

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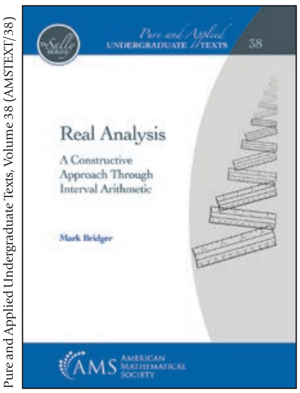

Real Analysis

A Constructive Approach Through Interval Arithmetic

by Mark Bridger

This book contains the clearest exposition of real analysis using constructivist principles that exists in English. The author understands how undergraduates think and meets them where they are. If you seek to avoid proofs by contradiction in your teaching of analysis, this text makes it possible. Every proof is constructive, in particular, existence is never established by proving that the assumption of nonexistence leads to a contradiction.

But the constructivist philosophy is not the most interesting feature of the book. Bridger develops the real numbers out of intervals of rational numbers using interval arithmetic. (This is an approach due to Gabriel Stolzenberg.) The appealing feature of this development is that it yields a clear analogy to scientific measurement. Any scientific measurement of known accuracy falls within an interval with rational endpoints because any measuring instrument yields an approximation. This makes the theory particularly appealing, and accessible, to students looking at further study in computer science or physical science. Computer scientists especially will realize that interval arithmetic, with its error bounds, is precisely what they need to trap an exact value inside a range of computed values. The constructivist law of ε-trichotomy will especially resonate with them: Given real numbers x, y, and $\varepsilon > 0$, either $x > y$, $x < y$, or x and y are within ε of one another.

There is a strong argument to be made that this approach through intervals will feel much more natural and satisfactory to physicists, engineers, applied mathematicians, and computer scientists than the standard mathematical approach. These students realize that every number they compute is an approximation and interval arithmetic tracks the error bounds for them. Even pure mathematicians who might not sympathize with the underlying philosophical principles of constructivism will be gratified by the satisfying solidity and practical utility of interval arithmetic.

Mathematics of Cellular Evolution and Some Biomedical Applications

Natalia L. Komarova

Background: Two Types of Genes

There are two types of genes whose evolution we would like to understand because of these genes' important role in cancer. *Tumor suppressor genes* protect cells from acquiring malignant properties. If a genetic mutation inactivates one of the copies of such a gene, the other copy will still protect the cell from becoming malignant. Therefore, in the context of this pathway, it takes two independent genetic "hits" to transform a cell. *Oncogenes* work differently. A specific mutation in any of the two copies of such a gene can turn on a malignant function, making the cell cancerous. It takes only a single genetic hit to turn on this mechanism of cancerous transformation.

Many cancers evolve as a chain of several events that can include mutations in one or both of these types of genes. In this talk we will look at the mathematical methods to study the evolution of cancerous populations and concentrate on these two gene types as building blocks of cancer.

The Moran Process of Cellular Turnover

Cellular populations are sometimes described by means of the so-called Moran process. Assume that there are N cells that undergo discrete rounds of divisions and deaths. In each round, one division and one death occur. There are two types of cells, which we will refer to as "wild type" (type A) and "mutant" (type B) cells. All wild type cells are characterized with a constant division rate, r_A, and a constant death rate, d_A. Similarly, mutants have a constant division rate, r_B, and a constant death rate, d_B, where in general, $r_A \neq r_B$ and/or $d_A \neq d_B$. For each update, a cell is selected for division with a probability that is proportional to its division rate. For example, if there are m mutants and $N-m$ wild type cells in the system, the probability that a mutant cell will divide is given by

$$P(\text{mut. div.}) = \frac{mr_B}{(N-m)r_A + mr_B},$$

and the probability that a wild type cell will divide is given by $P(\text{w.t. div.}) = 1 - P(\text{mut. div.})$. A division is followed by a death, with probabilities proportional to cells' death rates (formulated similarly). The newly produced offspring of the cell that divided then replaces the dead cell, and the total population remains at N. In the simplest case, the progeny cell retains the type of the parent cells. If mutations are included, the progeny cell may be of a different type compared to the parent cell.

If we are interested in oncogenes (or, more generally, the production of one-hit mutants), we can study the statistics and dynamics of mutant generation and spread; we can assume that the mutants can be advantageous (e.g. $r_B > r_A, d_B = d_A$), neutral ($r_B = r_A, d_B = d_A$), or disadvantageous (e.g. $r_B < r_A, d_B = d_A$). In the context of tumor suppressor genes (two consecutive mutations) we can study the timing of the generation of two-hit mutants.

This basic process is as well understood as it is unrealistic [4, 7]. Adding more complexities leads to interesting mathematical problems and opens up possibilities to answer some biologically relevant questions.

Spatial Moran Process

One important modification is to include a network of interactions. A reproducing cell can only replace a dying cell if they are connected by an edge of a network. In particular, a geometric network can be used to describe 1D, 2D (Figure 1), or 3D spatial systems. Do such spatial interactions speed up or slow down evolution? In the context of tumor-suppressor genes, the result is that under geometric networks, double-hit mutants are produced faster (and this can be explained intuitively) [1, 5]. When it comes to one-hit mutants (such as mutations in oncogenes), the answer is less straightforward: spatial systems may have fewer or more mutants compared to nonspatial systems of the same size.

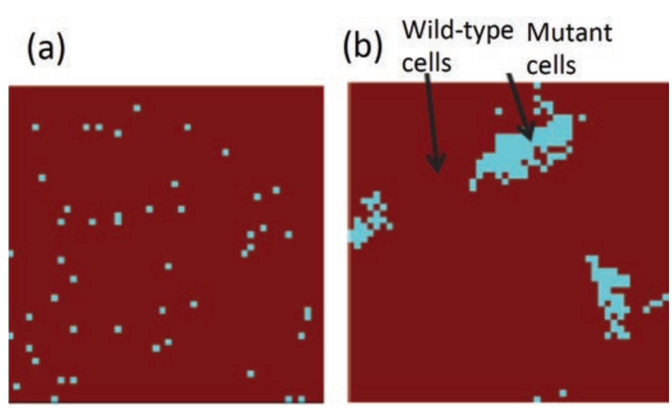

Figure 1. Nonspatial (a) and spatial (b) Moran process.

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DOI: https://doi.org/10.1090/noti2083

Cellular Hierarchies

Even if we don't consider mutations, not all cells are created equal. In real tissues, we can distinguish the so-called stem cells (SCs) and differentiated cells (DCs). Oversimplifying, we can say that SCs can divide into two SCs, two DCs, or one of each. Differentiated cells do not divide at all. What happens if we add cellular hierarchies to the basic Moran process (so, in the presence of a single type of mutation, there are now four types of cells)? Does this hierarchical structure change the speed of evolution? It turns out that it has an effect that is opposite of that described above. Cellular hierarchies slow down evolution considerably [6]. Perhaps this is one of the reasons the organs of multicellular organisms often have a hierarchical structure: this is a type of protection against cancer.

Cooperation and Defection

Cells in organs can cooperate. For example, suppose two consecutive mutations that occur in the same cell make this cell very advantageous. It could happen that two cells with complementary mutations, if they are found in each other's proximity, can share the products of the mutated genes, and both act as a very advantageous cell. Does cooperation speed up evolution? Yes, absolutely [3]. But cheating (defection) can speed up evolution even more!

Random Environments

So far, all the properties conferred by genetic mutations were fixed. For example, a disadvantageous mutant would always have a reduced division rate. This is a simplification which does not take into account the randomness of the environment. Depending on time and/or space, a mutant could have a higher or lower division rate compared to the wild type. What consequences does this have for the evolutionary process? Some very counterintuitive results can be found in this area [2], which will be presented in the lecture.

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Natalia L. Komarova

Credits

Figure 1 is from [5].
Photo of the author is courtesy of the author.

KEYSTONES

Probabilistic Inference for Trust

Andres Molina-Markham and Joseph J. Rushanan

Computer systems that need to account for uncertainty—such as those enabled by artificial intelligence (AI)—require novel approaches to evaluate their trustworthiness. These systems exhibit probabilistic behavior that depends on data from sources (sensors and computation) with varying degrees of trust, uncertain availability, and complex stochastic relations. Hence, describing the valid behavior of a probabilistic system and verifying that an implementation satisfies a formal specification require new approaches [1]. Furthermore, the problem becomes more challenging when adversaries can influence decisions (to induce erroneous detections or misinterpretation of the state of operations and more generally to force wrong decisions) by manipulating observations or the results of actions taken by an AI-enabled system (a system designed to perform tasks that normally require human intelligence, such as visual perception, threat assessment, or complex decision making). Therefore, evaluating trust in the presence of adversaries requires the triad of describing the relevant threats, establishing a set of assurance mechanisms to counter those threats, and characterizing valid behavior in a way that is amenable to verification. While a comprehensive solution to the problem of evaluating trustworthiness in adversarial settings is an open problem, its solution will rely on mathematics at various levels, from modeling threat and counterthreat behaviors and valid behavior of a system to designing robust protection mechanisms and Bayesian inference engines needed to estimate trust.

The MITRE Corporation is addressing the trust inference question in two specific AI applications. First, we have developed a trust inference engine for the widely important problem of trust in positioning, navigation, and timing (PNT) information. Such inference considers multiple sources of information, such as the Global Positioning System (GPS) and other sensor inputs, situational awareness information, and auxiliary sources (e.g., network data). The challenge is to fuse trust assumptions and assessments of these sources into useful assurance metrics that can be scrutinized and refined. Our solution, PNTTING (PNT Trust Inference Engine) facilitates this trust fusion according to probabilistic models with rigorous semantics, leveraging the emerging work in probabilistic programming languages (e.g., languages such as Anglican [2] or Gen [3]). PNTTING's probabilistic models describe relations between inputs and outputs and how inputs are transformed and combined. PNTTING encodes input transformations and other relations via probabilistic models developed by assurance model designers and PNT engineers.

Second, the work on PNTTING is being extended to address the problem of evaluating the extent to which we can trust the decisions made by autonomous vehicles (AVs) in adversarial environments. The goal of this new effort is to develop a similar formal framework and a probabilistic inference engine to assess AV decision-making platforms. Vehicles belong to a class of cyber-physical systems that can be exploited to cause serious harm. Thus, assessing the validity of an AV decision-making platform in adversarial conditions is a critical problem that requires modern probabilistic modeling concepts [4], as well as state-of-the-art

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Communicated by Notices *Associate Editor Emilie Purvine.*

DOI: https://dx.doi.org/10.1090/noti2081

probabilistic computing concepts [3], to attain rigor and practicality.

Our methods to evaluate trust leverage the vast research done in probabilistic modeling and inference [2], [3], [5], [6]. Adaptation of these methods leads to many open mathematical questions, including: What is the effect of our prior assumptions of trust? How much computation is needed for accurate trust inference? What is the dependence of this inference on the possible metrics for validity? Answering these questions requires logic, statistical methods, theory of algorithms, and advanced programming language concepts.

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Joseph J. Rushanan

Credits

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AMS Prizes & Awards

Levi L. Conant Prize

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years.

About this Prize

Levi L. Conant was a mathematician and educator who spent most of his career as a faculty member at Worcester Polytechnic Institute. He was head of the mathematics department from 1908 until his death and served as interim president of WPI from 1911 to 1913. Conant was noted as an outstanding teacher and an active scholar. He published a number of articles in scientific journals and wrote four textbooks. His will provided for funds to be donated to the AMS upon his wife's death.

Prize winners are invited to present a public lecture at Worcester Polytechnic Institute as part of their Levi L. Conant Lecture Series, which was established in 2006. Find and download videos of previous Conant Lectures at `www.wpi.edu/academics/math/news/lconant-series.html`.

The Conant Prize is awarded annually in the amount of US$1,000.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/conant-prize`.

Cole Prize in Algebra

This Prize recognizes a notable research work in algebra that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

About this Prize

This prize (and the Frank Nelson Cole Prize in Number Theory) was founded in honor of Professor Frank Nelson Cole upon his retirement after twenty-five years as secretary of the American Mathematical Society. Cole also served as editor-in-chief of the *Bulletin* for twenty-one years. The original fund was donated by Professor Cole from moneys presented to him on his retirement, and was augmented by contributions from members of the Society. The fund was later doubled by his son, Charles A. Cole, and supported by family members. It has been further supplemented by George Lusztig and by an anonymous donor.

The current prize amount is US$5,000 and the prize is awarded every three years.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/cole-prize-algebra`.

Mary P. Dolciani Prize for Excellence in Research

The AMS Mary P. Dolciani Prize for Excellence in Research recognizes a mathematician from a department that does not grant a PhD who has an active research program in mathematics and a distinguished record of scholarship. The primary criterion for the prize is an active research program as evidenced by a strong record of peer-reviewed publications.

Additional selection criteria may include the following:

- Evidence of a robust research program involving undergraduate students in mathematics;
- Demonstrated success in mentoring undergraduates whose work leads to peer-reviewed publication, poster presentations, or conference presentations;
- Membership in the AMS at the time of nomination and receipt of the award is preferred but not required.

About this Prize

This prize is funded by a grant from the Mary P. Dolciani Halloran Foundation. Mary P. Dolciani Halloran was a gifted mathematician, educator, and author. She devoted her life to developing excellence in mathematics education and was a leading author in the field of mathematical textbooks at the college and secondary school levels. Read more about her and the Foundation at `www.dolcianihalloranfoundation.org/meet-mary/`.

The prize amount is US$5,000, awarded every other year for five award cycles.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/dolciani-prize`.

Ulf Grenander Prize in Stochastic Theory and Modeling

The Grenander Prize recognizes exceptional theoretical and applied contributions in stochastic theory and modeling. It is awarded for seminal work, theoretical or applied, in the areas of probabilistic modeling, statistical inference, or related computational algorithms, especially for the analysis of complex or high-dimensional systems.

About this Prize

This prize was established in 2016 by colleagues of Ulf Grenander (1923–2016). Professor Grenander was an influential scholar in stochastic processes, abstract inference, and pattern theory. He published landmark works throughout his career, notably his 1950 dissertation, *Stochastic Processes and Statistical Interference* at Stockholm University, *Abstract Inference*, his seminal *Pattern Theory: From representation to inference,* and *General Pattern Theory*. A long-time faculty member of Brown University's Division of Applied Mathematics, Grenander received many honors. He was a Fellow of the American Academy of Arts and Sciences and the National Academy of Sciences and was a member of the Royal Swedish Academy of Sciences.

The current prize amount is US$5,000 and the prize is awarded every three years.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/grenander-prize`.

Bertrand Russell Prize of the AMS

The Bertrand Russell Prize honors research or service contributions of mathematicians or related professionals to promoting good in the world and recognizes the various ways that mathematics furthers human values.

About this Prize

The Bertrand Russell Prize of the AMS was established in 2016 by Thomas Hales. The prize looks beyond the confines of the profession to research or service contributions of mathematicians or related professionals to promoting good in the world. It recognizes the various ways that mathematics furthers fundamental human values. Mathematical contributions that further world health, our understanding of climate change, digital privacy, or education in developing countries are some examples of the type of work that might be considered for the prize.

The current prize amount is US$5,000, awarded every three years.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/russell-prize`.

Ruth Lyttle Satter Prize in Mathematics

The Satter Prize recognizes an outstanding contribution to mathematics research by a woman in the previous six years.

About this Prize

This prize was established in 1990 using funds donated by Joan S. Birman in memory of her sister, Ruth Lyttle Satter. Professor Birman requested that the prize be established to honor her sister's commitment to research and to encourage women in science. An anonymous benefactor added to the endowment in 2008.

The current prize amount is US$5,000 and the prize is awarded every two years.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/satter-prize`.

Albert Leon Whiteman Memorial Prize

The Whiteman Prize recognizes notable exposition and exceptional scholarship in the history of mathematics.

About this Prize

This prize was established in 1998 using funds donated by Mrs. Sally Whiteman in memory of her husband, Albert Leon Whiteman.

The US$5,000 prize is awarded every three years.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/whiteman-prize`.

Mathematics Programs that Make a Difference

This Award for Mathematics Programs that Make a Difference was established in 2005 by the AMS's Committee on the Profession to compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are replicable models.

About this Award

One program is selected each year by the AMS Committee on the Profession and is awarded US$1,000 provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

Preference is given to programs with significant participation by underrepresented minorities. Note that programs aimed at pre-college students are eligible only if there is a significant component of the program benefiting individuals from underrepresented groups at or beyond the undergraduate level.

Next Award: 2021

Nomination Deadline: September 15, 2020

The nomination procedure can be found at `https://www.ams.org/make-a-diff-award`. For questions, contact `aed-mps@ams.org`.

Joint Prizes & Awards

Birkhoff Prize in Applied Mathematics (AMS-SIAM George David Birkhoff Prize)

The Birkhoff Prize is awarded for an outstanding contribution to applied mathematics in the highest and broadest sense.

About this Prize

The prize was established in 1967 in honor of Professor George David Birkhoff, with an initial endowment contributed by the Birkhoff family and subsequent additions by others. The American Mathematical Society (AMS) and the Society for Industrial and Applied Mathematics (SIAM) award the Birkhoff Prize jointly.

The current prize amount is US$5,000, awarded every three years to a member of AMS or SIAM.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination, go to `https://www.ams.org/birkhoff-prize`.

Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student

The Morgan Prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who was enrolled as an undergraduate in December at a college or university in the United States or its possessions, Canada, or Mexico is eligible for the prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

About this Prize

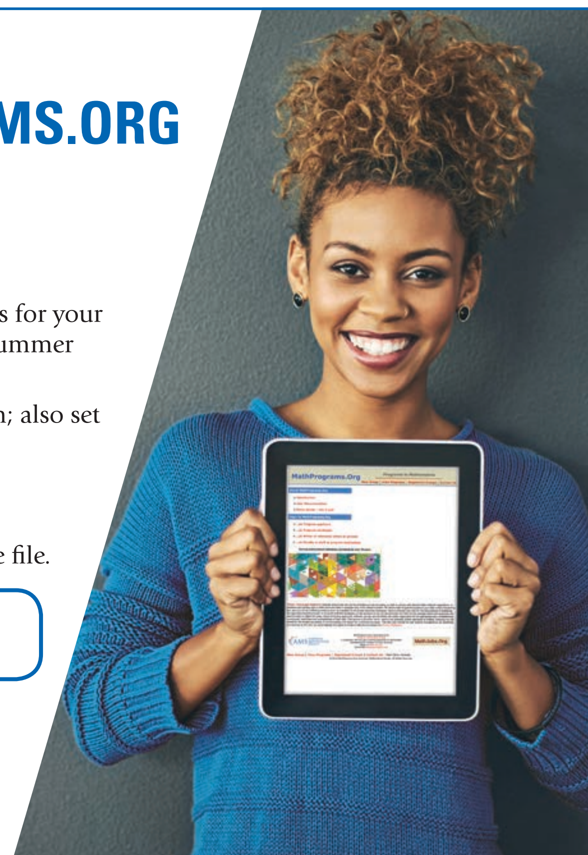
The prize was established in 1995. It is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. It is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

The current prize amount is US$1,200, awarded annually.

Next Prize: January 2021

Nomination Period: March 1–June 30, 2020

To make a nomination go to `https://www.ams.org/morgan-prize`.



Call for Nominations

2021

Frank and Brennie Morgan AMS-MAA-SIAM Prize

for Outstanding Research in Mathematics by an Undergraduate Student

The prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who is an undergraduate in a college or university in the United States or its possessions, Canada, or Mexico is eligible to be considered for this prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate; they cannot be written after the student's graduation. The research paper (or papers) may be submitted for the committee's consideration by the student or a nominator. Each submission for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research. Publication of research is not required.

The recipients of the prize are to be selected by a standing joint committee of the AMS, MAA, and SIAM. The decisions of this committee are final. Nominations for the 2021 Morgan Prize are due no later than June 30, 2020. Those eligible for the 2021 prize must have been undergraduates in December 2019.

Questions may be directed to:

James Sellers, Secretary
Mathematical Association of America

Penn State University
University Park, PA 16802

Telephone: 814-865-7528
Email: sellersj@psu.edu

Nominations and submissions should be sent to:

Carla Savage, Secretary
American Mathematical Society

Computer Science Department
North Carolina State University
Raleigh, NC 27695-8206

or uploaded via the form available at:
www.mathprograms.org

AMS American Mathematical Society | MAA Mathematical Association of America | siam

2020 Award for an Exemplary Program or Achievement in a Mathematics Department

The **Department of Mathematics at the Massachusetts Institute of Technology** is the recipient of the 2020 AMS Award for an Exemplary Program or Achievement in a Mathematics Department.

Figure 1. PRIMES students at an annual conference at MIT.

Citation

The 2020 Award for an Exemplary Program or Achievement in a Mathematics Department is presented to the Department of Mathematics at the Massachusetts Institute of Technology. The MIT Mathematics Department is being honored for its Program for Research in Mathematics, Engineering, and Science for High School Students (PRIMES), which provides significant research experiences and mathematics enrichment to high school students locally and globally, with particular attention to increasing the representation of women and underrepresented minorities.

PRIMES is a free, year-long research and enrichment program for high school students, created in the MIT Mathematics Department in October 2010. The program's innovative, year-long model for guiding high school student research is being used also in computer science, bioinformatics, computational and physical biology, genomics, and neuroscience. PRIMES students use their knowledge of mathematics and computer programming to solve problems related to cancer research, Internet security, traffic control, refugee migration, brain research, laser engineering, and many other applied fields. PRIMES Circle and MathROOTS help attract members of underrepresented groups to pursue careers in the STEM fields. A 2015 article about PRIMES, which sums up its experience and provides advice for setting up similar programs in other institutions, published in the *Notices of the AMS* (Pavel Etingof, Slava Gerovitch, and Tanya Khovanova, "Mathematical Research in High School: The PRIMES Experience," *Notices of the AMS*

62, no. 8 (2015)), has been translated into Mandarin and summarized in Spanish. In 2017, the PRIMES-Switzerland program was established at the University of Geneva and ETH in Zürich.

PRIMES includes five sections. MIT PRIMES offers research projects and guided reading to students living within driving distance from Boston. Program participants work with MIT researchers on exciting unsolved problems in mathematics, computer science, and computational biology. PRIMES-USA is a distance mentoring math research program for high school juniors from across the United States (outside of Greater Boston). PRIMES Circle and MathROOTS are math enrichment programs for high-potential high school students from underrepresented backgrounds or underserved communities. PRIMES Circle is a spring-term program for local students from Greater Boston. MathROOTS is a free two-week residential summer program. CrowdMath, run jointly with Art of Problem Solving, is a massive collaborative year-long online research forum open to all high school and college students around the world.

PRIMES has three main goals:

1. To give talented high school students a unique opportunity to experience the joy and beauty of mathematical research
2. To inspire them to pursue careers in the mathematical sciences
3. To diversify the pool of students interested in mathematics by providing additional opportunities for promising young women and underrepresented groups.

Between 2011 and 2019, 281 students participated in MIT PRIMES and PRIMES-USA. All of the 276 research projects completed were presented at nine annual PRIMES conferences, 179 research papers have been posted online, and at least 31 have been published in such journals as *Representation Theory*, *Journal of Algebra*, *Journal of Algebraic Combinatorics*, *Journal of Combinatorics*, *Journal of Integer Sequences*, *Electronic Journal of Combinatorics*, *International Journal of Game Theory*, *Transactions of the AMS*, *College Mathematics Journal*, *Topology and Its Applications*, *Involve*, *Math Horizons*, *Cell Reports*, *Letters in Biomathematics*, *Physical Review E*, and *PLoS Computational Biology*. Several PRIMES students have won prizes and awards at the MAA Undergraduate Student Poster sessions, at the Intel International Science and Engineering Fair, in the Siemens Competition in Math, Science, and Technology, and in the Intel/Regeneron Science Talent Search. Four Davidson Fellow Laureates and eight Davidson Fellows have been PRIMES participants.

From 2013 to 2019, eighty-four students completed the PRIMES Circle program, including sixty female, fourteen African American, and nine Latino students. From 2015 to 2019, one hundred students completed the MathROOTS program, including forty-four female, forty-seven African American, and forty-nine Latino students. Among the eighty participants in the 2015 to 2018 summer programs, forty-two were admitted to MIT, and twenty-eight enrolled. In 2019, forty out of 112 PRIMES students were female, and twenty-one were minority students

About the Program

In October of 2020 MIT PRIMES, founded by Pavel Etingof and Slava Gerovitch, will celebrate its tenth anniversary. It started as an experiment in year-long math research by high school students with just twenty-one local participants. The experiment proved very successful, with the program growing more than fivefold in ten years and expanding both nationally and internationally.

Figure 2. PRIMES Circle students Laura Clervil and Sekai Carr giving a talk at an annual conference at MIT.

Figure 3. MathROOTS students at a study session on the roof of the MIT Math Department's building.

PRIMES offers real, not toy, research projects to high school students and provides academic mentorship for a full year. The program builds collaborative teams that include faculty, postdoctoral researchers, graduate students, undergraduates, and high school students, promoting partnership and wider outreach in the mathematical sciences community.

Keys to the PRIMES success are thorough preparation, continuous review of research projects, and effective mentorship techniques. Choosing a research project for a high school student is no easy task. PRIMES's experience shows that most fruitful are the projects that have an *accessible beginning* with relatively simple initial steps; *flexibility* in switching among several related questions; *computer-assisted exploration* aimed at finding patterns and making conjectures; *faculty advisor involvement*; relation to the *mentor's own research area*; understanding of the *big picture and motivation*; a *learning component* that encourages the student to study advanced mathematics; and *doability* within a year-long time frame.

Effective mentorship involves striking a balance between guiding the student and allowing independent thinking, being attuned to the learning and research style of every student, and regularly reviewing the project progress and adjusting its scope, if needed. Head mentor Tanya Khovanova regularly meets with students to gather their feedback and help with communication and motivation issues.

PRIMES has accumulated experience in supervising both individual and group (two to three students) research projects, guided reading groups, a larger research group (a six-student computer algebra lab), and an open online forum with a varying group of participants (CrowdMath). The program regularly conducts both internal and external evaluations via surveys and interviews with student participants.

An award-winning student wrote in her testimonial: "PRIMES is an incredible opportunity that allows high schoolers to do what they would never normally have the chance to do: research, while also providing the guidance and encouragement that is crucial for success. Ultimately, PRIMES has truly cemented my interest in math, and it is for this reason that I would definitely encourage any student similarly passionate about mathematics to apply!"

Female and minority students from PRIMES Circle commented in their survey: "My experience in MIT PRIMES Circle has allowed me to appreciate mathematics from a new perspective and become fascinated by how beautifully simple a complex idea can become. The best part was walking out of the program with better developed critical thinking skills and a mathematical toolbox that I could apply to the real world." "To me, PRIMES Circle is a wonderful opportunity to learn what it is like to be a real mathematician."

Minority students attending MathROOTS also commented on their experience: "Personally, I thought it was awesome seeing people from a multitude of backgrounds all interested and passionate about math." "Before MathROOTS, I was unsure about MIT, but now, I feel like it's my home. I hope I get to return there soon."

About the Award

The Award for an Exemplary Program or Achievement in a Mathematics Department was established by the AMS Council in 2004 and was given for the first time in 2006.

This award recognizes a department that has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Departments of mathematical sciences in North America that offer at least a bachelor's degree in mathematical sciences are eligible. Through the generous support of an anonymous donor, the award carries a cash prize of US$5,000. The award is presented by the AMS Council acting on the recommendation of a selection committee. The members of the 2020 selection committee were George E. Andrews, Maria M. Klawe, Richard S. Laugesen, Brea Ratliffe (Chair), and Ulrica Y. Wilson.

Credits

Figures 1 and 2 are by Slava Gerovitch.
Figure 3 is by Sandi Miller.

2020 Award for Impact on the Teaching and Learning of Mathematics

Darryl Yong of Harvey Mudd College has been named the recipient of the 2020 AMS Award for Impact on the Teaching and Learning of Mathematics.

Darryl Yong

Citation

Dr. Darryl Yong is a professor of mathematics at Harvey Mudd College, where he also serves as the program director for the Mathematics Clinic. An accomplished mathematician who has written six books and several research papers that have appeared in top applied math and physics journals, Dr. Yong is also a prominent researcher in math education, with a scholarly focus on active and inquiry-based learning, inclusive pedagogy, and training of high school math teachers.

In 2007, Dr. Yong started a nonprofit professional development organization for math teachers called Math for America Los Angeles (MfA LA). This program has supported over 200 high school math and computer science teachers with multiyear fellowships for salary supplements, in addition to providing professional development opportunities and a supportive community. He is the primary author of four NSF Robert Noyce Scholarship Grants that have raised over US$12 million for MfA LA. Dr. Yong spent a sabbatical year teaching high school mathematics in the Los Angeles Unified School District, which he wrote about in a 2012 AMS *Notices* article entitled "Adventures in Teaching: A Professor Goes to High School to Learn about Teaching Math." He has also worked with the Teacher Leadership Program at the IAS/Park City Mathematics Institute since 2007 and has cotaught a math course for elementary and secondary math teachers that led to a book series published by the AMS containing teacher development materials using a problem-based approach.

At the college level, Dr. Yong has become an expert on inquiry-based learning methods and participated in a four-year controlled study of flipped classroom instruction supported by the NSF, which led to several research articles in conference proceedings and peer-reviewed journals. Dr. Yong is regarded by his colleagues at Harvey Mudd and the other Claremont Colleges as a gifted teacher who will continue to have a profound influence on how students and teachers perceive mathematics. In particular, he was the founding director of the Claremont Colleges Center for Teaching and Learning and served as the associate dean for diversity at Harvey Mudd from 2011 to 2016.

For his many sustainable and replicable contributions to mathematics and mathematics education at both the precollege and college levels, the AMS Committee on Education is delighted to award Dr. Darryl Yong the AMS Award for Impact on the Teaching and Learning of Mathematics.

Biographical Sketch

Darryl Yong earned his BS in mathematics at Harvey Mudd College, where he was also a piano performance major. He then earned a PhD in applied mathematics at the University of Washington under the supervision of Jirair Kevorkian. He served as a von Kármán Instructor at the California Institute of Technology from 2001 to 2003 and has worked at Harvey Mudd College since 2003. At Mudd, Dr. Yong is currently a professor of mathematics, associate dean for diversity and faculty development, and Mathematics Clinic program director.

His previous applied mathematics research focused on multiple-scale analysis of hyperbolic partial differential equations. His scholarly activities now focus on the retention and professional development of secondary school mathematics teachers and improving undergraduate mathematics education. He is regularly invited to speak and provide professional development for mathematics faculty on the teaching and learning of mathematics, broadening participation in STEM, and helping institutions build capacity for increasing diversity and inclusion.

He received a Caltech ASCIT Teaching Award in 2002, a Pomona Unified School District Community Service Award in 2012, and the Mathematical Association of America Southern California–Nevada Section Teaching Award in 2017.

Response

I am truly honored to receive this award and humbled to be included among the other award recipients. All of the different efforts that were mentioned above (Math for America Los Angeles, IAS/Park City Mathematics Institute, the Claremont Colleges Center for Teaching and Learning, and research projects) are/were collaborative efforts made possible by supportive and wonderful colleagues across many institutions. I am especially grateful to my colleagues at Harvey Mudd College for nominating me and supporting me in my work. And I am grateful to Kenneth I. and Mary Lou Gross and the AMS for their support of mathematics education at the precollege and college levels.

About the Award

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education (COE) in 2013. The award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education. Priorities of the award include recognition of (a) accomplished mathematicians who have worked directly with precollege teachers to enhance teachers' impact on mathematics achievement for all students, or (b) sustainable and replicable contributions by mathematicians to improve the mathematics education of students in the first two years of college. The US$1,000 award is given annually, and the recipient is selected by the COE. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen. The award is presented by the AMS COE acting on the recommendation of a selection subcommittee. For the 2020 award, the members of the subcommittee were:

- Douglas Ensley
- Michael Dorff
- Jon Wilkening (Chair)

A listing of the previous recipients of the Impact Award can be found on the AMS website at: `https://www.ams.org/ams-awards/impact`.

Credit

Photo of Darryl Yong is courtesy of Harvey Mudd College.

2020 Mathematics Programs That Make a Difference Award

The **Graduate Research Opportunities for Women** program is the recipient of the 2020 AMS Mathematics Programs That Make a Difference Award.

Citation

The American Mathematical Society, through its Committee on the Profession, is pleased to recognize the Graduate Research Opportunities for Women (GROW) Program with the 2020 Mathematics Programs That Make a Difference Award. GROW is an annual series of conferences that nurture, mentor and expose undergraduate women to the opportunities that await a career in mathematics. Funded by the National Science Foundation and participating universities, the GROW Program is in its fifth year and has served hundreds of participants. Over this short span, GROW has built a community which, as much as the conference programming itself, has helped to make the mathematics profession a more appealing place for women to live and work. Through feedback, GROW steadily improves and creates best practices for future iterations as well as for replication. Activities at GROW include research talks where scholars discuss not only their results but their varied routes through academics, giving a personal touch and dispelling the straight-and-narrow myth around career paths. There are also panel discussions about graduate admissions. Conference-goers come with questions about preparation, the importance (or not) of GRE scores, how to approach letter writers, and so forth. Other meet-and-greet activities aim to connect participants to experts in a potential field of interest. The community-building aspect of the gathering is crucial for female students in a majority-male profession; the feeling that one is not alone can boost confidence. The program includes inspirational talks by iconic female speakers that make a big impression on the participants. As one conference attendee who is now in graduate school writes in her support letter: “I gained confidence, personal and professional connections, and exposure to various careers in mathematics.… I met many women who assuaged my mounting fears about applying and succeeding in graduate school.… Sharing my fears and concerns about graduate school with other women who were either entering or attending graduate school was one of the most helpful aspects of GROW.” The AMS commends the GROW Program for its success in bringing more persons from underrepresented backgrounds into the mathematical profession.

GROW 2017. Left to right: Emmy Murphy, Ben Antieau, Bryna Kra.

GROW 2019 participants.

About the Program

The GROW workshop series encourages female-identifying undergraduates to consider research in mathematics as a discipline and a career. For the past five years, approximately eighty students have gathered over a weekend in October for a mixture of research talks, panel discussions, and opportunities to meet students and scholars from across the country. The participants share meals and have numerous opportunities for networking and mentoring. GROW is designed to encourage women to think and know about mathematics and to feel confident about the options for graduate students, as well as providing them with resources for their future success.

The first iteration of the GROW series was designed by Bryna Kra at Northwestern University, and she, along with more than forty volunteers, led organization of the conferences at Northwestern from 2015 to 2017. Since then, GROW has moved first to the University of Michigan in 2018, with Sarah Koch and Karen Smith as lead organizers, and then to the University of Illinois at Urbana-Champaign in 2019, with Zoi Rapti as the lead organizer. The next iteration will be at the University of Chicago in 2020. Over 350 students from across the United States have already participated in GROW, and participants have represented more than seventy-five undergraduate institutions.

The panel discussions cover what constitutes research in mathematics, with panelists sharing stories of their trajectories, an introduction to the varied options for careers in academia, and a wealth of information on the nuts and bolts of applying to graduate school in mathematics. A highlight of each meeting has been the Saturday evening dinner, with leading figures (including Alexandra Bellow, Dusa McDuff, Ingrid Daubechies, Kristin Lauter, and Marisa Eisenberg) giving inspirational talks.

About the Award

In 2005, the American Mathematical Society, acting upon the recommendation of its Committee on the Profession (CoProf), established the Mathematics Programs That Make a Difference Award in order to profile those programs that are succeeding and could serve as a model for others. Specifically, the committee seeks to honor programs that:

1. aim to bring more persons from underrepresented minority backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to an advanced degree in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are replicable models.

Preference is given to programs with significant participation by underrepresented minorities.

This recognition includes an award of US$1,000 provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

For a list of previous recipients of the Mathematics Programs That Make a Difference Award, see the AMS website at `https://www.ams.org/make-a-diff-award`.

—Elaine Kehoe with information from Bryna Kra

Credits

Photo from GROW 2017 is courtesy of the Department of Mathematics, Northwestern University.
Photo from GROW 2019 is courtesy of Zoi Rapti.

Read more about the Graduate Research Opportunities for Women Program on page 724.

GROWing a Graduate Cohort

Evelyn Lamb

Bryna Kra was frustrated. It was early 2015, and the math professor at Northwestern University had seen the number of women applying to her program decline for several years, even as the total number of applications was increasing. Anecdotally, her colleagues at other institutions were reporting similar problems. She wanted to do something to reverse the trend. After talking with other mathematicians at Northwestern, she decided to organize a conference specifically focused on encouraging women to apply for graduate school in math. A few months later, fifty undergraduate women descended on Evanston for the first iteration of the Graduate Research Opportunities for Women, or GROW, conference in October 2015. This year, it received the Programs that Make a Difference award from the American Mathematical Society. "It's an honor and rewarding to know that the program has been recognized only five years after it was established, and it makes it clear how valuable such programs are," Kra says. "My hope is that this will give the impetus for other institutions to take charge of the program in future years, hosting GROW and building more diverse departments."

In the five years since the first conference, GROW has taken place three times at Northwestern and once each at the University of Michigan and the University of Illinois at Urbana-Champaign. In October, it will move to the University of Chicago for the sixth iteration. Hundreds of young women have participated in GROW and been encouraged to consider graduate school in mathematics. "Ultimately, the main point of GROW is to increase the comfort that bright women undergraduates have with the idea of doing a doctorate in the mathematical sciences, that it is something available to them," says Ezra Getzler, one of Kra's colleagues at Northwestern and co-organizer of the conference when it took place there. Kra says that many alumnae have told her GROW gave them the nudge they needed to apply to graduate school.

Figure 1. GROW participants at the 2017 conference.

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DOI: https://dx.doi.org/10.1090/noti2088

Genesis

Putting on a conference for dozens of undergraduates in just a few months was no small feat. Kra and her co-organizers Getzler and Laura De Marco had the resources and connections to make GROW happen on their condensed timeline. They used funds from a Research Training Group (RTG) grant, in addition to university-level funding that they could secure on short notice. "Part of the purpose of an RTG is to create a vertical conveyor belt for mathematics in the United States, and so here we're dealing with a key transition point from undergraduate to graduate school," Getzler says. "It's a continual struggle to make sure that people are trained at the graduate level in mathematical sciences in the United States, and [GROW] is a very good way of increasing awareness among an important and to some degree underserved group."

Once the plans were in place, the organizers had to get the word out. "I sent information about the conference to everybody we knew," Kra says. "In two months, we had 120 applications from students at 40 schools." The first year, they accepted 50 students. In subsequent years, they increased attendance to 80 undergraduates to balance their goals of reaching as many people as possible while allowing as much one-on-one interaction as possible between attendees.

Organizers wanted to make sure to welcome students from communities that do not have good access to information about graduate school, including first-generation college students, women from underrepresented racial or ethnic groups, and those who come from socioeconomically disadvantaged backgrounds. "There are a lot of middle class assumptions baked into the PhD, which means there's a cultural divide for people not coming from that background. So I would hope that part of the role of GROW is to bridge that divide as well as the gender divide," Getzler says.

They advertised to departments all over the country, including historically Black colleges and universities, schools without graduate programs in math, and schools that rarely send students to graduate school in mathematics. They wanted to reach as broad a group as possible of interested undergraduates who could benefit from the conference. "I was extremely impressed at what a cross section of America these students were," Getzler says of the women who participated. (There have been some international participants as well, but the majority have been American women.)

The Conference Weekend

GROW is a weekend event with a mix of talks from invited speakers, panels about topics such as how to apply to graduate school and what to expect from a career in research mathematics, and less-structured time for one-on-one interactions. The organizers invited a large number of mentors—about one for every two students—to participate over the course of the weekend. Some mentors were on panels, and some were local professors and graduate students who were available for informal chats with undergraduates.

"I think here in this conference, anyone can approach anyone else," says Zoi Rapti, one of the organizers of GROW when it went to UIUC. She says the atmosphere at GROW, especially the amount of time given for undergraduates to talk with older students and professors, made them feel comfortable approaching speakers and asking questions.

"One thing that I have always done in my career, and will continue to do, is if I have to do something new, like apply to grad school, or apply for this job, or apply for this fellowship, I will always seek the counsel of older people who have done it before me," says Sarah Koch, who co-organized the conference when it moved to Michigan. "I think these conversations are really special and can make a huge difference in the direction that somebody wants to go."

One of the key events was the "nuts and bolts" panel about how to apply to graduate school. (The panel was so popular that the organizers made it significantly longer after the first year in response to feedback after the conference.) Panelists gave students information about how to apply and answered specific questions about everything from GRE scores to graduate stipends. A panel like that would have been helpful to Karen Smith, another Michigan conference organizer, as a student. "When I was an undergraduate, I had never heard of graduate school," she says. "Even after I had heard of it, I thought I would have to pay for it. I had never heard of getting funding for graduate school."

Figure 2. Sarah Koch describes her work to participants at GROW.

Figure 3. Bryna Kra talks with students at GROW.

As the daughter of an academic mathematician, Kra knew plenty about how graduate school worked and what a career as a researcher could look like. But when she was an undergraduate, she was nevertheless uncertain if going to graduate school would be the right choice for her. "It certainly didn't feel so welcoming that I'd never seen a class by a woman," she said. "I think it's important that people see that there are other women out there." She hopes that the conference gives women, whether they are already graduate school savvy or not, a way to see themselves as having the potential to have fulfilling careers in mathematics.

The organizers chose not to include presentations of undergraduate student research during the weekend in order to keep the focus more on the decision of whether to go to graduate school and how to get there. "It's not an undergraduate conference. It's a conference about going to graduate school in math," Koch says. "If we had the undergraduates present their work, I think that would have kind of detracted from getting together and thinking about going to graduate school. What does that look like mathematically? What does that look like practically?" Undergraduate presentations add a layer of stress for students, distracting them from those questions. "We wanted them to be looking forward and looking at the graduate students, looking at the postdocs, looking at the math talks," Koch says. "It's really about the next step for them instead of what they're already working on."

A highlight of each conference for Kra was the Saturday evening banquet, which featured a plenary talk by an august senior woman mathematician about her career journey. In 2015, the speaker was Alexandra Bellow, the first tenured woman mathematics professor at Northwestern. Eighty years old, she spoke softly, even with the microphone. "The room was so silent," Kra says. "Everybody was just glued to her stories. It was quite moving."

Evolution and GROWth

As the conference moves around to different universities, each organizing committee makes their own tweaks to GROW, and the experience will continue to evolve. "I think it's taken on its own life," Kra says. In 2018, after hosting three GROW conferences, she and her colleagues handed the duties off to Koch, Smith, and Mel Hochster at the University of Michigan.

Koch was a panelist and mentor at the 2017 conference. "It was such a positive, hopeful atmosphere," she says. She went back to Michigan enthusiastic about continuing the tradition. She and her co-organizers adapted the structure of previous GROW conferences to their own setting, modifying some parts of it to match their goals more closely. They kept Northwestern's emphasis on one-on-one mentoring opportunities and expanded the idea by creating booklets with pictures and short biographies of the allies and mentors who were attending the conference so students could get some relevant information, such as their institutions and research areas, before meeting them in person. "We only had a weekend with all these students, and we wanted them to get the most out of their experience," Koch says.

The Michigan organizers also decided to bring other scientists on board as they were planning the meeting. They consulted with researchers in the university's psychology department who study retention in STEM PhD programs. They wanted to get advice from the experts on what they could do to make the conference as effective as possible in encouraging interested women to apply for graduate school and helping them succeed when they got there.

The psychologists made a few recommendations. First, they recommended that the conference include people in a variety of roles on career panels, not just professors. The organizers included panelists from government agencies, the tech industry, and a nonprofit math education organization to broaden the perspectives offered. "It wasn't just about becoming a professor," Smith says.

They also broadened the scope of the colloquium-style talks, incorporating more applications of math to biology and social justice. In addition to standard theoretical math talks, they had a colloquium by an entrepreneur who designed an app that uses math to help people manage or prevent jet lag and one about gerrymandering, a hot-button political issue with connections to interesting problems and solutions in theoretical mathematics.

Second, the psychologists recommended an even greater focus on early career role models, "making sure that the participants saw people that were just a little bit older than they were succeeding in those roles," Smith says. "So instead of having a lot of programming where professors,

Figure 4. Ingrid Daubechies and Emmy Murphy serve on a panel at the 2017 GROW conference.

Figure 5. Dusa McDuff speaks to the 2016 GROW conference.

Figure 6. Participants at the 2015 GROW conference.

even female professors, were presenting their work, it was very important to have first-year graduate students presenting their work, people who would be just one step ahead."

When mathematicians at UIUC were approached about hosting GROW 2019, "of course we jumped on it," Rapti says. Jeremy Tyson, their department chair, attended the 2018 conference as a panelist and to help prepare for hosting the next year. The team at UIUC adopted the same conference structure and added more of an emphasis on helping undergraduate students learn about potential departments to apply to for graduate school. "They got to think about particular graduate programs, and maybe compare them and see what would be a good or a bad fit for them," Rapti says.

The UIUC GROW conference also continued Michigan's emphasis on graduate students mentoring undergraduates. One of the most popular events that year was a mentoring session run by the local Association for Women in Mathematics (AWM) chapter. Faculty members left the room so undergraduate and graduate students could talk privately about the graduate application process. "It was a huge success," Rapti says. "People just wouldn't leave the room."

The Student Experience

Priyanka Nanayakkara attended the first GROW conference as a sophomore at UCLA. She enjoys both math and writing and wanted to find a way to combine those interests, eventually deciding on statistics. "I was so focused on figuring out my major in undergrad, but the time to apply for grad school comes pretty quickly, so I think it was helpful to hear, 'You could really do this. It's not out of your reach. You're capable of it,'" she says.

Nanayakkara was impressed at the generosity of the mentors she talked with at the conference and believes they helped her see a future as an academic. "I knew it existed. I knew I wanted to go to grad school, but GROW helped me actually do it," she says. "That space to really feel confident was so useful, and I don't think it can be understated how valuable that is."

Nanayakkara is now working on a PhD in technology and social behavior at Northwestern as part of a joint program between the computer science department and communication studies. "I go to Northwestern now, and every time I walk past Tech [Technological Institute, a building on campus], I always think about eating lunch right there with all the people from GROW," she says. "I think about that conference as a turning point in my early college career."

She also feels like GROW helped her understand the importance of community in her academic discipline. "It was a really important step in introducing the idea of a cohort: these are the people who will be progressing with you throughout your life and career, and you'll see them again and again," Nanayakkara says. GROW's demographics give her hope that she can be part of an inclusive cohort as she continues in her career. "It has a community effect, and it shows you that the community can be diverse."

Emilee Cardin knew she wanted to go to graduate school when she applied for GROW in 2018. She was a student at the College of William and Mary and had participated in an REU at the University of Michigan–Dearborn the summer before she attended the conference. She was eager to continue studying math, but she worried that she might not get into the graduate programs she was interested in. "I felt like I was here to get as much advice and learn from as many people as possible," she says.

Friday afternoon, before GROW even officially started, she ended up in a conversation with Koch about recommendation letters. "I had a moment with Sarah, where she helped me take a breath and not be stressed out about it," she says. "She took the time to be helpful, even though I wasn't even a student there yet." Talking with Koch, Smith, and other Michigan faculty and students at GROW influenced her graduate application decisions. "I think it made me more confident in applying to Michigan," Cardin says. "I felt like it would be a more welcoming place after I went to GROW. It seemed less unapproachable."

Cardin is now in her first year of graduate work at Michigan. When we talked on the phone for this article, she was standing in the atrium where another favorite moment at GROW took place. She was sitting with a Boston College graduate student for lunch. "I remember her talking about how she loved it and she was so happy to have the people she was with [in graduate school], and I thought, 'Yes, that's what I'd like.'" She says the graduate school application process had felt like a constant competition, but talking with graduate students and faculty members about their experiences helped her prioritize the community she would find in graduate school and beyond. The Boston College student told her about how she was working to prioritize mental health in graduate school and how her cohort had helped her out when she needed it. "I hear those words ringing in my ears a lot," Cardin says.

Work/Life

"Work-life balance" can be a bit of a cliché, but the fact is that people of all genders have to consider how they will manage the demands of their careers, their other interests and hobbies, and the needs of their partners or family members when they are considering their potential career paths. Despite changes in American society in the past few decades, women still spend more time on childcare and caregiving for infirm relatives than men do, and they are more likely to face negative repercussions for it in professional settings.

It's no surprise, then, that conversations about family and career are common at GROW. "More 20-year-old women think about having a family than 20-year old men, or at least that is my impression," Getzler says. GROW

Figure 7. Students participate in a workshop at the 2015 GROW conference.

Figure 8. Laura De Marco talks about her work at the 2015 GROW conference.

provides a safe environment for women to talk with other women, both fellow students and women who are further along in their careers, about how they have managed or plan to manage the sometimes competing demands of career and family. "These things have to be thought over a bit more deeply when we're discussing women grad students," Getzler says.

Dusa McDuff and Ingrid Daubechies gave plenary addresses at GROW in 2016 and 2017, respectively. McDuff talked about taking time off from her career to raise children, and Daubechies spoke about weathering unexpected events in life that may mean you have to put your work on the back burner. "I think the students were really impressed by listening to these senior women talking about things that they also worry about," Kra says.

Cardin got to the conference preoccupied with getting into graduate school but says the panels and career talks helped her think about the process more holistically. "It was so good to hear you can be a human as well as being in a math graduate program," she says.

What About the Men?

Kra deliberately included men as panelists and mentors when she started GROW. Women are disproportionately asked to participate in service and outreach events in STEM fields. Even if the events are important and the women enthusiastic about participating, the volume of service requests can be a burden. "I did not want the conference to be more of a 'service tax' on women and other people from underrepresented groups," she says.

The organizers also wanted students to be able to interact with a diverse group of panelists and mentors. "One of the things [Kra] was very firm about is that these programs should expose prospective graduate students to a realistic atmosphere of what graduate school will be like," Getzler says. "For that reason, she did not want the weekend to be heavily dominated by contact only with other women."

Moreover, men need to be part of the solution if the proportion of women mathematicians is going to change. "By not including male grad students, we would miss out on future vectors for this kind of program," Getzler says. "But we indeed have male grad students—who have now become junior faculty—who have taken this level of activism across the country."

Kra also hoped the conference would provide a valuable change of perspective to men who participated. "The men came away with a lot from it," she says. "A colleague said to me, 'You know, I never knew what it felt like to be in the minority, and that was a great feeling to understand.'"

Getzler agrees that GROW has encouraged this kind of introspection among men who have participated. "I think a huge problem is that women have a tendency to be silenced if there are men in the room. And part of the power of the program is that now, since women are the vast majority, it's a corrective for the men in the room, to remind them that being a little bit quieter could be a good fit for everybody, that voices which aren't always encouraged to take charge do so," Getzler says.

One of the criticisms Kra read in a post-conference survey after the first conference was about the number of men involved. "For the future, I addressed that straight up from the beginning, saying men are involved because there are many men actively working to change who's in the room. We can't do it on our own, and it shouldn't just be on women," she says. "And we never got another complaint about it."

Broader Impacts

GROW is still a young conference. As such, there are no longitudinal studies tracking its effect on participants, though surveys done before and after each conference do show positive effects. Kra and other organizers believe it has already made a difference, not only to participants, but also to the schools that have hosted the conference and the broader profession.

As is the case with most conferences, person-to-person networking is one of the most valuable aspects. "GROW is exactly what it sounds like. It's GROW-ing a network, which I think is really awesome," Koch says.

Attendees have expanded their professional networks, creating connections with both their peers and more senior women mathematicians. "I'm still friends with some of the people I met at GROW," Cardin says.

GROW has also helped connect attendees to other existing programs and conferences for mathematicians. Nanayakkara learned about the AWM at GROW. When she got back to LA, she started looking up more information about the organization and learned about its annual essay contest for students. Her essay about Loyola Marymount University mathematician Alissa Crans won the undergraduate contest in 2016. "Before GROW, I don't think I really knew about the Association for Women in Mathematics. That was a direct result [of attending the conference]."

Cardin met mathematician Ruthi Hortsch, who works at Bridge to Enter Advanced Mathematics (BEAM), when she attended GROW. BEAM is a math education nonprofit organization for middle school students from underserved communities in New York City and Los Angeles. It provides enrichment for math-interested students and helps them progress in mathematics as they continue their education, primarily through summer programs.

"BEAM is probably one of the coolest math-related things I've ever done," Cardin says. After learning about BEAM from Hortsch, she spent part of the summer before starting graduate school in New York working at BEAM, and she has plans to join them again this year. Meeting Hortsch at GROW gave her "an immediate connection to something I fell in love with," she says.

Kra and other GROW organizers have also found that the conference has a positive effect on their departments. "I think it really did change the culture of the department," Kra says. Northwestern is creating a post-baccalaureate program for students from underrepresented groups in mathematics. "As much as it's a problem that there are very few women, it's a much bigger problem that there are so few people from other underrepresented groups," she says. A year-long program is very different from a weekend conference, but they are using some of their lessons from GROW as they shape the new program.

Both Northwestern and Michigan have seen the number of women who have applied to graduate school increase. "Our department got a lot out of it in many different ways, one of which was that we had this large number of really strong, really qualified female graduate student applicants last year," Koch says. "I wasn't really thinking of it as recruiting for Michigan per se," Smith says. "That was more of a side benefit for us."

Kra says the effect the conference has had at Northwestern is a reason it should continue to travel to different schools. "This is not any one department's problem; it's a problem in the field," Kra says. "If we fix it at Northwestern and get to parity, well that's great, but it doesn't do much for the field. I would rather see things fixed more broadly."

Evelyn Lamb

Credits

Figures 1–4 are courtesy of Mike Jue.
Figures 5–8 are courtesy of the Department of Mathematics, Northwestern University.
Author photo is courtesy of the author.

See the official announcement and citation for the 2020 AMS Award for Mathematics Programs that Make a Difference on page 722.

Mathematics People

Guth Awarded Mirzakhani Prize


Larry Guth

Larry Guth of the Massachusetts Institute of Technology has been awarded the newly named Maryam Mirzakhani Prize in Mathematics "for developing surprising, original, and deep connections between geometry, analysis, topology, and combinatorics, which have led to the solution of, or major advances on, many outstanding problems in these fields." The citation reads: "Guth has made spectacular contributions to many areas of mathematics, including systolic geometry, analysis, and combinatorics. He has developed surprising, original, and deep connections between geometry, analysis, topology, and combinatorics, leading to major advances or solutions for many outstanding problems in these fields. His accomplishments include the introduction of a new cell decomposition of Euclidean space, writing the authoritative book on the polynomial method, and creating a new induction on scales algorithm called the Bourgain-Guth method."

Guth received his PhD from MIT in 2005 under the supervision of Tomasz Mrowka. He held a postdoctoral position (2005–2006) and an assistant professorship (2006–2008) at Stanford University. He was assistant professor (2008–2011) at the University of Toronto, a member of the Institute for Advanced Study (2010–2011), and professor at the Courant Institute of New York University before joining MIT in 2012. His honors and awards include an NSF Postdoctoral Fellowship (2006–2008), a Sloan Fellowship (2010), the Salem Prize (2013), the MIT School of Science Prize for Excellence in Graduate Teaching (2015), a Clay Research Prize (with Nets Katz, 2015), the New Horizons in Mathematics Prize (2015), and the AMS Bôcher Prize (2020). He was named a Simons Investigator in 2014 and is a Fellow of the AMS and of the American Academy of Arts and Sciences. He is the author of the book *Polynomial Methods in Combinatorics*.

The prize, awarded by the National Academy of Sciences and formerly called the NAS Award in Mathematics, was renamed to honor the late Maryam Mirzakhani. It honors exceptional contributions to the mathematical sciences by a midcareer mathematician. It carries a cash award of US$20,000.

—From an NAS announcement

2020 AWM Prizes Awarded

The Association for Women in Mathematics (AWM) presented several awards at the Joint Mathematics Meetings held in Denver, Colorado, in January 2020.


Erika Camacho

Erika Camacho of Arizona State University was honored with the Louise Hay Award for Contribution to Mathematics Education "in recognition of her leadership and contributions as a mathematical scholar and educator." The prize citation reads: "Dr. Camacho has a passion for mentoring, especially the mentoring of underrepresented students. Her mentoring begins with her excitement for mathematics based in her research in mathematical physiology. This research involves developing mathematical models that describe the interactions of photoreceptors in the retina. Dr. Camacho brings graduate and undergraduate students into her research and also finds opportunities for students with other researchers.

"She created the Applied Mathematical Sciences Summer Institute and has codirected both this institute (2004–2007) and the Mathematical and Theoretical Biology Institute (2011–2013). Through these institutes and her other mentoring programs she has impacted over 600 undergraduates, including supervising the research of 89 of these students, with 30 receiving conference award recognitions.

"Through her work Dr. Camacho changes perceptions. Her own story is an existence proof that someone from an underprivileged and Latina background can earn a PhD in mathematics and be a successful mathematician. In over sixty-five plenary and panel presentations, she uses her story to inspire students to persevere and succeed in mathematics. Beyond presenting, Dr. Camacho meets with attendees individually afterwards to learn about their stories and give them advice based on their own interests and passions. By inspiring more women and members of underrepresented groups to continue in their mathematical pursuits, she enlarges the scope of what we see as successful mathematicians." Camacho received her PhD in 2003 from Cornell University under the direction of Richard H. Rand. Among her many recognitions are the SACNAS Distinguished Undergraduate Institution Mentor Award (2012), the Outstanding Latino/a Faculty in Higher Education: Research/Teaching in Higher Education (Research Institutions) (2018), the Presidential Award for Excellence in Science, Mathematics, and Engineering Mentoring (2014), and the American Association for the Advancement of Science Mentor Award (2019). She tells the *Notices:* "I grew up in East Los Angeles, the fourth of five children, where I was taught by Jaime Escalante (of *Stand and Deliver* fame). I enjoy spending time with my husband and three children."


Margaret Robinson

Margaret Robinson of Mount Holyoke College has been named the recipient of the 2020 M. Gweneth Humphreys Award for Mentorship of Undergraduate Women in Mathematics. According to the prize citation, "Margaret Robinson has been a mainstay of caring and thoughtful teaching and mentoring for many years at Mount Holyoke College, an institution whose mission is to educate women. Her focus is not just on the top students but on making a meaningful (and joyful) mathematical intervention for all the generations of learners that have crossed her path. As one student put it, 'she saw me in a way that no mathematics teacher had before.' Her impactful involvement in the Carleton Summer Math Program and the resounding response from a range of former mentees speak to her effectiveness and her ability to forge personal connections." Robinson received her PhD in 1986 from Johns Hopkins University under the supervision of Jun-Ichi Igusa. Her honors include the Mount Holyoke Faculty Teaching Award (2010), the NES/MAA Award for Distinguished Teaching of Mathematics (2012), and the MAA Haimo Award for Distinguished Teaching of Mathematics (2013). Robinson tells the *Notices*: "My favorite quote (that I tell my students and children) comes from *The Once and Future King* by T. H. White: '"The best thing for being sad," replied Merlin, beginning to puff and blow, "is to learn something. That's the only thing that never fails. … Learning is the only thing for you. Look what a lot of things there are to learn."'"

—*From AWM announcements*

2020 MAA Awards

The Mathematical Association of America (MAA) awarded several prizes at the Joint Mathematics Meetings in Denver, Colorado, in January 2020.


Vladimir Pozdnyakov


J. Michael Steele

Vladimir Pozdnyakov of the University of Connecticut and **J. Michael Steele** of the University of Pennsylvania were awarded the Chauvenet Prize for their article "Buses, Bullies, and Bijections," *Mathematics Magazine* 89 (2016), no. 3. The prize citation reads in part: "Pozdnyakov and Steele show the remarkable utility of bijections by considering seating assignments on a bus. Everyone has a designated seat, but all except the last passenger take seats at random. Then the final passenger—a bit of a bully—boards, not only wanting his own seat, but demanding that each subsequently displaced person finds his correct seat as well. What is the probability that the first person to board will need to change seats?

"The authors obtain the answer via a brute-force combinatorial argument, but then find the solution in an easier, more revealing way by making elegant use of permutation cycles. The authors then use bijections to derive even more surprising and beautiful results including the mean and variance of the number of cycles in a random permutation. This well-crafted paper, which introduces the reader to the theory of permutation patterns, flows naturally and easily, providing a journey that is interesting and insightful. This bus is available for all—professor and student alike—delighting the rider with the simple power of bijections."

Pozdnyakov received his PhD from the University of Pennsylvania in 2001 under the supervision of J. Michael Steele. He is currently professor of statistics and director of the Applied Financial Mathematics graduate program at Connecticut. He tells the *Notices:* "I'm an enthusiastic soccer player—an old one." J. Michael Steele received his PhD from Stanford University in 1975 under Kai Lai Chung. He is currently professor emeritus at the University of Pennsylvania. Steele and Pozdnyakov were jointly awarded the MAA's Carl Allendorfer Award in 2017. Steele tells the *Notices:* "I'm now retired from teaching, but I am still involved in writing. I am also passionate about languages and language learning. French is in focus for the moment, and most recently it absorbs four or more hours of my day."

Aubrey D. N. J. de Grey of SENS Research Foundation and AgeX Therapeutics has been awarded the 2020 David P.

Robbins Prize for his article "The Chromatic Number of the Plane Is at Least 5," *Geombinatorics* **28** (2018), no. 1, which addresses the question, What is the minimum number of colors needed to color the points of a Euclidean plane so that no two points at distance exactly 1 have the same color? This is "often known as the Hadwiger–Nelson problem; Hadwiger, several years earlier and for other reasons, had been the first to discuss the simplest coloring of the plane that demonstrates the upper bound." De Grey received his PhD in biology from the University of Cambridge. His research interests encompass the characterization of all the types of self-inflicted cellular and molecular damage that constitute mammalian aging and the design of interventions to repair and/or obviate that damage. He is particularly interested in combinatorics, especially graph theory.


Tim Chartier

Tim Chartier of Davidson College was awarded the 2020 Euler Book Prize for *Math Bytes* (Princeton University Press, 2014). The prize citation reads: "*Math Bytes* gives readers a taste of the mathematics and computing applications that underlie many aspects of everyday life. With a wide array of topics—including fractals, fonts, tweets, basketball, Google, digital images, movies, and more—the book exposes readers to a satisfying assortment of mathematical ideas, many of which will be new to nonmathematical audiences. That said, even more mathematically inclined readers should find plenty of interesting material, including new ways of thinking about and applying familiar mathematical concepts. Chartier's exposition is clear, accessible, and fun. Regular challenge problems encourage readers to explore for themselves the ideas introduced in the text. All in all, *Math Bytes* is an engaging and stimulating read that is sure to broaden horizons and increase appreciation for the ubiquitous and invaluable role of computational mathematics in modern society." Chartier received his PhD from the University of Colorado, Boulder. He specializes in numerical linear algebra, with his recent work focusing on data science. He has been a consultant on data analytics for ESPN, the *New York Times,* the US Olympic Committee, and teams in the NBA, NFL, and NASCAR. He was the first chair of the Advisory Council for the National Museum of Mathematics. In K–12 education, he has worked with Google and Pixar on their educational initiatives. He was the recipient of the MAA Daniel Solow Author's Award in 2019. Chartier tells the *Notices:* "My wife and I have professional training in mime, which includes master classes with Marcel Marceau. In fact, we paid for our wedding just over twenty-five years ago with a performance tour along the East Coast of the United States. We also have developed a mime show that introduces mathematical ideas and have performed it across the United States, as well as in Panama, South Korea, Japan, and Holland."

The 2020 Deborah and Franklin Tepper Haimo Awards for Distinguished College or University Teaching of Mathematics were awarded to **Federico Ardila** of San Francisco State University, **Mark Tomforde** of the University of Houston, and **Suzanne L. Weekes** of Worcester Polytechnic Institute.

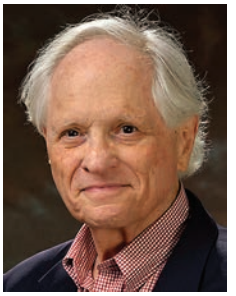
Ardila was recognized for inspiring students "from all walks of life to recognize and realize their potential in mathematics." He is a "leader in the movement to broaden and deepen diversity in research mathematics." He is director of the Mathematical Sciences Research Institute–Undergraduate Program (MSRI-UP), the largest Research Experiences for Undergraduates (REU) program in the United States and the one that serves the largest number of students from underrepresented groups. He conceived the SFSU-Colombia Combinatorics Initiative, through which he developed seven new courses to promote international scholarly collaboration among undergraduates and master's students, including many from underrepresented groups, at SFSU and the Universidad de Los Andes. He has published a wide range of expository articles in English, Spanish, and German. His YouTube channel contains more than 240 hours of freely available advanced mathematics, and his viewers come from over 150 countries. His article "Todos Cuentan: Cultivating Diversity in Combinatorics" was published in the *Notices* in November 2016. Ardila received his PhD from the Massachusetts Institute of Technology in 2003 under the direction of Richard P. Stanley. He is a Fellow of the AMS and of the Simons Foundation and the recipient of an NSF CAREER Award and of the Premio Nacional de Ciencias and the Premio Nacional de Matemáticas in Colombia. His research is in combinatorics and its connections to geometry, algebra, topology, and applications. He enjoys reading, *fútbol*, playing records, or playing marimba de chonta.


Mark Tomforde

Tomforde "has had a deep and positive impact at all levels of mathematics education." According to the prize citation, he has "recruited, retained, and mentored members of underrepresented groups spectacularly" at all levels, including by enrolling over seventy University of Houston students in the Math Alliance, the goal of which is to ensure that every underrepresented or underserved American student with talent and ambition has the opportunity to earn a doctoral degree in a mathematical science. He is a cofounder and coorganizer of Gulf States Math Alliance (GSMath), one of seven regional alliances, composed of members of the Math Alliance in Texas, Louisiana, and Mississippi. He facilitates and promotes associated opportunities in the Gulf Coast region. He developed the Cougars and Houston Area Math Program (CHAMP), working in collaboration with neighborhood high schools and middle schools to provide a wide variety of mathematical activities. CHAMP received the AMS Award for Mathematics Programs That Make a Difference in 2018, as well as a Phi Beta Kappa award for broadening participation in STEM. He developed a multifaceted collaboration between the University of Houston and Texas Southern University, recruited faculty members from Houston as Math Alliance members, has served as a Project NExT consultant, and maintains multiple websites with a wide variety of materials for faculty and students. Tomforde received his PhD from Dartmouth College under the supervision of Dana P. Williams in 2002, held a postdoctoral fellowship at the University of Iowa, and has also taught at the College of William and Mary. He was a Project NExT Fellow in 2002. He tells the *Notices:* "I am a cinephile, and in my spare time I enjoy watching a wide variety of movies from different genres."


Suzanne L. Weekes

Weekes "has had an extraordinary impact on the mathematics community via superlative teaching, advising, and mentoring of students and faculty at Worcester Polytechnic Institute (WPI), regionally, and nationally." She designed and organized the Applied and Industrial Mathematics Institute for Secondary Teaching at WPI, which offers workshops for high school mathematics teachers. As a member of the Math Advisory Group of Transforming Postsecondary Education in Mathematics (TPSE Math), she coorganized and hosted the New England Regional Meeting on Upper-Division Pathways at WPI, now a model for such workshops in other regions. She directed the Center for Industrial Mathematics and Statistics and also directed its WPI Research Experiences for Undergraduates (REU) Program in Industrial Mathematics and Statistics. She is a cofounder of the Mathematical Sciences Research Institute Undergraduate Program (MSRI-UP) and of the MAA's Preparation for Industrial Careers in Mathematical Sciences (PIC Math) program, which has increased awareness of nonacademic career options and preparing students for industrial careers. Weekes grew up in the Republic of Trinidad and Tobago, received her BS in mathematics from Indiana University, and earned her PhD in 1995 from the University of Michigan. She was awarded the M. Gweneth Humphries Award of the Association for Women in Mathematics (AWM) in 2019. Her research work is in numerical methods for differential equations, including applications to spatiotemporal composites/dynamic materials and cancer growth. She is involved in several initiatives connecting the academic mathematics community to mathematics and statistics work done in business, industry, and government, and with broadening the participation and success of students in mathematical sciences.


Gerald J. Porter

Gerald J. Porter of the University of Pennsylvania received the 2020 Gung and Hu Award for Distinguished Service to Mathematics for his service in "teaching, teacher education, research, MAA administration, and, most importantly of all, in leading the profession, especially the MAA, to value racial and gender diversity in all activities." The prize citation reads in part: "Jerry Porter has spent decades in service to the MAA. His service in terms of years and variety at the national level is extensive but his service and care for the organization goes far beyond the lengthy list of committees on which he served and positions he has held. His is the service that, while not appearing on any list, has made the difference in the MAA and our profession. He pursued this service while providing strong support to Executive Directors, learning and sharing his great expertise, and being a change agent in the areas in which he was involved. He has been a mentor to many young mathematicians and has nominated them for awards and committees, welcomed them at both section and national meetings, and shown by example the importance of inclusivity. For many years he was the only male member of the Joint Committee on Women; as always, Jerry strengthened MAA's role on this committee. Jerry welcomed the women and minorities who attend our meetings and encouraged them to take an active role in the Association." With Jim White, he directed the Interactive Mathematics Text Project, which funded the creation of computer laboratories in six colleges to encourage the creation of computer-based algebra materials in teaching. Porter received his PhD from Cornell University in 1963 under the direction of William Browder. He is a life member of the AMS and the MAA. Porter has been retired from teaching since 2006. He enjoys traveling and has visited about seventy-five countries and all fifty states in the United States. He is an avid photographer and has had six photo shows at Penn. Since his retirement he has audited courses and seminars at Penn, including courses in ethnomusicology, art, and literature. In June, he and his wife, Judy, will celebrate their sixtieth wedding anniversary. Together they funded the public lecture given each year at the Joint Mathematics Meetings with the goal of increasing public awareness and appreciation of mathematics.

—From MAA announcements

Bombieri Awarded Crafoord Prize in Mathematics


Enrico Bombieri

Enrico Bombieri of the Institute for Advanced Study has been awarded the 2020 Crafoord Prize in Mathematics "for outstanding and influential contributions in all the major areas of mathematics, particularly number theory, analysis and algebraic geometry." The prize is awarded by the Royal Swedish Academy of Sciences and the Crafoord Foundation; the disciplines rotate every year. The prize carries a cash award of 6 million Swedish krona (approximately US$618,000).

The prize citation reads in part: "Enrico Bombieri belongs to an increasingly rare group of mathematicians who can solve problems in almost all areas of mathematics. However, his greatest passion has always been number theory, which is the study of integers. He was just sixteen years old when he published his first work in number theory and, among other things, he is a leading expert on the Riemann hypothesis on the distribution of prime numbers.

"Enrico Bombieri has made significant contributions in algebra, advanced geometry, and complex analysis. He has also contributed to solving Bernstein's problem. This is a variation of Plateau's problem, about how to mathematically describe the shape of the soap film that forms when a wire frame is dipped into a soap solution."

Bombieri was born in 1940 in Milan, Italy, and received his PhD in 1963 from the Università degli Studi di Milano. He has been professor at the University of Pisa (1966–1974) and Scuola Normale Superiore, Pisa (1974–1977). He joined the faculty at the IAS in 1977, where he is now professor emeritus. He was awarded the Fields Medal in 1974. His honors also include the Feltrinelli Prize (1976), the Balzan Prize (1980), the Cavaliere di Gran Croce al Merito della Repubblica, Italy (2002), the Premio Internazionale Pitagora (2006), the AMS Joseph Doob Prize (2008), the King Faisal International Prize (2010), and the Lifetime Achievement Award of the Italian Scientists and Scholars of North America Foundation (2015). He is a member of the American Academy of Arts and Sciences, the National Academy of Sciences, and the Royal Swedish Academy of Arts and Sciences, among many others.

—From a Royal Swedish Academy announcement

Legatiuk Awarded Clifford Prize


Dmitrii Legatiuk

Dmitrii Legatiuk of Bauhaus-Universität Weimar has been selected as the recipient of the 2020 W. K. Clifford Prize for his "significant contributions in Clifford analysis, including interpolation of monogenic functions, quaternionic operator calculus, and construction of advanced numerical methods." Legatiuk's "interest in Clifford analysis, particularly its potential to solve difficult applied problems, has led him to such advances as using quaternionic operator calculus to construct representation formulas for solutions of boundary value problems in advanced elasticity theories, interpolation of monogenic functions by various tools, and developing a finite element exterior calculus based on script geometry. His interests span mathematics, computer science, and engineering, reflecting the broad applicability of Clifford algebras and echoing the wide-ranging interests of W. K. Clifford himself." Legatiuk earned his PhD from Bauhaus-Universität Weimar, where his doctoral research earned him the 2015 University Prize for Young Scientists.

The W. K. Clifford Prize is an international scientific prize intended to encourage young researchers to compete for excellence in research in theoretical and applied Clifford algebras and their analysis and geometry. It is awarded every three years at the International Conference on Clifford Algebras and Their Applications in Mathematical Physics, held this year in Hefei, China.

—G. Stacey Staples
Southern Illinois University

Chuzhoy Awarded NAS Held Prize

Julia Chuzhoy of Toyota Technological Institute, Chicago, has been named the recipient of the 2020 Michael and Sheila Held Prize of the National Academy of Sciences. According to the prize citation, Chuzhoy "has conducted influential work in the fields of graph algorithms, hardness of approximation, and structural graph theory, which have introduced powerful new techniques and resolved deep open questions.

"Chuzhoy and her coauthors achieved remarkable results in designing algorithms for graph routing problems, which are among the most studied and important problems in optimization. Insights from this work led to further significant impacts on structural graph theory, including an exponential strengthening of the parameters of the Excluded Grid theorem.

"Chuzhoy's work on graph routing problems settled central open questions in graph optimization and introduced powerful new graph decomposition and routing techniques, opening up the potential for future applications in algorithm design and structural graph theory. The improved parameters for the Excluded Grid theorem led to faster algorithms for a host of graph optimization problems, and stronger bounds for a number of graph theoretic results."

The prize carries a cash award of US$100,000. It honors outstanding, innovative, creative, and influential research in the areas of combinatorial and discrete optimization, or related parts of computer science, such as the design and analysis of algorithms and complexity theory.

—From an NAS announcement

Borodin and Viazovska Awarded Fermat Prize

Alexei Borodin of the Massachusetts Institute of Technology and **Maryna Viazovska** of the École Polytechnique Fédérale de Lausanne have been awarded the 2019 Fermat Prize for research in mathematics. Borodin was honored for the invention of integrable probability theory, a new area at the interface of representation theory, combinatorics, and statistical physics. Viazovska was honored for her original solution of the famous sphere packing problem in dimensions 8 and 24. The prize rewards mathematicians under forty-five years old whose research works are in number theory, analytic geometry, probability, and research related to the variational principles.

—Fermat Prize announcement

Haykazyan Awarded Emil Artin Junior Prize

Levon Haykazyan of Oxford Asset Management has been awarded the 2020 Emil Artin Junior Prize in Mathematics. Haykazyan was chosen for his paper "Spaces of Types in Positive Model Theory," *Journal of Symbolic Logic* **84** (2019).

Established in 2001, the Emil Artin Junior Prize in Mathematics carries a cash award of US$1,000 and is presented usually every year to a student or former student of an Armenian educational institution under the age of thirty-five for outstanding contributions to algebra, geometry, topology, and number theory—the fields in which Emil Artin made major contributions. The prize committee consisted of A. Basmajian, Y. Movsisyan, and V. Pambuccian.

—Victor Pambuccian
New College, Arizona State University

Borodin and Petrov Awarded 2020 Bernoulli Prize

Alexei Borodin of the Massachusetts Institute of Technology and **Leonid Petrov** of the University of Virginia have been awarded the 2020 Bernoulli Prize for an Outstanding Survey Article in Probability or Statistics. They were honored for their article "Integrable Probability: From Representation Theory to Macdonald Processes," *Probability Surveys* **11** (2014). The prize recognizes authors of an influential survey publication in the areas of probability and statistics.

—Bernoulli Society announcement

National Academy of Engineering Elections

The National Academy of Engineering (NAE) has elected eighty-seven new members and eighteen international members for 2020. Below are the new members whose work involves the mathematical sciences.

- **Graham V. Candler,** University of Minnesota, Minneapolis, for development and validation of computational models for high-fidelity simulation of supersonic and hypersonic interactions.
- **Kenneth C. Hall,** Duke University, for development of unsteady aerodynamic and aeromechanics theories and analysis for internal and external aerodynamic flows.
- **Mrdjan Jankovic,** Ford Motor Company, for contributions to nonlinear control theory and automotive technology.
- **Sallie Ann Keller,** University of Virginia, Charlottesville, for development and application of engineering and statistical techniques in support of national security and industry.
- **Ioannis G. Kevrekidis,** Johns Hopkins University, for research on multiscale mathematical modeling and scientific computation for complex, nonlinear reaction, and transport processes.
- **Tamara G. Kolda,** Sandia National Laboratories, for contributions to the design of scientific software, including tensor decompositions and multilinear algebra.

- **Muriel Médard,** Massachusetts Institute of Technology, for contributions to the theory and practice of network coding.
- **Jorge Nocedal,** Northwestern University, for contributions to the theory, design, and implementation of optimization algorithms and machine learning software.
- **Alexander A. Shapiro,** Georgia Institute of Technology, for contributions to the theory, computation, and application of stochastic programming.
- **Peter W. Shor,** Massachusetts Institute of Technology, for pioneering contributions to quantum computation.
- **Charles W. Wampler II,** General Motors Corporation, for leadership in robotic systems in manufacturing, mathematical methods for robot motion and machine design, and traction battery modeling.
- Elected as an international member was **Wolfgang Marquardt,** Forschungszentrum Jülich GmbH, Germany, for contributions to process systems engineering and large-scale computations and for national leadership in science/technology policy and management.

—From an NAE announcement

2020 Sloan Fellows

The Alfred P. Sloan Foundation has announced the names of 126 recipients of the 2020 Sloan Research Fellowships. Each year the foundation awards fellowships in the fields of mathematics, chemistry, computational and evolutionary molecular biology, computer science, economics, neuroscience, physics, and ocean sciences. Grants of US$75,000 for a two-year period are administered by each Fellow's institution. Once chosen, Fellows are free to pursue whatever lines of inquiry most interest them, and they are permitted to employ fellowship funds in a wide variety of ways to further their research aims.

Following are the names and institutions of the 2020 awardees in the mathematical sciences.

- **Jeff Calder,** University of Minnesota
- **Roger Casals,** University of California, Davis
- **Otis Chodosh,** Stanford University
- **Damek Davis,** Cornell University
- **Tarek M. Elgindi,** University of California, San Diego
- **Peter Hintz,** Massachusetts Institute of Technology
- **Robert Hough,** Stony Brook University
- **Hao Huang,** Emory University
- **Sebastian Hurtado-Salazar,** University of Chicago
- **Aleksandr Logunov,** Princeton University
- **Linquan Ma,** Purdue University
- **Sung-Jin Oh,** University of California, Berkeley
- **Weijie Su,** University of Pennsylvania
- **Omer Tamuz,** California Institute of Technology
- **Samuel Taylor,** Temple University
- **Ila Varma,** University of Toronto
- **Cynthia Vinzant,** North Carolina State University
- **Alexander Wright,** University of Michigan
- **Yao Yao,** Georgia Institute of Technology
- **Zhizhen Zhao,** University of Illinois, Urbana-Champaign

—From a Sloan Foundation announcement

Credits

Photo of Larry Guth is courtesy of the MIT Department of Mathematics.
Photo of Vladimir Pozdnyakov is courtesy of Zhanna Pozdnyakova.
Photo of J. Michael Steele is courtesy of Catalin Starica.
Photo of Tim Chartier is courtesy of Davidson College.
Photo of Suzanne Weekes is courtesy of Worcester Polytechnic Institute.
Photo of Gerald J. Porter is courtesy of Felice Macera.
Photo of Enrico Bombieri is courtesy of Cliff Moore.
Photo of Dmitrii Legatiuk is courtesy of Dmitrii Legatiuk.

Community Updates

AMS Department Chairs Workshop

The AMS held its annual Department Chairs Workshop on January 14, 2020, in Denver, Colorado, just prior to the start of the 2020 Joint Mathematics Meetings.

Workshop leaders were: Luca Capogna, Head, Department of Mathematical Sciences, Worcester Polytechnic Institute (WPI); Kevin Knudson, Chair, Department of Mathematics, University of Florida; Gloria Marí-Beffa, Associate Dean for the Natural, Physical and Mathematical Sciences, University of Wisconsin–Madison; and Jennifer Zhao, Associate Dean for the College of Arts, Sciences and Letters, University of Michigan–Dearborn.

What makes a chair different from any other engaged faculty member in the department? This workshop examined the chair's role in leading a department. The day was structured to include and encourage networking and sharing of ideas among participants. There were four sessions:

- Modernizing mathematics and mathematicians
- Evaluating teaching
- Difficult conversations
- The "entrepreneurial" mathematics department.

The 2020 workshop was attended by sixty-eight department chairs and leaders from across the country.

—Anita Benjamin
AMS Office of Government Relations

Every Generation Helps the Next

In 2017, an energetic group of twenty mathematicians at the American Mathematical Society launched a special fundraising campaign to create a new endowment called the Next Generation Fund. The goal of the Next Generation Fund is to support hundreds of early career mathematicians each year at modest but impactful levels.

Many donors contributed toward the $1.5 million matching gift challenge with donations that ranged from one dollar to more than two hundred thousand. Thanks to their personal commitment and community spirit, the campaign raised over three million dollars to inaugurate the Fund.

Initially, the Next Generation Fund will support programs such as travel grants to AMS meetings, Mathematics Research Communities, Child Care Grants, and Student Chapters. It is designed to be flexible over time to meet the changing needs of mathematicians as they begin their professional careers.

With the inaugural campaign now accomplished, the Next Generation Fund will continue to be a fundraising priority at the AMS. All members of the mathematics community are welcomed and encouraged to support it through their ongoing donations. For more information, contact the Development staff at 401-455-4111, or visit `www.ams.org/support`.

—Robin Marek

Deaths of AMS Members

K. Chandrasekharan, of Switzerland, died on April 13, 2017. Born on November 21, 1920, he was a member of the Society for 69 years.

Harry D. Huskey, of Santa Cruz, California, died on April 9, 2017. Born on January 19, 1916, he was a member of the Society for 74 years.

Credit

Photo is courtesy of Anita Benjamin.

NEWS

Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices *may be submitted to* notices@ams.org.

Early Career Opportunity

NSF Postdoctoral Research Fellowships

The National Science Foundation (NSF) awards Mathematical Sciences Postdoctoral Research Fellowships (MSPRF) in all areas of the mathematical sciences, including applications to other disciplines. Awards are either Research Fellowships or Instructorships. The Research Fellowship provides full-time support for any eighteen academic-year months in a three-year period. The Research Instructorship provides either two academic years of full-time support or one academic year of full-time and two academic years of half-time support. The deadline for proposals is **October 21, 2020**. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=5301&org=NSF.

—NSF announcement

Early Career Opportunity

NRC Research Associateship Programs

The National Academy of Sciences, Engineering, and Medicine offers postdoctoral and senior research awards on behalf of twenty-three US federal research agencies and affiliated institutions with facilities at over 100 locations throughout the United States and abroad. Applications are sought from highly qualified candidates, including recent doctoral recipients and senior researchers. Upcoming deadlines are **May 1, 2020**, and **August 1, 2020**. See sites.nationalacademies.org/pga/rap/.

—NRC announcement

The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at www.nsf.gov/dms *and for the Directorate of Education and Human Resources at* www.nsf.gov/dir/index.jsp?org=ehr. *To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at* www.nsf.gov/mps/dms/about.jsp.

Early Career Opportunity

NSF Research Training Groups in the Mathematical Sciences

The National Science Foundation (NSF) Research Training Groups in the Mathematical Sciences (RTG) program provides funds for the training of US students and postdoctoral associates through structured research groups that include vertically integrated activities spanning the entire spectrum of educational levels from undergraduate through postdoctoral. The deadline for full proposals is **June 2, 2020**. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=5732.

—NSF announcement

Early Career Opportunity

International Mathematics Competition for University Students

The Twenty-seventh International Mathematics Competition for University Students will be held July 24–30, 2020, at American University in Blagoevgrad, Bulgaria. Students completing their first, second, third, or fourth years of university education are eligible. See www.imc-math.org.uk/.

—John Jayne, University College London

IPAM Call for Proposals

The Institute for Pure and Applied Mathematics (IPAM) seeks program proposals from the mathematical, statistical, and scientific communities for long programs and workshops, to be reviewed at IPAM's Science Advisory Board meeting in November. For more information, go to www.ipam.ucla.edu/propose-a-program/ or contact the IPAM director at director@ipam.ucla.edu. Proposals should also address the inclusion of women and members of underrepresented minorities as speakers, organizers, and participants.

—IPAM announcement

Call for Papers for Haifa Workshop

The Twentieth Haifa Workshop on Interdisciplinary Applications of Graphs, Combinatorics, and Algorithms will be held June 7–9, 2020, at the University of Haifa, Caesarea Rothschild Institute. Contributed talks are invited. The workshop emphasizes the diversity of the use of combinatorial algorithms and graph theory in application areas. Abstracts of one to two pages should be sent by **April 27, 2020,** to `HaifaGraphWorkshop@gmail.com`. See the website `cri.hevra.haifa.ac.il`.

—Martin Golumbic, General Chair
University of Haifa

A New Panamanian Foundation for Mathematics

The Panamanian Foundation for the Promotion of Mathematics (FUNDAPROMAT) is a private nonprofit foundation whose mission is to promote the study of mathematics in the Republic of Panama. The official launch of the Foundation was on April 14, 2020, in Panama City and on April 15 in the province of Chiriquí, outside of the capital city. The keynote speaker of the event was Dr. Michael Dorff, president of the Mathematical Association of America (MAA), who gave his public presentation on "Math on Soap Bubbles."

The presentations to a general audience of the Program on Math Outreach are among the most popular events organized by the Foundation. The purpose of these math presentations given by prominent international mathematicians is to convince kids and adults of all ages that math is not only fun but it also has many interesting applications. We also want to inspire Panamanian youth to study math or to follow a scientific career. Talented public speakers like Eugenia Cheng, Robert Lang, and Colin Wright have travelled to Panama to share their passion for mathematics with the Panamanian population.

The Math Carnivals that highlight female Panamanian mathematicians are also in demand. The goal of these events is to inspire Panamanian youth, in particular girls, to study math by showing them real-life examples of female Panamanian mathematicians who are successful in their careers. The first Math Carnival took place on May 12, 2019, in the Biomuseum to celebrate the International Day for Women in Mathematics. These events are currently taking place once a month in different museums, parks, and malls of the Republic of Panama.

The Panamanian Foundation for the Promotion of Mathematics (FUNDAPROMAT) invites you to get involved. You can volunteer to visit our beautiful country and give a presentation to a general audience or participate in one of our math outreach events. You can donate funds to cover the expenses of running the Foundation. You can mail us math puzzles to use in our activities. You can share ideas of new ways to have a greater impact on a general audience and to successfully reach those who might not consider themselves math enthusiasts. If you know of someone who might be interested in sponsoring our educational efforts in Panama and would benefit from having an international presence, please do not hesitate to contact us.

You can follow us on Instagram as @fundapromat and through our public Facebook page known as "Fundapromat." You can also access more information on our website `www.fundapromat.org` or you can email us directly at `info@fundapromat.org`.

—Jeanette Shakalli, PhD

Classified Advertising

Employment Opportunities

CANADA

Brock University
Department of Mathematics & Statistics
Faculty Position in Mathematics of Data Science

The Department of Mathematics and Statistics at Brock University invites applications for a probationary tenure-track position as Assistant Professor in the mathematics of data science, effective July 1, 2020.

About the position

The Department currently offers a BSc and an MSc in Mathematics and Statistics and is preparing to launch a new trans-disciplinary BSc in Data Sciences and Analytics as well as a new PhD Program in Computational, Mathematical, and Statistical Sciences jointly with the Department of Computer Science. The Department will also be contributing to the development and delivery of engineering programs at Brock University. The successful candidate is expected to have a primary role in the development of these current and new programs in mathematics, data sciences and engineering.

Qualifications

The successful candidate must have recently completed a PhD (since 2015) in an area of mathematics related to data science (e.g., computational discrete mathematics, algorithmic and computational optimization, analysis of complex networks, etc.) by the time of the appointment, a record of or a strong potential for excellent and independent research that will attract external funding and an active program of fundamental and applied research in mathematical methods in data and information processing with applications to science and engineering problems. The successful candidate must demonstrate the ability to teach both undergraduate and graduate courses in theoretical and computational discrete mathematics with applications to the data sciences, and the ability and the commitment to supervise graduate students.

Applicants should submit through the Brock Careers website at the link below by March 31, 2020 (indicating file number stated above), a cover letter, a curriculum vitae, samples of recent publications, statements of research and teaching interests: `https://brocku.wd3.myworkdayjobs.com/brocku_careers/job/St-Catharines-Main-Campus/Assistant-Professor--Mathematics-of-Data-Science_JR-1005178`.

Learn more about Brock University by visiting `www.brocku.ca`.

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The *Notices* Classified Advertising section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

The 2020 rate is $3.65 per word. Advertisements will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted. There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: August 2020—May 15, 2020; September 2020—June 17, 2020; October 2020—July 17, 2020; November 2020—August 19, 2020; December 2020—September 16, 2020.

US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to `classads@ams.org`.

Brock University
Department of Mathematics & Statistics
Faculty Position in Financial Mathematics and Mathematical Modeling

The Department of Mathematics and Statistics at Brock University invites applications for a probationary tenure-track position as Assistant Professor in Financial Mathematics and Mathematical Modeling, effective July 1, 2020.

About the position

The Department currently offers a BSc and an MSc in Mathematics and Statistics. The successful candidate is expected to have a primary role in the development of a new undergraduate program in Financial Mathematics, sustain and expand existing financial math courses, and teach undergraduate courses in the MICA (Mathematics Integrated with Computers and Applications) concentration. The successful candidate will also build a modern transdisciplinary research program which will strengthen and broaden the Department's MSc program, as well as contribute to a proposed joint PhD program with the Department of Computer Science. Commitment to develop new graduate courses and supervise graduate students in the areas of financial mathematics and mathematical modeling is anticipated.

Qualifications

The successful candidate must have recently completed a PhD (since 2015) with expertise in analytical and computational aspects of Financial Mathematics and computer-based Mathematical Modeling. A strong record of independent research in these areas which will attract external funding is desired. The successful candidate must have a demonstrated teaching ability.

Applicants should submit through the Brock Careers Website at the link below by March 31, 2020, (indicating file number stated above), a cover letter, a curriculum vitae, samples of recent publications, statements of research and teaching interests: `https://brocku.wd3.myworkdayjobs.com/brocku_careers/job/St-Catharines-Main-Campus/Assistant-Professor--Financial-Mathematics---Mathematical-Modeling_JR-1005179-1`.

Learn more about Brock University by visiting `www.brocku.ca`.

09

Brock University
Faculty Position in Data Science and Statistical Theory

The Department of Mathematics and Statistics at Brock University invites applications for a probationary tenure-track position as Assistant Professor in data science and statistical theory, effective July 1, 2020.

About the Position

The Department currently offers a BSc and MSc in Mathematics and Statistics and is preparing to launch a new trans-disciplinary BSc in Data Sciences and Analytics as well as a new PhD Program in Computational, Mathematical, and Statistical Sciences jointly with the Department of Computer Science. Statistics graduates within the department have an outstanding placement record as professional statisticians with high reputation, working in diverse fields. The successful candidate will bring expertise in Data Science and Statistical Theory, providing a bridge between the existing Statistics program and the new Data Science program and will help to translate the former's success to the latter.

Qualifications

The successful candidate must have recently completed a PhD (since 2015) in statistics (or a related field) by the time of the appointment, a proven record of research excellence, and an active research program that will attract external funding. The successful candidate must demonstrate the ability to teach both undergraduate and graduate courses in statistical theory (e.g., experiment design, sampling theory) and will be tasked with modernizing these courses and integrating them with the new Data Science program. The candidate must demonstrate the ability to supervise graduate students and will have a primary role in extending our existing MSc in Statistics to include a stream in Data Science.

Applicants should submit through the Brock Careers website at the link below by March 31, 2020 (indicating file number stated above), a cover letter, a curriculum vitae, samples of recent publications, statements of research and teaching interests: `https://brocku.wd3.myworkdayjobs.com/brocku_careers/job/St-Catharines-Main-Campus/Assistant-Professor--Data-Science---Statistical-Theory_JR-1005177`.

Learn more about Brock University by visiting `www.brocku.ca`.

10

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

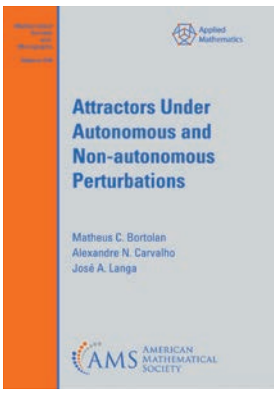
Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Ms. Erica Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

01

New Books Offered by the AMS

Analysis



Attractors Under Autonomous and Non-autonomous Perturbations

Matheus C. Bortolan, *Universidade Federal de Santa Catarina, Florianópolis SC, Brazil*, **Alexandre N. Carvalho**, *Universidade de São Paulo, São Carlos SP, Brazil*, and **José A. Langa**, *Universidad de Sevilla, Seville, Spain*

This book provides a comprehensive study of how attractors behave under perturbations for both autonomous and non-autonomous problems. Furthermore, the forward asymptotics of non-autonomous dynamical systems is presented here for the first time in a unified manner.

When modelling real world phenomena imprecisions are unavoidable. On the other hand, it is paramount that mathematical models reflect the modelled phenomenon, in spite of unimportant neglectable influences discounted by simplifications, small errors introduced by empirical laws or measurements, among others.

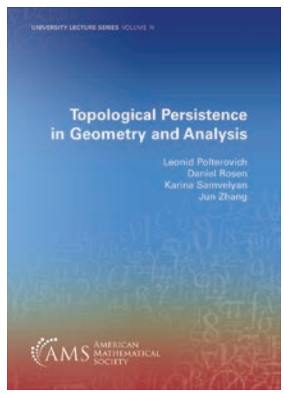
The authors deal with this issue by investigating the permanence of dynamical structures and continuity properties of the attractor. This is done in both the autonomous (time independent) and non-autonomous (time dependent) framework in four distinct levels of approximation: the upper semicontinuity, lower semicontinuity, topological structural stability and geometrical structural stability.

This book is aimed at graduate students and researchers interested in dissipative dynamical systems and stability theory, and requires only a basic background in metric spaces, functional analysis and, for the applications, techniques of ordinary and partial differential equations.

Mathematical Surveys and Monographs, Volume 246
July 2020, 254 pages, Hardcover, ISBN: 978-1-4704-5308-4, 2010 *Mathematics Subject Classification:* 34D45, 35B41; 37C70, 35B20, 37D15, List US$140, **AMS members US$112**, **MAA members US$126**, Order code SURV/246

bookstore.ams.org/surv-246

Geometry and Topology



Topological Persistence in Geometry and Analysis

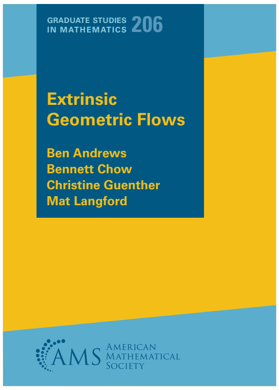
Leonid Polterovich, *Tel Aviv University, Israel*, **Daniel Rosen**, *Ruhr-Universität Bochum, Germany*, **Karina Samvelyan**, *Tel Aviv University, Israel*, and **Jun Zhang**, *Université de Montréal, Canada*

The theory of persistence modules originated in topological data analysis and became an active area of research in algebraic topology. This book provides a concise and self-contained introduction to persistence modules and focuses on its interactions with pure mathematics, bringing the reader to the cutting edge of current research. In particular, the authors present applications of persistence to symplectic topology, including the geometry of symplectomorphism groups and embedding problems. Furthermore, they discuss topological function theory, which provides new insight into oscillation of functions. The book is accessible to readers with a basic background in algebraic and differential topology.

This item will also be of interest to those working in applications.

University Lecture Series, Volume 74
June 2020, 140 pages, Softcover, ISBN: 978-1-4704-5495-1, LC 2019059052, 2010 *Mathematics Subject Classification:* 55U99, 58Cxx, 53Dxx, List US$55, **AMS members US$44**, **MAA members US$49.50**, Order code ULECT/74

bookstore.ams.org/ulect-74



Extrinsic Geometric Flows

Ben Andrews, *The Australian National University, Canberra, Australia*, **Bennett Chow**, *University of California, San Diego, La Jolla, CA*, **Christine Guenther**, *Pacific University, Forrest Grove, OR*, and **Mat Langford**, *University of Tennessee, Knoxville, TN*

Extrinsic geometric flows are characterized by a submanifold evolving in an ambient space with velocity determined by its extrinsic curvature. The goal of this book is to give an extensive introduction to a few of the most prominent extrinsic flows, namely, the curve shortening flow, the mean curvature flow, the Gauß curvature flow, the inverse-mean curvature flow, and fully nonlinear flows of mean curvature and inverse-mean curvature type. The authors highlight techniques and behaviors that frequently arise in the study of these (and other) flows. To illustrate the broad applicability of the techniques developed, they also consider general classes of fully nonlinear curvature flows.

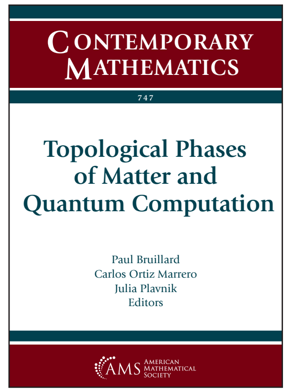
The book is written at the level of a graduate student who has had a basic course in differential geometry and has some familiarity with partial differential equations. It is intended also to be useful as a reference for specialists. In general, the authors provide detailed proofs, although for some more specialized results they may only present the main ideas; in such cases, they provide references for complete proofs. A brief survey of additional topics, with extensive references, can be found in the notes and commentary at the end of each chapter.

Graduate Studies in Mathematics, Volume 206
May 2020, 790 pages, Hardcover, ISBN: 978-1-4704-5596-5, LC 2019059835, 2010 *Mathematics Subject Classification:* 53C44, 58J35, 53A07, 52A20, 35K20, List US$98, **AMS members US$78.40**, **MAA members US$88.20**, Order code GSM/206

bookstore.ams.org/gsm-206

New in Contemporary Mathematics

Applications



Topological Phases of Matter and Quantum Computation

Paul Bruillard, *Pacific Northwest National Laboratory, Richland, WA*, **Carlos Ortiz Marrero**, *Pacific Northwest National Laboratory, Richland, WA*, and **Julia Plavnik**, *Indiana University, Bloomington, IN*, Editors

This volume contains the proceedings of the AMS Special Session on Topological Phases of Matter and Quantum Computation, held from September 24–25, 2016, at Bowdoin College, Brunswick, Maine.

Topological quantum computing has exploded in popularity in recent years. Sitting at the triple point between mathematics, physics, and computer science, it has the potential to revolutionize sub-disciplines in these fields. The academic importance of this field has been recognized in physics through the 2016 Nobel Prize. In mathematics, some of the 1990 Fields Medals were awarded for developments in topics that nowadays are fundamental tools for the study of topological quantum computation. Moreover, the practical importance of this discipline has been underscored by recent industry investments.

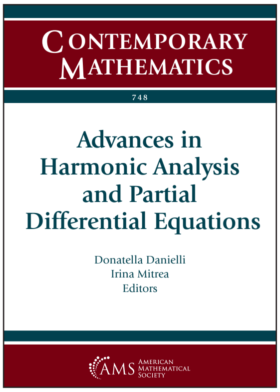
The relative youth of this field combined with a high degree of interest in it makes now an excellent time to get involved. Furthermore, the cross-disciplinary nature of topological quantum computing provides an unprecedented number of opportunities for cross-pollination of mathematics, physics, and computer science. This can be seen in the variety of works contained in this volume. With articles coming from mathematics, physics, and computer science, this volume aims to provide a taste of different sub-disciplines for novices and a wealth of new perspectives for veteran researchers. Regardless of your point of entry into topological quantum computing or your experience level, this volume has something for you.

Contemporary Mathematics, Volume 747
May 2020, 240 pages, Softcover, ISBN: 978-1-4704-4074-9, LC 2019040079, 2010 *Mathematics Subject Classification:* 81R50, 16D90, 81T05, 20G42, 18D10, 19D23, List US$120,

AMS members US$96, **MAA members US$108**, Order code CONM/747

bookstore.ams.org/conm-747

Differential Equations



Advances in Harmonic Analysis and Partial Differential Equations

Donatella Danielli, *Purdue University, West Lafayette, IN*, and **Irina Mitrea**, *Temple University, Philadelphia, PA*, Editors

This volume contains the proceedings of the AMS Special Session on Harmonic Analysis and Partial Differential Equations, held from April 21–22, 2018, at Northeastern University, Boston, Massachusetts.

The book features a series of recent developments at the interface between harmonic analysis and partial differential equations and is aimed toward the theoretical and applied communities of researchers working in real, complex, and harmonic analysis, partial differential equations, and their applications.

The topics covered belong to the general areas of the theory of function spaces, partial differential equations of elliptic, parabolic, and dissipative types, geometric optics, free boundary problems, and ergodic theory, and the emphasis is on a host of new concepts, methods, and results.

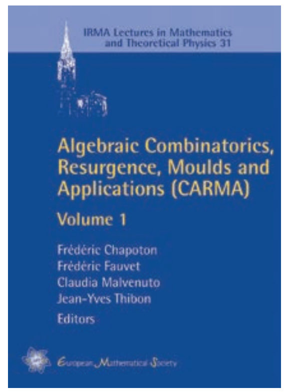
This item will also be of interest to those working in analysis.

Contemporary Mathematics, Volume 748
May 2020, 210 pages, Softcover, ISBN: 978-1-4704-4896-7, LC 2019040080, 2010 *Mathematics Subject Classification:* 31A10, 33C10, 35G20, 35P20, 35S05, 39B72, 42B35, 46E30, 76D03, 78A05, List US$120, **AMS members US$96**, **MAA members US$108**, Order code CONM/748

bookstore.ams.org/conm-748

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA): Volume 1

Frédéric Chapoton, *Université de Strasbourg, France*, **Frédéric Fauvet**, *Université de Strasbourg, France*, **Claudia Malvenuto**, *Universitá di Roma La Sapienza, Italy*, and **Jean-Yves Thibon**, *Université Paris-Est Marne-la-Vallée, France*, Editors

This is volume 1 of a 2-volume work comprising a total of 14 refereed research articles which stem from the CARMA Conference (Algebraic Combinatorics, Resurgence, Moulds and Applications), held at the Centre International de Rencontres Mathématiques in Luminy, France, from June 26–30, 2017. The conference did notably emphasize the role of Hopf algebraic techniques and related concepts (e.g. Rota-Baxter algebras, operads, and Ecalle's mould calculus) which have lately proved pervasive in combinatorics, but also in many other fields, from multiple zeta values to the algebraic study of control systems and the theory of rough paths.

The volumes should be useful to researchers or graduate students in mathematics working in these domains and to theoretical physicists involved with resurgent functions and alien calculus.

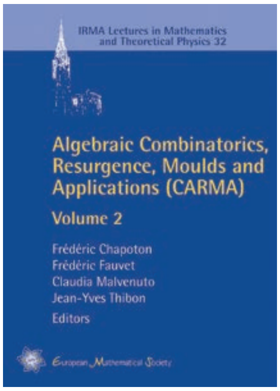
This item will also be of interest to those working in discrete mathematics and combinatorics.

A publication of the European Mathematical Society. Distributed within the Americas by the American Mathematical Society.

IRMA Lectures in Mathematics and Theoretical Physics, Volume 31
March 2020, 354 pages, Hardcover, ISBN: 978-3-03719-204-7, 2010 *Mathematics Subject Classification:* 05Exx, 81T15, 81T18, 81Q30, 34C20, 37C10, 18D50, 34M40, 34M60,

11M32, 30D60, List US$68, **AMS members US$54.40**, Order code EMSILMTP/31

bookstore.ams.org/emsilmtp-31



Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA): Volume 2

Frédéric Chapoton, *Universitë de Strasbourg, France*, **Frédéric Fauvet**, *Université de Strasbourg, France*, **Claudia Malvenuto**, *Universitá di Roma La Sapienza, Italy*, and **Jean-Yves Thibon**, *Université Paris-Est Marne-la-Vallée, France*, Editors

This is volume 2 of a 2-volume work comprising a total of 14 refereed research articles which stem from the CARMA Conference (Algebraic Combinatorics, Resurgence, Moulds and Applications), held at the Centre International de Rencontres Mathématiques in Luminy, France, from June 26–30, 2017. The conference did notably emphasize the role of Hopf algebraic techniques and related concepts (e.g. Rota-Baxter algebras, operads, and Ecalle's mould calculus) which have lately proved pervasive in combinatorics, but also in many other fields, from multiple zeta values to the algebraic study of control systems and the theory of rough paths.

The volumes should be useful to researchers or graduate students in mathematics working in these domains and to theoretical physicists involved with resurgent functions and alien calculus.

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A publication of the European Mathematical Society. Distributed within the Americas by the American Mathematical Society.

IRMA Lectures in Mathematics and Theoretical Physics, Volume 32
March 2020, 396 pages, Hardcover, ISBN: 978-3-03719-205-4, 2010 *Mathematics Subject Classification:* 05Exx, 81T15, 81T18, 81Q30, 34C20, 37C10, 18D50, 34M40, 34M60, 11M32, 30D60, List US$68, **AMS members US$54.40**, Order code EMSILMTP/32

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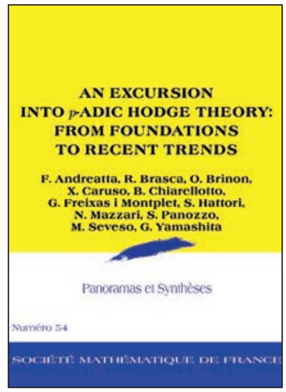


An Excursion into *p*-Adic Hodge Theory

From Foundations to Recent Trends

Fabrizio Andreatta, *Universita degli Studi di Milano, Milano, Italy*, **Riccardo Brasca**, *Institut de Mathématiques de Jussieu, Université Paris Diderot, Paris, France*, **Olivier Brinon**, *Université de Bordeaux, Talence, France*, **Xavier Caruso**, *Université de Bourdeaux, Talence, France*, **Bruno Chiarellotto**, *University of Padova, Italy*, **Gerard Freixas i Montplet**, *Institut de Mathématiques de Jussieu, Université Paris Diderot, Paris, France*, **Shin Hattori**, *Tokyo City University, Tokyo, Japan*, **Nicola Mazzari**, *Université de Bourdeaux, Talence, France*, **Simone Panozzo**, *Universita degli Studi di Milano, Milano, Italy*, **Marco Seveso**, *Universita degli Studi di Milano, Milano, Italy*, and **Go Yamashita**, *Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan*

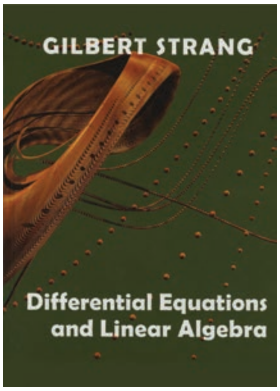
This volume offers a progressive and comprehensive introduction to *p*-adic Hodge theory. It starts with Tate's works on *p*-adic divisible groups and the cohomology of *p*-adic varieties which constitutes the main concrete motivations for the development of *p*-adic Hodge theory. It then moves smoothly to the construction of Fontaine's *p*-adic period rings and their apparition in several comparison theorems between various *p*-adic cohomologies. Applications and generalizations of these theorems are subsequently discussed. Finally, Scholze's modern vision on *p*-adic Hodge theory, based on the theory of perfectoids, is presented.

This item will also be of interest to those working in geometry and topology.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Panoramas et Synthèses, Number 54
January 2020, 268 pages, Softcover, ISBN: 978-2-85629-913-5, 2010 *Mathematics Subject Classification:* 14F30, 14F40, 11G25, 11F80, List US$75, **AMS members US$60**, Order code PASY/54

bookstore.ams.org/pasy-54



Differential Equations and Linear Algebra

Gilbert Strang, *Massachusetts Institute of Technology*

Differential equations and linear algebra are the two crucial courses in undergraduate mathematics. This new textbook develops those subjects separately and together. The complete book is a year's course and includes Fourier and Laplace transforms, as well as the Fast Fourier Transform and Singular Value Decomposition.

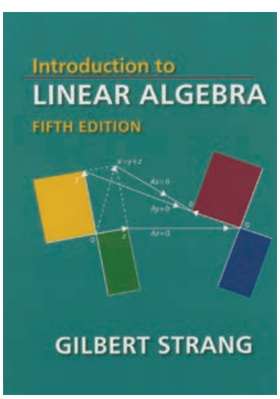
Undergraduate students in courses covering differential equations and linear algebra, either separately or together, will find this material essential to their understanding.

This item will also be of interest to those working in differential equations.

A publication of Wellesley-Cambridge Press. Distributed within the Americas by the American Mathematical Society.

The Gilbert Strang Series, Volume 1
January 2014, 502 pages, Hardcover, ISBN: 978-0-9802327-9-0, 2010 *Mathematics Subject Classification:* 15–01; 34–01, List US$87.50, **AMS members US$70**, Order code STRANG/1

bookstore.ams.org/strang-1



Introduction to Linear Algebra

Fifth Edition

Gilbert Strang, *Massachusetts Institute of Technology*

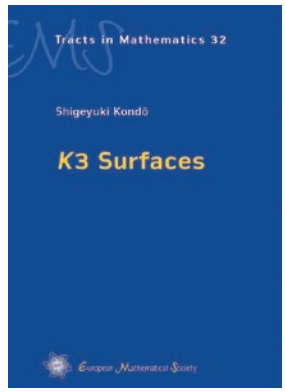
This book is designed to help students understand and solve the four central problems of linear algebra that involve: (1) linear systems, (2) least squares, (3) eigenvalues, and (4) singular values.

The diagram on the front cover shows the four fundamental subspaces for the matrix A. Those subspaces lead to the Fundamental Theorem of Linear Algebra: (1) The dimensions of the four subspaces, (2) The orthogonality of the two pairs, and (3) The best bases for all four subspaces.

A publication of Wellesley-Cambridge Press. Distributed within the Americas by the American Mathematical Society.

The Gilbert Strang Series, Volume 2
May 2016, 573 pages, Hardcover, ISBN: 978-0-9802327-7-6, 2010 *Mathematics Subject Classification:* 15–01, List US$95, **AMS members US$76**, Order code STRANG/2

bookstore.ams.org/strang-2



*K*3 Surfaces

Shigeyuki Kondo, *Nagoya University, Japan*

*K*3 surfaces are a key piece in the classification of complex analytic or algebraic surfaces. The term was coined by A. Weil in 1958, a result of the initials Kummer, Kähler, Kodaira, and the mountain K2 found in Karakoram. The most famous example is the Kummer surface discovered in the 19th century. *K*3 surfaces can be considered as a 2-dimensional analogue of an elliptic curve, and the theory of periods—called the Torelli-type theorem for *K*3 surfaces—was established around 1970. Since then, several pieces of research on *K*3 surfaces have been undertaken and more recently *K*3 surfaces have even become of interest in theoretical physics.

The main purpose of this book is an introduction to the Torelli-type theorem for complex analytic *K*3 surfaces, and its applications. The theory of lattices and their reflection groups is necessary to study *K*3 surfaces, and this book introduces these notions. The book contains, as well as lattices and reflection groups, the classification of complex analytic surfaces, the Torelli-type theorem, the subjectivity of the period map, Enriques surfaces, an application to the moduli space of plane quartics, finite automorphisms of *K*3 surfaces, Niemeier lattices and the Mathieu group, the automorphism group of Kummer surfaces and the Leech lattice.

The author seeks to demonstrate the interplay between several sorts of mathematics and hopes the book will prove helpful to researchers in algebraic geometry and related areas, and to graduate students with a basic grounding in algebraic geometry.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Tracts in Mathematics, Volume 32
April 2020, 250 pages, Hardcover, ISBN: 978-3-03719-208-5, 2010 *Mathematics Subject Classification:* 14J28, 14C34,

14J10, 14J15, 14J50, 32G20, List US$88, **AMS members US$70.40**, Order code EMSTM/32

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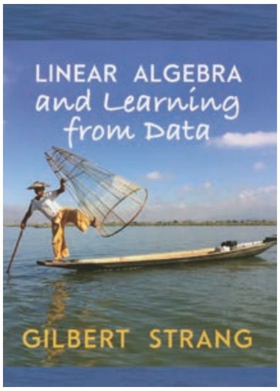


Linear Algebra and Learning from Data

Gilbert Strang, *Massachusetts Institute of Technology*

This is a textbook to help readers understand the steps that lead to deep learning. Linear algebra comes first, especially singular values, least squares, and matrix factorizations. Often the goal is a low rank approximation A = CR (column-row) to a large matrix of data to see its most important part. This uses the full array of applied linear algebra, including randomization for very large matrices.

Then deep learning creates a large-scale optimization problem for the weights solved by gradient descent or better stochastic gradient descent. Finally, the book develops the architectures of fully connected neural nets and of Convolutional Neural Nets (CNNs) to find patterns in data.

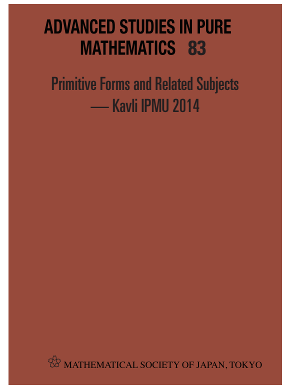
This item will also be of interest to those working in applications.

A publication of Wellesley-Cambridge Press. Distributed within the Americas by the American Mathematical Society.

The Gilbert Strang Series, Volume 3
January 2019, 432 pages, Hardcover, ISBN: 978-0-692-19638-0, 2010 *Mathematics Subject Classification:* 15–01; 68–01, List US$95, **AMS members US$76**, Order code STRANG/3

bookstore.ams.org/strang-3

Differential Equations



Primitive Forms and Related Subjects—Kavli IPMU 2014

Kentaro Hori, *Kavli Institute for the Physics and Mathematics of the Universe (IPMU)*, **Changzheng Li**, *Sun Yat-sen University*, **Si Li**, *Tsinghua University*, and **Kyoji Saito**, *Kavli Institute for the Physics and Mathematics of the Universe (IPMU)*, Editors

This volume contains the proceedings of the conference "Primitive Forms and Related Subjects", held at the Kavli Institute for the Physics and Mathematics of the Universe (IPMU), the University of Tokyo, February 10–14, 2014.

The principal aim of the conference was to discuss the current status of rapidly developing subjects related to primitive forms. In particular, Fukaya category, Gromov-Witten and FJRW invariants, mathematical formulation of Landau-Ginzburg models, and mirror symmetry were discussed. The conference had three introductory courses by.experts and 12 lectures on more advanced topics. This volume contains two survey articles and 11 research articles based on the conference presentations.

This item will also be of interest to those working in geometry and topology.

Published for the Mathematical Society of Japan by Kinokuniya, Tokyo, and distributed worldwide, except in Japan, by the AMS.

Advanced Studies in Pure Mathematics, Volume 83
December 2019, 415 pages, Hardcover, ISBN: 978-4-86497-085-3, 2010 *Mathematics Subject Classification:* 53Dxx; 14N35, List US$82, **AMS members US$65.60**, Order code ASPM/83

bookstore.ams.org/aspm-83

NEW BOOKS

Discrete Mathematics and Combinatorics

Séminaire Bourbaki

Volume 2017–2018 Exposés 1136-1150

This 70th volume of the Bourbaki Seminar gathers the texts of the 15 survey lectures delivered during the year 2017/2018. Among the topics addressed, the reader will find the Zimmer program, Deligne-Lusztig varieties, cocompact-convex groups, Soergel bimodules, the Gan-Gross-Prasad conjectures, exponential sums, Bernoulli convolutions, the Navier-Stokes equations, the combinatorics of matroids, the dynamics of Schrödinger equations, the asymptotic distribution of Frobenius eigenvalues, stability conditions in birational geometry, the Monge-Ampére equation, harmonic maps in negative curvature, and the Fargues-Fontaine curves.

This item will also be of interest to those working in geometry and topology.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 414
January 2020, 626 pages, Softcover, ISBN: 978-2-85629-915-9, 2010 *Mathematics Subject Classification:* 22E40, 37C85, 37C40, 53C24, 20C20, 20C33, 20G05, 20G40, 18E30, 37B05, 20H10, 53C35, 51E24, 22F30, 37D40, 57S30, 20C08, 20F55, 11F70, 11F67, 11F27, 11T24, 11L05, 11G25, 14F05, 14F20, 28A80, 42A85, 35Q30, 60F99, 82C22, 05A99, 05E99, 14F43, 14F99, 14M25, 14T05, 35P20, 35J10, 35S15, 58J50, 11B05, 14D20, 14E05, 14J28, 35J96, 35B65, 35J60, 53C43, 31B05, 31B25, 31B35, 58E20, 14G20, 11F80, 14G22, List US$120, **AMS members US$96**, Order code AST/414

bookstore.ams.org/ast-414

Meetings & Conferences of the AMS

May Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 110 in the January 2020 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Georgia Benkart, University of Wisconsin–Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: lapidus@math.ucr.edu; telephone: 951-827-5910.

Meetings in this Issue

See www.ams.org/meetings *for the most up-to-date information on the meetings and conferences that we offer.*

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

El Paso, Texas

University of Texas at El Paso

September 12–13, 2020

Saturday – Sunday

Meeting #1159

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: June 2020
Program first available on AMS website: July 28, 2020
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 41, Issue 3

Deadlines

For organizers: Expired
For abstracts: July 14, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Caroline Klivans, Brown University, *Title to be announced*.
Brisa Sanchez, Drexel University, *Title to be announced*.
Alejandra Sorto, Texas State University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic, geometric and topological combinatorics (Code: SS 6A), **Art Duval**, University of Texas at El Paso, **Caroline Klivans**, Brown University, and **Jeremy Martin**, University of Kansas.

Algebraic structures in topology, logic, and arithmetic (Code: SS 3A), **John Harding**, New Mexico State University, and **Emil Schwab**, The University of Texas at El Paso.

Commutative Algebra (Code: SS 12A), **Sara Faridi**, Dalhousie University, and **Susan Morey**, Texas State University.

Fixed point theory and its applications (Code: SS 5A), **Monther R. Alfuraidan**, King Fahd University of Petroleum & Minerals, KSA, **Mohamed A. Khamsi**, The University of Texas at El Paso, **Poom Kumam**, King Mongkut's University of Technology, Thonburi, Thailand, and **Osvaldo Mendez**, The University of Texas at El Paso.

Free Resolutions, Combinatorics, and Geometry (Code: SS 17A), **Anton Dochtermann**, Texas State University, and **Sean Sather-Wagstaff**, Clemson University.

Geometric Inequalities and Nonlinear Partial Differential Equations (Code: SS 20A), **Joshua Flynn**, University of Connecticut, **Jungang Li**, Brown University, and **Guozhen Lu**, University of Connecticut.

Geometry of Submanifolds and Integrable Systems (Code: SS 21A), **Magdalena Toda** and **Hung Tran**, Texas Tech University.

Groups and Their Cohomological Invariants in Arithmetic and Geometry (Code: SS 13A), **Stefan Gille** and **Nikita Karpenko**, University of Alberta, and **Jan Minac**, Western University.

High-Frequency data analysis and applications (Code: SS 1A), **Maria Christina Mariani** and **Michael Pokojovy**, University of Texas at El Paso, and **Ambar Sengupta**, University of Connecticut.

Leibniz Algebras and related topics (Code: SS 7A), **Guy Biyogmam**, Georgia College and State University, and **Jerry Lodder**, New Mexico State University.

Low-dimensional topology and knot theory (Code: SS 4A), **Mohamed Ait Nouh** and **Luis Valdez-Sanchez**, University of Texas at El Paso.

Methods and applications in data Science (Code: SS 9A), **Sangjin Kim**, **Ming-Ying Leung**, **Xiaogang Su**, and **Amy Wagler**, The University of Texas at El Paso.

Nonlinear analysis and optimization (Code: SS 2A), **Behzad Djafari-Rouhani**, University of Texas at El Paso, and **Akhtar A. Khan**, Rochester Institute of Technology.

Non-Linear Evolution Equations (Code: SS 19A), **Irena Lasiecka** and **Roberto Triggiani**, University of Memphis, and **Xiang Wan**, George Washington University.

Numerical partial differential equations and applications (Code: SS 10A), **Son-Young Yi** and **Xianyi Zeng**, The University of Texas at El Paso.

Recent advances in scientific computing and applications (Code: SS 11A), **Natasha Sharma**, University of Texas at El Paso, and **Annalisa Quaini**, University of Houston.

Recent Developments in Commutative Algebra (Code: SS 15A), **Louiza Fouli** and **Jonathan Montaño**, New Mexico State University.

Statistical methodology and applications (Code: SS 8A), **Ori Rosen** and **Suneel Chatla**, University of Texas at El Paso.

Stochastic Modeling in Mathematical Biology (Code: SS 16A), **Mary Ballyk**, New Mexico State University, **Si Tang**, Lehigh University, and **Jianjun Paul Tian**, New Mexico State University.

Theoretical and Computational Studies of PDEs Related to Fluid Mechanics (Code: SS 14A), **Phuong Nguyen**, Texas Tech University, **Andrei Tarfulea**, Louisiana State University, and **Kazuo Yamazaki**, Texas Tech University.

Undergraduate Teaching and Learning of Mathematics (Code: SS 18A), **Paul Dawkins** and **Samuel Obara**, Texas State University.

State College, Pennsylvania

Pennsylvania State University, University Park Campus

October 3–4, 2020

Saturday – Sunday

Meeting #1160

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: August 2020
Program first available on AMS website: August 25, 2020
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 41, Issue 3

Deadlines

For organizers: Expired
For abstracts: August 11, 2020

MEETINGS & CONFERENCES

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Melody Chan, Brown University, *Title to be announced*.
Steven J. Miller, Williams College, *Title to be announced*.
Tadashi Tokieda, Stanford University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Mathematical Modeling of Infection (Code: SS 21A), **Jessica M. Conway**, Pennsylvania State University, **Troy Day**, Queen's University, and **Timothy C. Reluga**, Pennsylvania State University.

Algebraic and Analytic Theory of Elliptic Curves (Code: SS 22A), **Alina Cojocaru**, University of Illinois at Chicago, **Seoyung Kim**, Queen's University, **Steven J. Miller**, Williams College, and **Jesse Thorner**, University of Florida.

Algebraic Singularities in Arbitrary Characteristic (Code: SS 12A), **Rankeya Datta**, University of Illinois at Chicago, and **Takumi Murayama**, Princeton University.

Analytic Number Theory (Code: SS 11A), **Angel V. Kumchev**, Towson University, and **Siddhi S. Pathak** and **Robert C. Vaughan**, Pennsylvania State University.

Automorphic Forms and Galois Representations (Code: SS 3A), **Jim Brown**, Occidental College, and **Krzysztof Klosin**, Queens College, CUNY.

Cluster Algebras and Plabic Graphs (Code: SS 16A), **Chris Fraser**, University of Minnesota, and **Max Glick**, Google Inc.

Combinatorics and Computing (Code: SS 17A), **Saúl A. Blanco**, Indiana University, and **Charles Buehrle**, Notre Dame of Maryland University.

Commutative Algebra and Connections to Algebraic Geometry and Combinatorics (Code: SS 4A), **Ayah Almousa**, Cornell University, and **Kuei-Nuan Lin**, Pennsylvania State University, Greater Allegheny.

Configuration Spaces across Combinatorics and Topology (Code: SS 18A), **Florian Frick** and **Michael Harrison**, Carnegie Mellon University.

Conservation Laws and Nonlinear Wave Equations (Code: SS 14A), **Alberto Bressan**, Pennsylvania State University, **Geng Chen**, University of Kansas, and **Qingtian Zhang**, West Virginia University.

Drinfeld Modules, Modular Varieties and Arithmetic Applications (Code: SS 10A), **Mihran Papikian**, Pennsylvania State University, and **Dinesh Thakur**, University of Rochester.

Geometric Dynamics (Code: SS 23A), **Mark Levi**, Pennsylvania State University, and **Sergei Tabachnikov**, Pennsylvania State University.

Geometry and Arithmetic of Algebraic Varieties (Code: SS 8A), **Jack Huizenga**, **John Kopper**, and **John Lesieutre**, Pennsylvania State University.

Geometry of Groups and 3-manifolds (Code: SS 2A), **Abhijit Champanerkar**, College of Staten Island and The Graduate Center, CUNY, and **Hongbin Sun**, Rutgers University.

Homological Methods in Algebra (Code: SS 5A), **Ela Celikbas** and **Olgur Celikbas**, West Virginia University, and **Saeed Nasseh**, Georgia Southern University.

Legendrian Knots and Surfaces (Code: SS 6A), **Honghao Gao**, Michigan State University, and **Dan Rutherford**, Ball State University.

Nonlinear Scientific Computing and Applications (Code: SS 1A), **Wenrui Hao**, Pennsylvania State University.

q-Series and Related Areas in Combinatorics and Number Theory (Code: SS 7A), **George Andrews**, **David Little**, and **Ae Ja Yee**, Pennsylvania State University.

Recent Developments in Gauge Theory (Code: SS 19A), **Siqi He**, Stony Brook University, and **Ákos Nagy**, Duke University.

Recent Probabilistic Advances in Mathematical Physics (Code: SS 20A), **Alexei Novikiov, Izabella Stuhl**, and **Yuri Suhov,** Pennsylvania State University.

Turbulence and Mixing in Fluid Dynamics (Code: SS 15A), **Yuanyuan Feng, Anna Mazzucato,** and **Alexei Novikov,** Pennsylvania State University.

Variational Aspects of Geometric Analysis (Code: SS 13A), **Jeffrey Case**, Pennsylvania State University, **Casey Kelleher,** Princeton University, **Chao Li**, Princeton University, and **Siyi Zhang**, University of Notre Dame.

Chattanooga, Tennessee

University of Tennessee at Chattanooga

October 10–11, 2020

Saturday – Sunday

Meeting #1161

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: August 2020

Program first available on AMS website: September 1, 2020
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 41, Issue 4

Deadlines

For organizers: Expired
For abstracts: August 18, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Giulia Saccà, Columbia University, *Title to be announced*.

Chad Topaz, Williams College, *Title to be announced*.

Xingxing Yu, Georgia Institute of Technology, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Active Learning Methods and Pedagogical Approaches in Teaching College Level Mathematics (Code: SS 20A), **Hashim A. Saber**, University of North Georgia.

Advances in Graph Theory (Code: SS 5A), **Xiaofeng Gu**, University of West Georgia, and **Dong Ye**, Middle Tennessee State University.

Advances in Image Reconstruction Algorithms for Inverse Tomography Problems (Code: SS 22A), **Sanwar Uddin Ahmad**, Colorado State University, and **Taufiquar R Khan**, Clemson University.

Advances in the Modeling and Computation of Fluid Flows and Fluid-Structure Interactions (Code: SS 11A), **Jin Wang** and **Eleni Panagiotou**, University of Tennessee at Chattanooga.

Applicable Analysis of PDE Systems which Govern Fluid Flows and Flow-Structure Interactions (Code: SS 12A), **Pelin Guven Geredeli**, Iowa State University, and **George Avalos**, University of Nebraska-Lincoln.

Applied Knot Theory (Code: SS 4A), **Jason Cantarella**, University of Georgia, **Eleni Panagiotou**, University of Tennessee at Chattanooga, and **Eric Rawdon**, University of St Thomas.

Boundary Value Problems for Differential, Difference, and Fractional Equations (Code: SS 2A), **John R Graef** and **Lingju Kong**, University of Tennessee at Chattanooga, and **Min Wang**, Kennesaw State University.

Coding Theory, Cryptography, and Number Theory (Code: SS 16A), **Ryann Cartor**, **Shuhong Gao**, **Kevin James,** and **Felice Manganiello**, Clemson University.

Commutative Algebra (Code: SS 1A), **Simplice Tchamna**, Georgia College, and **Lokendra Paudel**, University of South Carolina, Salkehatchie.

Convexity and Probability in High Dimensions (Code: SS 21A), **Steven Hoehner**, Longwood University, **Stanislaw Szarek,** Case Western Reserve University, and **Elisabeth Werner**, Case Western Reserve University.

Geometric and Topological Generalization of Groups (Code: SS 19A), **Bikash C Das**, University of North Georgia.

Homological Commutative Algebra (Code: SS 9A), **Hugh Geller**, **James Gossell**, and **Sean Sather-Wagstaff**, Clemson University.

Interactions Between Algebra, Geometry and Topology in Low Dimensions (Code: SS 6A), **Alex Casella** and **Lorenzo Ruffoni**, Florida State University at Tallahassee, and **Michelle Chu**, University of Illinois at Chicago.

Mathematics in Industry and National Laboratories (Code: SS 15A), **Samantha Erwin** and **John Gounley,** Oak Ridge National Laboratory.

Modern Applied Analysis (Code: SS 8A), **Boris Belinskiy**, University of Tennessee at Chattanooga.

Nonstandard Elliptic and Parabolic Regularity Theory with Applications (Code: SS 10A), **Hongjie Dong**, Brown University, and **Tuoc Phan**, University of Tennessee, Knoxville.

Polynomials, Approximation Theory, and Potential Theory (Code: SS 13A), **Aaron Yeager**, College of Coastal Georgia, and **Erik Lundberg**, Florida Atlantic University.

Probability and Statistical Models with Applications (Code: SS 7A), **Sher Chhetri**, University of South Carolina, Sumter, and **Cory Ball**, Florida Atlantic University.

Quantitative Approaches to Social Justice (Code: SS 17A), **Chad Topaz**, Williams College.

Structural and Extremal Graph Theory (Code: SS 3A), **Hao Huang**, Emory University, and **Xingxing Yu**, Georgia Institute of Technology.

Title to be Announced (Code: SS 18A), **Giulia Saccà**, Columbia University.

Topological Data Analysis and Artificial Intelligence (Code: SS 14A), **Vasilios Maroulas**, University of Tennessee, Knoxville, **Farzana Nasrin,** University of Tennessee, Knoxville, **Eleni Panagiotou,** University of Tennessee at Chattanooga, and **Theodore Papamarkou,** Oak Ridge National Laboratory.

Salt Lake City, Utah

University of Utah

October 24–25, 2020

Saturday – Sunday

Meeting #1162

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: August 2020
Program first available on AMS website: September 17, 2020
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: Volume 41, Issue 4

Deadlines

For organizers: Expired
For abstracts: September 1, 2020

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Bhargav Bhatt, University of Michigan, Ann Arbor, *Title to be announced*.
Jonathan Brundan, University of Oregon, Eugene, *Title to be announced*.
Andrei Okounkov, Columbia University, *Title to be announced* (Erdős Memorial Lecture).
Mariel Vazquez, University of California, Davis, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic combinatorics and applications in harmonic analysis (Code: SS 4A), **Joseph Iverson** and **Sung Y. Song**, Iowa State University, and **Bangteng Xu**, Eastern Kentucky University.

Building bridges between commutative algebra and nearby areas (Code: SS 6A), **Benjamin Briggs** and **Josh Pollitz**, University of Utah.

Commutative Algebra (MSC 13) (Code: SS 5A), **Adam Boocher**, University of San Diego, **Eloísa Grifo**, University of California, Riverside, and **Jennifer Kenkel**, University of Michigan.

Extremal Graph Theory (Code: SS 1A), **József Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

Geometry and Representation Theory of Quantum Algebras and Related Topics (Code: SS 7A), **Mee Seong Im**, United States Military Academy, West Point, **Bach Nguyen**, Temple University, and **Arik Wilbert**, University of Georgia.

Monoidal Categories in Representation Theory (associated with the Invited Address by Jon Brundan) (Code: SS 2A), **Jonathan Brundan**, **Ben Elias**, and **Victor Ostrik**, University of Oregon.

PDEs, data, and inverse problems (Code: SS 3A), **Vira Babenko**, Drake University, and **Akil Narayan**, University of Utah.

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday

Meeting #1163

Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: October 2020

Program first available on AMS website: November 1, 2020

Program issue of electronic *Notices*: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 16, 2020

For abstracts: To be announced

Atlanta, Georgia

Georgia Institute of Technology

March 13–14, 2021

Saturday – Sunday

Meeting #1164

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Differential Graded Methods in Commutative Algebra (Code: SS 1A), **Saeed Nasseh**, Georgia Southern University, and **Adela Vraciu**, University of South Carolina, Columbia.

Providence, Rhode Island

Brown University

March 20–21, 2021

Saturday – Sunday

Meeting #1165

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative Algebra, **Laura Ghezzi**, Department of Mathematics, New York City College of Technology-CUNY, **Saeed Nasseh**, Georgia Southern University, and **Oana Veliche**, Northeastern University.

Recent Advances in Schubert Calculus and Related Topics (Code: SS 2A), **Cristian Lenart** and **Changlong Zhong**, State University of New York at Albany.

Cincinnati, Ohio

University of Cincinnati

April 17–18, 2021

Saturday – Sunday

Meeting #1166

Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Francisco, California

San Francisco State University

May 1–2, 2021

Saturday – Sunday

Meeting #1167

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Differential Geometry and Geometric PDE, **Alfonso Agnew**, **Nicholas Brubaker**, **Thomas Murphy**, **Shoo Seto**, and **Bogdan Suceavă**, California State University, Fullerton.

Grenoble, France

Université de Grenoble-Alpes

July 5–9, 2021

Monday – Friday

Meeting #1168

Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: September 16, 2020
For abstracts: To be announced

Buenos Aires, Argentina

The University of Buenos Aires

July 19–23, 2021

Monday – Friday

Meeting #1169

Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: Not applicable
Program issue of electronic *Notices*: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: To be announced
For abstracts: To be announced

Buffalo, New York

University at Buffalo (SUNY)

September 18–19, 2021

Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Omaha, Nebraska

Creighton University

October 9–10, 2021

Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Albuquerque, New Mexico

University of New Mexico

October 23–24, 2021

Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Inverse Problems, **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico.

Mobile, Alabama

University of South Alabama

November 20, 2021

Saturday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 5–8, 2022

Wednesday – Saturday
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2021
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023

Wednesday – Saturday
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2022
Program first available on AMS website: To be announced
Program issue of electronic *Notices*: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

POSITION AVAILABLE

ASSOCIATE EXECUTIVE DIRECTOR

for Meetings and Professional Services

AMERICAN MATHEMATICAL SOCIETY

Applications are invited for the position of Associate Executive Director for Meetings and Professional Services. Come be part of an extraordinary collection of talent who support the Society's extensive activities. This is an exciting opportunity to have a direct impact on the ways we advance research and create connections in the mathematics community.

The Associate Executive Director heads the Meetings and Professional Services division and is part of the executive leadership team. Departments in the division support a variety of AMS meetings, programs, and activities that engage our members and the entire mathematical community. This robust range of activities includes meetings such as the Joint Mathematics Meetings, projects such as the Annual Survey and Mathjobs.org, membership activities such as the AMS Graduate Student Chapters, and activities such as our AMS Mathematics Research Communities and the AMS Fellows program, as well as a number of education initiatives, various travel support programs, and several outreach activities.

Responsibilities of the Associate Executive Director include:

- Developing and implementing long-range plans for all parts of the division
- Overseeing departments in the division, including budgetary planning and control
- Leadership and vision to ensure existing AMS programs optimize their impact, as well as in creating, planning, and implementing new programs
- Collaborating with other mathematical organizations
- Representing the division with AMS governance and the mathematical community
- Working closely with senior executive staff, as well as department directors across the organization, to ensure excellence and professionalism

Candidates should have an earned doctorate in one of the mathematical sciences as well as administrative experience. A strong interest in professional programs and services is essential, as is experience with grant writing. This position reports to the AMS Executive Director and also interacts with the AMS governance on the Council and Board of Trustees.

This position is full time, located in our Providence, RI headquarters. Salary is negotiable and will be commensurate with experience. Inquiries about the position are encouraged. Please contact exdir@ams.org. This position is open until filled. Please submit letter of interest, CV, and three professional references to be considered for the position.

AMS American Mathematical Society

Advancing research. Creating connections.

The American Mathematical Society is committed to creating a diverse environment and is proud to be an equal opportunity employer. All qualified applicants will receive consideration for employment without regard to race, color, religion, gender, gender identity or expression, sexual orientation, national origin, genetics, disability, age, veteran status, or immigration status.

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