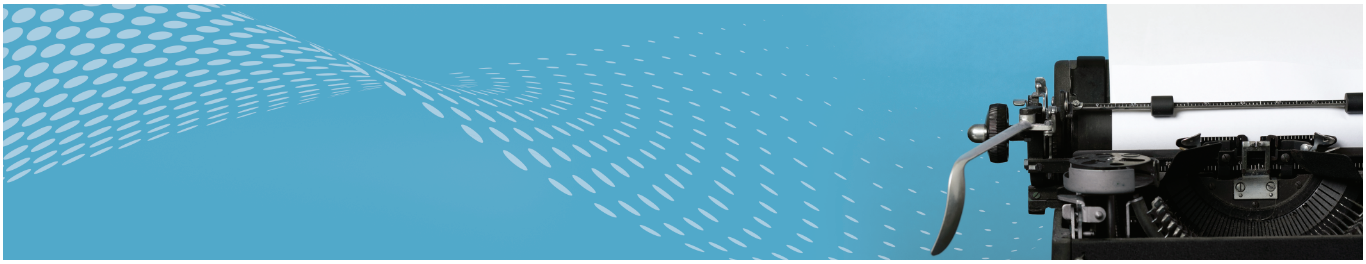

On the Kinematic Formula in the Lives of the Saints

Danny Calegari

Saint Sebastian was an early Christian martyr. He served as a captain in the Praetorian Guard under Diocletian until his religious faith was discovered, at which point he was taken to a field, bound to a stake, and shot by archers "till he was as full of arrows as an urchin[1] is full of pricks[2]." Rather miraculously, he made a full recovery, but was later executed anyway for insulting the emperor. The transpierced saint became a popular subject for Renaissance painters, e.g., Figure 1.

The arrows in Mantegna's painting have apparently arrived from all directions, though they are conspicuously grouped around the legs and groin, almost completely missing the thorax. Intuitively, we should expect more arrows in the parts of the body that present a bigger cross section. This intuition is formalized by the claim that a subset of the surface of Saint Sebastian has an area proportional to its expected number of intersections with a random line (i.e., arrow). Since both area and expectation are additive, we may reduce the claim (by polygonal approximation and limit) to the case of a flat triangular Saint Sebastian, in which case it is obvious.

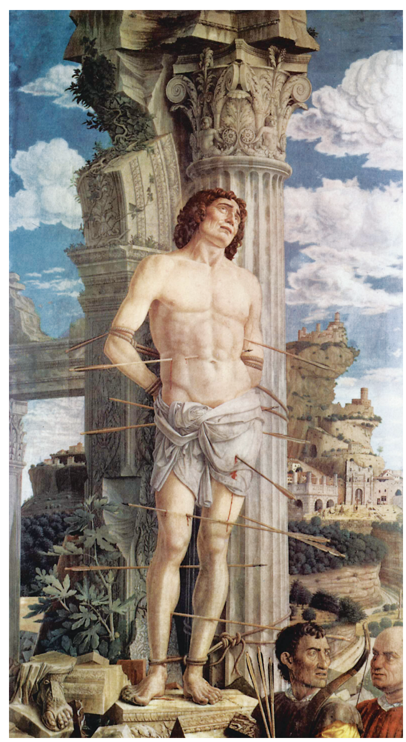

Figure 1. Andrea Mantegna's painting of Saint Sebastian.

This is a 3-dimensional version of the classical *Crofton formula*, which says that the length of a plane curve is proportional to its expected number of intersections with a random line. A famous application is known as "Buffon's needle". In the Euclidean plane we consider the family of

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[1] *hedgehog*

[2] *quills*

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all horizontal lines whose y coordinate is an integer, and we drop a needle of length 1 so that it lands somewhere at random. Then the probability that the needle crosses one of the lines is $2/\pi$ (i.e., the average of $|\sin(\theta)|$, where θ is the angle the needle makes with the horizontal). In fact, the only relevance of the straightness of the needle is that a straight needle of length 1 can cross at most one of the horizontal lines in at most one point. If you trade in your needle for a stick of spaghetti of the same length (i.e., "Buffon's noodle") and then cook the spaghetti and drop it at random in the plane, the expected number of crossings with the horizontal lines is still $2/\pi$, no matter what complicated physical process determines its shape.

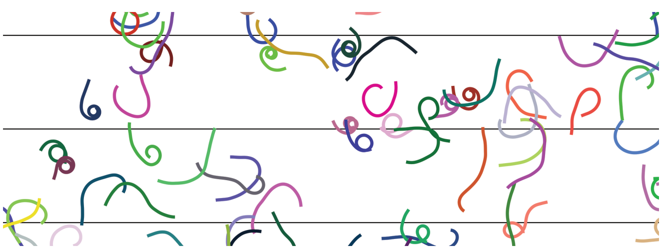

Figure 2. A random noodle of length 1 has $2/\pi$ expected crossings with the horizontal lines.

Crofton's formula can be greatly generalized. For example, Kang and Tasaki give a formula for the expected number of intersections of a (real) surface S in $\mathbb{CP}^n$ with a random hyperplane. The number of such intersections is

$$\frac{1}{2\pi}\int_S 1+\cos^2(\theta_x)\,d\mathrm{area}(x),$$

where for a point $x \in S$ the "Kähler angle" θ_x measures how far TS is from being complex: if v_1, v_2 is an orthonormal basis for T_xS, then $\cos(\theta_x)$ is the inner product of iv_1 with v_2. For example, if S is a holomorphic curve, then $\theta = 0$ and the area of S is π times the degree, whereas for S equal to the "standardly embedded" $\mathbb{RP}^2$ one has $\theta = \pi/2$ and the area is 2π.

One of the most beautiful generalizations is the so-called *kinematic formula* of Chern (actually, a more general formula still, valid for manifolds with boundary among other things, was obtained seven years earlier by Federer). "Kinematic" here refers to the group G of isometries of Euclidean space. Given submanifolds M^p and N^q of Euclidean $\mathbb{R}^n$, a modest improvement of Crofton's formula says

$$\int_{g\in G}\mathrm{vol}_{p+q-n}(M\cap gN)\,dg = c\,\mathrm{vol}_p(M)\mathrm{vol}_q(N),$$

where vol_i denotes i-dimensional volume, and c is a universal constant depending only on the dimensions n, p, q, and the normalization of the measure on G (the actual computation of c was first carried out by Santaló, who showed it is a ratio of products of Euclidean unit spheres of various dimension). Chern generalizes this as follows: for every even number $e \le p+q-n$ he proves there is a formula

$$\int_{g\in G}\mu_e(M\cap gN)\,dg = \sum_{\text{even } i\le e} c_i\mu_i(M)\mu_{e-i}(N)$$

where c_i are universal constants depending only on n, p, q, e, and the μ_e are certain geometric invariants of M and N concentrated in "codimension e" (so to speak). Furthermore, Chern's formula is valid in the "noodly" sense that the invariants μ_i depend only on the intrinsic Riemannian geometry of the manifolds, and *not* on how they are isometrically embedded in Euclidean space.

The geometric invariants μ_e appear in Weyl's (closely related) formula for the volume of a (sufficiently narrow) tube of radius ρ about a k-dimensional submanifold X of $\mathbb{R}^n$. Let $T(X,\rho)$ denote the tube around X of radius ρ, where ρ is sufficiently small depending on X. Then

$$\begin{aligned}&\mathrm{vol}(T(X,\rho))\\ &= O_{n-k}\sum_{\text{even } e\le k}\frac{(e-1)(e-3)\cdots 1}{(n-k+e)(n-k+e-2)\cdots(n-k)}\mu_e(X)\rho^{n-k+e},\end{aligned}$$

where O_i is the volume of the unit sphere in Euclidean $\mathbb{R}^i$, and the $\mu_e(X)$ are certain integrals over X of invariant polynomials in the matrix entries of the second fundamental form. Evidently the leading order term for small ρ is just $O_{n-k}\mathrm{vol}_k(X)\rho^{n-k}$; i.e., $\mu_0(X) = \mathrm{vol}_k(X)$. One may think of Weyl's formula as a special case of (Federer's version of) the kinematic formula, taking $M = X$ and N equal to a ball B_ρ of small radius ρ. Then $\mu_i(B_\rho)$ is (up to a constant) just the appropriate power of ρ.

Weyl's formula is *exact*, but only valid for ρ small enough so that the exponential map is an embedding on the normal disk bundle of radius ρ. What the formula really computes is the integral over this normal disk bundle of the pullback of the Euclidean volume form under the exponential map. For larger ρ one must compute volume with multiplicity (where different sheets of the image of the exponential map overlap each other) and with sign (where the Jacobian of the exponential map has negative sign). For accuracy we should call this the *algebraic volume* of $T(X,\rho)$, to distinguish it from the naive volume. Figure 3 shows a region where three local sheets of the exponential map overlap, two with positive sign and one negative.

One can think of Weyl's formula as a Taylor series, but if so it has two properties that are rather startling at first glance:

1. the nonzero terms all have the same parity (equal to the codimension of X); and
2. there are only *finitely many* nonzero terms!

Actually a little reflection makes these properties appear less preposterous. Firstly, if we express $T(X,\rho)$ as the integral of a density $T(X,\rho,v)$ over vectors v in the unit normal bundle ν, then if we expand $T(X,\rho,v)$ for any fixed v as a

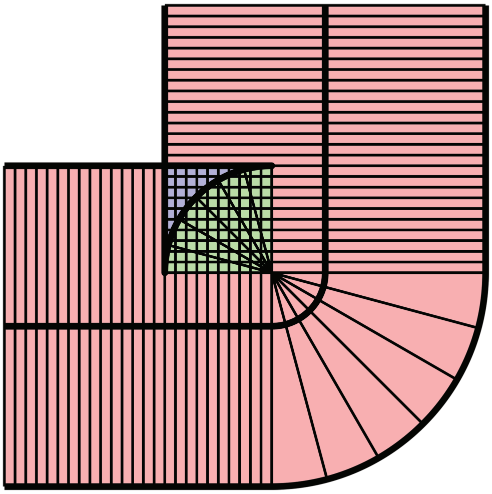

Figure 3. A 'tube' of sufficiently large radius has volume which must be counted with multiplicity and with sign. In this figure, the red region is counted with multiplicity 1, the green region with multiplicity $1+1-1$, and the blue region with multiplicity $1+1$.

power series in ρ we will have $T(X,\rho,-v) = T(X,-\rho,v)$. Thus only the terms with an even exponent will survive after integrating over v.

Secondly, if we approximate X by a polyhedron Y, it is obvious that each codimension e simplex σ contributes an expression of the form $c_e\,\alpha(\sigma)\mathrm{vol}_{k-e}(\sigma)\rho^{n-k+e}$, where c_e is a constant depending only on n,k,e, and where $\alpha(\sigma)$ (the *angle excess*) is the volume of a certain (dual) immersed spherical polyhedron whose faces are inductively associated to higher-dimensional simplices τ incident to σ. Here it is important that we are computing the *algebraic* volume — the exponential map on the "normal bundle" of a polyhedron Y is singular for any positive ρ unless Y is totally geodesic. In any case, this argument shows that $\mathrm{vol}(T(Y,\rho))$ is a polynomial in ρ of degree n and so—taking limits—is $\mathrm{vol}(T(X,\rho))$.

Now, as remarked above, the μ_e are integrals of invariant polynomials of even degree in the coefficients of the second fundamental form. In particular, they can be expressed purely in terms of the intrinsic Riemannian metric for M. When k is even, the highest order term $\mu_k(X)$ is especially nice: the Chern–Gauss–Bonnet formula gives the identity

$$\mu_k(X) = \frac{(2\pi)^{k/2}}{(k-1)(k-3)\cdots 1}\chi(X),$$

where χ is the Euler characteristic of X. Actually, this can be seen directly, at least for the case of a hypersurface X. When ρ is very large, $T(X,\rho)$ is approximately equal to a ball of radius ρ. To compute the algebraic volume $\mathrm{vol}(T(X,\rho))$ we must figure out the multiplicity of a typical point very far from X. Think of $X\times\pm1$ as the unit normal bundle to X. The copy $X\times 1$ is oriented like X, and $X\times -1$ is oriented oppositely. Define $N_\rho : X\times\pm1 \to \mathbb{R}^n$ by $N_\rho(x,\pm1) = \exp(\pm\rho v)$, where v is the positive unit normal to X at x. For big ρ, it is approximately true that N_ρ maps $X\times 1$ (resp., $X\times -1$) to the sphere of radius ρ by $\rho\cdot G$ (resp., $-\rho\cdot G$) where G is the Gauss map. If k is even the antipodal map has degree -1. Since the components of $X\times\pm1$ have opposite orientations, it follows that the oriented degree of both maps is equal to $\deg(G) = \chi(X)/2$. Thus the coefficient of ρ^n in Weyl's formula is $O_n\chi(X)$, and by running this argument in reverse, we obtain a proof of the Chern–Gauss–Bonnet formula!

Taking $e = p+q-n$ in Chern's formula one has the rather extraordinary conclusion that if one drops p- and q-dimensional noodles M and N randomly into $\mathbb{R}^n$ the expected Euler characteristic of their intersection depends only on the intrinsic Riemannian geometry of M and N. Saints alive!

AUTHOR'S NOTE. Chern's formula is proved in the paper "On the Kinematic Formula in Integral Geometry," *J. Math. Mech.* **16** (1966), no. 1, 101–118. Federer's paper "Curvature measures," *Trans. AMS* **93** (1959), 418–491, is also recommended.

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