

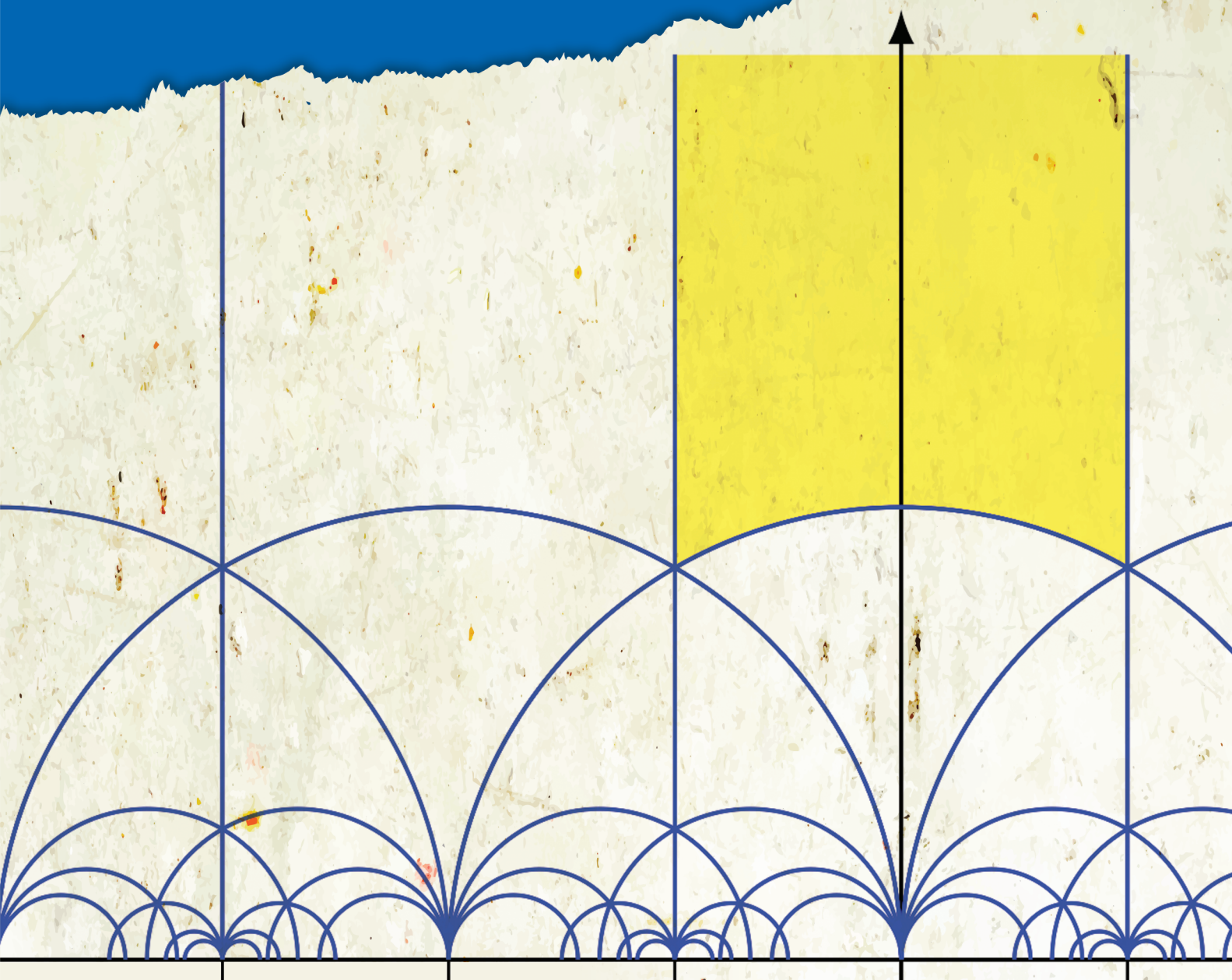


# Notices

of the American Mathematical Society

February 2021

Volume 68, Number 2





# SUPPORT THE 2020 FUND

*Join us on a new path with the 2020 Fund.*

In the summer of 2020, in solidarity with the Black Lives Matter movement, the AMS took pause with **#ShutDownSTEM** to acknowledge our history of racist behavior and develop strategies for addressing systemic inequities in our mathematics community.

Our Board of Trustees and Council decided to take new steps to promote equitable opportunities for the advancement of Black mathematicians, one of which was creating—and donating to—the AMS 2020 Fund, a new endowed fund. The 2020 Fund will support a portfolio of future opportunities for advancing and celebrating the careers of Black mathematicians and is part of our AMS' long-term commitment to an equitable future. To the AMS donors who have already begun to support the 2020 Fund: **THANK YOU!**

We know there is much to learn, and we recognize that reducing the effect of systemic inequities takes consistent and conscious effort. Generous support of the 2020 Fund is an important step on the path toward encouraging and facilitating full participation of all mathematicians in our shared mission of advancing research and creating connections.

As our AMS works to establish the goals and initiatives the Fund will support, we hope you will join us on the path to ensuring equity for Black mathematicians.

To read more, please go to  
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# A WORD FROM...

Robin Wilson, Professor of Mathematics<sup>1</sup>



Courtesy of Tom Zasadzinski



Sankofa is an Akan word from the people of Ghana, Liberia, and Sierra Leone. The literal translation of the word the Sankofa symbol is used to represent is “it is not taboo to go back and fetch what you forgot,” which embodies the idea of learning from the past as we move forward to the future [Tem20]. The Akan people inhabit the regions of West Africa where some of the largest slave trading outposts operated for hundreds of years during the Transatlantic slave trade when tens of millions were captured and sold into slavery in the US, the Caribbean, and Central and South America. This year, during Black History Month, we are reminded like Sankofa of the importance of learning from the past to ensure a strong future.

The Sankofa symbol is one of many Adinkra symbols, each conveying its own message of traditional wisdom. They appear as the name of a type of graph being studied in a research group led by Sylvester James Gates as a part of the African Diaspora Joint (ADJOINT) Mathematics Workshop at MSRI. Dr. Edray Goins of Pomona College and Hélène Barcelo,

Deputy Director of MSRI, introduce this new and promising program that works to increase the productivity and visibility of African American mathematicians by hosting small collaborative research groups for two weeks over the summer along with support for continued collaboration during the academic year.

The feature articles in this issue provide just a sample of the scholarship and breadth of mathematics research being produced today by Black mathematicians. Their research interests and accomplishments are substantial enough to be highlighted in any issue of the *Notices*, yet the authors were generous enough to share their work in this special issue. In this issue Professor Gerard Awanou from the University of Illinois at Chicago introduces us to the study of partial differential equations of Monge-Ampere type that arise in the study of geometric optics. Aaron Pollack, Assistant Professor at the University of California at San Diego, explains automorphic forms on exceptional groups in a way that can be appreciated by mathematicians outside of number theory and algebra. And Joel Nagloo, Assistant Professor of mathematics at CUNY Bronx Community College and Graduate Center, shares with us an overview of the study of differential equations using a model-theoretic approach.

The year 2021 will mark 96 years since Elbert F. Cox became the first Black person to earn a PhD in mathematics, and 76 years since Euphemia Lofton Haynes became the first Black woman to earn a PhD in mathematics. In this issue, in the spirit of honoring the legacy of those that came before us, we celebrate the life of a prominent member of the second



**Figure 1.** The Sankofa symbol, generally depicted as a mythic bird flying forward with its head turned backward with an egg, embodies the importance of learning what the past can teach us as we move forward toward the future.

Robin Wilson is a professor of mathematics at California Polytechnic State University, Pomona. His email address is [robinwilson@cpp.edu](mailto:robinwilson@cpp.edu).

<sup>1</sup>The opinions expressed here are not necessarily those of the *Notices* or the AMS.



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generation of Black mathematicians. Dr. James Ashley Donaldson was a giant, not only in stature but in spirit. He was a distinguished mathematician and activist whose accomplishments range from starting the PhD program at Howard University to successfully advocating against the AMS reciprocity agreement with the South African Mathematical Sciences Association (SAMSA) during the anti-apartheid movement of the 1970s. Dr. Donaldson was one of the earliest and most vocal champions of equity and inclusion in the mathematical sciences, and his life and work should be remembered.

The autobiographical article by Nathaniel Whitaker tells a compelling story about growing up in a supportive Black community during segregation, and his path from being a graduate student at UC Berkeley to the University of Massachusetts, Amherst, where he is now Department Chair. His story reveals important insights into successful practices for recruiting and supporting Black students in mathematics. For example, the story of his early life provides a view into the complicated history of this country's education system. When writing about his experience as a graduate student, Whitaker shares insight into the efforts of Maxwell Reade at the University of Michigan and Leon Henkin at UC Berkeley to bring African American graduate students into their mathematics PhD programs in record numbers in the 1970s and 1980s. The impact of their efforts is still paying off today and can provide ideas for graduate programs looking to include more Black students in their programs.

The Transatlantic slave trade has had long-lasting effects on both sides of the Atlantic. On one side of the Atlantic, Black people suffered from the loss of nationality, erasure of language, culture, and all liberty. On the other side of the Atlantic the loss of at least 10 million men, women, and children on the African continent has yet to be fully studied and understood. Black communities on both sides are connected by this reality. The Masamu Program was developed by Dr. Overtoun Jenda of Auburn University and colleagues from SAMSA in 2010 to support and enhance research collaborations between African and American mathematicians. The program, highlighted here, has made an impact on research and development of talent in both the US and Sub-Saharan Africa. The Masamu Program only scratches the surface of Dr. Jenda's work in mentorship and outreach; in this issue, we also highlight his 2020 Presidential Award for Excellence in Science, Mathematics, and Engineering Mentoring with an article written by his colleagues about the impact he has had.

Now, more than ever it seems, many instructors are looking for suggestions on how to support Black students and other students of color in their classrooms. In the Early Career Section, whose February theme is "opportunities for diversity in the classroom," Ranthony Edmonds and John Johnson, Candice Price, and Colleen Duffy offer three articles that provide concrete suggestions for ways we can build courses in our curriculum where students can see themselves and windows through which they can see the cultures of others. In their mathematics education article, James A. Mendoza Álvarez and Minerva Cordero share their own thoughts on this timely topic. They draw on important frameworks from education literature and suggest practical recommendations: changing the messaging we give to students about who can do mathematics, identifying with our students, recognizing the humanity in our students, providing opportunities for collaboration, presenting mathematics through personally meaningful contexts, and practicing an asset-based approach that values the knowledge students bring into the classroom. These practices support the most marginalized students that we teach, and at the same time promote success for all students, even the most privileged.

This past year has been an especially difficult one for the Black community. The disproportionate consequences of COVID-19 have unmasked the intersection of far too many inequities facing Black Americans. Last Spring, the brutal killing of Amaud Aubrey was witnessed on video, Breonna Taylor was killed by police, and when the video of George Floyd's murder went viral, people across the globe stood up to say that enough is enough. Individuals and institutions around the country, including the AMS, were forced to reflect on their own historic roles in perpetuating racism, commit to working against it, and admit that there is no place for neutrality on the issue of racial justice [Rob20, CBMS20].

What each Black History Month issue of the *Notices* teaches us is that despite a deep history of the systemic anti-Black racism within institutional spaces in mathematics, Black mathematicians have been present in the American mathematics community from its beginning [Rob20]. We have not always been welcomed or included, yet we have persisted and made

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significant contributions to the field of mathematics. Black mathematicians have also provided leadership and have been a moral compass in the struggle to create a mathematical community free of racism where wider communities of color are not only welcome but able to thrive. These stories also tell of allies that have used their voices and institutional power to stand up against injustice. We all suffer from the burden of racism in this country. It will take collective effort to lift this historic burden off of our backs. I hope you enjoy this issue, and hope that this collection of articles fosters professional dialogue and forward-thinking policies that will continue to help us inch closer towards the goal of an equitable, inclusive, and just mathematics community.

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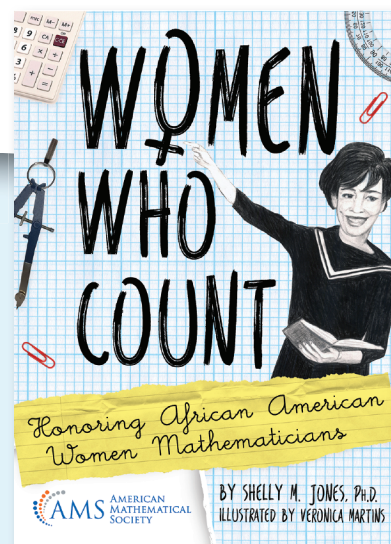
## Women Who Count

Honoring African American Women Mathematicians

Shelly M. Jones, *Central Connecticut State University, New Britain, CT*

Tessellations, palindromes, tangrams, oh my! *Women Who Count: Honoring African American Women Mathematicians* is a children's activity book highlighting the lives and work of 29 African American women mathematicians, including Dr. Christine Darden, Mary Jackson, Katherine Johnson, and Dorothy Vaughan from the award-winning book and movie *Hidden Figures*. Although the book is geared toward children in grades 3–8, it is appropriate for all ages.

2019; 138 pages; Softcover; ISBN: 978-1-4704-4889-9; List US\$15; AMS members US\$12; MAA members US\$13.50; Order code MBK/124





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# LETTERS TO THE EDITOR



## Letter to the Editor

I hereby wish to inform the mathematical community of my resignation as Editor-in-Chief of the *Journal of Nonlinear Mathematical Physics*, which I submitted to the president/owner of Atlantis Press, Zeger Karssen, on 30 September 2020. The entire Editorial Board has also since resigned. My resignation is in protest to the policies of Atlantis Press, in particular the policy that requires all authors who wish to publish in this journal to pay a compulsory article processing fee (750 euros) that was introduced against my will on the 1st of September 2020. Unfortunately, I am legally bound by a contract with Atlantis Press to continue to serve as Editor-in-Chief until 30 March 2021.

Feel free to contact me if you have any questions: Euler199@gmail.com.

—Norbert Euler, outgoing Editor-in-Chief,  
Journal of Nonlinear Mathematical Physics,  
Jinan University, Guangzhou, China

## Letter to the Editor

In a letter to the editor (November 2020), Charles H. Jones asks why the editors of the *Notices* did not change “300 BC” to “300 BCE” in Sergei Gelfand’s article “A Word From...” But Gelfand’s article has a prominent disclaimer that the opinions in the article are not necessarily those of the *Notices* or the AMS. Whether Gelfand wrote “300 BC” out of personal religious conviction or some other reason, the editors did the right thing by not overriding the author’s choice.

—Timothy Chow  
Princeton, NJ

## Letter to the Editor

I am writing to express my thanks to AMS President Jill Pipher, the AMS Council, and all other members of the Governance and Executive Staff for creating the AMS’s Action Plan to reckon with past racist actions by the society and to take action to ensure more equitable treatment of African American mathematicians. I especially appreciate the work Catherine A. Roberts, Dylan Thurston, Talitha Washington, and Ravi Vakil did to make this happen. Thanks also to AMS members like Noah Snyder who helped advocate that the AMS take this step.

\*We invite readers to submit letters to the editor at [notices-letters@ams.org](mailto:notices-letters@ams.org).

The AMS Council’s action was a very quick response to requests that the society address its historical treatment of African Americans. Within about a week of receiving email requests, the AMS Council met and created the plan. The speed of the response speaks positively to how our society currently responds to community needs, especially those concerning diversity and inclusion. By comparison, about 20 years ago (in 1996), Lee Lorch suggested in print that the AMS should issue an apology along the lines of the action plan, and his suggestion was met with silence. The difference between Lorch’s reception and the reception current members received reflects well on current and recent AMS leadership.

(The Action Plan is described here: <https://www.ams.org/about-us/understanding-ams-history>.)

—Jesse Kass  
Associate Professor  
University of South Carolina

## Revival of the Encyclopedia of Mathematics

The Encyclopedia of Mathematics Wiki<sup>1</sup> (EoM) is, as most readers of this text probably already know, an open access resource designed specifically for the mathematics community. With more than 8,000 entries, illuminating nearly 50,000 notions in mathematics, the Encyclopedia of Mathematics was the most up-to-date graduate-level reference work in the field of mathematics.<sup>2,3</sup>

From its start in 2011, the EoM had to cope with the problem that the mathematics formula code was only available through png images, based on a former CD edition from 2002, because the TeX code was lost by the former publishers of the EoM.

This problem concerned about 270,000 formulas, which, due to the missing TeX code, needed to be completely retyped whenever they were edited. Therefore, over the course of two decades, the EoM has become more and

<sup>1</sup>[https://encyclopediaofmath.org/wiki/Main\\_Page](https://encyclopediaofmath.org/wiki/Main_Page)

<sup>2</sup>The EoM is based on a book version “Encyclopaedia of Mathematics,” edited by Michiel Hazewinkel. Its last print, consisting of 13 volumes, was published in 2002.

<sup>3</sup>A statistical EoM example by Boris Tsirelson ([https://encyclopediaofmath.org/wiki/User:Boris\\_Tsirelson#Some\\_statistics](https://encyclopediaofmath.org/wiki/User:Boris_Tsirelson#Some_statistics)): Measure space (50,000+ views); Standard Borel space (12,000+ views); Analytic Borel space (5,000+ views); Universally measurable (5,000+ views); Measure space (20,000+ views); Standard probability space (6,000+ views); Measure algebra (measure theory) (7,000+ views).



more out of date, as the loss of the TeX codes has made it difficult to update the 8,000 articles of the EoM.

This problem was recently solved. There were three categories of formulas with missing TeX code:

1) During the last years, about 60% of all formulas had already been manually translated into TeX by worldwide volunteers cooperating with EoM.

2) For the majority of formulas, old markup typesets in an nroff-like style became available, however with no interpreter. Recently, an interpreter for these markup pages has been devised allowing it to automatically translate, mostly error-free, the image-based code into TeX.

3) Finally, there were the remaining around 60,000 formulas, for which there were no markup and no manual translations.

Ulf Rehmann, professor at Bielefeld University and editor in chief of EoM, has organized the automatic translation for most pages as described in 2). Maximilian Janisch, student at the University of Zürich, has translated the formulas of type 3) into TeX semi-automatically (i.e. the formulas were translated with machine learning, but the translations were checked twice manually). Now, an almost completely TeXified version of the EoM is available online.<sup>1</sup>

The Revival of the EoM: Long story short, the renewal of the EoM articles is now possible without tedious manual retyping of the formulas. It would be great if many mathematicians started using this chance in order to bring the EoM back up to date.

—Ulf Rehmann  
Editor in Chief of EoM

Maximilian Janisch  
Mathematics student at the University of Zürich

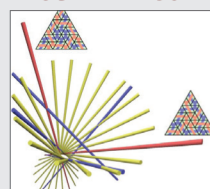
## Semester Program: Combinatorial Algebraic Geometry

February 1 - May 7, 2021

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### PROGRAM DESCRIPTION



Research in combinatorial algebraic geometry utilizes combinatorial techniques to answer questions about geometry. It also uses geometric methods to provide powerful tools for studying combinatorial objects.

Much research in combinatorial algebraic geometry relies on mathematical software to explore and enumerate combinatorial structures and compute geometric invariants. The development of new mathematics software is therefore prioritized in the program.

This program will bring together experts in both pure and applied parts of mathematics as well mathematical programmers, all working at the confluence of discrete mathematics and algebraic geometry, creating an environment conducive to interdisciplinary collaboration.

### Affiliated Workshops (all only offered virtually):

- Introductory Workshop: Combinatorial Algebraic Geometry (Feb 1-5, 2021)
- Sage/Oscar Days for Combinatorial Algebraic Geometry (Feb 15-19, 2021)
- Geometry and Combinatorics from Root Systems (Mar 22-26, 2021)
- Algebraic Geometry and Polyhedra (Apr 12-16, 2021)

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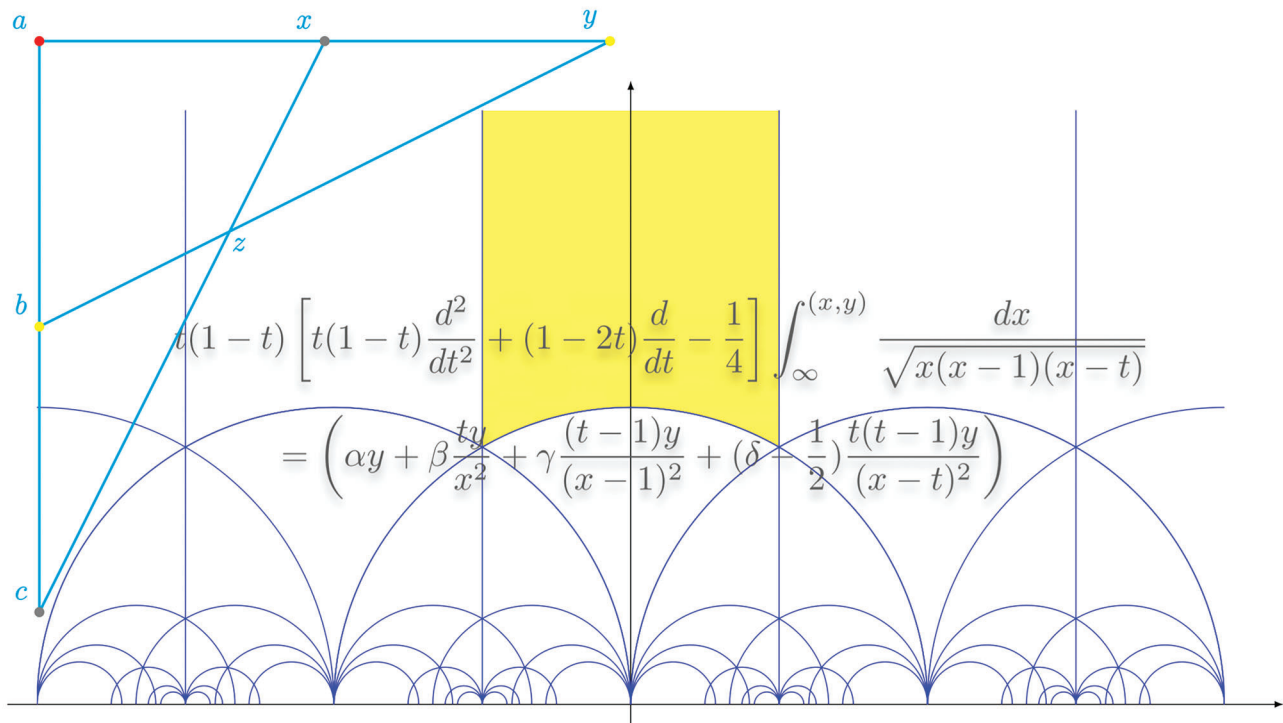
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# Model Theory and Differential Equations



Joel Nagloo

## Introduction

The model-theoretic approach to the study of differential equations has a long and rich history beginning with A. Robinson [Rob59]. The theory of differentially closed fields of characteristic 0,  $DCF_0$ , has been studied intensively and has played an important role in the internal development of geometric model theory. It is also behind one of the most spectacular applications of logic to number theory; namely, E. Hrushovski's celebrated proof of the function field Mordell-Lang conjecture. Furthermore, the study of the theory  $DCF_0$  has led to substantial

development in a Galois theory for differential equations and its applications.

Nevertheless, only very recently have the techniques from model theory been used to study *classical* differential equations. First in the work of the author and A. Pillay on the Painlevé transcendents [NP14], [NP17] and then in that of J. Freitag and T. Scanlon [FS18] on the differential equation satisfied by the modular  $j$ -function. More recently, in joint work with G. Casale and J. Freitag [CFN20], the author has also studied the differential equations satisfied by the Fuchsian automorphic functions and in the process proved an old claim of P. Painlevé (1895).

In this article, we give an overview of those recent applications of model theory to the study of differential equations. The focus will be on the role of the classification problem for strongly minimal sets and on results in functional transcendence. Unavoidably, many other interesting and important aspects of the interaction between model theory and differential algebra will be omitted.

Joel Nagloo is an associate professor of mathematics at CUNY Bronx Community College and Graduate Center. His email address is joel.nagloo@bcc.cuny.edu.

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## Differential Algebraic Geometry

Differential algebraic geometry, which has its origin at the beginning of the 1930s, was founded by J. Ritt and E. Kolchin. Although not widely known, it gives a general *algebraic* setting for the study of differential equations and the approach is similar to that of the study of polynomial equations in algebraic geometry. We will in this article focus on *ordinary* differential equations. Moreover, we will say a few words at the end about the setting of partial differential and difference equations. The standard reference for this section is Kolchin's book [Kol73]. All fields will be assumed to be of characteristic 0.

**Definition 1.** A *differential field*  $(K, \delta)$  is a field  $K$  equipped with a derivation  $\delta : K \rightarrow K$ , i.e., an additive group homomorphism satisfying the Leibniz rule  $\delta(xy) = x\delta(y) + y\delta(x)$ .

The field of *constants*  $C_K$  of  $K$  is defined set theoretically as  $\{x \in K : \delta(x) = 0\}$ . We usually write  $x'$  for  $\delta(x)$  and  $x^{(n)}$  for  $\underbrace{\delta \cdots \delta}_{n} \delta(x)$ .

**Example 1.**  $(\mathbb{C}(t), d/dt)$  is the field of rational functions over  $\mathbb{C}$  in a single indeterminate, where in this case the field of constants is  $\mathbb{C}$ .

Associated with a differential field  $(K, \delta)$  is the *ring of differential polynomials*  $K\{\mathbf{X}\}$  in  $m$  differential variables  $\mathbf{X} = (X_1, \dots, X_m)$ . An element of  $K\{\mathbf{X}\}$  is called a *differential polynomial* over  $K$  and is simply a regular polynomial with coefficients in  $K$  but in variables  $\mathbf{X}, \mathbf{X}', \mathbf{X}^{(2)}, \dots$ . We use here the notation  $\mathbf{X}^{(n)} = (X_1^{(n)}, \dots, X_m^{(n)})$ . If  $f \in K\{\mathbf{X}\}$ , then the order of  $f$ , denoted  $\text{ord}(f)$ , is the largest  $n$  such that for some  $i$ ,  $X_i^{(n)}$  occurs in  $f$ .

**Example 2.**  $f(X) = (X')^2 - 4X^3 - tX$  is a differential polynomial in  $\mathbb{C}(t)\{X\}$  and  $\text{ord}(f) = 1$ .

As one can see, if  $f \in K\{\mathbf{X}\}$ , then  $f(\mathbf{X}) = 0$  is an ordinary (algebraic) differential equation. More generally, by a *Kolchin closed* subset of  $K^n$ , we mean the common zero set of a finite system of differential polynomial equations, i.e., a set of the form

$$V(S) = \{y \in K^n : f(y) = 0 \text{ for all } f \in S\},$$

where  $S \subset K\{\mathbf{X}\}$  is a finite subset. The Kolchin closed sets are the basic closed sets in the Kolchin topology and are the analogues of the basic closed sets in the Zariski topology. A *Kolchin constructible set* is simply a boolean combination of Kolchin closed sets.

Given a differential field  $(K, \delta)$ , it follows that the derivation  $\delta$  uniquely extends to the algebraic closure  $K^{\text{alg}}$  of  $K$ . However, in order for Kolchin closed sets to necessarily have points whose coordinates are from the underlying field, a much stronger condition than algebraic closedness is needed.

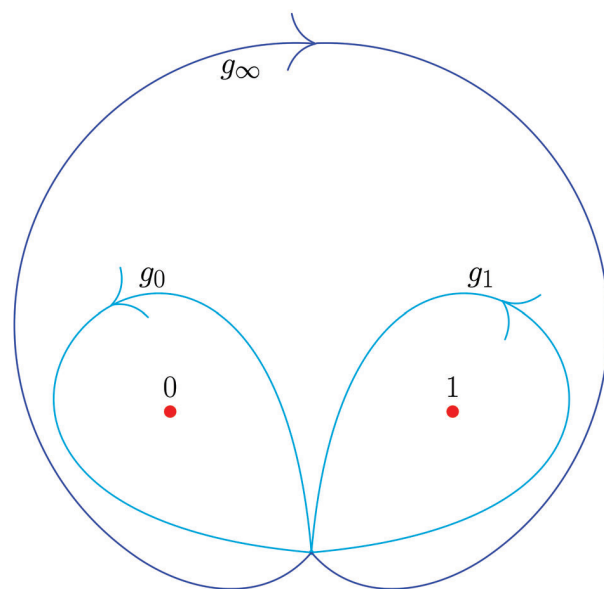
**Definition 2.** A differential field  $(K, \delta)$  is said to be *differentially closed* if for every  $f, g \in K\{X\}$  such that  $\text{ord}(f) > \text{ord}(g)$ , there is  $y \in K$  such that  $f(y) = 0$  and  $g(y) \neq 0$ .

Differential algebraic geometry as developed by Kolchin studies Kolchin closed sets in a differentially closed field. At this point, let us mention that Kolchin closed sets can have very rich algebraic structure. Take for example the field of constants: if  $K$  is differentially closed, then from Definition 2 we see that  $C_K$  is an algebraically closed field. Less obvious is that  $C_K$  is indeed the only algebraically closed subfield of  $K$  that is given by a differential equation. Another interesting well-known example is that of a homogeneous linear differential polynomial

$$f(X) = X^{(n)} + a_{n-1}X^{(n-1)} + \cdots + a_1X' + a_0X, \quad a_i \in K.$$

One has that the associated Kolchin closed set (in a differentially closed field  $K$ ) is a vector space over  $C_K$ .

Kolchin's approach has been instrumental in the development of a Galois theory for differential equations that solidifies and extends the Picard-Vessiot theory for linear differential equations. For example, the fact that, in a differentially closed field  $K$ , the Galois group of a linear differential equation is a linear algebraic group defined over  $C_K$  has been generalized in Kolchin's strongly normal theory using algebraic groups as the Galois group of so-called logarithmic equations.



**Figure 1.** Loops  $g_0, g_1, g_\infty$  in the complex plane around the singularities  $0, 1, \infty$  of the hypergeometric equation. The differential Galois group is the Zariski closure in  $GL_2(\mathbb{C})$  of the monodromy group of the equation.

Kolchin's Galois theory for differential equations has helped answer questions related to the study of the structure of Kolchin constructible sets. Moreover, it can be



argued that the point of view of Kolchin's theory coincides with that of the model theory of differentially closed fields. This is of course our approach to studying differential equations.

## Model Theory

For the topics covered in this article, we recommend D. Marker's book [Mar02]. The starting point in model theory is the notion of a model of a first-order theory. Here by a *first-order theory*  $T$  we mean a set of axioms (or more accurately first-order sentences) in a fixed language  $L$ . The language  $L$  is simply a set of constant symbols, function symbols, and relation symbols. We assume throughout that the language is countable.

**Example 3.** A familiar example is  $T_G$ , the theory for groups which consist of the usual axioms for groups expressed using the language  $L_G = (e, *)$  together with the logical symbols  $=, (, ), \exists$ , and  $\forall$ .

A *structure* for a language  $L$ , or an  $L$ -*structure* for short, is a set together with interpretations for each symbol in  $L$ . A *model* of a theory  $T$  is simply an  $L$ -structure in which the axioms are true. In Example 3, we see that both  $(\mathbb{N}, 0, +)$  and  $(\mathbb{Z}, 0, +)$  are  $L_G$ -structures, moreover only the latter is a model of  $T_G$ .

The notion of a (well-formed) formula extends that of an axiom, whereby free variables, that is, those not quantified upon, are allowed. Continuing with Example 3, we see that a well-formed formula with free variable  $X$  is  $\phi(X) := \forall Y (X * Y = Y * X)$ . For a model  $G$  (i.e., a group), if  $C(G)$  denotes the set of elements of  $G$  which satisfy the formula  $\phi(X)$ , then we have that  $C(G)$  is the center of  $G$ . The center,  $C(G)$ , is an example of a definable set.

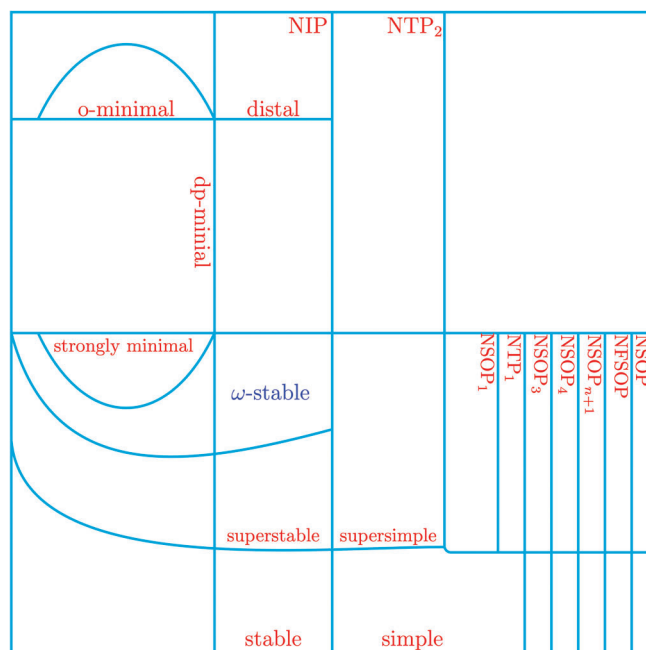
**Definition 3.** A *definable set*  $Y \subset M^n$  is a set of the form

$$Y = \{y \in M^n : \phi(y) \text{ is true}\},$$

where  $\phi$  is a formula in  $L$  with  $n$  free variables.

**Remark 1.** For any subset  $A \subset M$  of a model, one can extend the language  $L$  by adding a constant symbol for each element  $a \in A$ . One usually denotes the new language obtained by  $L_A$ . If in Definition 3 one replaces  $L$  by  $L_A$  for some  $A \subset M$ , then one obtains the definition of an  $A$ -definable set or more precisely a definable set with *parameters* from the set  $A$ .

For a fixed theory  $T$  a major goal of model theory is to study *all* definable sets in some/any model of  $T$ . This of course would be hopeless unless one could identify classes of structures where there is some control over the definable sets. In model theory, this leads to the distinction between "tame" and "wild" structures or theories. In this article we discuss two notions of tameness, namely quantifier elimination and  $\omega$ -stability. There are many more natural



**Figure 2.** The model theory universe as described at forkinganddividing.com. In the sea of theories, an  $\omega$ -stable theory is ideally placed in the center left of the natural "tame/wild" divide.

"tame" versus "wild" distinctions and some are illustrated in Figure 2.

A theory  $T$  is said to have *quantifier elimination* if for every formula  $\phi(\mathbf{X})$  there is a quantifier-free formula  $\psi(\mathbf{X})$  such that the two define the same definable set. It hence follows that for theories with quantifier elimination the definable sets are defined using "simple" formulas.

A theory  $T$  is  $\omega$ -stable if every definable set  $X$  can be given an intrinsic ordinal-valued dimension called the Morley rank, denoted by  $RM(X)$ . In rough terms, the inductive definition is as follows:  $RM(X) = 0$  if  $X$  is finite, and  $RM(X) \geq \alpha + 1$  if there are pairwise disjoint definable subsets  $X_i$  of  $X$  for  $i = 1, 2, \dots$  such that each  $RM(X_i) \geq \alpha$  for all  $i < \omega$  (one extends the definition naturally to limit ordinals). We set  $RM(X) = \alpha$  if  $RM(X) \geq \alpha$  but not  $\geq \alpha + 1$ . Using this rank, one can define in  $T$  a good notion of independence and dimension analogous to the notion of linear independence and basis in the study of vector space.

The theory of algebraically closed fields of characteristic zero,  $ACF_0$ , with the obvious axioms given in the language of rings  $L_R = (+, -, \cdot, 0, 1)$  has both quantifier elimination and is  $\omega$ -stable. In this setting quantifier elimination is equivalent to the Chevalley-Tarski theorem that over an algebraically closed field the projection of a constructible set is constructible. The Morley rank of a definable set (so a constructible set) corresponds to the transcendence degree of a generic point, while the independence notion is equivalent to algebraic independence.

## The Theory $DCF_0$

Let us bring together the ideas of the first two sections. We refer the reader to [MMP96] for historical background and additional details. In the context of differentially closed fields, the relevant language is  $L_\delta = (+, -, \cdot, \delta, 0, 1)$ , the language of differential rings, and we denote by  $DF_0$  the theory of differential fields of characteristic zero. The axioms of  $DF_0$  consist of the axioms for fields and the axioms for the derivation  $\delta$ .

Now, for each  $n$ ,  $d_1$ , and  $d_2 \in \mathbb{N}$ , one can write down an axiom (in  $L_\delta$ ) that asserts that if  $f$  is a differential polynomial of order  $n$  and degree at most  $d_1$  and  $g$  is a nonzero differential polynomial of order less than  $n$  and degree at most  $d_2$ , then there is a solution to  $f(X) = 0$  and  $g(X) \neq 0$ . The theory obtained by adding to  $DF_0$  all these axioms is called the theory of differentially closed fields of characteristic 0,  $DCF_0$ . This theory sits on the tame side of many of the most important dividing lines in model theory as shown by Blum [Blu69].

**Theorem 1.** *The theory  $DCF_0$  eliminates quantifiers and is  $\omega$ -stable.*

For the remainder of the article  $\mathcal{U}$  will denote a saturated<sup>1</sup> model of  $DCF_0$ .

Quantifier elimination means that a definable set  $Y \subseteq \mathcal{U}^n$ , definable over a differential subfield  $K$  of  $\mathcal{U}$ , is nothing more than a Kolchin constructible set over  $K$ . On the other hand, as discussed above,  $\omega$ -stability means (among other things) that any definable set has a well-defined ordinal-valued Morley rank. The independence notion in  $\mathcal{U}$  is as follows:  $\mathbf{a}$  is *independent* from  $\mathbf{b}$  over  $K$  if  $K\langle \mathbf{a} \rangle$  is algebraically disjoint from  $K\langle \mathbf{b} \rangle$  over  $K$ . Here  $K\langle \mathbf{a} \rangle = K(\mathbf{a}, \mathbf{a}', \mathbf{a}^{(2)}, \dots)$  denotes the differential field generated by  $\mathbf{a}$  over  $K$ .

Along with the Morley rank, we also have another invariant for definable sets called the *order*. For  $\mathbf{a} \in \mathcal{U}^n$  and  $K < \mathcal{U}$ , we define  $\text{ord}(\mathbf{a}/K)$  to be the transcendence degree of the field  $K\langle \mathbf{a} \rangle$  over  $K$ . If  $Y \subseteq \mathcal{U}^n$  is definable over  $K$ , we define  $\text{ord}(Y) = \sup\{\text{ord}(\mathbf{a}/K) : \mathbf{a} \in Y\}$ . One can show that  $\text{RM}(Y)$  is always less than or equal to  $\text{ord}(Y)$ . Furthermore,  $\text{RM}(Y) < \omega$  if and only if  $\text{ord}(Y) < \omega$ . We will later see examples of Kolchin closed sets for which the Morley rank is strictly less than the order.

**Definition 4.** Let  $Y \subseteq \mathcal{U}^n$  be a definable set.

1.  $Y$  is said to be *finite dimensional (or rank)* if it has finite order, i.e.,  $\text{ord}(Y) < \omega$ .
2.  $Y$  is said to be *strongly minimal* if it is infinite and for every definable subset  $Z \subseteq Y$ , either  $Z$  or  $Y \setminus Z$  is finite.

<sup>1</sup>Saturation is a notion of “largeness” which mimics the idea that an algebraically closed field of uncountable transcendence degree over the prime field is large/rich.

If  $Y$  is strongly minimal, then it has Morley rank one. Strongly minimal sets determine, in a precise manner (not to be discussed in this article), the structure of *all* finite-dimensional definable sets. This fact, which follows from very general model-theoretic considerations, holds in any  $\omega$ -stable theory and is obtained in part using the robust notion of independence.

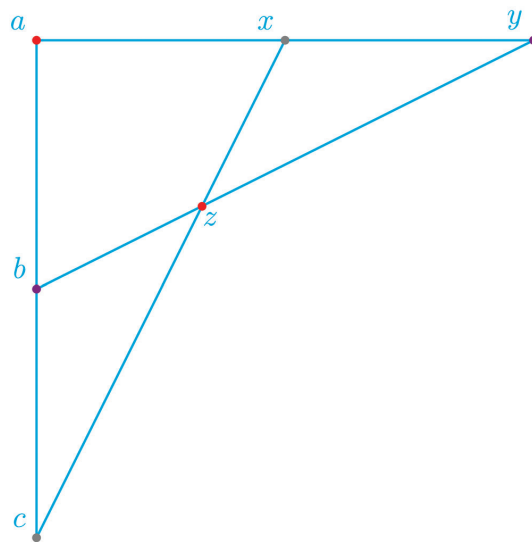
Notice that if  $Y$  is a definable set with  $\text{ord}(Y) = n$ , then  $Y$  is strongly minimal if and only if  $Y$  cannot be written as the disjoint union of definable sets of order  $n$ , and for *any* differential field  $K$  over which  $Y$  is defined and element  $y \in Y$ , then  $\text{tr.deg}(K\langle y \rangle/K) = 0$  or  $n$ .

**Example 4.** The field of constants  $C_{\mathcal{U}}$  is strongly minimal.

**Example 5.** If  $f$  is an absolutely irreducible polynomial over  $\mathcal{U}$  in two variables, then the subset  $Y$  of  $\mathcal{U}$  defined by  $f(y, y') = 0$  is strongly minimal, of order 1.

It is quite a difficult task to show that the set defined by a given differential equation is strongly minimal. Indeed, except for limited or special cases, no general tools are available. For example, we refer the reader to Section 5.17 of [MMP96] for the (tedious) calculations involved in showing that the subset of  $\mathcal{U}$  defined by  $\{yy'' = y', y' \neq 0\}$  is strongly minimal, of order 2.

Nevertheless, the goal of understanding all definable sets in  $DCF_0$  goes through a complete understanding of the strongly minimal sets. A considerable amount of work, beginning in the 1990s, has been devoted to just that. The deepest result in that direction, due to E. Hrushovski and Z. Sokolovic [HS94], concerns the classification of strongly minimal sets that have “nontrivial” structures.



**Figure 3.** Presence of a definable group: in a nongeometrically trivial strongly minimal set one can find a group configuration. Each point has (Morley) rank 1, each line rank 2, and any three noncollinear points are independent.

**Definition 5.** Let  $Y$  be a strongly minimal set defined over a differential field  $K$ . Then  $Y$  is said to be *geometrically trivial* if for any  $y, y_1, \dots, y_n \in Y$  if  $y \in K\langle y_1, \dots, y_n \rangle^{\text{alg}}$ , then there is  $1 \leq i \leq n$  such that  $y \in K\langle y_i \rangle^{\text{alg}}$ .

In essence, a geometrically trivial set can have at most a “binary” structure. The field of constants  $C_U$  is not geometrically trivial. The same is true of definable groups (i.e., definable sets equipped with definable group structures).

The work of Hrushovski and Sokolovic did not attempt to classify geometrically trivial strongly minimal sets. On the other hand, a key step in their work and which builds on those of A. Buium [Bui92] was the identification of some “exotic” differential algebraic groups (i.e., definable groups where the underlying definable set is Kolchin closed not Zariski closed).

**Theorem 2.** Let  $A$  be an abelian variety over  $\mathcal{U}$ . We identify  $A$  with its set  $A(\mathcal{U})$  of  $\mathcal{U}$ -points. Let  $A^\sharp$  be the Kolchin closure of the torsion subgroup of  $A$ . Then:

1.  $A^\sharp$  is a differential algebraic group and is Zariski dense in  $A$ .
2. If  $A$  is a simple abelian variety that does not descend to  $C_U$ , then  $A^\sharp$  is strongly minimal.

The group  $A^\sharp$  is called the *Manin kernel* of  $A$ . One remarkable property of  $A^\sharp$  is that every definable subset of it is a finite Boolean combination of cosets of definable subgroups. The result of Hrushovski and Sokolovic is that, up to equivalence, the field of constants  $C_U$  and the groups  $A^\sharp$  cover all the nongeometrically trivial examples!

**Theorem 3** (The trichotomy theorem). If  $Y \in \mathcal{U}^n$  is strongly minimal, then exactly one of the following hold:

1.  $Y$  is geometrically trivial, or
2. (Group-like)  $Y$  is nonorthogonal to the Manin kernel  $A^\sharp$  of some simple abelian variety  $A$  that does not descend to  $C_U$ , or
3. (Field-like)  $Y$  is nonorthogonal to the field of constants  $C_U$ .

We say that  $Y$  and  $Z$  (both strongly minimal) are *nonorthogonal* if there is some infinite definable relation  $R \subset Y \times Z$  such that  $\pi_{Y|_R}$  and  $\pi_{Z|_R}$  are finite-to-one functions. Here  $\pi_Y : Y \times Z \rightarrow Y$  and  $\pi_Z : Y \times Z \rightarrow Z$  denote the projections to  $Y$  and  $Z$ , respectively. It is not hard to see that nonorthogonality is indeed an equivalence relation on strongly minimal sets. Furthermore, if  $Y$  and  $Z$  are nonorthogonal strongly minimal sets, then  $\text{ord}(Y) = \text{ord}(Z)$ .

The work of Hrushovski and Sokolovic was never published. Moreover, an alternate proof of the characterization of the field-like strongly minimal sets—a key step—has appeared in the work of A. Pillay and M. Zeigler [PZ03].

A good summary of the proof of Theorem 3 can be found in [NP17, Section 2.1].

There are other interesting and important consequences of the trichotomy theorem that are not apparent but worth mentioning. Firstly, if  $A^\sharp$  is the Manin kernel of a simple abelian variety  $A$  that does not descend to  $C_U$ , then  $\text{ord}(A^\sharp) \geq 2$ . Hence, strongly minimal sets of order 1 are either geometrically trivial or nonorthogonal to  $C_U$ . Secondly, strongly minimal sets that are defined over  $C_U$  and of order  $\geq 2$  are geometrically trivial! This surprising fact was somewhat forgotten for a while but now plays a crucial role in some of the applications of the theory to functional transcendence as we shall see in the next section.

Finally, it is worth mentioning that strong minimality is closely related to Painlevé’s notion of *irreducibility* of differential equations. Roughly speaking, a differential equation is irreducible if none of its solutions are “known” special functions. Establishing irreducibility, which goes through establishing strong minimality, has been part of long-standing open conjectures in the theory of nonlinear special function.

## Trivial Pursuits and Applications

As we have seen, the trichotomy theorem, which gives a very general classification theorem for strongly minimal sets, has nothing to say about geometrically trivial strongly minimal sets. Understanding these strongly minimal sets, or trivial pursuits (as coined by J. Baldwin and L. Harrington), is one of the most important open problems in the study of  $DCF_0$ . But to this date very little progress has been made.

For a while it was conjectured that all geometrically trivial strongly minimal sets would have no (or very little) structure: for any element  $y$  of a trivial strongly minimal set  $Y$  only finitely many other elements of  $Y$  are interalgebraic with  $y$ . More precisely,

**Definition 6.** Let  $Y$  be a strongly minimal set defined over a differential field  $K$ . Then  $Y$  is said to be  $\omega$ -categorical if for any tuple  $\mathbf{b}$  from  $\mathcal{U}$ , the set  $K\langle \mathbf{b} \rangle^{\text{alg}} \cap Y$  is finite.

If a strongly minimal set is  $\omega$ -categorical, then it is geometrically trivial. A beautiful result of E. Hrushovski [Hru95] is that the converse holds for order 1 strongly minimal sets (cf. [Pil02, Cor 1.82] and [FM17] for a generalization).

**Theorem 4.** Let  $Y \subset \mathcal{U}^n$  be an order 1 geometrically trivial strongly minimal set. Then  $Y$  is  $\omega$ -categorical.

This result of Hrushovski gave rise to a conjecture about geometrically trivial strongly minimal sets of arbitrary order: *in differentially closed fields, every geometrically trivial strongly minimal set is  $\omega$ -categorical*. This was proven to be false at this level of generality in [FS18] using the order 3

differential equation satisfied by the modular  $j$ -function (see below). The following interesting question remains.

**Question 1.** Are all order 2 geometrically trivial strongly minimal sets  $\omega$ -categorical?

At this point, let us mention that if a strongly minimal set  $Y$  has  $\text{ord}(Y) = n$  and is defined over  $K$ , then  $\omega$ -categoricity can be translated to the following strong transcendence statement: there is an  $m \in \mathbb{N}$  such that if  $y_1, \dots, y_k \in Y$  are distinct and satisfy  $\text{tr.deg}(K\langle y_1, \dots, y_k \rangle / k) = nk$ , then for any other  $y \in Y$ , except for at most  $mk$ , we have that  $\text{tr.deg}(K\langle y_1, \dots, y_k, y \rangle / k) = n(k+1)$ . It follows that establishing strong minimality, geometric triviality, and  $\omega$ -categoricity can be seen as part of a strategy to tackle number-theoretic/functional transcendence type results for the solutions of the differential equations. As such, a positive answer to the above question is of great interest. We will now illustrate this by looking at several recent applications of the model-theoretic approach, in particular the trivial pursuits, to some classical differential equations.

**The generic Painlevé transcendents.** The Painlevé equations are second-order ordinary differential equations and come in six families  $P_I - P_{VI}$ , where  $P_I$  consists of the single equation

$$\frac{d^2 y}{dt^2} = 6y^2 + t,$$

and  $P_{II} - P_{VI}$  come with some complex parameters. They were isolated in the early part of the 20th century by P. Painlevé, with refinements by B. Gambier and R. Fuchs, as those ODE's of the form  $y'' = f(y, y', t)$  (where  $f$  is rational over  $\mathbb{C}$ ) which have the Painlevé property: any local analytic solution extends to a meromorphic solution on the universal cover of  $P^1(\mathbb{C}) \setminus S$ , where  $S$  is the finite set of singularities of the equation. The equations have arisen in a variety of important physical applications including, for example, statistical mechanics, general relativity, and fiber optics.

**Example 6.** The second Painlevé equation  $P_{II}(\alpha)$  is given by

$$\frac{d^2 y}{dt^2} = 2y^3 + ty + \alpha,$$

where  $\alpha \in \mathbb{C}$ . The equation appears quite prevalently in random matrix theory (cf. [FW15]).

Painlevé believed that, at least for general values of the parameters, the set defined by the equations would be strongly minimal. This was proven to be true in a series of papers by K. Okamoto, K. Nishioka, M. Noumi, H. Umemura, and H. Watanabe (cf. [Oka99] for a survey). In particular, the first Painlevé equation is strongly minimal and in the case of the second Painlevé equation, they proved that  $P_{II}(\alpha)$  is strongly minimal if and only if

$\alpha \notin \frac{1}{2} + \mathbb{Z}$ . By a *generic* Painlevé equation we mean one equation among the family  $P_I - P_{VI}$ , such that all the corresponding complex parameters are transcendental and algebraically independent over  $\mathbb{Q}$ . So  $P_{II}(\pi)$  is a generic equation. The works of Watanabe and others hence give that all the generic Painlevé equations are strongly minimal. They left wide open the question of the fine structure of the definable sets. We now have a full answer.

**Theorem 5.** Suppose  $y_1, \dots, y_n$  are distinct solutions of one of the generic Painlevé equations. Then  $y_1, y'_1, \dots, y_n, y'_n$  are algebraically independent over  $\mathbb{C}(t)$ , i.e.,

$$\text{tr.deg}(\mathbb{C}(t)(y_1, y'_1, \dots, y_n, y'_n) / \mathbb{C}(t)) = 2n.$$

In particular the generic Painlevé equations are all  $\omega$ -categorical. K. Nishioka [Nis04] proved the result for  $P_I$  using differential algebra. However his calculations and techniques do not seem to generalize to the other equations. The author, in [Nag20] and before that in joint work with Pillay in [NP14], proved the result for all the other equations using model theory. The proofs rely heavily on earlier work [NP17] in which the trichotomy is used to show that the generic Painlevé equations are all geometrically trivial.

The model-theoretic approach has also allowed us to show that the generic equations from most distinct Painlevé families are orthogonal. Work is currently underway towards obtaining a full classification of algebraic relations between solutions of the Painlevé equations. As of now, except for the second Painlevé equation (where for example the author showed geometrically that triviality holds if and only if  $\alpha \notin \frac{1}{2} + \mathbb{Z}$ ), the study of the non-generic Painlevé equations is wide open. The following is an example of the most basic question one would like to answer.

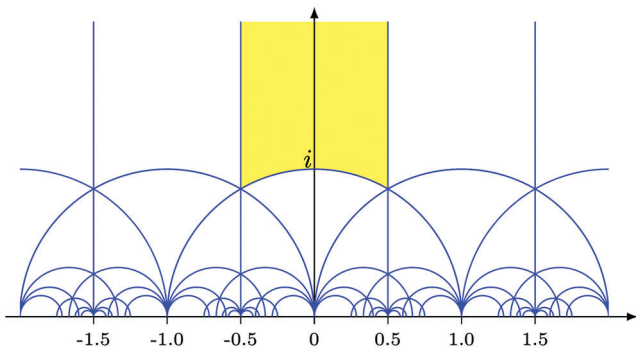
**Question 2.** For which values of the parameters of a fixed Painlevé equation is it true that if  $y_1, \dots, y_n$  are distinct solutions (not in  $\mathbb{C}(t)^{\text{alg}}$ ), then

$$\text{tr.deg}(\mathbb{C}(t)(y_1, y'_1, \dots, y_n, y'_n) / \mathbb{C}(t)) = 2n?$$

**Fuchsian automorphic functions.** We now consider the most natural generalizations of the trigonometric and elliptic functions (i.e., the periodic functions).

Let  $\Gamma \subset \text{PSL}_2(\mathbb{R})$  be a Fuchsian group, that is, assume that  $\Gamma$  is a discrete subgroup of  $\text{PSL}_2(\mathbb{R})$ . A point  $\tau \in \mathbb{H} \cup \mathbf{P}^1(\mathbb{R})$  is said to be a cusp if its stabilizer group  $\Gamma_\tau = \{g \in \Gamma : g \cdot \tau = \tau\}$  has infinite order. We also assume throughout that  $\Gamma$  is of first kind (i.e., its limit set is  $\mathbf{P}^1(\mathbb{R})$ ) and of genus zero (i.e.,  $\Gamma \backslash \mathbb{H}$  can be compactified to a compact Riemann surface of genus 0). An *automorphic function*  $f$  for  $\Gamma$  is a function on the complex upper half plane  $\mathbb{H}$ ,





**Figure 4.** The fundamental domain for the action of a Fuchsian group of the first kind—here  $PSL_2(\mathbb{Z})$ —has a finite number of generators and is of finite volume.

such that<sup>2</sup>

$$f(g \cdot \tau) = f(\tau) \quad \text{for all } g \in \Gamma \text{ and } \tau \in \mathbb{H},$$

and such that  $f$  is meromorphic at every cusp of  $\Gamma$ . The collection  $\mathcal{A}_0(\Gamma)$  of all automorphic functions for  $\Gamma$  is a field and is generated (over  $\mathbb{C}$ ) by some automorphic function called an *hauptmodul* or *uniformizer* for  $\Gamma$ . We will denote by  $j_\Gamma(t)$  one such fixed *hauptmodul*.

It is a classical fact that  $j_\Gamma(t)$  satisfies a third-order ordinary differential equation of Schwarzian type,

$$S_t(y) + (y')^2 R_{j_\Gamma}(y) = 0. \quad (\star)$$

Here  $S_t(y) = \left(\frac{y''}{y'}\right)' - \frac{1}{2} \left(\frac{y''}{y'}\right)^2$  denotes the Schwarzian derivative ( $' = \frac{d}{dt}$ ) and

$$R_{j_\Gamma}(y) = \frac{1}{2} \sum_{i=1}^r \frac{1 - \alpha_i^{-2}}{(y - a_i)^2} + \sum_{i=1}^r \frac{\beta_i}{y - a_i}$$

with  $a_1, \dots, a_n$  and  $\beta_1, \dots, \beta_n$  real numbers depending on  $\Gamma$  and  $j_\Gamma$ . Every solution in  $\mathcal{U}$  of the Schwarzian equation  $(\star)$  can be taken to be of the form  $j_\Gamma(g \cdot t)$  for some  $g \in GL_2(\mathbb{C})$ .

**Example 7.** If  $\Gamma = PSL_2(\mathbb{Z})$ , then the classical modular  $j$ -function

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots,$$

where  $q = e^{2\pi i \tau}$ , is an *hauptmodul*. In this case the differential equation is given with

$$R_j(y) = \frac{y^2 - 1968y + 2654208}{y^2(y - 1728)^2}.$$

P. Painlevé in 1895 again claimed that the set defined by equation  $(\star)$  would be strongly minimal. K. Nishioka proved that the *hauptmodul*  $j_\Gamma$  does not satisfy any algebraic differential equation of order two or less over  $\mathbb{C}(t, e^{\lambda t})$  for any  $\lambda \in \mathbb{C}$ . He also obtained a very weak form of

<sup>2</sup>Throughout  $g \cdot \tau$  will denote the action of an element of  $GL_2(\mathbb{C})$  by linear fractional transformation.

the Painlevé assertion in the case of triangle groups. The first real progress was made by J. Freitag and T. Scanlon [FS18] in their work on the modular  $j$ -function (they did not know of Painlevé's claim then).

**Theorem 6.** Let  $\Gamma = PSL_2(\mathbb{Z})$ . Then the set defined by the Schwarzian equation  $(\star)$  is strongly minimal and geometrically trivial but not  $\omega$ -categorical.

Their proof relies on a deep functional transcendence result of J. Pila [Pil13] called the modular Ax-Lindemann-Weierstrass theorem with derivatives (see below).

**Remark 2.** Granted that strong minimality holds, it is rather unsurprising that the definable set in the case of the  $j$ -function is not  $\omega$ -categorical. Indeed, for each  $n \in \mathbb{N}$  we have the classical modular polynomials  $\Phi_n(X, Y) \in \mathbb{Z}[X, Y]$  that relate solutions of the equation for  $j$ : if  $g_1$  and  $g_2$  are in the same coset of  $GL_2(\mathbb{Q})$ , then  $\Phi_n(j(g_1 \cdot t), j(g_2 \cdot t)) = 0$  for some  $n$ .

For a while the result of Freitag and Scanlon seemed to have shut the door on a possible classification of geometrically trivial strongly minimal sets. However, it turned out that studying the Schwarzian equation  $(\star)$  in its full generality has allowed us to place the case  $\Gamma = PSL_2(\mathbb{Z})$  in context. A natural and key question is the following: is there a way to explain the existence of the modular polynomials? The answer is again very classical and is brought to light through the notion of commensurability.

Recall that two subgroups  $G$  and  $H$  of  $PSL_2(\mathbb{R})$  are commensurable, denoted by  $G \sim H$ , if their intersection  $G \cap H$  has finite index in both  $G$  and  $H$ . For a Fuchsian group  $\Gamma$ , let  $\text{Comm}(\Gamma)$  be the commensurator of  $\Gamma$ , namely

$$\text{Comm}(\Gamma) = \{g \in PSL_2(\mathbb{R}) : g\Gamma g^{-1} \sim \Gamma\}.$$

If  $g \in \text{Comm}(\Gamma) \setminus \Gamma$ , then one has that the intersection  $\Gamma_g = g\Gamma g^{-1} \cap \Gamma$  is a Fuchsian group of first kind and with the same set of cusps as  $\Gamma$ . But the functions  $j_\Gamma(t)$  and  $j_\Gamma(g^{-1}t)$  are respective uniformizers for  $\Gamma$  and  $g\Gamma g^{-1}$ . It follows that they also are automorphic functions for  $\Gamma_g$ . The work of H. Poincaré gives that any two automorphic functions for a Fuchsian group are algebraically dependent over  $\mathbb{C}$ . So there is a polynomial  $\Phi_g \in \mathbb{C}[X, Y]$ , such that  $\Phi_g(j_\Gamma(t), j_\Gamma(g \cdot t)) = 0$ . Such a polynomial is called a  $\Gamma$ -special polynomial.

So if  $\Gamma$  has infinite index in  $\text{Comm}(\Gamma)$ , then there are infinitely many  $\Gamma$ -special polynomials. In particular, if one can prove strong minimality, then non- $\omega$ -categoricity would follow. It turns out that groups  $\Gamma$  having this “infinite index” property are well known in group theory.

Let  $F$  be a totally real number field, and let  $A$  be a quaternion algebra over  $F$  that is ramified at exactly one infinite place. Let  $\rho$  be the unique embedding of  $A$  into  $M_2(\mathbb{R})$ , and let  $\mathcal{O}$  be an order in  $A$ . The image  $\rho(\mathcal{O}^\times)$  of the norm-one group of  $\mathcal{O}$  under  $\rho$  is a discrete subgroup of  $SL_2(\mathbb{R})$ .

We denote by  $\Gamma(A, \mathcal{O})$  the Fuchsian group obtained under the projection in  $PSL_2(\mathbb{R})$  of the group  $\rho(\mathcal{O}^1)$ .

**Definition 7.** A Fuchsian group  $\Gamma$  is said to be arithmetic if it is commensurable with a group of the form  $\Gamma(A, \mathcal{O})$ .

The modular group  $PSL_2(\mathbb{Z})$  and its finite index subgroups are the most well-known examples of arithmetic groups. We have the following deep result of G. Margulis.

**Theorem 7.** *The group  $\Gamma$  is arithmetic if and only if it has infinite index in  $Comm(\Gamma)$  and so there are infinitely many  $\Gamma$ -special polynomials.*

The work of the author with G. Casale and J. Freitag [CFN20] completely proves Painlevé's claim and provides a striking connection between categoricity and arithmeticity.

**Theorem 8.** *Let  $\Gamma$  be a Fuchsian group of first kind and genus zero, and let  $X_\Gamma$  be the set defined by the Schwarzian equation  $(\star)$ . Then:*

1.  $X_\Gamma$  is strongly minimal and (so) geometrically trivial.
2.  $X_\Gamma$  is  $\omega$ -categorical if and only if  $\Gamma$  is nonarithmetic.

The techniques in the proof of Theorem 8 rely on differential Galois theory, monodromy of linear differential equations, the study of algebraic and Liouvillian solutions, differential algebraic work of Nishioka towards the Painlevé irreducibility of certain Schwarzian equations, and considerable machinery from the model theory of differentially closed fields. The following question can be seen as the next major challenge in the classification of geometrically trivial strongly minimal sets in differentially closed fields.

**Question 3.** In  $DCF_0$ , does every non- $\omega$ -categorical strongly minimal set arise from an arithmetic Fuchsian group in this way?

Finally, let us mention that the above work on fully classifying the structure of the definable sets associated with the Schwarzian equation  $(\star)$  has been used in [CFN20] to give a proof of the Ax-Lindemann-Weierstrass theorem with derivatives for  $\Gamma$ : let  $V \subset \mathbb{A}^n$  be an irreducible algebraic variety defined over  $\mathbb{C}$  such that  $V(\mathbb{C}) \cap \mathbb{H}^n \neq \emptyset$  and  $V$  projects dominantly to each of its coordinates (each coordinate function is nonconstant). Let  $t_1, \dots, t_n$  be the functions on  $V$  induced by the canonical coordinate functions on  $\mathbb{A}^n$ . We say that  $t_1, \dots, t_n$  are  $\Gamma$ -geodesically independent if there are no relations of the form  $t_i = g \cdot t_j$ , where  $i \neq j$  and  $g \in Comm(\Gamma)$ .

**Theorem 9.** *With the notation (and assumption  $V(\mathbb{C}) \cap \mathbb{H}^n \neq \emptyset$ ) as above, suppose that  $t_1, \dots, t_n$  are  $\Gamma$ -geodesically independent. Then the  $3n$  functions*

$$j_\Gamma(t_1), j'_\Gamma(t_1), j''_\Gamma(t_1), \dots, j_\Gamma(t_n), j'_\Gamma(t_n), j''_\Gamma(t_n)$$

*(defined locally) on  $V(\mathbb{C})$  are algebraically independent over  $\mathbb{C}(V)$ .*

As mentioned earlier, J. Pila [Pil13] had already proved the result for  $PSL_2(\mathbb{Z})$ . J. Freitag and T. Scanlon [FS18] established the same for arithmetic subgroups of  $PSL_2(\mathbb{Z})$ . The Ax-Lindemann-Weierstrass theorem (mostly without derivatives) has also been proved by various authors in the more general context of Shimura varieties. The work in [CFN20] differs from all the above in that it does not use a tool called *o-minimality* (originating in model theory) and also tackles the nonarithmetic groups as well as the derivatives of the functions all at once.

## Beyond $DCF_0$

We end by saying a few words about the partial differential and the difference equations settings. We denote by  $DCF_{0,m}$  the theory of differentially closed fields of characteristic 0 with  $m$  commuting derivations (partial context) and by  $ACFA$  the theory of algebraically closed fields with a generic automorphism (difference context). The theory  $DCF_{0,m}$  is also  $\omega$ -stable and has quantifier elimination. However, strongly minimal sets do not fully capture the complexity of all definable sets. There are so-called infinite rank regular types<sup>3</sup> that do so. The trichotomy theorem is yet to be fully established in that setting. On the other hand,  $ACFA$  is not  $\omega$ -stable but is rather a so-called *simple theory* (characterized by existence of a good notion of independence). Furthermore, although definable sets are still given by simple enough formulas,  $ACFA$  does not have full quantifier elimination. A version of the trichotomy theorem does hold in that setting and the study of  $ACFA$  has been very successfully used to obtain new results in number theory and algebraic dynamics.

However in both cases, except for a few examples, applications to the study of classical equations is yet to be undertaken. There are obvious candidates that would mirror the situation of  $DCF_0$ . In  $DCF_{0,m}$ , tackling the generalized Schwarzian equations for uniformizers for Shimura varieties is of great interest. In  $ACFA$ , proving that the  $q$ -Painlevé equations are rank 1 is a challenge. These difference equations are discrete analogues of the classical Painlevé equations. In fact, in many real world problems, the Painlevé equations arise from a limiting process, starting with the  $q$ -Painlevé equations. We expect that, as with  $DCF_0$ , important model-theoretic questions about the structure of definable sets can be formed and answered by studying these concrete differential and difference equations.

<sup>3</sup>One such infinite rank regular type also exists for  $DCF_0$ . However the finite rank part of the theory is where most of the complexity lies.

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# Computational Nonimaging Geometric Optics: Monge-Ampère



*Gerard Awanou*

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## 1. Overview

The goal of computational nonimaging geometric optics is the efficient design of optical lenses and mirrors for the accurate control of light. Light waste in the United States is equivalent to 72.9 million mwh of unnecessary electricity generated at a cost of \$6.9 billion a year [10] and the amount of CO<sub>2</sub> generated in that process is equivalent to 9.5 million cars on the roads. Light pollution also has adverse health impacts on wildlife and humans. Other examples where an accurate control of light is required include projection displays, laser weapons, concentrated solar energy, and medical illuminators. Freeform illumination design, i.e., with no a priori symmetry assumption, often leads to numerically solving a nonlinear second order

partial differential equation of Monge-Ampère type with nonlocal boundary conditions. For a review of other approaches we refer to [8, section 2].

Let  $\Omega$  and  $\Omega^*$  be two bounded convex domains of  $\mathbb{R}^d$ . We are interested in the redistribution of an incoming source of light with density  $f \in L^1(\Omega)$ ,  $f \geq 0$ , by a surface defined by a function  $u$  on  $\Omega$ , into a prescribed irradiance described by a density  $R \in L^1(\Omega^*)$ ,  $R > 0$ . Conservation of energy requires  $\int_{\Omega} f(x)dx = \int_{\Omega^*} R(p)dp$ . In the case the surface represents a mirror, light is reflected and we will say that we have a reflector problem. In the case of a lens, light is transmitted with a new direction of travel, i.e., the light is refracted. We will refer to this as a refractor problem. One often makes the assumption of an idealized point light source. Another design we will consider is based on the assumption that the incoming light is collimated, i.e., has parallel rays. As for the target, when it is very far from the source, the light output can be described with a set of directions on the unit sphere. This is referred to as a far field problem. The combination of these design constraints leads to the type of problems we consider in

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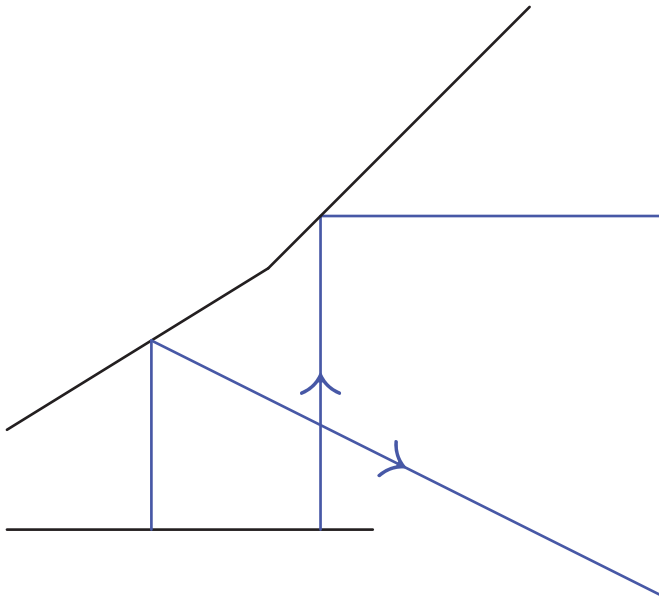
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**Figure 1.** Convex reflector for the density  $R$  discretized with two Dirac masses; cf. section 2.5. The target  $\Omega^*$  is not shown.

this review, i.e., the parallel near field reflector problem, the point source far field refractor problem, etc.

For the parallel far field reflector problem,  $\Omega \subset \mathbb{R}^d$  and  $\Omega^*$  is identified with a subset of  $\mathbb{R}^d$  which is the stereographic projection of a domain of the unit sphere in  $\mathbb{R}^{d+1}$ . In this case, a ray originating at  $x \in \Omega$  is reflected by the mirror described by the graph of  $u$  into the point  $Du(x)$ , the gradient of  $u$  at  $x$ . It is shown for example in [19] that one can choose  $u$  convex solving the Monge-Ampère equation

$$R(Du(x)) \det D^2u(x) = f(x) \text{ in } \Omega, \quad (1.1)$$

with the natural boundary condition

$$Du(\Omega) = \Omega^*. \quad (1.2)$$

For a smooth function  $u$ ,  $D^2u = (\partial^2 u / (\partial x_i \partial x_j))_{i,j=1,\dots,d}$  denotes its Hessian matrix and  $\det D^2u$  is its determinant. Figure 1 illustrates a convex reflector which redirects a parallel light beam into a finite number of directions of the unit sphere.

Problem (1.1)–(1.2) also appears in optimal transport problems as we discuss in section 3. In general, problems in geometric optics lead to more general Monge-Ampère equations

$$\det[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du), \quad T(\cdot, u, Du)(\Omega) = \Omega^* \quad (1.3)$$

which may not have interpretations as optimal transport problems. The numerical resolution of (1.1) in conjunction with the Dirichlet boundary condition

$$u = g \text{ on } \partial\Omega \quad (1.4)$$

for a function  $g$  continuous on  $\partial\Omega$  has been the subject of several reviews; cf. [17] for the latest.

Our focus in this review is on numerical methods for (1.3). We first start with the model problem (1.1)–(1.2) in section 2. The discretizations are based on the kind of solutions for (1.1), classical solutions and various notions of weak solutions. At this point we mainly consider the recent discretization of (1.2) from [2]. We introduce the setting of generated Jacobian equations for (1.3) in section 3 where we review two notions of weak solutions for (1.3). We then discuss the convergence analysis of some of the methods in that setting in section 4. It is here that we review other discretizations of the second boundary condition. We conclude with a list of possible future directions.

This review focuses on computational aspects of generated Jacobians. For insights about the general theory, and applications beyond optics, we refer to the excellent recent review [11].

## 2. Numerical Methods for the Second Boundary Value Problem for the Monge-Ampère Equation

The constraint (1.2) is referred to as the second boundary value condition for (1.1) because it was studied much later than the Dirichlet boundary condition (1.4). For a smooth strictly convex function  $u$ , (1.2) was shown in [19] to be equivalent to

$$Du(\partial\Omega) = \partial\Omega^*, \quad (2.1)$$

which looks more like a boundary condition and nonlocal. Several approaches have been proposed to enforce (1.2) in a numerical scheme. We review most of them in section 4. Below we focus on discretizations of the differential operator in conjunction with the approach through asymptotic cones of [2] for enforcing (1.2). The constraint (1.2) can also be enforced directly by seeking piecewise linear functions with points in  $\Omega^*$  as their piecewise gradients; cf. section 2.5.

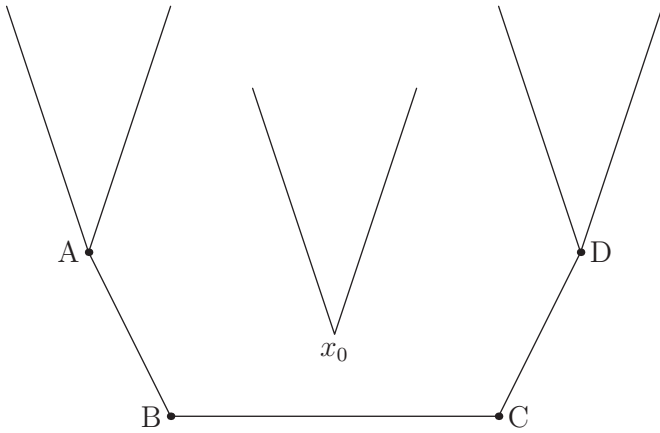
Next, we interpret (1.1)–(1.2) as a problem in the geometry of convex surfaces. We recall that a set  $K \subset \mathbb{R}^d$  is a cone if  $tx \in K$  for all  $t \geq 0$  and  $x \in K$ . We associate to the domain  $\Omega^*$  the cone

$$K_{\Omega^*} = \bigcap_{p \in \Omega^*} \{(x, z) \in \mathbb{R}^d \times \mathbb{R}, z \geq p \cdot x\}.$$

Given a convex function  $u$  on  $\Omega$ , recall that its epigraph is the convex set

$$M = \{(x, z) \in \mathbb{R}^d \times \mathbb{R}, z \geq u(x)\}.$$

The convex hull  $M_*$  of  $M$  and the set  $(x_0, u(x_0)) + K_{\Omega^*}$  for  $x_0 \in \partial\Omega$  defines an infinite convex hypersurface whose boundary defines a convex function  $\tilde{u}$  on  $\mathbb{R}^d$ . For any  $y \in M_*$ ,  $y + K_{\Omega^*} \subset M_*$ . The convex function  $u$  is said to have asymptotic cone  $K_{\Omega^*}$  if  $\tilde{u} = u$  on  $\Omega$ . See Figure 2 for an illustration.



**Figure 2.** The convex hull  $M$  of  $\{A, B, C, D\}$  defines a piecewise linear convex function on a finite interval. The convex hull  $M_*$  of  $M$  and the cone  $x_0 + K_{\Omega^*}$  with  $\Omega^* = (-3, 3)$  defines a piecewise linear convex function on the real line with asymptotic cone  $K_{\Omega^*}$ .

Problem (1.1)–(1.2) has the formulation: find a convex function  $u$  on  $\Omega$  with asymptotic cone  $K_{\Omega^*}$  such that (1.1) holds. If  $u$  has asymptotic cone  $K_{\Omega^*}$ , it is shown in [2] that

$$\tilde{u}(x) = \inf_{s \in \Omega} u(s) + \sup_{p \in \Omega^*} (x - s) \cdot p, x \notin \Omega. \quad (2.2)$$

For example, if  $\Omega = (-1, 1)$  and  $\Omega^* = (-1/2, 1/2)$ , then  $K_{\Omega^*}$  is the epigraph of the function  $y = |x|/2$ . Examples of functions with asymptotic cone  $K_{\Omega^*}$  are given by  $u_1(x) = |x|/2$  and  $u_2(x) = 0$  for  $-1 \leq x \leq 1$  with  $u_2(x) = |x|/2 - 1/2$  for  $x \notin \Omega$ .

In the sequel we will approximate the convex domain  $\Omega^*$  by polygons  $K^* \subset \overline{\Omega^*}$ . The resulting approximate problems are shown to be convergent in [2]. Assuming now for simplicity that  $K^* = \overline{\Omega^*}$ , it can be shown [2] that problem (1.1)–(1.2) is equivalent to finding a convex function  $u$  on  $\Omega$  which extends to a convex function on  $\mathbb{R}^d$  by (2.2) and such that (1.1) holds. Let  $a_j^*$ ,  $j = 1, \dots, N$ , denote the vertices of  $K^*$ . It can then be shown that

$$\tilde{u}(x) = \inf_{s \in \Omega} u(s) + \max_{j=1, \dots, N} (x - s) \cdot a_j^*, x \notin \Omega. \quad (2.3)$$

The infimum can be further restricted to boundary points of a computational mesh.

**2.1. Standard discretizations.** By standard discretizations, we refer to discretizations based on the interpretation of the solution  $u$  of (1.1) as a classical  $C^2(\Omega)$  solution. In that case

$$\det D^2 u = \operatorname{div}((\operatorname{cof} D^2 u) Du),$$

where  $\operatorname{cof} A$  denotes the cofactor matrix of the matrix  $A$  and  $\operatorname{div}$  denotes the divergence operator. Thus (1.1) can be seen as a nonlinear Poisson equation. Pretty much methods developed for elliptic problems can be applied to (1.1)–(2.2). The resulting nonlinear discrete equations may have multiple solutions and cannot be solved by a

vanilla Newton's method when the goal is to reproduce a nonsmooth solution. Iterative methods which preserve a notion of discrete convexity can be used. A particular solution was selected in [19] in a least squares setting with a mixed approximation, i.e., the introduction of new variables  $m = Du$  and  $P = Dm$ . Therein, the second boundary condition was also enforced in a least squares setting. A least squares solution in  $\mathbb{R}^N$  of a system of linear equations  $Ax = b$  is a vector  $x$  which minimizes  $\|b - Ax\|^2$  for the Euclidean norm  $\|\cdot\|$  on  $\mathbb{R}^N$ .

**2.2. Semidiscretizations for Aleksandrov solutions.** The semidiscrete problem considered here is obtained by approximating the density  $f$  with a sum of Dirac masses  $f_M = \sum_{i=1}^M \mu_i \delta_{x_i}$  for an integer  $M$ , weights  $\mu_i \geq 0$ , and  $x_i \in \Omega$ . We will assume that  $R = 1$ . For illustration, we consider a one-dimensional Monge-Ampère equation, i.e., find a convex function  $u$  on  $(0, 1)$  such that in a weak sense

$$u'' = \sum_{i=1}^M \mu_i \delta_{x_i} \text{ in } (0, 1),$$

and for all  $x \in (0, 1)$  we have  $u'(x) \in (-1, 2)$ , i.e.,  $u'(0, 1) = (-1, 2)$ .

For  $u$  smooth and a Borel set  $B \subset (0, 1)$ , the Monge-Ampère measure associated to  $u$  is defined as  $M[u](B) = \int_B u''(x) dx$ . By the change of variable  $x \rightarrow \gamma(x) = u'(x) = p$  (gradient mapping) we obtain  $M[u](B) = \int_{\gamma(B)} dp$ . Next, we replace  $\gamma(x)$  by the subgradient mapping for nonsmooth convex solutions

$$\partial u(x_0) = \{p \in \mathbb{R} : u(x) \geq u(x_0) + p(x - x_0) \text{ for all } x \in (0, 1)\}.$$

For  $v(x) = |x - 1/2|$ , we have  $\partial v(x_0) = \{-1\}$ ,  $x_0 < 1/2$ ,  $\partial v(1/2) = [-1, 1]$ , and  $\partial v(x_0) = \{1\}$ ,  $x_0 > 1/2$ .

For a Borel set  $B$ , we have  $M[u](B) = |\partial u(B)|$ , where for a set  $S$ ,  $|S|$  denotes its Lebesgue measure. By Aleksandrov solutions, we mean a convex function  $u$  such that  $\partial u(0, 1) = (-1, 2)$  and, for all Borel sets  $B \subset (0, 1)$ , we have  $M[u](B) = \sum_{i=1}^M \mu_i \delta_{x_i}(B)$ . We require the compatibility condition  $\sum_{i=1}^M \mu_i = |(-1, 2)| = 3$ .

If we assume that the points  $x_i$  are equidistributed, i.e.,  $x_{i+1} - x_i = h$ , then for  $1 \leq i \leq M$  with  $x_0 = 0$  and  $x_{M+1} = 1$

$$M[u](\{x_i\}) = \frac{u_{i+1} - u_i}{h} - \frac{u_i - u_{i-1}}{h} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h},$$

where  $u_i \approx u(x_i)$ . The main contribution in [2] is the observation that the above formula can also be used at  $x_1$  and  $x_M$  provided that one uses the extension formula (2.3) which gives here for  $i = 0, M + 1$

$$u_i = \min \{ u_1 + \max(- (x_i - x_1), 2(x_i - x_1)), u_M + \max(- (x_i - x_M), 2(x_i - x_M)) \}.$$

Solutions can be shown to be unique up to a constant. One can impose the constraint  $u_0 = \alpha$  for an arbitrary number  $\alpha$ . The value at  $x_{M+1}$  is given by the above formula.

The discretization just described generalizes a similar discretization proposed for the Dirichlet problem in [18]. Details for arbitrary dimensions will be reported elsewhere.

**2.3. Medius approach: Lattice basis reduction.** It is possible to exploit the arithmetic of two-dimensional Cartesian meshes for an efficient resolution of the nonlinear discrete system obtained from a semidiscretization for Aleksandrov solutions, though one has to relax the convexity criterion. The scheme we describe can be seen as a finite difference version of the one described in the previous section. It can be implemented through an efficient adaptive algorithm. Again, we present a variant of an existing scheme [6]. The modification was crucial for the proof of existence and uniqueness of a solution to the discrete problem [2].

The discrete operator is written as a minimization problem over subsets of the mesh. The mesh is identified with a tree and the adaptive algorithm selects subtrees [5], hence the name lattice basis reduction. By adaptivity here, we mean that the operator is evaluated in a cheap and smart way. Let  $h$  be a small positive parameter, and let  $\mathbb{Z}_h^2 = \{mh, m \in \mathbb{Z}\}$  denote the orthogonal lattice with mesh length  $h$ . The set of mesh points is given by  $\Omega_h = \Omega \cap \mathbb{Z}_h^2$ . Computations are made with a finite subset of the mesh. Let  $V$  denote a finite set of nonzero elements of  $\mathbb{Z}^2$  such that if  $e \in V$ ,  $-e \in V$ . It is further assumed that elements of  $V$  have coprime coordinates and span  $\mathbb{R}^2$ , and  $V$  contains the elements of the canonical basis of  $\mathbb{R}^2$  and a normal to each side of the target polygonal domain  $K^*$ . We also require that  $V$  contains  $\{(a, b) \in \{-1, 0, 1\}^2, ab \neq 0\}$ .

Let  $(e_1, e_2)$  denote the canonical basis of  $\mathbb{R}^2$ , and let

$$\partial\Omega_h = \{x \in \Omega_h \text{ such that for some } i = 1, 2, \\ x + he_i \notin \Omega_h \text{ or } x - he_i \notin \Omega_h\}.$$

We define for a function  $v_h$  on  $\mathbb{Z}_h^2$ ,  $e \in \mathbb{Z}^2$ , and  $x \in \Omega_h$

$$\Delta_{he} v_h(x) = v_h(x + he) - 2v_h(x) + v_h(x - he).$$

We are interested in mesh functions on  $\Omega_h$  which are extended to  $\mathbb{Z}_h^2$  using

$$\tilde{u}(x) = \inf_{s \in \partial\Omega_h} u(s) + \max_{j=1, \dots, N} (x - s) \cdot a_j^*, x \notin \Omega, \quad (2.4)$$

and are discrete convex in the sense that  $\Delta_{he} v_h(x) \geq 0$  for all  $x \in \Omega_h$  and  $e \in V$ . A uniform limit of mesh functions which are discrete convex in the sense above and solve suitable discrete Monge-Ampère equations is a convex function. It is a result implicit in convergence studies of discretizations of (1.1) with  $R = 1$ .

Next, we consider a local version of a symmetrization of a discrete version of the subgradient

$$D_L v_h(x) = \{p \in \mathbb{R}^2, 2p \cdot (he) \leq \Delta_{he} v_h(x) \forall e \in L\}.$$

Define a basis of  $\mathbb{Z}^2$  as a pair  $(e_1, e_2) \in (\mathbb{Z}^2)^2$  such that  $|\det(e_1, e_2)| = 1$ . A superbase of  $\mathbb{Z}^2$  is a triplet

$(e_0, e_1, e_2) \in (\mathbb{Z}^2)^3$  such that  $e_0 + e_1 + e_2 = 0$ , and  $(e_1, e_2)$  is a basis of  $\mathbb{Z}^2$ . The Monge-Ampère operator with lattice basis reduction in the case  $R = 1$  is defined as

$$MA_{LBR} u_h(x) = \min_{\substack{L=(e_0, e_1, e_2) \in V^3 \\ \text{superbase}}} |D_L u_h(x)|.$$

The discrete problem consists in finding a discrete convex mesh function  $u_h$  such that

$$MA_{LBR} u_h(x) = \int_{E_x} f(t) dt, x \in \Omega_h, \quad (2.5)$$

where  $E_x = x + [-h/2, h/2]^d$  is a cube centered at  $x$  with  $E_x \cap \Omega_h = \{x\}$ . The unknowns in the above equation are the mesh values  $u_h(x)$ ,  $x \in \Omega_h$ . For  $x \notin \Omega_h$ , the value  $u_h(x)$  needed for the evaluation of  $D_L v_h(x)$  is obtained from the extension formula (2.4). Here we made the simplifying assumption that  $\Omega^* = K^*$  so that conservation of energy holds. Again here we impose the constraint  $u_h(x_0) = \alpha$  for an arbitrary real number  $\alpha$  and with  $x_0 \in \Omega_h$  for all  $h$ . A damped Newton's method can be used for solving the nonlinear equations for  $f > 0$ .

**2.4. Approach through viscosity solutions.** The notion of viscosity solution for (1.1) is based on comparisons with smooth test functions. Aleksandrov solutions of (1.1) are equivalent to viscosity solutions when the right-hand side  $f$  is continuous and positive [13]. For  $R = 1$ , it was shown in [4] through a perturbation argument that the equivalence also holds.

For solutions of schemes to converge to a viscosity solution, it is convenient that the scheme satisfies a monotonicity property allowing comparison with smooth test functions. This often requires writing discretizations in a specific form and, for schemes which violate the monotonicity condition, it is very difficult to prove convergence in the setting of viscosity solutions. For example, the scheme (2.5) may require a numerical integration leading to nonmonotone schemes. However, we believe that convergence can still be proven in the setting of Aleksandrov solutions through a perturbation argument taking advantage of convergence results for (2.5) which are essentially the same as the ones discussed in [2]. In fact, a nonmonotone approximation of  $MA_{LBR} u_h(x)$  through a standard discretization of the gradient was actually considered in [6].

The geometric content of solutions to the Monge-Ampère equation is lost in the viscosity solution setting. It is unlikely to explain the behavior of standard discretizations for (1.1)–(1.2) and nonsmooth solutions. We refer to [7, 12] for explicit monotone discretizations of (1.1). We do not discuss this further since their analysis in conjunction with (2.4) is similar to the analysis of the effect of numerical integration for (2.5), a topic we wish to discuss in a separate work.

**2.5. Semidiscretizations for Brenier solutions.** Here we assume that the density  $R$  is approximated by a sum of Dirac masses  $\sum_{i=1}^M r_i \delta_{P_i}$  for  $P_i \in \Omega^*$  and  $r_i \in \mathbb{R}$  for all  $i$ . Energy conservation reads  $\sum_{i=1}^M r_i = \int_{\Omega} f(x) dx$ . In the case of one Dirac mass  $r_i \delta_{P_i}$  the surface which reflects all rays with direction  $(0, \dots, 0, 1) \in \mathbb{R}^{d+1}$  from  $\Omega$  into a direction of the unit sphere with stereographic projection the point  $P_i$  is, by Snell's law, given by a plane  $x \cdot P_i - b_i$  for a parameter  $b_i$ . The reflector is then given by the graph of the convex function

$$u_M(x) = \max_{i=1, \dots, M} x \cdot P_i - b_i,$$

with rays in the region

$$W_i(b) = \{x \in \Omega, x \cdot P_i - b_i \geq x \cdot P_j - b_j \text{ for all } j = 1, \dots, M\},$$

reflected in the direction  $P_i$ . We thus need

$$\int_{W_i(b)} f(x) dx = r_i, i = 1, \dots, M,$$

which is the nonlinear equation to be solved for  $b_i, i = 1, \dots, M$ . The constraint (1.2) is enforced implicitly in the sense that by construction  $\partial u_M(\Omega) \subset \Omega^*$  when  $\Omega^*$  is convex.

### 3. Generated Jacobian Equations

Generated Jacobian equations are a class of prescribed Jacobian equations, i.e., one seeks a mapping  $T$  between two bounded domains  $\Omega$  and  $\Omega^*$  of  $\mathbb{R}^d$  whose Jacobian  $\det DT(x)$  is prescribed by an equation

$$\det DT(x) = \psi(x, T(x))$$

for a given function  $\psi$  on  $\Omega \times \Omega^*$ . An example of such a mapping is the optimal transport map, discussed above, between two measures supported respectively on  $\Omega$  and  $\Omega^*$ . In that case, one requires  $\psi \geq 0$  and the mapping  $T$  is generated by a convex function  $u$  on  $\Omega$  in the sense that  $T(x) = Du(x)$ . In geometric optics problems,  $\psi \geq 0$  as well and the mapping  $T_u$  taking a light described by  $x \in \Omega$  into a point  $T_u(x)$  on the target is also generated by a scalar function  $u$  on  $\Omega$  which describes the optical surface and solves the generated Jacobian equation

$$\det DT_u(x) = \psi(x, u(x), T_u(x)), \quad T_u(x) = T(x, u(x), Du(x)), \quad (3.1)$$

where now  $\psi$  and  $T$  are functions on  $\Omega \times \mathbb{R} \times \Omega^*$  which take values in  $\mathbb{R}$  and  $\mathbb{R}^d$ , respectively. We assume in this paper that  $\psi$  is separable in the sense that

$$\psi(x, u, p) = \frac{f(x)}{R(T(x, u, p))}$$

for positive functions  $f \in L^1(\Omega)$  and  $g \in L^1(\Omega^*)$ . This structural assumption encompasses applications in geometric

optics. The second boundary value problem for the generated Jacobian equation (3.1) is to prescribe in addition the image of  $\Omega$  by  $T_u$ , i.e.,

$$\begin{aligned} R(T_u(x)) \det DT_u(x) &= f(x), x \in \Omega, \\ T_u(\Omega) &= \Omega^*. \end{aligned} \quad (3.2)$$

The transformation  $T$  and the "potential"  $u$  are now related through a generating function  $G : \bar{\Omega} \times \bar{\Omega}^* \times \mathbb{R}^+ \mapsto \mathbb{R}$  and  $T(x, u, p)$  is obtained by solving the system

$$D_x G(x, T, Z) = p, \quad G(x, T, Z) = u,$$

where  $Z$  is an additional unknown. It is assumed that the above system has a unique solution, that  $G$  is sufficiently smooth and strictly decreasing with respect to  $z$ . When  $G(x, y, z) = x \cdot y + \log z$ , we obtain  $T(x, y, p) = p$ , i.e.,  $T_u(x) = Du(x)$  and  $\psi$  does not depend on  $u(x)$ . The same holds for optimal transport problems with a general cost function  $c(x, y)$ , with compatible assumptions on  $c : \Omega \times \Omega^* \rightarrow \mathbb{R}$ , in which case  $G(x, y, z) = c(x, y) + \log z$ . The mapping  $T_u$  is then the optimal transport map, i.e.,

$$T_u(x) = \arg \min_T \int_{\Omega} c(x, T(x)) f(x) dx,$$

where the infimum is taken over mappings  $T$  which push forward the measure with density  $f$  onto the measure with density  $g$ , that is, mappings  $T$  which satisfy

$$\int_{\Omega} \phi(T(x)) f(x) dx = \int_{\Omega^*} \phi(y) g(y) dy$$

for all continuous functions  $\phi : \Omega^* \rightarrow \mathbb{R}$ .

For weak solutions of (3.2) one has, as in the case of (1.1)–(1.2), the Aleksandrov theory and the Brenier formulation.

The functions  $x \mapsto G(x, \cdot, \cdot)$  play the role hyperplanes play as support functions in the theory of convex functions. Given  $y_0 \in \Omega^*$  and  $\lambda_0 \in \mathbb{R}^+$ , the function  $x \mapsto G(x, y_0, \lambda_0)$  is said to be a  $G$ -support to a function  $u : \Omega \rightarrow \mathbb{R}$  at  $x = x_0 \in \Omega$  if  $u(x) \geq G(x, y_0, \lambda_0) \forall x \in \Omega$  with equality at  $x = x_0$ . The function  $u$  is said to be  $G$ -convex if it has a  $G$ -support at all points  $x \in \Omega$ . Equivalently  $u$  is  $G$ -convex if and only if there exists a set  $\mathcal{A} \subset \Omega^* \times \mathbb{R}^+$  such that

$$u(x) = \sup_{(y, \lambda) \in \Omega^* \times \mathbb{R}^+} G(x, y, \lambda).$$

The  $G$ -subdifferential of  $u$  at  $x_0 \in \Omega$  is defined as the set-valued function

$$\partial_G u(x_0) = \{y \in \Omega^*, \exists \lambda_0 \in \mathbb{R}^+ \text{ such that } G(x, y, \lambda_0) \text{ is a } G\text{-support to } u \text{ at } x_0\}.$$

It is known that for a  $G$ -convex function  $u$ , the set  $\partial_G u(E) = \bigcup_{x \in E} \partial_G u(x)$  is measurable when  $E$  is measurable ([1, Lemma 2.1] and [20, pp. 12–13]). Moreover the



set function

$$M[u](E) = \int_{\partial_G u(E)} g(p) dp$$

is a Radon measure. A weak solution of (3.2) in the sense of Aleksandrov is a  $G$ -convex function  $u$  such that

$$M[u](E) = \int_E f(x) dx \text{ for all Borel sets } E \subset \Omega.$$

Thus we have the necessary condition

$$\int_{\Omega} f(x) dx = \int_{\Omega^*} g(p) dp.$$

The tracing map of  $u$  for  $y_0 \in \Omega^*$  is defined as

$$\tau_G u(y_0) = \{x_0 \in \Omega, \exists \lambda_0 \in \mathbb{R}^+ \text{ such that } G(x, y_0, \lambda_0) \text{ supports } \phi \text{ at } x_0\}.$$

Note that  $\tau_G$  is the inverse of the  $G$ -subdifferential, i.e.,  $\tau_G u(y_0) = (\partial_G u)^{-1}(y_0)$ , and can be interpreted as the set of directions from which light emanating from the origin is redirected in the direction  $y$ .

For a subset  $F \subset \Omega^*$  we define  $\tau_G u(F) = \bigcup_{y \in F} \tau_G u(y)$ , and for a  $G$ -convex function  $u$  we define the set function

$$\eta_u(F) = \int_{\tau_G u(F)} f(x) dx. \quad (3.3)$$

A weak solution of (3.2) in the sense of Brenier is a  $G$ -convex function  $u$  such that

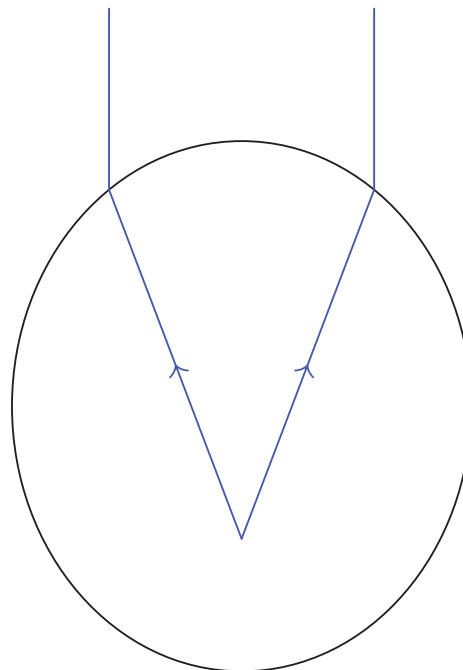
$$\eta_u(F) = \int_F g(p) dp \text{ for all Borel sets } F \subset \Omega^*. \quad (3.4)$$

Explicit expressions of the generating function  $G$  and the terms for the differential equation (1.3) can be found in [20, 8]. For the far field reflector problem with an incoming parallel light beam, as in section 2.5, the surface  $x \mapsto x \cdot P_i - b_i$  gives the generating function  $G(x, P_i, \lambda_0) = x \cdot P_i + \log \lambda_0$ . In general, the generating function describes a basic optical surface which converts light from  $x \in \Omega$  into  $y \in \Omega^*$ . The optical surface can be made of ellipses, parabolas, hyperbolas, Cartesian ovals, etc. Figure 3 shows an ellipse refracting light from a point light source into a uniform direction.

#### 4. Convergence of Numerical Methods for Generated Jacobian Equations

There are two types of convergence to be addressed: convergence of an iterative method for solving the discrete nonlinear system resulting from a discretization and convergence of the numerical solution to the exact solution.

Many of the developments have taken place with discretizations of (1.2) different from the approach through asymptotic cones taken in section 2. We review them below. It is our goal to systematically extend the Aleksandrov solution approach and the lattice basis reduction approach to generated Jacobian equations.



**Figure 3.** An example of a surface with a uniform refracting property.

##### 4.1. Other discretizations of the second boundary condition.

**4.1.1. Defining function of the target domain.** Let  $H$  be a defining function of  $\Omega^*$ , i.e.,  $\Omega^* = \{x \in \mathbb{R}^d, H(x) < 0\}$ . The boundary condition  $Du(\Omega) = \Omega^*$  can be shown to be equivalent to

$$H(Du) = 0 \text{ on } \partial\Omega$$

for a defining function  $H$  of  $\Omega^*$ ; cf. for example [7].

Let  $d_{\partial\Omega^*}$  denote the distance function to the boundary  $\partial\Omega^*$  of  $\Omega^*$ , i.e.,  $d_{\partial\Omega^*}(x) = \inf_{z \in \partial\Omega^*} \|x - z\|$ . An example of defining function is given by the signed distance-to-the-boundary defined as

$$\delta_{\partial\Omega^*}(x) = -d_{\partial\Omega^*}(x), \quad x \in \Omega^*,$$

$$\text{and } \delta_{\partial\Omega^*}(x) = d_{\partial\Omega^*}(x), \quad x \notin \Omega^*.$$

If the goal is to prove convergence to a viscosity solution, one chooses a monotone discretization of  $H(Du) = 0$  as in [7, 12].

**4.1.2. Iterated projection algorithm.** Numerical experiments in [9] suggested that the following iterative method converges to a solution of (1.1)–(1.2). Let  $n$  denote the outward normal to  $\partial\Omega$ . We consider the sequence  $u^k$  defined by

$$R(Du^k) \det D^2 u^{k+1} = f \text{ in } \Omega,$$

$$\frac{\partial u^{k+1}}{\partial n} = (\text{Proj}_{\partial\Omega^*} Du^k) \cdot n \text{ on } \partial\Omega,$$

$$u^{k+1}(x_0) = \alpha$$

for  $x_0 \in \partial\Omega$  and  $\alpha$  a real number. Here for a vector  $v \in \mathbb{R}^d$ , we define

$$\text{Proj}_{\partial\Omega^*} v = \inf_{y \in \partial\Omega^*} \|y - v\|.$$

**4.1.3. Enforced in a least squares sense.** Introducing new variables  $m = Du$  and  $P = Dm$ , (2.2) becomes  $m(\partial\Omega) = \partial\Omega^*$ . In [19] (1.1) is written in terms of the new variables and solved along with the constraint  $m(\partial\Omega) = \partial\Omega^*$  in a least squares setting.

**4.1.4. Enforced throughout the source domain.** It has been suggested in [12] that instead of enforcing  $H(Du) = 0$  on  $\partial\Omega$ , one can enforce  $H(Du) = 0$  on  $\Omega$ . The motivation was to get convergence results to a viscosity solution of (1.1)–(1.2). A similar idea was previously used in [15] where the authors sought a piecewise linear convex approximation with the requirement that its piecewise gradients be vectors in  $\Omega^*$  enforced as a constraint in an optimization scheme.

Although we have assumed in this paper that  $\Omega^*$  is convex, several of the discretizations proposed for (1.2) should work for nonconvex domains, an exception being the approach based on a defining function of  $\Omega^*$ .

**4.2. Numerical methods for generated Jacobian equations.** A general approach for handling (1.3) was initiated in [8] for the point source near field reflector and refractor problems. It is based on the iterative projection algorithm for handling the second boundary condition.

A priori the semidiscrete approximations with Brenier solutions can be applied to generated Jacobian equations. But this raises the practical issue of how to compute the analogues of the sets  $W_i(b)$  described in section 2.5. For far field problems a computational geometry approach was used in [16].

**4.3. Convergence of iterative methods.** No convergence analysis has been reported for the iterated method based on projections proposed in [9]. Damped Newton methods have also been used. A damped Newton's method is a variant of Newton's method for which the Jacobian matrix is multiplied by a damping factor, with the goal of having convergence of the iterates independent of the closeness to the solution of an initial guess.

For the semidiscrete problems with Brenier solutions, a convergence analysis for a damped Newton's method was given in [14]. It does not cover for example the far field refractor problem which is included in the class of generated Jacobian equations for which convergence of an iterative method is proven in [1].

**4.4. Convergence of discretizations.** Not much is known about convergence of standard discretizations for (1.1)–(1.2). For smooth solutions and the Dirichlet problem, existence of a solution can be proven for various discretizations but for  $h$  sufficiently small. The least squares method for the discretization of (1.1)–(1.4) has not been analyzed for smooth solutions. We gave a theory of

convergence of standard discretizations for Aleksandrov solutions of the Dirichlet problem for the Monge-Ampère equation [3]. It is based on the assumption that computers do not see the difference between a computational domain and a fictitious subdomain arbitrarily close. It would be interesting to have a theory without that assumption.

The convergence of semidiscrete approximations for Aleksandrov and Brenier solutions is central to the theory of generated Jacobian equations [20]. For the medius approach convergence of the discretization was proven in [6, 2]. We note that convergence of the discretization was also proven for both approaches [15, 12] where the second boundary condition is enforced throughout the domain.

**4.5. Performance of the numerical methods.** There has not been a comparative numerical study of discretizations for (1.1)–(1.2). No numerical experiments were reported in [2, 12]. The approach through asymptotic cones of [2] should yield results similar to the ones reported in [6] for the medius approach. Methods based on standard discretizations may not be very efficient. A possible exception is the least squares approach which has been applied to a variety of optics problems. In [15], it was reported that a method based on standard discretizations of the gradient and Hessian vastly outperforms an analogue which is provably competent. It is reasonable to expect that methods based on the iterative projection algorithm would be less efficient than a more direct approach. Understanding the mechanisms of the not so efficient methods could give a better understanding of the computational process and may lead to more efficient numerical methods.

## 5. Possible Future Directions

Computational nonimaging geometric optics based on Monge-Ampère equations is like a painting which is largely incomplete. The main open issue is to what extent existing methods can be made more efficient. Several possible moves or combinations of the ideas discussed above are possible. For example, one can adapt to the second boundary value problem methods proposed for the Dirichlet problem [17]. A possible direction is to adapt advances in computational optimal transport to generated Jacobian equations. Also, the state of the art in computational mathematics such as fast solvers and adaptive methods have not been applied to geometric optics problems. There are also many unanswered questions which deal with the analysis of several of the numerical methods that have been proposed. In addition to several of the issues mentioned above, we give several more examples.

1. Problems with loss of energy, i.e., when only part of the radiation is transmitted, and problems with multiple sources or extended fields and systems with two lenses could be addressed with recent advances on proven convergent numerical methods.

2. The semigeostrophic flow equations were formulated as a coupled system consisting of the Monge-Ampère equation with the second boundary condition and a transport equation. It would be interesting to see how recent advances can be used for its numerical resolution.
3. There is no theory for viscosity solutions of generated Jacobian equations. The special case (1.1)–(1.2) has been recently solved in [12].

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# Exceptional Groups and Their Modular Forms



Aaron Pollack

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## 1. Introduction

It is an old theorem of Lagrange that every nonnegative integer can be expressed as the sum of four squares of integers. That is, if  $n \in \mathbf{Z}_{\geq 0}$ , then there exist integers  $x_1, x_2, x_3, x_4$  so that  $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$ . For the sake of comparison, note that 7 is not the sum of three squares, so not every integer can be expressed as  $x_1^2 + x_2^2 + x_3^2$ .

If  $n \in \mathbf{Z}_{\geq 0}$  and  $m > 0$  is a positive integer, denote by  $r_m(n)$  the number of ways one can express  $n$  as the sum of  $m$  squares of integers. That is, define  $r_m(n)$  to be the size of the set

$$\{(x_1, x_2, \dots, x_m) \in \mathbf{Z}^m : x_1^2 + x_2^2 + \dots + x_m^2 = n\}.$$

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Then Lagrange's theorem says that  $r_4(n) \geq 1$  for every positive integer  $n$ . In fact, there is a beautiful formula due to Jacobi for  $r_4(n)$ . Denote by  $\sigma(n) = \sum_{d|n} d$  the sum of the divisors of the integer  $n$ . Then  $r_4(n) = 8\sigma(n)$  if  $n$  is odd and  $r_4(n) = 24\sigma(n)$  if  $n$  is even.

Where does such a nice formula come from? The contemporary explanation is that this formula comes from the theory of **modular forms**. Consider the power series  $\sum_{n \geq 0} r_4(n)q^n \in \mathbf{Z}[[q]]$  in the variable  $q$ . If  $z$  is in the complex upper half-plane  $\mathfrak{h} = \{x + iy : x, y \in \mathbf{R}, y > 0\}$  and  $q = e^{2\pi iz}$ , then  $\theta_4(z) := \sum_{n \geq 0} r_4(n)q^n$  becomes a holomorphic function on  $\mathfrak{h}$ . The function  $\theta_4(z)$  is an example of a modular form (defined below), which amounts to the fact that  $\theta_4(z + 1) = \theta_4(z)$  and  $\theta_4\left(-\frac{1}{4z}\right) = (4z)^2\theta_4(z)$  as functions on  $\mathfrak{h}$ . The first functional equation is obvious but the second is not. These symmetries go a long way to proving the relationship between  $r_4(n)$  and the divisor function  $\sigma(n)$ .

Note that  $z \mapsto z + 1$  and  $z \mapsto -\frac{1}{4z}$  are the linear fractional transformations of  $\mathfrak{h}$  associated to the matrices  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



and  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . There is a finite index subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbf{Z})$  so that  $\theta_4(z)$  satisfies a functional equation associated to every element  $\gamma \in \Gamma$ . Namely, if  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , then  $\theta_4\left(\frac{az+b}{cz+d}\right) = (cz+d)^2\theta_4(z)$ . In this way, one can think of  $\theta_4(z)$  as a special function associated to the group  $\mathrm{SL}_2$ . The reader can see [Zag08] and [Ser73] for an introduction to modular forms, and in particular [Zag08, Section 3] for a proof of Jacobi's formula via  $\theta_4(z)$ .

The modular form  $\theta_4(z)$  is a special example of what is called an **automorphic form**. Automorphic forms are functions that have large groups of discrete symmetries and satisfy particular types of differential equations. (The function  $\theta_4(z)$  satisfies the Cauchy-Riemann equations.) This article is about automorphic forms, with a special emphasis on those automorphic forms whose groups of symmetries are connected to the exceptional Lie groups. We do not suppose the reader knows anything about modular forms or exceptional groups, only the representation theory of compact groups and a little algebraic geometry.

## 2. Modular Forms

We now dig into the definition of modular forms. Suppose that  $k > 0$  is a positive integer and  $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$  is a finite index subgroup. We assume that there exists a positive integer  $N$  so that  $\Gamma$  contains the subgroup  $\Gamma(N)$  of  $\mathrm{SL}_2(\mathbf{Z})$  consisting of matrices congruent to 1 modulo  $N$ , i.e.,  $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a, d \equiv 1 \pmod{N}, b, c \equiv 0 \pmod{N} \right\}$ . A modular form of weight  $k$  for  $\Gamma$  is a holomorphic function  $f : \mathfrak{h} \rightarrow \mathbf{C}$  that is semi-invariant under  $\Gamma$  and doesn't grow too quickly. More precisely, a holomorphic function  $f : \mathfrak{h} \rightarrow \mathbf{C}$  is a modular form of weight  $k$  for  $\Gamma$  if

1.  $f(\gamma z) = (cz+d)^k f(z)$  for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ ,
2. the function  $y^{k/2}|f(z)|$  grows at most polynomially with  $y$  on  $\Gamma \backslash \mathfrak{h}$ . (Note that  $y^{k/2}|f(z)|$  is  $\Gamma$ -invariant, as  $\mathrm{Im}(\gamma z) = \mathrm{Im}(z)|cz+d|^{-2}$ .)

We denote by  $M_k(\Gamma)$  the space of modular forms of weight  $k$  for  $\Gamma$ . For  $k$  and  $\Gamma$  fixed, the space  $M_k(\Gamma)$  is a finite-dimensional (sometimes 0) complex vector space.

Note that if  $f : \mathfrak{h} \rightarrow \mathbf{C}$  is a holomorphic function, then  $f(z)dz$  is  $\Gamma$ -invariant if and only if  $f(\gamma z) = (cz+d)^2 f(z)$ , so weight-2 modular forms are in bijection with a subspace of the holomorphic differential-one forms on  $Y_\Gamma := \Gamma \backslash \mathfrak{h}$ .

One of the first things that must be said about modular forms is that they have a Fourier expansion: suppose for simplicity that  $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ . Then the elements  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma$  for  $n \in \mathbf{Z}$  and thus the condition  $f(\gamma z) = (cz+d)^k f(z)$  becomes  $f(z+n) = f(z)$ . As  $f$  is holomorphic, this implies that  $f(z) = \sum_{n \in \mathbf{Z}} a_f(n) e^{2\pi i n z}$  for complex numbers  $a_f(n) \in \mathbf{C}$ . As the reader can immediately check, condition 2 above implies that  $a_f(n) = 0$  if  $n$  is negative. That is,  $f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z}$ .

**2.1. Examples.** The special function  $\theta_4(z)$  described in the introduction is a modular form of weight 2. More generally, if  $m \geq 0$  is a nonnegative integer, then one can define  $\theta_m(z) = \sum_{n \geq 0} r_m(n) q^n$ . When  $m$  is even,  $\theta_m(z)$  is a modular form of weight  $m/2$ .

A simpler set of examples of modular forms are the so-called Eisenstein series  $E_k(z)$ . Suppose  $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$  is fixed, and denote  $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}$ . If  $k \geq 4$ , one defines

$$E_k(z; \Gamma) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-k}. \quad (1)$$

In the sum above,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and the condition  $k \geq 4$  ensures that the sum converges absolutely to a holomorphic function on  $\mathfrak{h}$ . Note that if  $\gamma_\infty \in \Gamma_\infty$ , then  $\gamma_\infty \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ , so that the sum (1) is well-defined.

These functions have simple Fourier expansions. For example, if  $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ , then  $E_k(z) := E_k(z; \Gamma)$  has Fourier expansion

$$E_k(z) = 1 + \alpha_k \sum_{n \geq 1} \left( \sum_{d|n} d^{k-1} \right) q^n$$

for a nonzero rational number  $\alpha_k$ . By relating  $\theta_4(z)$  with a certain weight-2 Eisenstein series  $E_2(z; \Gamma)$ , one can prove Jacobi's formula for  $r_4(n)$ .

The examples above all have the feature that the constant term  $a_f(0) \neq 0$ , so that the function  $y^{k/2}|f(z)|$  grows as  $y \rightarrow \infty$ . The subspace of modular forms that *decay* as  $y \rightarrow \infty$  occupies a special place in the theory. More precisely, the noncompact complex curve  $Y_\Gamma$  can be compactified to a curve  $X_\Gamma$ . The set  $X_\Gamma \setminus Y_\Gamma$  is finite, and is called the cusps of the modular curve  $X_\Gamma$ . The modular forms  $f$  of weight  $k$  for which the function  $y^{k/2}|f(z)|$  on  $Y_\Gamma$  decays towards the cusps are called cusp forms, denoted by  $S_k(\Gamma) \subseteq M_k(\Gamma)$ . Ramanujan defined the following beautiful cusp form of weight  $k = 12$ :

$$\Delta(z) := q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n.$$

The numbers  $\tau(n)$  are by definition the complex numbers that make this an equality. It is not obvious that  $\Delta(z)$  is a modular form. Another fact that is not clear from the definition is that  $\tau(mn) = \tau(m)\tau(n)$  if the positive integers  $m$  and  $n$  are relatively prime. To give the reader a sense for a simple statement which is already not known, we remark that it is conjectured that  $\tau(n) \neq 0$  for all  $n$ .

**2.2. The group  $\mathrm{SL}_2$ .** As suggested in the introduction, modular forms should be thought of as functions associated to the group  $\mathrm{SL}_2$ . To make this precise, recall that the action of  $\mathrm{SL}_2(\mathbf{R})$  on  $\mathfrak{h}$  via linear fractional transformations is transitive, and that the stabilizer in  $\mathrm{SL}_2(\mathbf{R})$  of the point  $\sqrt{-1} \in \mathfrak{h}$  is the special orthogonal group of size two:

$$\mathrm{SO}(2) = \left\{ k_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbf{R} \right\}. \quad (2)$$

That is, one has an  $\mathrm{SL}_2(\mathbf{R})$ -equivariant identification  $\mathfrak{h} \simeq \mathrm{SL}_2(\mathbf{R})/\mathrm{SO}(2)$ .

For  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$  and  $z \in \mathfrak{h}$ , set  $j(g, z) = cz + d$ . The reader can easily check that for  $g_1, g_2 \in \mathrm{SL}_2(\mathbf{R})$ ,  $j(g_1 g_2, z) = j(g_1, g_2 z) j(g_2, z)$ . Now if  $f \in M_k(\Gamma)$  is a weight- $k$  modular form, then one can define a closely related function  $\phi_f : \Gamma \backslash \mathrm{SL}_2(\mathbf{R}) \rightarrow \mathbf{C}$ . Namely, set

$$\phi_f(g) = j(g, i)^{-k} f(g \cdot i).$$

Then if  $\gamma \in \Gamma$  and  $z = g \cdot i$ ,

$$\begin{aligned} \phi_f(\gamma g) &= j(\gamma g, i)^{-k} f(\gamma \cdot i) \\ &= j(\gamma, z)^{-k} j(g, i)^{-k} j(\gamma, z)^k f(z) = \phi_f(g) \end{aligned}$$

so that  $\phi_f$  is left- $\Gamma$ -invariant. Similarly, one can easily verify that if  $k_\theta \in \mathrm{SO}(2)$  as in (2), then  $\phi_f(g k_\theta) = e^{ik\theta} \phi_f(g)$  for all  $g \in \mathrm{SL}_2(\mathbf{R})$ .

To summarize, set  $G = \mathrm{SL}_2(\mathbf{R})$  and fix  $\Gamma \subseteq G$  and  $k$  an integer. Then one can consider the space  $M_{k,\infty}(\Gamma)$  of smooth functions  $\varphi : G \rightarrow \mathbf{C}$  defined as follows.

**Definition 1.** A smooth function  $\varphi : G \rightarrow \mathbf{C}$  is in  $M_{k,\infty}(\Gamma)$  if

1.  $\varphi(\gamma g) = \varphi(g)$  for all  $g \in G$  and  $\gamma \in \Gamma$ ,
2.  $\varphi(g k_\theta) = e^{ik\theta} \varphi(g)$  for all  $g \in G$  and  $k_\theta \in \mathrm{SO}(2)$ ,
3.  $\varphi \in C^\infty(\Gamma \backslash G)$  has “moderate growth” (condition 2 above) in an appropriate sense.

We have just defined an injection  $M_k(\Gamma) \hookrightarrow M_{k,\infty}(\Gamma)$ . The space  $M_k(\Gamma)$  can then be picked out inside  $M_{k,\infty}(\Gamma)$  by finding the functions that (essentially) satisfy the Cauchy-Riemann equations.

Put in this context, one can immediately generalize: what are the elements of  $M_{k,\infty}(\Gamma)$  (for some  $k$ ) that do not come from holomorphic functions on  $\mathfrak{h}$ ? How about analogously defined spaces for groups other than  $\mathrm{SL}_2(\mathbf{R})$ ? Both of these generalizations are important. The relevant space of functions are called “automorphic forms.” We will concentrate our exposition on the latter question of generalizing from  $\mathrm{SL}_2$  to other groups.

### 3. Automorphic Forms

So that the reader can get some sense of the broader picture, let us briefly say a bit about automorphic forms more generally. As mentioned above, one arrives at the notion of automorphic forms by appropriately replacing the pair  $(\mathrm{SL}_2(\mathbf{R}), \Gamma)$  by other pairs  $(G, \Gamma_G)$  in the definition of  $M_{k,\infty}(\Gamma)$  above, where  $G$  is a reductive Lie group (reviewed momentarily) and  $\Gamma$  is an appropriate discrete subgroup of  $G$ .

**3.1. Reductive groups.** At first pass, reductive groups are perhaps best understood by example, as opposed to by definition. Set  $I_{p,q} = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$  and  $J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$ . Some examples of reductive Lie groups  $G$  are:

- $\mathrm{GL}_n(\mathbf{R})$ ,
- $\mathrm{SL}_n(\mathbf{R})$ ,
- $\mathrm{SO}(p, q) = \{g \in \mathrm{SL}_{p+q}(\mathbf{R}) : g I_{p,q} g^t = I_{p,q}\}$ , the special orthogonal group of a vector space with a non-degenerate quadratic form of signature  $(p, q)$ ,
- $U(p, q) = \{g \in \mathrm{GL}_{p+q}(\mathbf{C}) : g I_{p,q} \bar{g}^t = I_{p,q}\}$ , the similarly defined unitary group, and
- $\mathrm{Sp}_{2n}(\mathbf{R}) = \{g \in \mathrm{SL}_{2n}(\mathbf{R}) : g J_n g^t = J_n\}$ , the group of automorphisms of a  $2n$ -dimensional real vector space preserving a nondegenerate alternating bilinear form.

It is good to compare and contrast reductive groups with compact Lie groups. Recall that the compact finite-dimensional Lie groups are classified, and many come in infinite families such as the special orthogonal group  $\mathrm{SO}(n)$  and the unitary group  $U(n)$ . If  $K$  is a compact Lie group, then

- any finite-dimensional representation of  $K$  can be written as a finite sum of irreducible representations, and
- every irreducible (unitary) representation of  $K$  is finite dimensional.

A reductive Lie group is a group  $G$  that possesses this first property, but almost never possesses the second: if  $G \neq \{1\}$  is a *noncompact* reductive Lie group, there are no faithful finite-dimensional irreducible unitary representations<sup>1</sup> of  $G$ . All compact Lie groups are reductive, but automorphic forms are generally only interesting to consider for noncompact groups  $G$ . The reader should keep in mind  $G = \mathrm{SL}_n(\mathbf{R})$  and  $G = \mathrm{SO}(p, q)$  with  $pq \neq 0$  as examples of reductive groups.

**3.2. Spaces of functions.** For reductive groups  $G$ , most of the discrete subgroups  $\Gamma_G$  that arise in the theory of automorphic forms are what are known as *arithmetic* subgroups, and they have the property that  $\Gamma_G \backslash G$  is sometimes (but not usually) compact, but always has finite  $G$ -invariant volume. Without giving a precise definition, one should think of  $\Gamma_G$  as being defined in the same way as  $G$  except with the real numbers  $\mathbf{R}$  replaced by the integers  $\mathbf{Z}$ . For example, if  $G = \mathrm{SL}_n(\mathbf{R})$ , then  $\Gamma_G$  could be  $\mathrm{SL}_n(\mathbf{Z})$  ( $= \mathrm{SL}_n(\mathbf{R}) \cap M_n(\mathbf{Z})$ ) or one of its finite index subgroups.

With  $G$  and  $\Gamma_G$  fixed, one can consider various spaces of functions on the manifold  $\Gamma_G \backslash G$ , for example  $L^{2,\infty}(\Gamma_G \backslash G)$ , the space of smooth, complex-valued  $L^2$ -functions on  $\Gamma_G \backslash G$ . One also considers the space  $L^{mg,\infty}(\Gamma_G \backslash G)$ , the smooth, moderate-growth functions on  $G$ . The group  $G$  acts on  $L^{2,\infty}(\Gamma_G \backslash G)$  via right translation:  $(g \cdot \varphi)(x) = \varphi(xg)$ , and this action affords an infinite-dimensional unitary representation of  $G$ .

<sup>1</sup>If  $G = G_1 \times G_2$  with  $G_1$  compact, then  $G$  has nonfaithful irreducible unitary representations via projection onto  $G_1$ .

The automorphic forms  $\mathcal{A}(G; \Gamma_G)$  are, by definition, a certain dense, nice subspace of  $L^{mg, \infty}(\Gamma_G \backslash G)$ . To pick out this subspace, one imposes two extra conditions on functions  $\varphi \in L^{mg, \infty}(\Gamma_G \backslash G)$ . One condition is the analogue of condition 2 in Definition 1 above, and the other says that  $\varphi$  satisfies sufficiently many differential equations that come from the group theory of  $G$ .

For the sake of completeness, we now spell out these conditions, although the reader might wish to skip to section 4. To make explicit the first condition, recall that every reductive Lie group  $G$  has a maximal<sup>2</sup> compact subgroup  $K$ , and any two maximal compact subgroups of  $G$  are conjugate. Fixing one such  $K$ , we can restrict the  $G$ -action to  $K$  and find the functions  $\varphi \in L^{mg, \infty}(\Gamma_G \backslash G)$  whose translates by  $K$  make up a finite-dimensional vector space. These functions are called  $K$ -finite. This is the analogue of condition 2 in Definition 1 above.

The second condition involves the universal enveloping algebra  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  of the complexified Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  of  $G$ . This infinite-dimensional noncommutative  $\mathbb{C}$ -algebra acts on  $C^\infty(\Gamma_G \backslash G)$  by differentiating the right-translation action of  $G$ . The center  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$  of  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  is a commutative  $\mathbb{C}$ -algebra of finite type. For example, if  $\mathfrak{g} = \mathfrak{gl}_n$ , then

$$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{C}[t_1, \dots, t_n]^{S_n},$$

the symmetric polynomials in the variables  $t_1, t_2, \dots, t_n$ . One can further consider functions  $\varphi$  that are annihilated by an ideal of  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$  of finite codimension. Or in other words, for which  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \cdot \varphi$  is finite dimensional. Such functions are said to be  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite.

Putting everything together, set  $\mathcal{A}(G; \Gamma_G)$  as the space of smooth, moderate growth functions  $\varphi : \Gamma_G \backslash G \rightarrow \mathbb{C}$  that are  $K$ -finite and  $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ -finite. These are the  $\Gamma_G$ -invariant automorphic forms on  $G$ .

## 4. $L$ -Functions

One of the most important aspects of automorphic forms is that they connect apparently disparate areas of mathematics. One way of making this connection precise is to use  $L$ -functions.

The reader is probably familiar with the Riemann zeta function  $\zeta(s)$ . Recall that if  $s \in \mathbb{C}$  has  $\operatorname{Re}(s) > 1$ , then one defines

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p (1 - p^{-s})^{-1},$$

where the latter product is over the prime numbers  $p$ .  $L$ -functions are generalizations of the Riemann zeta function. It is a famous fact that  $\zeta(s)$  has a meromorphic continuation to the complex plane with a simple pole at  $s = 1$ , and satisfies a functional equation relating  $s$  to  $1 - s$ . More specifically, define  $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ ; then  $\xi(s)$  has

<sup>2</sup>I.e., a compact subgroup that is not strictly included in any other compact subgroup.

simple poles at  $s = 1$  and  $s = 0$ , is analytic elsewhere in the complex plane, and satisfies the exact functional equation  $\xi(s) = \xi(1 - s)$ .

More generally, if  $\{a_n\}_{n \geq 1}$  is a sequence of complex numbers, and if the sum  $\sum_{n \geq 1} a_n n^{-s}$  converges absolutely for  $\operatorname{Re}(s) > s_0$ , one can then consider this function of  $s$ ; it is called a Dirichlet series. An  $L$ -function is a very special type of Dirichlet series, and they are assigned to all sorts of “number-theoretic data,” including modular forms and automorphic forms. Besides being representable as a sum  $\sum_{n \geq 1} a_n n^{-s}$ ,  $L$ -functions share two other properties with the Riemann zeta function:

1. They can be representable as a product over primes  $p$ , called an “Euler product.”
2. They satisfy a functional equation relating  $s$  to  $k - s$  for some real number  $k$ .

The Euler product comes from a multiplicativity property of the numbers  $a_n$ . Specifically, suppose that  $a_1 = 1$  and  $a_{nm} = a_n a_m$  when  $n$  and  $m$  are relatively prime. Then, in the range of absolute convergence, one has

$$\begin{aligned} \sum_{n \geq 1} a_n n^{-s} &= \prod_p (1 + a_p p^{-s} + a_{p^2} p^{-2s} + \dots) \\ &= \prod_p \left( \sum_{k \geq 0} a_{p^k} p^{-ks} \right). \end{aligned}$$

**4.1. Automorphic  $L$ -functions.** Denote by  $\chi_4$  the unique nontrivial character on  $(\mathbb{Z}/4\mathbb{Z})^\times$ , i.e.,  $\chi_4(n) = 1$  if  $n \equiv 1 \pmod{4}$  and  $\chi_4(n) = -1$  if  $n \equiv 3 \pmod{4}$ . The simplest example of an  $L$ -function, beyond the Riemann zeta function, is

$$\begin{aligned} L(\chi_4, s) &:= 1 - 3^{-s} + 5^{-s} - 7^{-s} + 9^{-s} + \dots \quad (3) \\ &= \prod_{p \equiv 1(4)} (1 - p^{-s})^{-1} \prod_{p \equiv 3(4)} (1 + p^{-s})^{-1}. \end{aligned}$$

More generally, suppose  $N \geq 1$  is a positive integer and  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$  is a character, i.e.,  $\chi(m_1 m_2) = \chi(m_1) \chi(m_2)$  for integers  $m_1, m_2$  prime to  $N$ . Then one can extend  $\chi$  to a function  $\mathbb{Z} \rightarrow \mathbb{C}$  as  $\chi(m) = 0$  if  $m$  and  $N$  share a nontrivial common factor, and define

$$L(\chi, s) = \sum_{n \geq 1} \chi(n) n^{-s} = \prod_{(p, N) = 1} (1 - \chi(p) p^{-s})^{-1}.$$

When  $\chi = \chi_4$  one obtains the  $L$ -function (3).

It is an important fact that these  $L$ -functions have meromorphic (analytic) continuation and functional equation. In fact, Dirichlet [Dir37] used the properties of these  $L$ -functions to prove that if  $a$  and  $N$  are relatively prime positive integers, then there are infinitely many prime numbers congruent to  $a$  modulo  $N$ .

The above examples have the property that the coefficients  $a_n$  are *completely multiplicative*, i.e.,  $a_{mn} = a_m a_n$  for all  $m, n$ , not just when  $m$  and  $n$  are relatively prime. A more complicated example comes from the modular

forms. Suppose  $f(z) = \sum_{n \geq 1} a_n q^n$  is a modular form of weight  $k$ . For example,  $f(z)$  could be Ramanujan's function  $\Delta(z)$ . The space of cusp forms  $S_k(\mathrm{SL}_2(\mathbf{Z}))$  has a preferred basis, called the basis of Hecke eigenforms, and if  $f(z)$  is such an eigenform, then one can associate an  $L$ -function to  $f$ . Specifically, suppose that  $f(z)$  is normalized so that  $a_1 = 1$ . Then one defines  $L(f, s) = \sum_{n \geq 1} a_n n^{-s}$ . Setting  $\Lambda(f, s) = \pi^{-s} \Gamma(s) L(f, s)$ , then this function satisfies the functional equation  $\Lambda(f, s) = \Lambda(f, k - s)$ . In the special case of modular forms  $f$  as above, the Euler product expansion of  $L(f, s)$  looks as follows:

$$L(f, s) = \prod_p (1 - a_p p^{-s} + p^{k-1} p^{-2s})^{-1}.$$

The above  $L$ -function is defined as a product over all the prime numbers  $p$ . It is technically very convenient in the theory of automorphic forms to also consider products as above over all by finitely many primes, whereby obtaining what are called partial  $L$ -functions. Throughout the rest of the text we blur the distinction between these different scenarios: some of the statements below are only correct when  $L$ -functions are replaced by partial  $L$ -functions.

**4.2. Motivic  $L$ -functions.** One can associate  $L$ -functions not only to modular forms, but more generally, to automorphic forms  $\varphi$ . However, one of the most basic ways to construct  $L$ -functions is via Galois theory. Suppose  $K$  is a number field, with  $K/\mathbf{Q}$  a Galois extension, and  $\rho : \mathrm{Gal}(K/\mathbf{Q}) \rightarrow \mathrm{GL}_n(\mathbf{C})$  is a representation. One can construct an  $L$ -function  $L(\rho, s)$  associated to this data as follows:

$$L(\rho, s) = \prod_p \det(1 - \rho(\mathrm{Frob}_p) p^{-s})^{-1}.$$

Here  $\mathrm{Frob}_p$  is a certain conjugacy class of  $\mathrm{Gal}(K/\mathbf{Q})$  associated to the prime  $p$  of  $\mathbf{Z}$ , and note that the factor  $\det(1 - \rho(\mathrm{Frob}_p) p^{-s})$  is well-defined because the determinant makes the choice of representative in the conjugacy class irrelevant.

These  $L$ -functions  $L(\rho, s)$  are known to have meromorphic continuation in  $s$ , as a consequence of results of Dirichlet and Brauer. Their holomorphy in  $s$  is wide open.

**Conjecture 2** (The Artin conjecture). *Suppose  $K/\mathbf{Q}$  is a Galois extension and  $\rho : \mathrm{Gal}(K/\mathbf{Q}) \rightarrow \mathrm{GL}_n(\mathbf{C})$  is an irreducible, nontrivial representation. Then  $L(\rho, s)$  is an entire holomorphic function of  $s$ .*

Algebraic-geometric objects also have associated  $L$ -functions. More precisely, one can associate  $L$ -functions to the étale cohomology of smooth projective varieties over a number field. The first important example is that of an elliptic curve  $E$  over  $\mathbf{Q}$ . For all but finitely many primes  $p$ , one can reduce  $E$  modulo  $p$  to obtain a curve over the finite field  $\mathbf{F}_p$ . Counting points modulo  $p$ , define the

integer  $a_p(E)$  as  $a_p(E) = p + 1 - \#E(\mathbf{F}_p)$ .<sup>3</sup> One can package the integers  $a_p(E)$  together as

$$L(E, s) = \prod_p (1 - a_p(E) p^{-s} + p^{1-2s})^{-1}. \quad (4)$$

This is the  $L$ -function of an elliptic curve  $E$ .

Note that the form of the Euler product for  $L(E, s)$ , i.e., the right-hand side of (4), is the same as that of a weight-2 modular form. One says that the modular form  $f$  of weight 2 is associated to  $E$  if  $L(f, s) = L(E, s)$ ; equivalently, if  $a_p(f) = a_p(E)$  for all  $p$ . That every rational elliptic curve is associated to some modular form of weight 2 was called the Taniyama-Shimura-Weil conjecture, and was proved (for semistable<sup>4</sup>) elliptic curves in the famous papers [Wil95] and [TW95] of Wiles and Taylor-Wiles. We direct the reader to Ribet's article [Rib95] for the history of this problem and its connection to Fermat's Last Theorem.

Suppose  $X$  is a smooth projective algebraic variety over  $\mathbf{Q}$  of dimension  $n$  and  $0 \leq k \leq 2n$  is a nonnegative integer. Then one can assign an  $L$ -function  $L(H^k(X), s)$  to the étale cohomology  $H_{\mathrm{et}}^k(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$ . For example, if  $X = E$  is an elliptic curve and  $k = 1$ , then  $L(H^1(E), s) = L(E, s)$  as defined above.

As mentioned above, it is via  $L$ -functions that number theorists find a bridge between disparate areas of mathematics: algebraic geometry and automorphic forms. For example, here is an important conjecture.

**Conjecture 3** (Langlands [Lan80]). *Suppose  $X$  is a smooth projective variety over  $\mathbf{Q}$  and  $0 \leq k \leq 2 \dim(X)$ . Suppose that the étale cohomology  $H_{\mathrm{et}}^k(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)$  affords an irreducible representation of  $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  of dimension  $N$ . Then there is an automorphic form  $\varphi$  on  $\mathrm{GL}_N$  so that  $L(H^k(X), s) = L(\varphi, s)$ .*

## 5. Exceptional Groups

Having defined modular forms and automorphic forms, let us now turn our attention to exceptional groups. The Lie groups one meets in practice are often defined as automorphisms of particular algebraic structures. For example,  $\mathrm{Sp}(2n)$  is the automorphism group fixing a symplectic form on a  $2n$ -dimensional vector space, and  $\mathrm{SO}(p, q)$  is the automorphism group fixing a quadratic form of signature  $(p, q)$ . These algebraic structures and associated Lie groups come in infinite families, e.g.,  $\mathrm{Sp}(2n)$  for  $n = 1, 2, 3, 4, \dots$ . Because of this fact, they are called *classical groups*.

It turns out that there are very interesting, *exceptional* algebraic structures. The exceptional algebraic groups are defined to be automorphism groups of these structures. More precisely, suppose  $V$  is a vector space and  $\{\xi_\ell\}_\ell$  is

<sup>3</sup>It is a theorem of Weil that  $a_p(E)$  satisfies the bound  $|a_p(E)| \leq 2\sqrt{p}$ . In other words,  $\#E(\mathbf{F}_p)$  is approximately  $\#\mathbf{P}^1(\mathbf{F}_p)$ , and  $a_p(E)$  measures the discrepancy.

<sup>4</sup>The semistability condition was later removed in work of Breuil-Conrad-Diamond-Taylor.



a finite list of tensors, i.e.,  $\xi_\ell \in \bigotimes_k \text{Sym}^{\alpha_k}(\bigwedge^k V)$ . Associated to this data, one can consider the group  $G \subseteq \text{GL}(V)$  that fixes the tensors  $\xi_\ell$  for all  $\ell$ . For example, if  $V$  is  $2n$ -dimensional, then for an appropriate  $\omega \in \bigwedge^2 V$ ,  $\text{Sp}(2n)$  is the subgroup of  $\text{GL}(V)$  that fixes  $\omega$ .

If one writes down some random collection of tensors  $\xi_\ell$  as above, then usually the group  $G$  will be trivial, i.e.,  $G = \{1\}$ . However, it sometimes (very rarely) happens that one can write down tensors  $\xi_\ell$  so that  $G$  is nontrivial, and in fact has positive dimension. When these are not the classical groups, one gets the exceptional groups. In this section, we will define these algebraic structures (the  $\xi_\ell$ ) and their automorphism groups.

**5.1. The octonions and  $G_2$ .** We begin with the simplest case, that of the octonions and the exceptional group  $G_2$ . The octonions are an 8-dimensional  $\mathbf{R}$ -vector space  $\Theta$ , that comes equipped with a multiplication  $\Theta \otimes \Theta \rightarrow \Theta$ , written as  $x, y \mapsto xy$ . This multiplication is neither commutative nor associative. However, it does have some redeeming qualities.

First, there is a multiplicative identity  $1 \in \Theta$ , and  $x \cdot 1 = 1 \cdot x = x$  for all  $x \in \Theta$ . Most importantly, there is a quadratic norm  $n_\Theta : \Theta \rightarrow \mathbf{R}$  satisfying  $n_\Theta(xy) = n_\Theta(x)n_\Theta(y)$ . The group  $G_2$  is defined as the group fixing this multiplication on  $\Theta$ :

$$G_2 = \{g \in \text{GL}(\Theta) : g(1) = 1 \text{ and } g(x \cdot y) = gx \cdot gy\}. \quad (5)$$

To make this more concrete, we give the reader an explicit description of  $\Theta$ . Namely, one defines  $\Theta = M_2(\mathbf{R}) \oplus M_2(\mathbf{R})$ , a direct sum of two copies of the  $2 \times 2$ -matrices. Suppose  $(x_1, y_1)$  and  $(x_2, y_2)$  are in  $\Theta$ . Then  $n_\Theta(x_1, y_1) = \det(x_1) - \det(y_1)$  is the quadratic norm. The multiplication is defined as

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + y_2^*y_1, y_2x_1 + y_1x_2^*).$$

Here if  $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is a  $2 \times 2$  matrix, then  $m^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  so that  $m + m^* = \text{tr}(m)1_2$  and  $mm^* = \det(m)1_2$ . The involution  $*$  on  $M_2(\mathbf{R})$  extends to one on  $\Theta$ , with similar properties:  $(x, y)^* = (x^*, -y)$ . One has  $(x, y) + (x, y)^* = \text{tr}(x)1$  and  $(x, y)(x, y)^* = n_\Theta((x, y))$ .

The group  $G_2$  defined by (5) is a noncompact Lie group of dimension 14. There is also a compact form of the group  $G_2$ , which is defined as the automorphisms of an algebra  $\Theta^c$ , closely related to  $\Theta$ .

To define  $\Theta^c$ , first recall Hamilton's quaternions

$$\mathbf{H} = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$$

with multiplication  $ij = k$ ,  $jk = i$ ,  $ki = j$ . Denote by  $*$  :  $\mathbf{H} \rightarrow \mathbf{H}$  the involution on  $\mathbf{H}$  defined as

$$(w + xi + yj + zk)^* = w - xi - yj - zk,$$

where  $w, x, y, z \in \mathbf{R}$ . Now define  $\Theta^c = \mathbf{H} \oplus \mathbf{H}$  with multiplication

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_2^*y_1, y_2x_1 + y_1x_2^*).$$

The algebra  $\Theta^c$  is isomorphic to  $\Theta$  over  $\mathbf{C}$ , i.e.,  $\Theta^c \otimes \mathbf{C} \simeq \Theta \otimes \mathbf{C}$ . The group  $G_2^c$  is

$$G_2^c = \{g \in \text{GL}(\Theta^c) : g(1) = 1 \text{ and } g(x \cdot y) = gx \cdot gy\}. \quad (6)$$

**5.2. The groups  $F_4$  and  $E_6$ .** The group  $G_2$  is the smallest of the exceptional groups. The other exceptional groups are labeled  $F_4, E_6, E_7, E_8$ . They sit in a chain  $G_2 \subseteq F_4 \subseteq E_6 \subseteq E_7 \subseteq E_8$ , and are of dimensions 14, 52, 78, 133, 248, respectively.

We now define  $F_4$  and  $E_6$ . To do this, let  $H_3(\Theta)$  denote the Hermitian  $3 \times 3$  matrices with coefficients in the octonions  $\Theta$ . That is,

$$H_3(\Theta) = \left\{ \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} : c_j \in \mathbf{R}, x_k \in \Theta \right\}.$$

Like the octonions, the 27-dimensional vector space  $H_3(\Theta)$  has an algebraic structure that has surprisingly many automorphisms. More precisely,  $H_3(\Theta)$  has a cubic determinant map: if  $X \in H_3(\Theta)$ ,

$$X = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix},$$

set

$$\det(X) = c_1c_2c_3 - c_1n_\Theta(x_1) - c_2n_\Theta(x_2) - c_3n_\Theta(x_3) + \text{tr}_\Theta((x_1x_2)x_3).$$

The group  $E_6$  is defined as the subgroup of  $\text{GL}(H_3(\Theta))$  that fixes this determinant map:

$$E_6 = \{g \in \text{GL}(H_3(\Theta)) : \det(gX) = \det(X) \forall X \in H_3(\Theta)\}.$$

The group  $F_4$  is the subgroup of  $E_6$  that fixes the element  $1_3 = \text{diag}(1, 1, 1) \in H_3(\Theta)$ .

More precisely, there are different "forms" of these exceptional groups. Recall that if  $H_1$  and  $H_2$  are real Lie groups, then one says that  $H_2$  is a "real form" of  $H_1$  if  $\text{Lie}(H_1) \otimes \mathbf{C} \simeq \text{Lie}(H_2) \otimes \mathbf{C}$ . Every reductive Lie group has a real form that is compact, and also has a real form that is split. The groups  $G_2, F_4, E_6$  that we described above are these split forms. Replacing  $H_3(\Theta)$  with  $H_3(\Theta^c)$  in the definitions of  $F_4$  and  $E_6$  yields different real forms of these groups. By contrast with the case of  $G_2$ , the forms of  $F_4$  and  $E_6$  defined in this way are not compact.

**5.3. The group  $E_7$ .** To define the group  $E_7$ , one proceeds as follows. For ease of notation, set  $J = H_3(\Theta)$ . This notation comes from the fact that  $J$  is what is called a "Jordan algebra." Now set  $W = \mathbf{R} \oplus J \oplus J^\vee \oplus \mathbf{R}$ , where  $J^\vee$  is the linear dual of  $J$ . This space  $W$  is 56-dimensional, and comes equipped with a nondegenerate symplectic form

and a particular quartic form  $Q$  (homogeneous of degree four) which were defined by Freudenthal. The group  $E_7$  can be defined as the subgroup of  $\mathrm{GL}(W)$  that fixes these two forms on  $W$ .

**5.4. The exceptional group  $E_8$ .** Every reductive Lie group acts on its Lie algebra by automorphisms. And conversely, if  $\mathfrak{g}$  is a (reductive) Lie algebra, then one can define a corresponding Lie group  $G$  as

$$G = \{g \in \mathrm{GL}(\mathfrak{g}) : [gX, gY] = g[X, Y] \ \forall X, Y \in \mathfrak{g}\}.$$

All the reductive Lie algebras with the exception of  $E_8$  have a nontrivial finite-dimensional representation on a vector space smaller than its Lie algebra. For  $E_8$ , its smallest nontrivial representation is the adjoint action on its Lie algebra, so in a sense one must define  $E_8$  through this action. Fortunately, it is not so difficult to define the Lie algebra  $\mathfrak{e}_8$ .

One can do this as follows [Rum97]. Denote by  $V_3$  the three-dimensional representation of  $\mathfrak{sl}_3$ . Then

$$\mathfrak{e}_8 = (\mathfrak{sl}_3 \oplus \mathfrak{e}_6) \oplus V_3 \otimes J \oplus (V_3 \otimes J)^\vee. \quad (7)$$

This is a  $\mathbf{Z}/3$ -grading with  $\mathfrak{sl}_3 \oplus \mathfrak{e}_6$  in degree zero,  $V_3 \otimes J$  in degree one, and  $(V_3 \otimes J)^\vee$  in degree two. The subalgebra  $\mathfrak{sl}_3 \oplus \mathfrak{e}_6$  acts on  $V_3 \otimes J$  and  $(V_3 \otimes J)^\vee$  in the obvious manner. For an analogous construction of the Lie algebra  $\mathfrak{g}_2$ , see [FH91, p. 358]. In fact, one can construct all the exceptional Lie algebras using an analogue of (7).

## 6. Automorphic Forms on Exceptional Groups

Having described exceptional groups, and automorphic forms in general, let us now consider automorphic forms on exceptional groups.

**6.1. Holomorphic modular forms on  $E_7$ .** We first turn our attention to the exceptional group  $E_7$ , as it possesses “modular forms” completely analogous to the classical holomorphic modular forms that we described in the first section. Let  $J = H_3(\mathcal{O}^c)$ , and let  $G$  denote the exceptional group of type  $E_7$  defined as in subsection 5.3. The group  $G$  is neither split nor compact. For this group, if  $K_G$  denotes the maximal compact of  $G(\mathbf{R})$ , the symmetric space  $G(\mathbf{R})/K_G$  has a  $G(\mathbf{R})$ -invariant complex structure.

Denote by  $J_+$  the subset of  $J$  consisting of elements of the form  $Y^2$  for some  $Y \in J$  with  $\det(Y) \neq 0$ . (If  $Y \in J$ , then its square  $Y^2$ —in the sense of usual matrix multiplication—is also an element of  $J$ .) The symmetric space  $G(\mathbf{R})/K_G$  can be identified with

$$\mathfrak{h}_J = \{Z = X + iY : X \in J, Y \in J_+\}.$$

Consequently, there is an action of  $G(\mathbf{R})$  on  $\mathfrak{h}_J$ , which one should think of as generalized (exceptional) linear fractional transformations.

As the reader will no doubt have noticed, the space  $\mathfrak{h}_J$  is very similar to the complex upper half-space  $\mathfrak{h}$ . And indeed, the group  $G$  possesses special automorphic forms—

the modular forms—that are close cousins of the classical modular forms on  $\mathrm{SL}_2$  defined in section 2. Just as in the classical case, there is a function  $j : G(\mathbf{R}) \times \mathfrak{h}_J \rightarrow \mathfrak{h}_J$  satisfying  $j(g_1 g_2, Z) = j(g_1, g_2 Z) j(g_2, Z)$  for all  $g_1, g_2 \in G(\mathbf{R})$ . To define the modular forms, suppose  $\Gamma$  is an arithmetic subgroup of  $G(\mathbf{R})$  and  $k \geq 0$  is an integer. A *modular form* of weight  $k$  for  $\Gamma$  is a holomorphic function  $f : \mathfrak{h}_J \rightarrow \mathbf{C}$  on  $\mathfrak{h}_J$  satisfying  $f(\gamma Z) = j(\gamma, Z)^k f(Z)$  for all  $\gamma \in \Gamma$  and for which the  $\Gamma$ -invariant function

$$|\det(Y)^{k/2} f(Z)| = |j(g, i1_3)^{-k} f(g \cdot i1_3)|$$

is of moderate growth on  $G(\mathbf{R})$ .

These modular forms have a “ $q$ -expansion,” completely analogous to the Fourier expansion of classical modular forms. Assume for simplicity that  $\Gamma = G(\mathbf{Z})$  is the maximal arithmetic subgroup of  $G(\mathbf{R})$ . Fix a certain lattice  $J_{\mathbf{Z}}$  in  $J$ , and let  $J_{\mathbf{Z}}^+$  denote the intersection of  $J_{\mathbf{Z}}$  with the closure of  $J_+$  in  $J_{\mathbf{R}}$ . If  $f$  is a modular form for  $\Gamma$ , then

$$f(Z) = \sum_{T \in J_{\mathbf{Z}}^+} a_f(T) e^{2\pi i \mathrm{tr}(TZ)} \quad (8)$$

for complex numbers  $a_f(T)$ ; this is its Fourier expansion.

Some examples of modular forms on  $G$  can be found in work of Bailly [Bai70], Kim [Kim93], and Kim-Yamauchi [KY16]. However, let us reiterate that very little is known about them. For example, the  $L$ -functions  $L(f, s)$  of modular forms on  $G$  are known to have meromorphic continuation in  $s$  [Lan71]. However, the conjectured finer analytic properties of these  $L$ -functions, such as the finiteness of its poles, are not known.

**6.2. Quaternionic modular forms.** Of the exceptional Dynkin types,  $E_7$  is special in that the types  $G_2, F_4, E_6$ , and  $E_8$  do not have real forms  $G$  so that the associated symmetric space  $X_G$  has “modular forms” similar to the classical holomorphic modular forms.<sup>5</sup> It turns out there is a certain type of nonholomorphic “quaternionic” modular forms on one of the real forms of each of the exceptional Dynkin types

$$G_2 \subseteq F_4 \subseteq E_6 \subseteq E_7 \subseteq E_8. \quad (9)$$

These were defined by Gross and Wallach [GW96] and Gan-Gross-Savin [GGS02], and have subsequently been studied by Gan, Loke, Weissman, and the author. Among their nice features are that they have a Fourier expansion with some of the same good properties of classical modular forms [Pol20], and that they behave well under pull-back in the sequence (9).

Instead of being  $\mathbf{C}$ -valued functions on an exceptional group  $G$ , quaternionic modular forms are naturally valued in the finite-dimensional representations of the compact Lie group  $\mathrm{SU}(2)/\{\pm 1\}$ . More precisely, if  $n \geq 1$  is

<sup>5</sup>There is a real form of  $E_6$  whose symmetric space has hermitian structure, but the automorphic forms on this  $E_6$  do not have Fourier expansions similar to that of (8).

an integer, a quaternionic modular form of weight  $n$  on a quaternionic exceptional group  $G$  is a function  $f : \Gamma \backslash G \rightarrow \text{Sym}^{2n}(\mathbb{C}^2)$  satisfying conditions analogous to those of Definition 1, together with being annihilated by a specific first-order linear differential operator  $\mathcal{D}_n$ , i.e.,  $\mathcal{D}_n f = 0$ . Here  $\text{Sym}^{2n}(\mathbb{C}^2)$  is the  $(2n+1)$ -dimensional irreducible representation of  $\text{SU}(2)$ . Similar to how the Cauchy-Riemann solutions force the nice form of the Fourier expansion of holomorphic modular forms, being annihilated by  $\mathcal{D}_n$  forces the quaternionic modular form  $f$  to have a robust Fourier expansion. The robustness of the Fourier expansion allows one to ask (and answer<sup>6</sup>) questions such as “Do there exist quaternionic modular forms on  $E_8$  and  $G_2$  whose Fourier coefficients are all integers?”

One beautiful example of such a quaternionic modular form is a function  $\theta_{\min}$  on  $E_8$  of weight 4, defined and studied by Gan [Gan00]. The function  $\theta_{\min}$  can be thought of as an  $E_8$ -analogue of the classical theta function  $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ .

Another source of examples of these quaternionic modular forms comes from Eisenstein series associated to holomorphic cusp forms. Precisely, if  $n \geq 6$  and  $\varphi$  is a classical holomorphic modular cusp form of weight  $3n$ , then one can construct out of  $\varphi$  an Eisenstein series  $E(\varphi)$  on  $G_2$  that gives a modular form of weight  $n$ . The fact that  $\varphi$  is holomorphic is crucial to  $E(\varphi)$  being annihilated by  $\mathcal{D}_n$ .

### 6.3. Automorphic forms on split exceptional groups.

Our treatment of automorphic forms in this article has been biased towards those that are most similar to the classical holomorphic modular forms that one first meets, in terms of having a robust Fourier expansion. This bias is a bit out-of-line with the development of the theory of automorphic forms. In this final subsection, we go back to the mainstream and highlight some work on automorphic forms on split exceptional groups, for which there is not a similarly robust theory of the Fourier expansion. In no way should our selection of topics here be considered exhaustive.

**6.3.1.  $L$ -functions.** Relatively speaking, much more is known about the  $L$ -functions of automorphic forms on split exceptional groups than on their nonsplit counterparts. To explain the state of affairs in just a little more detail, recall that split reductive algebraic groups  $G$  possess *generic* cusp forms. Roughly speaking, a generic cusp form is a cuspidal automorphic form which possesses nontrivial averages over large unipotent groups  $U'$ . That is, if  $\varphi$  is cuspidal and generic, then the integral  $\int_{(U' \cap \Gamma) \backslash U'} \varphi(ug) du$  is not identically 0 as a function of  $g \in G$ . Here, for the more knowledgeable reader, we point out that  $U' = [U, U]$  is the commutator subgroup of a maximal unipotent group  $U$ .

Generic cuspidal automorphic forms are very far from being holomorphic: many *one-dimensional* unipotent averages of  $\varphi$  vanish identically if  $\varphi$  is a holomorphic cuspidal modular form. Similarly, certain three-dimensional unipotent averages of  $\varphi$  vanish identically if  $\varphi$  is a quaternionic cuspidal modular form, in the sense of subsection 6.2.

The  $L$ -functions of generic cuspidal automorphic forms on the split exceptional groups  $G_2, F_4, E_6$ , and  $E_7$  have been studied by Ginzburg, Ginzburg–Rallis, and Piatetski-Shapiro–Rallis–Schiffmann. For example, in [Gin95] it is proved that the so-called standard  $L$ -function  $L(\pi, \text{Std}, s)$  of cuspidal generic automorphic representations on split  $E_7$  has meromorphic analytic continuation with at most two simple poles.

**6.3.2. Theta functions.** As a rule of thumb, constructing automorphic forms is difficult. The first explicit examples one tends to write down are often constructed using  $\theta$  functions and their generalizations. For example, taking the classical  $\theta$  function  $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$  and considering  $\theta(z)^4$  gives the modular form  $\theta_4(z)$  of section 1.

The function  $\theta(z)$  has the property that many of its Fourier coefficients are 0: the coefficient of  $q^N$  is nonzero only when  $N$  is a square. Many reductive groups, and in particular the split exceptional groups  $F_4, E_6, E_7$ , and  $E_8$ , possess analogues of this function  $\theta(z)$ . Such an analogue goes by the moniker of a *minimal* automorphic form, and possesses the property that very many of its generalized Fourier coefficients are 0. We direct the reader to work of Ginzburg–Rallis–Soudry [GRS97] and Ginzburg for results on these minimal automorphic forms.

Besides  $\theta$  functions themselves, one can use theta functions as integral kernel functions to create other automorphic forms. Without going into the details of this procedure, let us simply mention the very interesting examples of Rallis–Schiffmann, Li–Schwermer, and Gan–Gurevich–Jiang, which use such a construction to create special automorphic forms on the split exceptional group  $G_2$ .

**6.3.3. Further directions.** Finally, we mention that there has been some activity to relate the Fourier coefficients of automorphic functions on split exceptional groups to quantities that arise in string theory. We direct the interested reader to the book [FGKP18] and the numerous references contained therein.

## A. Technical Details on Exceptional Groups

In this appendix we include a few technical details on exceptional groups that might be useful to the interested reader.

**A.1. Real forms of exceptional groups.** In section 5 we defined various forms of the exceptional groups. We remark that the forms of  $G_2, F_4, E_6, E_7, E_8$  that are defined there are the simply connected algebraic groups. (The

<sup>6</sup>The answer is “Yes.”

algebraic groups  $G_2$ ,  $F_4$ , and  $E_8$  are both adjoint and simply connected, while the simply connected  $E_6$  and  $E_7$  have centers the cyclic groups of orders 3 and 2, respectively.) We connect what was written above to other standard notation for these groups.

- The real form of  $F_4$  defined in section 5 using  $H_3(\Theta)$  is the split form  $F_{4(4)}$  if  $\Theta$  is the split octonions and is the compact form  $F_{4(-52)}$  if defined using  $H_3(\Theta^c)$ .
- The real forms of  $E_6$  defined in section 5 are the split form if defined using  $H_3(\Theta)$  and the form  $E_{6(-26)}$  if defined using  $H_3(\Theta^c)$ .
- The real forms of  $E_7$  defined in section 5 are the split form if defined using  $J = H_3(\Theta)$  and the form  $E_{7(-25)}$  if defined using  $J = H_3(\Theta^c)$ .
- The real forms of  $E_8$  defined in section 5 are the split form if defined using  $J = H_3(\Theta)$  and the form  $E_{8(-24)}$  if defined using  $J = H_3(\Theta^c)$ .

In section 5 it was mentioned that every exceptional Lie algebra can be constructed in a form analogous to (7). To do this, one lets the Jordan algebra (or precisely, cubic norm structure)  $J$  vary, and the Lie algebra  $\mathfrak{e}_6$  is replaced with the Lie algebra  $\mathfrak{m}(J)$  of the group of determinant-preserving linear maps on  $J$ . One obtains a Lie algebra

$$\mathfrak{g}(J) = (\mathfrak{sl}_3 \oplus \mathfrak{m}(J)) \oplus V_3 \otimes J \oplus (V_3 \otimes J)^\vee, \quad (10)$$

where now  $J$  is an arbitrary cubic norm structure, instead of just the exceptional cubic norm structure  $H_3(\Theta^c)$ . If  $J$  varies over cubic norm structures with positive-definite trace forms, one constructs (uniformly) the so-called quaternionic forms of the exceptional groups. Details can be found in, for example, [Pol20]. With  $J = H_3(\mathbf{H})$ ,  $\mathfrak{g}(J)$  is the quaternionic  $\mathfrak{e}_7$  (also known as  $\mathfrak{e}_{7(-5)}$ ); if  $J = H_3(\mathbf{C})$ , then  $\mathfrak{g}(J)$  is the quasisplit and quaternionic  $\mathfrak{e}_6$  (also known as  $\mathfrak{e}_{6(2)}$ ); and if  $J = H_3(\mathbf{R})$ , then  $\mathfrak{g}(J)$  is the split  $\mathfrak{f}_4$ .

**A.2. The definition of  $E_7$ .** It is not difficult to be even more precise about the definition of the exceptional groups  $E_7$  and  $E_8$  given above. To do so, we require one additional piece of notation for the Jordan algebra  $J = H_3(\Theta)$ ; namely, a symmetric bilinear map  $\times : J \otimes J \rightarrow J^\vee$  defined as follows. Polarizing the determinant map  $\det : J \rightarrow \mathbf{R}$ , there is a symmetric trilinear map  $(\cdot, \cdot, \cdot)$  on  $J$  satisfying  $(X, X, X) = 6 \det(X)$ . Now, for  $X, Y$  in  $J$ , define  $X \times Y \in J^\vee$  (the linear dual of  $J$ ) via  $(X \times Y)(Z) = (X, Y, Z)$ . Set  $X^\# = \frac{1}{2}X \times X$ . There is an identically defined map  $\times : J^\vee \otimes J^\vee \rightarrow J$ .

With this bit of notation, we can now precisely define the symplectic and quartic forms on  $W_J$  fixed by  $E_7$ . For  $E_7$ , suppose  $(a, b, c, d) \in W$ , so that  $a, d \in \mathbf{R}$ ,  $b \in J$ , and  $c \in J^\vee$ , and similarly for  $(a', b', c', d')$ . Then the symplectic form  $\langle \cdot, \cdot \rangle$  on  $W$  is given by  $\langle (a, b, c, d), (a', b', c', d') \rangle = ad' - (b, c') + (b', c) - da'$  and the quartic form  $Q$  on  $W$  is given

by

$$Q((a, b, c, d)) = (ad - (b, c))^2 + 4a \det(c) + 4d \det(b) - 4(c^\#, b^\#).$$

**A.3. The definition of  $E_8$ .** It is also possible to concisely define the Lie bracket on  $\mathfrak{e}_8$ . Because  $V_3$  is a representation of  $\mathfrak{sl}_3$ , there are identifications  $\bigwedge^2 V_3 \simeq V_3^\vee$  and  $\bigwedge^2 V_3^\vee \simeq V_3$ . Now if  $u, u' \in V_3$  and  $X, X' \in J$ , then  $[u \otimes X, u' \otimes X'] = (u \wedge u') \otimes (X \times X')$ , considered as an element of  $(V_3 \otimes J)^\vee$ . The Lie bracket  $[\cdot, \cdot] : (V_3 \otimes J)^\vee \otimes (V_3 \otimes J)^\vee \rightarrow V_3 \otimes J$  is defined analogously. Finally, the map  $(V_3 \otimes J) \otimes (V_3 \otimes J)^\vee$  induced by the Lie bracket is defined using the maps  $V_3 \otimes V_3^\vee \simeq \text{End}(V_3) \rightarrow \mathfrak{sl}_3$  and a bilinear map  $\Phi^0 : J^\vee \otimes J \rightarrow \mathfrak{e}_6$  defined as

$$\Phi_{c,b}^0(X) = -c \times (b \times X) + (c, X)b + \frac{1}{3}(b, c)X.$$

Here  $c \in J^\vee$  and  $b, X \in J$ .

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# EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section. Next month's theme will be expanding your teaching repertoire.



## Opportunities For Diversity in the Classroom

### An Opportunity for Inclusion: A Course in the History of Mathematics that Includes Mathematical Contributions of Non-European Culture

*Candice Price*

#### Motivation

Throughout the course of history, mathematics has changed the way people view the world. A course in history of mathematics is a cross between a history course and a mathematics course, drawing on sciences, anthropology, sociology, and the languages, where assignments focus on writing rather than problem solving. In many American colleges and universities, this course only highlights the contributions of individuals with European roots, with an overwhelming number of them identifying as male. While this exclusion of important narrative supports the incorrect and naive view that other cultures did not contribute to the growth and expansion of mathematical thought, it also creates a cultural idea of what a mathematician looks like: white and male. As a Black woman, I understand the age-old issue of imposter syndrome and the phrase “we cannot be what we cannot see.” Because of this, I sought to create a course that included a critical look at the “culture” of the mathematics community in America while also highlighting the contributions of cultures and people outside of the Eurocentric gaze.

The year before planning to teach a History of Mathematics course at the University of San Diego, my home institution at the time, I discussed the course with Gail Tang, who was teaching the course at the University of La Verne. Tang shared several resources including the book *The Crest of the Peacock: Non-European Roots of Mathematics* by George Gheverghese Joseph. While looking through several

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resources as possible material for the course, I centered the flow and narrative of the course around this text.

### Designing My Course

A history of math course can be taught in many different ways. One path can be to focus on the chronological history of mathematics starting with the mathematics of ancient civilizations and ending at the beginning of the 19th century. Or one's course can take a biographical viewpoint and cover the contributions of well-known mathematicians. But without a concrete plan for inclusion, these two paths can often lead to a course with little to no discussion on the contributions of women and non-European mathematicians.

To mitigate the erasure of contributions from minoritized groups in mathematics, I consciously mapped out my course objectives to make sure that inclusion and diversity were included in my pedagogy. Thus, based on an email thread and conversations with Ron Buckmire, Omraya Ortega, and Gail Tang about readings for their history of math courses at their respective institutions, I mapped out a plan for how the course would go. One of my learning outcomes/goals for this course was to include discussions on the culture of mathematics from anthropological and sociological viewpoints. I also decided to focus on mathematical contributions of cultures outside of the Eurocentric viewpoint.

As a mathematician whose primary research area is not math history, I was able to learn along with the students and engage with them in meaningful discussions. As a Black female, I was inspired when learning about the contributions of Africans to mathematics.

### Designing a Future Course

While I truly enjoyed teaching this course, there is always room for reflection and improvement. Building on the way I taught the course previously, I would continue to utilize *The Crest of the Peacock* as the primary text while adding course readings from other books as well. *Mathematics Across Cultures: The History of Non-Western Mathematics*, edited by Helaine Selin, is a collection of short articles on the mathematical contributions of various cultures and includes discussions of mathematics through the lens of anthropology and communication studies. One popular textbook that would be useful for its exercises is Victor Katz's *A History of Mathematics*, especially those exercises in the first chapter on Ancient Mathematics. Other texts that could be included are *Africa and Mathematics: From Colonial Findings Back to the Ishango Rods* by Dirk Huylebrouck, which includes the rich history of mathematics across the African continent and African cultures, and *The Universal History of Numbers: From Prehistory to the Invention of the Computer* by Georges Ifrah, which has beautiful illustrations and is centered around mathematical tools and counting.

Because my course is designed as a discussion-based writing course, integrating the areas of anthropology, history, languages, and mathematics, the course can benefit from the inclusion of lectures by scholars of the civilizations and communities that the course features. The hope is that these scholars will share with the students what these communities were like during the time the mathematics being highlighted was developed and discovered.

In my previous course, I included a guest lecturer during the sections discussing contributions of African cultures to mathematics. African History scholar T. J. Tallie guest lectured and taught the students how to count and say common phrases like "Hello" and the days of the week in isiZulu. Tallie also discussed with the students how colonization impacted the mathematics of different parts of Africa. For example, the introduction of written language led to some spoken words being lost or having their meaning changed. While I know this is not limited to just mathematics—Tallie shared a great anecdote with us about how there was no word for "Sunday" but that one was created to mean "the day you go to church"—one may never fully know what mathematical knowledge was forgotten or hidden or lost. Yet, what must be realized is that even though originally spoken information may be lost, this information may still have an impact as a reference in other words or works.

George Gheverghese Joseph puts wonderfully into words what can be felt at the end of a course of this nature: "As our knowledge develops, differences in historical perspectives emerge. And to this extent that different views of the past affect our perception of ourselves and of the outside world, history becomes an important point of reference in understanding the clash of cultures and of ideas" [3]. Including the non-European roots of mathematics gives all students in these courses the opportunity to learn about and acknowledge these important contributions.

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## Intersections of Mathematics and Society

Ranthy A. C. Edmonds  
and John H. Johnson Jr.

### Introduction

At the undergraduate level, most mathematics programs and courses don't devote time to discussing how mathematical communities are formed and maintained. Instead, mathematics is often presented as a collection of "eternal truths and objective algorithms" that are discovered (or invented) and simply passed on from one generation to the next. Outside of perhaps one history of math course, little regard is usually given to the larger social and cultural milieu that supports and sustains mathematical communities. We claim this is one—but not the only—reason many students and practitioners feel "disconnected/isolated" in math. This is especially true for Black, Indigenous, and other people of color.

In our course, *Intersections of Mathematics and Society: Hidden Figures* at The Ohio State University, we directly addressed the connection between the creation of mathematics, its developments and applications, and society. We also emphasized the importance of a strong mathematical identity as students try to join, be accepted, and valued as a member of various mathematical communities. To do this, we centered our focus on mathematical community via the *Hidden Figures* text by Margot Shetterly [She16].

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Our effort ties into the larger landscape of instructors using culturally competent pedagogy. Culturally relevant practices are tied to three overarching instructional pillars: (1) academic achievement, (2) cultural competence, and (3) sociopolitical consciousness. While the first pillar is already a staple of the college mathematics classroom, the subsequent two may feel less familiar in a mathematical context.

Culturally competent practices give students a way to develop their own mathematical identity in addition to providing insights into the lived experiences of others within the mathematical community. Sociopolitical consciousness provides students with an understanding of the interplay between mathematics and the social and political issues that impact local communities and the world at large. These perspectives allow for a holistic view of mathematics, where students consider not only the conceptual frameworks that permeate our courses, but the historical and political context in which mathematics is created and used.

In this article, we highlight tips that may be helpful for those considering designing a new course or redesigning an existing course at their institutions so that it incorporates cultural-competent practices. These tips are related to elements of our course that we found the most successful for creating the learning environment we desired, and that can be implemented in any course, regardless of the content. We also describe our hidden figures course in more detail.

### Tips for Course (re)Design

In this section we provide tips for course design related to components from our course that were inspired by a focus on the intersections of mathematics and society, and rooted in the three pillars of culturally competent pedagogy highlighted above. We focus on (a) embedding reflection within a course, (b) involving students in outreach and service learning, and (c) highlighting external voices.

#### *Embedding Reflection within the Course*

- Intentionally include reflective (or extension) questions in computational assignments. In a calculus course, for example, a question could be included in an assignment about derivatives that asks students to find three real-world applications of derivatives in their major field.
- Consider adding ongoing reflections related to a course theme you want to emphasize throughout the term. (In our course we had several service-learning specific reflections, the first asking students to reflect on their own experiences with service learning along with the potential impact of their outreach on the local community, and the last asking them to reflect on the process of developing STEM programming and on the utility of service learning to increase access to mathematics.)
- Incorporate reflection into the course by utilizing online forums for discussion. Be clear about the

rules and expectations for communication in online environments; include examples of appropriate and inappropriate responses.

- Make sure that most online reflections involve a component that requires students to respond to each other. Give students time to do this intentionally. (In our course, students had one day to respond to two additional classmates after their initial posting was due.) In a large class, break students into smaller groups online, and have them respond to their group members after an initial posting.

### **Outreach and Service Learning**

- Try not to reinvent the wheel. You do not have to start your community engagement efforts from scratch! The best way to start is by plugging in to existing efforts. Most local school districts and community centers (like the library system in Columbus) have STEM and/or college readiness initiatives and are often in search of volunteers. Many corporate websites have an outreach and community engagement tab, with a list of activities and a point of contact.
- Consider utilizing an appropriate person within your department or college or undergraduate students to help you with coordination. This helps avoid the need to manage the moving parts of setting up volunteer opportunities alone.
- Outreach can be more than student presentations. It is possible to create mathematical course “artifacts” that community organizations can use for STEM education. (This was one adaptation for our course due to COVID-19.)
- Have students workshop a service-learning project in class. This class time should be structured with clear objectives for each period in terms of progress. (We carefully scaffolded the development of our students’ service-learning programming, beginning with students brainstorming their ideas as individuals, sharing these ideas, and then determining several programming ideas that incorporated everyone’s vision.)
- Include assessments (brief follow-up surveys, online discussion board posts, etc.) that require students to reflect on outreach and service-learning experiences in a meaningful way. As an aside, an intentional cycle of service and reflection is the key difference between service learning and experiential learning.

### **Collaborations and External Voices**

- Invite other instructors to speak to your course to fill gaps in your knowledge about the intersections of history, culture, and STEM. (We had a professor from the Department of Women, Gender, and Sexuality Studies come and speak to our students

about the historical context of the *Hidden Figures* story, which led to a deeper understanding and appreciation of the mathematical contributions of the women in the text.)

- Invite STEM professionals to discuss their career, research, and the mentors and community that aided them on their path. Ideal speakers will be able to speak to ideas and experiences related to course themes.
- Consider adding “nontraditional” STEM books as part of required reading in the course. (We had our students pick three stories in the MAA and AMS’s *Living Proof: Stories of Resilience Along the Mathematical Journey* [HLPT19] as a precursor to a course discussion about the myth of a “math person” and challenges that can arise when pursuing mathematics.)
- Incorporate a “mathematician of the week” as part of your course. Two great sources for contemporary mathematicians are [Lathisms.org](http://Lathisms.org) and [MathematicallyGiftedandBlack.com](http://MathematicallyGiftedandBlack.com).

Integrating reflective components into a course provides an avenue to address all three pillars of culturally competent practices. Intentionally structured discussion prompts offer the opportunity to deepen conceptual understanding of course concepts, as well as to position course ideas within larger social and political frameworks. We found that these reflective components greatly added to a sense of community in our course, and allowed for the modeling of appropriate ways to engage in intellectual discourse related to diversity and inclusion in mathematics.

Outreach and service-learning opportunities allow students to interact and serve communities similar to or different than the ones they grew up in. When facilitated appropriately, these experiences increase the cultural competencies of our students. Incorporating experiential learning also has the effect of highlighting the value of service as part of science.

Using colleagues and community members as instructional resources is a valuable practice when developing curricula that are culturally responsive. It alleviates the pressure of becoming an expert in historical and contemporary subject matters outside of mathematics, increases instructor and student knowledge of local and academic resources, and increases exposure to the intersection of mathematical ideas and society.

### **Our Hidden Figures Course**

Both authors were inspired after reading the text, *Hidden Figures: The American Dream and the Untold Story of the Black Women Mathematicians Who Helped Win the Space Race* by Margot Shetterly [She16]. Shetterly’s book provides a brilliant view of how society and mathematical and scientific advancements were intertwined with the story of four black women mathematicians—Dorothy Vaughan, Mary Jackson,

Katherine Johnson, and Christine Darden—roughly during the period from 1940–1970 in the United States. Socially, in the US, this period of time covers World War II, segregation throughout the US and Jim Crow laws in the American South, postwar civil rights protests and activism, and culminates in the Second Reconstruction (or Civil Rights Movement and accompanying legislation).

Scientifically, this period of time covers the development of the atomic bomb, the growth and expansion of the aeronautics industry (such as the National Advisory Committee for Aeronautics (NACA), the precursor organization to NASA), and the Space Race which contributed to the rapid growth of the mathematical (and scientific) communities whose expansion provided the foundations for much of our current technology.

Due to our focus on the intersections between mathematics and society, our course is positioned as a special topics course within the Department of Mathematics. We believe this course is well positioned to serve as a model for a contemporary history of math course at other institutions; components of the course can be included in regular course offerings, and the course themes and components could form the basis for a redesigned introductory seminar.

These themes motivated the design of the course around three major outcomes for our students:

1. Understand the role of mathematical communities, how they are formed and maintained, how membership is mediated, how community members are “valued” and promoted, and the mathematics and computational tools that they use.
2. To interview and engage with “local Hidden Figures” from the Greater Columbus Community, to learn how they apply mathematics in their own professions and understand how personal and professional communities have played a role in their career.
3. Create local mathematical communities of their own, emphasizing the importance of mathematical communication along with community service related to service learning. (The implementation of community service was ultimately changed and modified in the Spring 2020 pilot with the switch to remote instruction due to COVID-19.)

Below we briefly describe some of the activities that addressed the outcomes noted above. In particular, the first half of the semester of the course addressed outcomes 1 and 2, while the second half of the semester addressed outcome 3. For general information about the course please visit [math.osu.edu/courses/2010s](http://math.osu.edu/courses/2010s).

### First Half of Semester

Students were given a brief biography of each local Hidden Figure and asked to rank order the local Hidden Figures they would like to interview. Based on students’ surveyed preferences, students were arranged into small groups and each group was assigned to a local Hidden Figure. We provided guidance (for example, template emails and

suggested timeline) on how the groups should make contact with their assigned local Hidden Figures and handle scheduling.

Students interviewed their local Hidden Figures in order to complete two group projects. The first project required students to analyze their local Hidden Figures’ backgrounds and professional journeys using intersectionality and community as a framework.

In the second project students created a report about the mathematical tools used by their local Hidden Figures appropriate for a lay mathematical audience. To help the students understand these mathematical tools, the instructors held several consultation sessions with each of the groups to provide necessary mathematical background and point to supplemental resources. Tools ranged in scope from Catia 3D software modeling used in engineering, to techniques of logistic regression and data visualization techniques using Microsoft Excel.

Students were also issued a Post 1460 Versalog Slide Rule, an engineering-grade computational device comparable to what the Hidden Figures in the text would have used at NACA/NASA. We dedicated a significant portion of class time to understand their construction (they use a logarithmic scale to perform operations such as multiplication and division) and operation. Students also read the *Hidden Figures* text, and completed weekly online reading quizzes and discussion-board posts related to the readings. This allowed us to focus on different mathematical ideas present in the book outside of our dedicated discussion days, while still creating a rich dialogue about the text.

### Second Half of Semester

The second half of the semester focused on the service-learning portion of the course. We established a community partnership with the Columbus Metropolitan Library (CML). Students would deliver an original STEM program related to course material to library patrons at select branches. During the second half of the semester, we allotted two days of class time a week so that students could work on these projects together in class, keeping the third day for discussions about assigned readings.

Ultimately, students developed a worksheet that would teach basic computations with the slide rule (multiplication and division), as well as a brief presentation on select historical facts about certain Hidden Figures in the text for students at the local library branches. As a result of COVID-19, instead of facilitating the worksheet and presentation in person, students delivered a “program kit” that could be administered by library personnel who had access to suitable training regarding the slide rules.

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Ranthony A. C. Edmonds



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## Designing a Course Connecting Mathematics with Latin American Cultures

Colleen Duffy

We teach what we love; I love both mathematics and exploring other cultures. I had a strong desire to combine my interest and studies in Latin American cultures with mathematics, and hence the course entitled *Mathematics in Latin American Cultures* was born. Its course catalog description is:

This course introduces important mathematical concepts and topics, such as number and arithmetic systems, symmetry, and data structure, using the cultural lenses of pre-Columbian Latin American indigenous cultures.

In this article we will explore how my interest in such a course developed, the logistics of designing a new course, and how to connect mathematical ideas with cultural components.

### Background

I have always had a strong interest in exploring other cultures, peoples, and places. My family hosted an exchange student from Chiapas, Mexico, my senior year of high school, and she became a member of our family. Through several trips to visit her, I have been able to see first-hand

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several historical sites of the Maya peoples and to see some of how indigenous peoples today are connected to them. I double majored in mathematics and Spanish in university. During my semester abroad in Salamanca, Spain, I took a course on Spanish and Portuguese colonialism in Latin America. And, during graduate school, I went to Peru where I visited several Incan historical sites including Macchu Picchu. These experiences naturally led me to consider how I could incorporate a new perspective, and specifically an indigenous Latin American perspective, into a mathematics course. Furthermore, I have been able to enrich the course with stories and pictures from my travels. While my background prepared me for teaching a course based on Latin American culture, there are a lot of resources available to get you started in developing your own course if you do not have such a background. I mention some resources below.

### Logistics

Our university and department espouses interdisciplinary collaborations in course work and research, so institutional support was not a concern in deciding to develop the course. At the end of my first year at the university I talked with the chair of my department to get his support in developing this course. I then found a colleague, Dr. Manny Fernández, in the Latin American and Latinx Studies (LAS) program who was interested in co-developing the course with me. It was invaluable to have his expertise on the cultural side of things, and fun to collaborate on ideas. We also brought two students into the project: a mathematics education major with a Spanish minor as well as an LAS major. We received financial support through our Office of Research and Sponsored Programs to pay the students a stipend to work on developing the course with us. Our university also now has funding for faculty to develop or revamp a course; so, it is worth checking into funding options. We developed it over the course of the year 2009–2010, and offered the course the following spring 2011.

The University of Wisconsin–Eau Claire, like many colleges, has a liberal education math course focused on introducing a variety of mathematical topics that the students have likely not seen before; here the course is entitled *Introduction to Mathematical Thinking*. This type of course provides a good platform to try a new course. There is a lot of paperwork involved in getting a new course in the catalog, so we decided to pilot our course as a special section of *Introduction to Mathematical Thinking* first. Because we have a lot of flexibility as to topics covered in the course, this was possible for us. Starting by introducing a project or a couple of topics in a course would also be a way to build up to designing a new course. Another possibility would be to offer a seminar or honor's course. Fernández and I co-taught the first two iterations of the course. Student interest was high enough that the mathematics department and the LAS program were interested in having it be its own course after that. It is housed in the math department



(so I now teach it alone), but the course counts as elective credit in the LAS major or minor as well as satisfying our math requirement in the liberal education core. We get a variety of students in the course, not only students in LAS. We have offered one section a year since the course was first introduced.

## Topics

In this section we highlight topics covered in this course, but also how to connect these mathematical concepts to cultural content in general. Most of these mathematical topics were chosen because of their natural occurrence in the culture being studied, but in some cases the mathematical topics were chosen and then the cultural connection was made. We also make an effort in the course to form direct cultural connections among the cultures we are studying, the modern cultures of those peoples and places, and the students' own culture.

The main mathematical topics covered in the course are: history of zero, base conversion, modular arithmetic, symmetry, groups, basic data structures, cryptography, Fibonacci sequence, and scaling factors. Throughout the course we also discuss more general topics such as the need for multidisciplinary collaboration, modern research in the area, and the need to analyze claims in light of both mathematics and culture. As stated above, the course uses the cultural lenses of pre-Columbian Latin American indigenous cultures to study mathematics. We decided to focus on the Maya, the Inca, and the Nasca.

## Maya

The Maya were a Mesoamerican civilization that built on earlier cultures such as the Olmec and shared many traits with other peoples in the region. They used a vigesimal (base 20) number system. In the course we use this to introduce converting numbers between bases and doing arithmetic in other bases. We start with base 20, but we also discuss using other bases such as base 60 as it relates to time and base 40 because of its application in a future unit. We also build into this a discussion on the history of 0 [Kap99].

The Maya studied astronomy and used three separate calendars: sacred, solar, and historical. The first two are based on regular cycles of length 13 and 20 and the last uses a modified vigesimal system. Calendars are a great place to introduce modular arithmetic. We apply modular arithmetic in calculating future and past days. For example, many stelae reference an event date based on how many days or years after the installation of a ruler or another important occurrence, and so we want to calculate on what date that occurred. One great source of dates in other calendar systems is the Fourmilab website [Swi15]. The students also enjoy making a direct cultural connection by calculating their birth date in the Maya calendar.

Art plays an important role in culture, and is a good topic through which to study symmetry. We discuss symmetry, frieze patterns, and wallpaper patterns and look at examples of textiles to identify the symmetries and patterns. Symmetry also gives us an example of a mathematical group. The students learn the definition of a group and that symmetry and modular arithmetic are examples. This is abstract and so one of the harder topics for students, but we find it worthwhile to give them an introduction to abstract mathematical ideas through the lens of tangible objects found throughout other cultures.

One of the movies we watch in class is *Breaking the Maya Code* [Leb08]. This movie shows the lengthy process and multidisciplinary, collaborative nature of deciphering the Mayan written language. It gives the students a sense of how mathematicians can contribute to other fields of study as well as how pattern finding was important.

## Inca

Less is known about the Inca as the accounts we have are colonial and post-colonial. We first learn about the quipu, focusing on quipus which contain numerical data. Quipus are knotted cords that were used to carry information across the vast empire. There are still a lot of open questions about what information quipus contain (just numerical data or also history and stories) and how to read them. Almost no quipus have been tied to written records or been conclusively interpreted. Quipus generally have two organizational structures: cross-categorization and hierarchical. A good resource is the book *Mathematics of the Incas: Code of the Quipu* by mathematician Marcia Ascher and anthropologist Robert Ascher [AA97]. In the course we talk about organizing data in tables as well as learning about tree diagrams. A database of quipus that you can have



Figure 1. Quipu by a student group in Spring 2019.

students look at (or try to analyze) is the Khipu Database Project [Urt20]. We do not try to fit in some statistical analysis with this topic, but one could. The head of the database project, Gary Urton, also has several books out on the subject. At the end of the unit, students create their own numerical data quipu, and the other students attempt to decipher its meaning.

While it is a loose connection, we also relate quipus to cryptography asking the question of how to break the “code” that the quipu is to us as well as asking how one could repurpose the quipu as a cipher method.

The Inca also used an object called a yupana, which many experts think was used as an abacus. Others think it was used as a game, or perhaps for both purposes. The few colonial mentions of it reference the numbers 1, 2, 3, and 5, and hence we cover the Fibonacci sequence and golden ratio. There are several hypotheses out there on how to use the yupana. We learn and analyze (in terms of mathematics and culture) a few theories, as well as have the students develop their own.

## Nasca

The German mathematician Maria Reiche spent most of her career studying the Nazca lines in Peru [Rei49]. Because of the aridity of the desert, these etchings in the sand have been preserved for over 2000 years. The geoglyphs include lines, geometric shapes, and zoomorphic designs. Reiche believed that the glyphs were tied to astronomy; there are other more widely-accepted theories today as researchers have studied them more. We discuss scaling factors and their use in marking distances and angles with a piece of string. We also look at how Reiche used astronomy and what calculations need to be done to see if the lines today line up with the stars around 2000 years ago. Many of the designs and the concept of using lines appear in other Andean cultures, so ethnographic allegory is another tool we discuss. The students end the unit by creating their own designs depicting something relevant in their culture today.

Throughout the topics, we discuss how the needs and interests of a community influence the math and science



**Figure 2.** Nasca-style line drawings by student groups in Spring 2014.

which develop. For example, agriculture often leads to a study of astronomy which may lead to a calendar or architecture built to line up with celestial events. While we do not delve into astronomy or architecture in our course, those are other good areas to relate to mathematics. Having a vast empire necessitates having a method of communication that can be easily transported, and resources dictate the medium. Consider what conceptual ideas and skill sets you want to impart upon the students, and look for ways they appear in culture.

## Conclusion

Find something you are passionate about when designing a new course. Also, consider where it will fit into the curriculum in your department or other departments. Collaborate or consult with someone whose primary studies include the culture or topic. Creating a course is very enriching for the professor, but also for the students who get to see a new viewpoint of mathematics. Designing a mathematics course with a focus on culture is an opportunity to teach students that mathematics is more varied than they have seen before. It also is an opportunity to teach an appreciation and respect for indigenous (or other) culture(s). Mathematics is influenced by culture and culture influences the mathematics developed.

Developing this course and other work I have done since then has encouraged me to look at ways to incorporate history and culture into each math class I teach.

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Colleen Duffy

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# I Am a Black Mathematician<sup>1</sup>

John Urschel

I am an outlier. And in my opinion, it has nothing to do with what I have done as an athlete.

I am a black mathematician.

I am an African-American male raised by a single mother. Those facts make me an anomaly, statistically unlikely to love math, let alone to be a doctoral candidate at MIT.

My mother's parents were born and raised in the South, during a time when schools were segregated. My grandmother's classroom was one big room located in the basement of their church, led by a teacher who taught all grades. She only completed the eighth grade, because that was the highest level taught where she lived in South Carolina. My grandfather, as the oldest of his siblings, had to drop out of school to find work to help support his family in Alabama. He eventually got his high school diploma at the age of 25. They both left home and moved to the North, where they became blue-collar workers. Although they possessed only very basic educational skills, they believed that a better education was essential for their children.

My mother attended public schools where the goal was just to get by, not to expand a student's mind. Her high school had high dropout and teen pregnancy rates. Fortunately, her math teachers recognized that she had an unusual aptitude and placed her in classes above her grade level. By the time she entered her senior year, she had taken all of the available math classes her school had to offer. Her math teacher, Mr. Stern, took the initiative to enroll her in a college calculus class and arranged for the school to pay her tuition. She was valedictorian of her graduating class. Even so, her guidance counselor encouraged her to become a secretary. Instead of following a path in mathematics, she attended university and became a nurse. Still, her love of math never died.

When I was a young child, walking down the street or riding in the car, she and I would frequently play games. Every week, we had family night, where we would play board games. I thought we were just having fun. I never saw it as learning, but learning I was. My mom would buy me countless educational games and books along with whatever action figure was popular at the time. She gave me a balance of education, fun, and sports.

As early as the first grade, my mom was contacted by my teacher and informed that she believed I had problems "processing," which was a politically correct way of

saying that I was unable to grasp the material. The teacher's suggestion was to test me so that they could place me in a class where the material was not so strenuous. As one of my biggest advocates, my mother agreed that I should be tested, because she had no doubt I knew the material. At home, I was an avid reader and devoured algebraic math books faster than my mom could get me new ones. After the teacher saw my result on the test, the teacher suggested that I be moved up a grade level. My mom said no. From her meetings, my mom believed that my teacher saw me as a child from a single-parent household and viewed my shyness as a sign that I was a typical minority child unable to keep up in a classroom setting. My mom withdrew me from the school and looked for educational environments that would not prejudge my abilities based on the color of my skin. She did not want me labeled as a "lesser-than."

In that respect she succeeded—so much so that I can say I have never in my life felt that the color of my skin has ever affected my math, nor how I have viewed myself as being perceived. In many ways, my experience and view of the world as an African-American is disjoint from my mother's. Where she sees racism, I often see fairness. Where I see a struggling student, she sees a minority who has been implicitly told their whole life that they are "lesser-than."

I sometimes struggle to reconcile my experience with her worldview—and also with the realities of the field. I know that the color of skin has nothing to do with the ability to do mathematics, and yet when I look at the top mathematics departments in the US, I cannot help but notice that many do not have a single African-American professor. Since it is ridiculous to think that all of the most brilliant mathematical minds born in the US are Caucasian, this leaves us with the sad truth that talented African-Americans are being left behind.

The optimist in me says that change has already occurred, and it will be more and more apparent as time goes on. But sometimes, I find too much truth in my mother's sentiments. Sometimes, I find myself meeting with young African-American would-be mathematicians, hearing them ask how I have managed to get to where I am, and watching them hold back tears when talking about being behind or feeling like they cannot succeed because they do not have the background that the "elite" young talent in their classes have had. It is a sobering experience, and I cannot help but feel a sense of privilege for being unable to relate to it personally.

As an outlier, I have a responsibility to set a good example for young people everywhere who have mathematical talent but may feel like they cannot succeed because they do not look like those who have succeeded before. I have a responsibility to succeed, not just for myself, but for my mother, my grandparents, and every minority who feels like the field is closed to them. I am all too aware that as a

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mathematician, the color of my skin means both nothing and all too much.



John Urschel

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## Othering and Such Climatic Joy Killers<sup>2</sup>

Arlie O. Petters

I remember being giddy with excitement to attend the welcoming reception for my entering class of math graduate students. I walked into the room and heads turned towards me. Feeling out of place, I walked over to two student-looking faces. One happened to be a fourth-year graduate math student and the other was a first-year like me. I introduced myself and, because I wanted a quick exit, I asked the more senior student how to get to the main math office. He told me that when I walked out the door, I should make a left, walk down the hallway, make another left, and it would be on my right. "Or, you could tie a rope to the ceiling and swing over to the other side," he said with a mischievous grin. The first-year student turned red with embarrassment. It did not matter whether the senior student thought of me as a monkey in a tree, Tarzan, or something else; his decision to engage in an unnecessary framing that could provoke a negative stereotype was telling. I quickly responded, "I see that you're going to be an asshole," and I walked out of the room. Here I was, looking forward to being part of a new community of mathematicians and then being made to feel unwelcome at the onset.

I went directly to my apartment and started packing. My mind was racing and I was angry: "To hell with them. They turned around looking at me because I am a person of color. I am leaving this place. To hell with these people." As

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<sup>2</sup>(Petters) This story is excerpted from "Belonging" in *Notices of the American Mathematical Society*, February 2018, pp. 120–123.

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I started calming down, a counterintuitive thought occurred to me: "What if the others in the room weren't like him? What if they turned around and looked at me because they don't often see someone like me in an entering class and were curious to get to know me? ... If I leave, this guy will win. I refuse to let him win." My psychological bounce back was that he had brought the fight to me, and I refused to cower in fear or run away in anger. I had briefly allowed him to hijack and taint my perspective. And, even worse, by allowing him to make me angry, I had given him power over me in that moment. Never again. The emotional-intelligence battle was on. Would I have had such a fight-back spirit in the academic sphere if from pre-kindergarten my sense of self had been chipped away, bit by bit, by individual and institutional racism? I doubt it. Fortunately, I was raised until the age of 15 in Belize by a loving and resilient grandmother who strengthened me internally, fortifying my identity and allowing me to maintain its structural integrity in the face of undermining forces.

I was not naive about the epiphany that caused me to stay. My hypothesis that most people in the room were not like him needed to be tested. But I had enough internal energy and grit to hold on to it by blind faith in the short term. The energy sustained me through the long hours of hard work needed to perform very well on my homework sets. And the grit enabled me to bear the anxiety that maybe most people in the environment did not really care for my being there and did not think much of me intellectually. In my case, I was fortunate to discover with time that most of the people were not like that graduate student. I had a perceptive and supportive thesis advisor and a positive interaction with the majority of the other math and physics graduate students and faculty. That young man had acted as if he owned the place. To me, he had a warped sense of belonging and entitlement that made him feel confident enough to treat me in a demeaning way without consequences.

I wish I could tell you that my experience was an anomaly. Over the years I have mentored a host of underrepresented minority students and listened to their experiences. They range from regular racial micro-aggression, through "oppressive othering," to more overt examples, like being the only one not invited to a bus outing organized by fellow math graduate students. A sense of belonging involves one's personal belief that one is an accepted member of an academic community whose presence and contributions are valued. This is important not only for the mathematics community but also for education and our society at large. At the convocation for Duke's entering 2017 undergraduate class, Stephen Nowicki emphasized to our students:

*We only learn best from each other and teach each other well if we all feel like we belong. We can only achieve the excellence that lies in the potential of the different people and perspectives, the different*



*aspirations and ideas we've brought together at Duke, if everyone feels equally that Duke belongs to them.*

*There's another important thing to understand about what it means to belong, which is that "belonging" does not mean "conforming." ... The excellence of this place emerges from the very different kinds of people who join our community. To diminish those differences through conformity would only diminish our excellence.*

If we truly believe that diversity in all its dimensions is a key driver of excellence in our educational institutions and increases the probability of intellectual breakthroughs, then we cannot ignore the implicit biases directed toward underrepresented minorities and women. Actionable first steps a department can take as part of fostering a welcoming culture are to assign thoughtfully chosen mentors to incoming students and faculty; to advocate inclusion, acceptance, and understanding; and to promote effective ways to engage diversity. Imagine for a moment that you are a newcomer. Having someone in your department teach you the ropes and advise you from their own experiences is part of an onboarding that tells you from the beginning that you matter. Usually it is through such a relationship that your trust in the environment grows. By trust, I mean that you can allow yourself to be intellectually vulnerable without fear that your admission of the need for help or clarity will be attached to your race, ethnicity, gender identity, or social-class history. For example, you can feel secure enough to admit that you have certain gaps in your math background and allow the mentor to assist you with filling them. And you can ask faculty and seminar speakers questions about mathematical issues that are unclear to you.

For many underrepresented minorities and women, the issue of belonging in mathematics has been a continued fundamental challenge. I believe that an integral part of keeping our field vibrant and relevant is for its participants to welcome everyone, knowing that anyone can get better at mathematics through an ample commitment of time and energy by teacher and student. Equally important, one should not only be welcoming at the door but also give people a chance to add value inside. Belonging is indeed a foundational human need, which when nurtured can bring out the best in all of us, enabling our community to maximize its excellence. In the end, mathematicians are the custodians of mathematics. The onus is on us.



Arlie O. Petters

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# James A. Donaldson— Black Mathematician, Advocate, Gentleman (1941–2019)

*Fern Y. Hunt, Robin T. Wilson,  
and Daniel A. Williams*

## Introduction

In reviewing the participation and contribution of African Americans to the US mathematical community, the achievements of the last 100 years present a stunning contrast to the previous 300 years. Indeed, the institutions of slavery in the Western Hemisphere and the century of segregation and discrimination that plagued African Americans after the Civil War posed insurmountable barriers to most individuals of talent and inclination. Thus, it is interesting to note that much of the progress towards greater diversity and inclusion that we have seen until now in the American mathematical community can be tied to the African American mathematicians who came of age in the 1940s, 1950s, and 1960s. Using the moral energy and political opening created by the civil rights movement, these men and women established programs that made space

for younger African Americans to join the mathematics community and have their efforts better recognized and rewarded. Among the leaders of this generation, 6 foot 5 James Ashley Donaldson stands tall. In 1969, he helped start the National Association of Mathematicians (NAM), whose goal was to end the shameful collusion and direct acts of discrimination against Black mathematicians and other mathematicians of color so common in the 1940s, 1950s, and even 1960s [Don89a, Kas19, Lor94, NGRS80]. Donaldson continued his support and leadership of NAM by serving as the first editor of the NAM newsletter and the director of communications. He also was a member of its Board of Directors in the years 1984–1994.

His signal achievement however was the establishment of a mathematics PhD program at Howard University, the first such program at a historically Black university. As he recounted in his HistoryMakers interview [HMY], the idea originated with James Cheek, then president of Howard, who embarked on an ambitious plan to bolster science and engineering at the university. In his years as chair (1972–1990), Donaldson recruited and hired young PhDs from leading universities to create what is now a flourishing mathematics department. During the last two decades the department has produced an average of 2.5 PhDs per year and in 2019 they graduated seven students.<sup>1</sup>

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<sup>1</sup>From Professor Dennis Davenport, the Director of Graduate Studies for the Howard Mathematics Department.

Donaldson became the third African American (David Blackwell and J. Ernest Wilkins being the first and second) elected to the AMS Council. He was also elected second vice president of the Mathematical Association of America [NGRS80]. From the earliest days of his career, Donaldson had a broad interest in and connection to the mathematical community in the African diaspora. This came to the fore in 1974 when he and American University statistician Mary Gray waged a successful campaign to derail a proposed reciprocal relationship between the AMS and the apartheid-era South African Mathematical Society [Pit88].

In the 1990s Donaldson's academic career entered a new phase as he took on major administrative responsibilities. An active alumnus of Lincoln University and member of its Board of Trustees, Donaldson was recruited to serve as acting president of the university during 1998–1999 and then returned to Howard to serve as Dean of the College of Liberal Arts and Sciences in 1999–2012. During Donaldson's tenure as Dean, a group led by a Howard University team of scientists, anthropologists, and archeologists wrote a report on the excavation, analysis, and history of the prerevolutionary New York African burial ground, the oldest and largest urban gravesite of free and enslaved Africans in the country. This project, which was of deep national significance, was made possible as a result of the financial and administrative support that Donaldson obtained from Howard University.

## Early Years

James A. Donaldson was born in 1941 to Oliver and Audrey Donaldson, the 8th of 11 children, on a farm in Madison County near Tallahassee, Florida. He was taught by his uncle Enoch Donaldson to read, write, and calculate long before entering elementary school. He was educated in a segregated school system in Madison County, where his first eight years of education took place in a rural two-room school house with grades one through four in one room and grades five through eight in the other. The Supreme Court struck down segregated public schools in 1954, when Donaldson was in the 10th grade. Despite the ruling, many questions remained regarding the fate of the segregated educational system, and after high school Donaldson was encouraged by a teacher to leave the south and go north for his college education. Donaldson followed this advice and entered Lincoln University in Pennsylvania in 1957, graduating with a BA in mathematics in 1961. He went on to do graduate work in mathematics at the University of Illinois in Urbana-Champaign, and obtained a PhD in 1965 in partial differential equations. Upon graduation from UIUC, he held a series of faculty positions



**Figure 1.** James Donaldson with his brothers. From left to right: Joseph, Clarence, James, Albert, Enoch, Chandler, Rufus, and Oliver on the Donaldson Farm in Madison County, Florida, in the 1970s.

starting at Howard University, then the University of Illinois at Chicago, and the University of New Mexico, before returning to Howard University in 1971 [Don].

In the rest of this article we will hear from a number of people touched during his rich and distinguished career. We begin with Mary Gray, founder of the Association for Women in Mathematics and its first president. We next hear from Chandler Davis, Professor Emeritus of the University of Toronto, who shares remembrances from his early activist years, and then follow with a contribution from Donaldson's former Howard colleague Professor Isom Herron of Rensselaer Polytechnic Institute, and then Daniel Williams, Associate Professor at Howard University. Donaldson was a skilled and devoted teacher and mentor. Contributions from his former PhD student Sean Brooks of Coppin State University and from Fern Hunt (scientist emeritus) of the National Institute of Standards and Technology complete this brief portrait of a truly impactful life.

## Sometimes You Just Need to Do Something

### Mary Gray

Jim Donaldson was a force for activism for human rights and an inspiration and fellow agitator with many of us of his generation. Back in the 1970s we decided to organize to challenge discriminatory practices in the mathematical community. The National Association of Mathematicians



(NAM) and the Association for Women in Mathematics (AWM) were founded with the idea of encouraging and assisting minorities and women to study mathematics and to consider and prepare for a career in the field. It was apparent certainly to Jim and to me that the reigning superstructure was probably not going to do much, so we decided that more people interested in progress on this and other progressive fronts needed to be elected to the AMS Council to provide some institutional support and action. With lots of assistance from supporters of diverse backgrounds such as Jim, I was fortunate to be elected to the Council and subsequently as Vice President.

At the Council, my ideas were often not well received. So, when the issue of the reciprocity agreement with the South African Mathematical Society came up, no one was surprised when I suggested that such an agreement with an organization which was a part of official apartheid should be turned down; nor was I surprised at the opposition this encountered. The South African government enforced a rigid and brutal regime of racial separation and subjugation known as apartheid. It treated nonwhites as if they were not human beings entitled to rights, a situation which was similar to what had taken place in the past in the US and is still echoed in spots around the world today.

The question of whether individual members of a math society should be held responsible for the policies of their government was raised of course. Did we want to be accountable for US government policies (remember Vietnam and the still existing widespread discrimination here)? At least we should be held responsible for not agitating against them and working to improve the situation, but some cases can be so abhorrent—as with apartheid—that personal responsibility must be assigned as part of massive international condemnation. We argued that action must be taken, even if it is as minor as denying reciprocity membership. Even if we cannot do much, we cannot do nothing when that might be considered as tacit endorsement of these policies. We needed to make clear that no matter how committed we might be to the principles of the participation of individuals in the worldwide community of mathematics, the situation in South Africa was beyond what could be tolerated. Sometimes principles of encouraging participation may conflict with principles of condemnation, and the time had come to condemn apartheid.

But the battle at the AMS Council would have been lost were it not for Jim's eloquence when invited to address the Council. For me, the resulting victory established an alliance that lasted nearly 50 years, and in the mathematical community spurred awareness of ethical responsibilities of mathematicians in the US and in countries around the world who faced violations of their human rights. As a



**Figure 2.** From left to right: Wilbur Smith, Gerald Chachere, Evelyn Green, Rogers Newman, Robert Bozeman, Ulysses Bail, and James Donaldson at a NAM meeting at Elizabeth City State University.

result of this battle, council-mandated policies improved in a number of ways.

Jim and I often joined forces on local issues on which we were both working. Among these occasions were gatherings of the AWM within larger mathematics conferences. Lots of men were supporters of the AWM, some openly and some more clandestinely except perhaps at the AWM parties that became a feature of AMS meetings. Jim was always with us, not only on opening up mathematics to more minorities and women at all levels, but on other matters of ethical concern like military collaboration. Through the years, when some of my minority PhD students felt discouraged by the difficulties they still faced, I pointed them to Jim, who was always willing to engage, helping them persevere to eventual success. Thanks, Jim, for all that you have done.

## Donaldson Steps Up

### *Chandler Davis*

Jim Donaldson told a story of a turning point in his youth. As a student (and student athlete) he was walking home one afternoon after a few beers, when a car passing on the street slowed down, and some young White guys hollered insults. Jim answered clearly and audibly, "So's your mother." The car had been speeding on, but instead it stopped in front of him and the passengers got out. It happened that Jim was carrying a substantial length of 2-by-4 for some repair project. He stood his ground, hefting the lumber thoughtfully. The White guys, seeing that he might



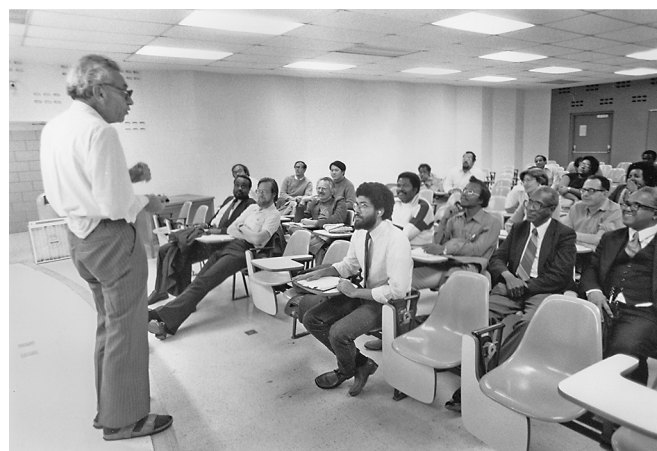
be hard to subdue and anyway would likely dent their car before they did, got back in the car and drove away. Jim reflected on this. He decided, "Hey, that was not good. They could have followed me in the car, found out where I live, and made all sorts of trouble." And he said to himself, "If I had not had those beers, I wouldn't have said that." He instantly became a teetotaler. It wasn't many years after that that I got to know Jim, and he had clearly built a personality fitting with that. He didn't give an inch in his beliefs, but he was not one to waste his passion on needless confrontation.

Some of us mathematicians had been dedicated activists against the war in Vietnam, and although few of us did this style of mathematics we flippantly called ourselves the Bourbaki Brigade, named after the then popular style of abstract mathematics. Now Jim Donaldson was one of the most visible spokespeople for the Bourbaki Brigade, and let me point out that among those few he was the only African-American and the only one without academic tenure. But calm. Despite the outwardly light-hearted tone of the young group, the late 1960s were ominous.

By 1968, public support for the war collapsed and, because of the draft, most college-aged men faced the threat of being forced into the military and into the ongoing carnage in Southeast Asia. The Bourbaki Brigade briefly participated in an anti-war demonstration in Chicago at the 1968 Democratic convention on its first day before going on to an AMS meeting in Madison, Wisconsin. The events that transpired in Chicago in the days after they left were a turning point in the history of the Democratic Party and of the country. We refer you to the many books and films about this time.

How did the anti-war math contingent carry on after Chicago? With a motion to the Business Meeting at the AMS Winter Meeting in New Orleans. According to its schedule, the AMS was to hold its 1969 Summer Meeting in Chicago. In protest against the city government's violent repression of anti-war protests in 1968, our motion was to cancel the reservation and hold the Summer Meeting in Cincinnati instead. Not the sort of last-minute reconsideration the officers could readily accept. There was a floor fight, and some hard feelings; but our side won, and Cincinnati it was.

In the margins of the New Orleans Meeting, some of the anti-war activists got together and brought into existence a new organization, the Mathematicians Action Group (MAG). An auspicious debut, with that nontrivial victory in the Business Meeting. I missed the New Orleans excitement, but I was part of all the subsequent history of MAG, as was Jim Donaldson. Amorphous though the group was, it had a philosophy and an identity, and we tried to stick



**Figure 3.** Paul Erdős delivers colloquium talk to the mathematics department. Erdős visited Howard at the invitation of his collaborator Neil Hindman (seated in front row, second from the left). Jim Donaldson is on the extreme left.

together. Resistance to the Vietnam War, and appeals to the profession not to put itself in the service of the war-makers, continued, along with other issues. Our resistance to racism was often centered in NAM, in which Jim played a major part.

Jim sometimes saw the way to unity when others didn't, but he didn't dictate it; sometimes he saw sensible solutions which we learned only when we asked him for them. Let me just share one incident from the MAG years. Two of our most active comrades, with many (figurative) hash marks for past service to the cause, were much respected and appreciated by all of us, except by each other. We were distressed every so often by the way they would lash out. After one such set-to, Jim Donaldson smiled at them and said gently, "What's with you two? Come on! You know you're both going to heaven." In other words, the issue dividing them was one that didn't need to be settled. Couldn't have been said better. I speak for many in saying "Thank you, Jim."

## Jim Donaldson: A Remembrance

### Isom Herron

I first met Jim in 1973 while, as a postdoctoral fellow in applied mathematics at Caltech, I began looking for a tenure track position. One of the places I visited on that application tour was Howard University. Jim was chair of the mathematics department. We met in his office in a WWII-era, seemingly temporary, building. It was made clear to

me that though a permanent position was not inevitable, I would be given a fair shot at Howard. That was one of the hallmarks of his character; Jim was a man of his word.

He had been tasked with building a PhD program at Howard and he envisioned me as contributing to the applied wing of that effort. My training was in asymptotic and perturbation methods. I was impressed then that he brought an analytical strength to his work. I saw that he had a copy of the classic *A Course of Modern Analysis* by Whittaker and Watson [WW96], which was an underpinning of much of the work I did. He was also involved in the application of semigroup methods in partial differential equations. He often gave tribute to Einar Hille for this, as well as for the personal mentorship he had showed to Jim. I believe Jim met Hille while they both were at the University of New Mexico; one a recent PhD, the other a recently retired senior Professor.

Though the PhD program of Howard was still to be finally approved, the awarding of Masters degrees had gone on for generations. Jim had a prospective Masters student, Peter Philip, whom he turned over to me. That was a tough act to follow. Jim had taught Peter Philip, and though I myself had no direct experience of Jim's teaching, this young man was a living example of its success, someone who had already achieved a very strong Bachelors degree. Anecdotally, he told me that Jim could arrive in the classroom with just a stick of chalk and develop important theory, for example about differential equations, in a cogent and dependable way. Peter Philip went on to write his Masters thesis, entitled *Green's Function for a Singular Boundary Value Problem* in 1975, making use of work by Natterer. The main result was a necessary and sufficient condition for the Green's matrix to exist for a singular system of ordinary differential equations with a singular boundary condition. We could not find this in any books at the time.

Later in my career, I was asked to organize a workshop for underrepresented minority students at the SIAM Annual Meeting in Portland, Oregon, in 2004. The occasion was called Diversity Day. Jim, on his own dime, attended to help mentor these students. He gave a talk at the closing dinner entitled "Some exceptional mathematical scientists I have known." Indeed, Jim could say this because he knew, besides those mathematicians I have already mentioned, most of the living African Americans to receive a PhD in mathematics. I remember well the day when at the Department of Mathematics at Howard University, he arranged for J. Ernest Wilkins and David Blackwell to both be in attendance at a dedicatory event for Dr. Elbert Cox, the first African American to receive a PhD in mathematics. His interest in the history of African American mathematics continued and in his later years he wrote about the

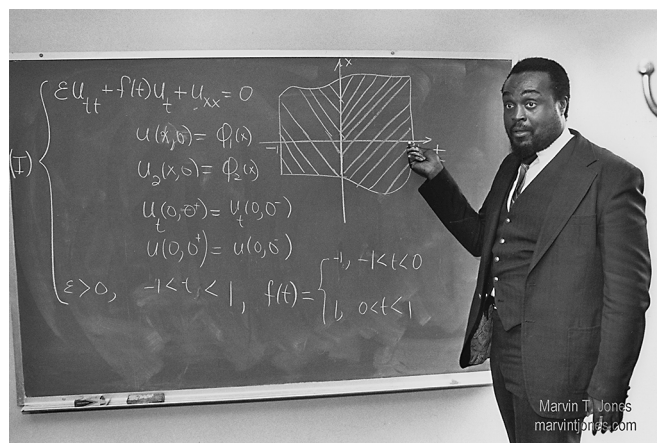


Figure 4. James Donaldson at the chalkboard.

place of African Americans in the growth of mathematics [Don89b].

Being in middle management, as a department chair, Jim faced the inevitable pressures of his position. This has given one of my former colleagues there to remark: He avoided offending people by harsh word or by jarring deed. His ideal was that of a surgeon skillfully cutting a patient so that there was no expression of pain, and that was for the patient's well-being and benefit. Jim asked me to act as chair for a couple of semesters when he was on leave. Since he asked the second time, I know I had fulfilled some of his expectations.

To the outside world of mathematics, his monumental accomplishment was in establishing the PhD program in the Howard University Department of Mathematics. Clearly such an achievement took the vision and commitment of Dr. James Cheek, then the president of Howard. Still it lay with Jim to encourage implementation, generating administration support in the recruitment of faculty and students, building seminars, promoting conferences, obtaining clerical support, and all the while supervising the mathematics education needed for students from premed to engineering.

In summary, the legacy of Jim Donaldson is undying. In his intellectual interactions with his colleagues, he freely shared his ideas, and succeeded in several productive collaborations. He readily encouraged his colleagues in their research pursuits, and he was a beacon to students who looked at mathematics as a possible career.

## A Mathematician and Leader

### Daniel A. Williams III

The chair of a department at Howard University is by no means a figurehead post. Despite the fact that his most productive years appeared to coincide with his early years as chair of the mathematics department, I cannot help but believe that his mathematical productivity had to have suffered from the relentless difficult-to-delegate administrative tasks that fell in his lap on a daily basis. I suspect however that he recognized that his leadership abilities were too greatly needed by the communities he served, and he accepted the price it exacted. The same statement can be repeated regarding his stint as acting president of Lincoln University (he refused to be considered for the position permanently), and even more so as dean of the College of Arts and Sciences at Howard. Nevertheless he had a string of mathematical papers refereed in some of the most highly regarded journals during his years as chair of the Howard mathematics department. A MathSciNet search shows that Donaldson authored or co-authored 20 publications during the years 1967–2004. The years that I worked with him mathematically were during the eight-year hiatus he had from administration between the chairmanship and deanship.

Donaldson invited me to work with him on a problem in the theory of water waves that I believe was originally suggested to him by Avner Friedman. Although this subject fell broadly within my area of interest then, I was not specifically knowledgeable about it at the time. Most of his papers were single authored, but he also had a few collaborators over the years, so I believe he also enjoyed working on mathematics with others. I considered it my good fortune to be invited to work with him, because I certainly enjoyed the time we worked together. We would get together once or twice a week to discuss the progress each had made. We shared the chore of writing it up in  $\text{\TeX}$  as we progressed.

If you assume the depth of the water in a domain  $\Omega_B$ , where  $B$  is a function describing the boundary (8), (9) and the amplitude relative to the wavelength of surface waves is small, we get the following model. We are required to find the velocity potential  $\Phi(x, y, t)$  satisfying the boundary-value problem

$$\Phi_{xx} + \Phi_{yy} = 0 \quad \text{in } \Omega_B \times \mathbb{R}^+, \quad (1)$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on } \Gamma_B, \quad (2)$$

$$\Phi_y = \eta_t, \quad \Phi_t + \eta = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \quad (3)$$

or equivalently

$$\Phi_y + \Phi_{tt} = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \quad (4)$$

$$\Phi(x, 0, 0) = F_0(x) \quad \text{on } \Gamma, \quad (5)$$

$$\Phi_t(x, 0, 0) = F_1(x) \quad \text{on } \Gamma, \quad (6)$$

and where  $y = \eta(x, t)$  is the equation of the free surface,

$$\Gamma = \{(x, 0) : x \in \mathbb{R}\}, \quad (7)$$

$$\Gamma_B = \{(x, y) : y = -\epsilon B(x)\}, \quad (8)$$

$$\Omega_B = \{(x, y) : -\epsilon B(x) < y < 0, x \in \mathbb{R}\}. \quad (9)$$

The initial conditions (5), (6) result from specifying the initial surface elevation  $\eta(x, 0)$  and the initial velocity potential  $\Phi(x, y, 0)$ . Some other assumptions made are that  $\epsilon > 0$  in (8) and (9) is small, and both the density of the fluid and acceleration due to gravity are constants equated to one.

In the linear theory of shallow water, the hyperbolic initial-value problem plays a crucial role:

$$W_{tt}^0 - \epsilon(BW_x^0)_x = 0, \quad (10)$$

$$W^0(x, 0) = F_0(x), \quad W_t^0(x, 0) = F_1(x). \quad (11)$$

The partial differential equation in the system above is called the “shallow water” equation. Analogous to (3), we can define the free surface of this system by  $\eta^0(x, t) = -W_t^0(x, t)$ . A fundamental problem in the shallow water theory is to determine in terms of  $\epsilon$  a bound for the error which results when  $\eta^0$  is used to approximate  $\eta$ . This is the mathematical justification to which the title of the paper refers in [DW93], under less restrictive conditions than had previously been achieved.

This problem had been studied and solved in the simple harmonic case by Marvin Shinbrot. Later others (see the references in [DW93]) contributed to this question under a variety of other limiting conditions. Our contribution was to address the problem in a Sobolev space where the region of the fluid has finite depth and the bottom is not horizontal.

Our first paper inspired a second shorter paper in which we considered a Dirichlet-Neumann problem for Laplace’s equation over a region that is a horizontal slab. This led to an investigation and results for an abstract version of a Dirichlet-Neumann operator that had been introduced and studied by W. Craig and H. Yosihara about a decade earlier. We tied the results back to the justification of the shallow water theory.

Understandably to those who knew him, he would frequently publicly joke that his administrative position was a step down from being a professor. At ceremonial occasions such as commencement, where he was obliged to give speeches to a wide audience, he loved to insert a mathematics problem at the beginning of his speech for





**Figure 5.** James Donaldson (right; chair of the Howard Department of Mathematics, 1972–1990) strolls the college yard with David Blackwell (left; chair of the Howard Department of Mathematics, 1947–1954).

students to ponder during the (nonmathematical!) talk, and then present his solution (invariably something that did not require any mathematical machinery, but rather a clever use of geometry or logic to solve elegantly) by way of closing his speech.

His impact and influence as a leader was so immense in the mathematical and nonmathematical communities, locally at Howard and broadly in his political activities, that it overshadowed the recognition of his mathematical contributions. Nevertheless his research was respected by prominent mathematicians and I am certain he would be disappointed if his mathematical contributions were not acknowledged. I am honored to have been allowed to contribute to this aspect of this memorial tribute.

## Jim Donaldson, a Black American Master

### *Sean Brooks*

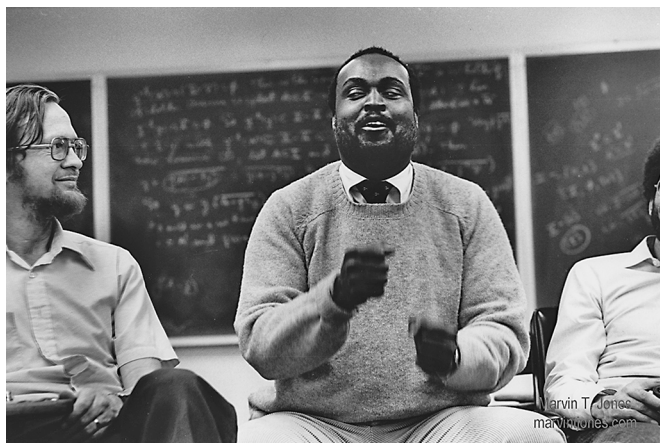
James Ashley Donaldson was my friend and mentor. As a towering figure, both literally and figuratively, he was a beautiful Black Man, and he was larger than life. Dr. Donaldson was a significant mathematical analyst and applied mathematician. He was a multidimensional humanitarian, a spiritual man, and a standing member of the African Methodist Episcopal Church. He was also a mathematician and educator who understood the buzz words “achievement gap” and “underrepresented minorities” and defied them both through his teaching, mentoring, and research.

I arrived at Howard University in the Fall of 1995, as a graduate student in mathematics. My mentor from Coplin State University, Dr. Genevieve Knight, informed Dr. Donaldson that I was there. Shortly after, a faculty member said to me “Professor Donaldson is looking for you.” I started asking other graduate students, “How does he look? Who should I be looking for?” The students described him as a big man. Still, this description did not fully prepare me for his presence. Well above average height, he was muscular, and had a complexion as perfectly dark brown as mine. Until that moment, I had never met a mathematician that possessed these physical attributes. Soon after our introduction, I was a student in his Partial Differential Equations course. He achieved completely coherent lectures as effortlessly as Charlie Parker played the saxophone. I had seen many good lectures in mathematics, but never so engaging, coherent, and comprehensive mathematical lectures, including proofs, without notes or text. I knew I was witnessing a true Black American mathematician. His uncompromisingly excellent teaching was inspirational and he provided an impeccable role model for all of his students, especially those who were underrepresented.

Naturally, as one of my thesis advisors, Professor Donaldson became my mentor. He believed in the whole person. He was a consumer of the arts, a producer and curator of history, and a purveyor of vocational skills. This gave him the ability to see a solution to a problem through many prisms. Indeed, he was a first-rate mathematical analyst who believed in the fundamentals, but effectively employed abstraction. Donaldson understood that in the fundamentals lie innovation and creativity, and there is no bridge to an achievement gap without them. This was on display countless times as he interacted with students. I recall when one of the undergrads was going to Carnegie Mellon for graduate school, Donaldson noticed some gaps in his knowledge and immediately set up meetings with the student to address the issue. The student went on to earn his PhD from Carnegie Mellon. A great mentor garners trust, and Donaldson sustained that trust through his strong sense of empathy.

The first person I ever heard refer to Professor James A. Donaldson as a “master” was his colleague, friend, and collaborator Dr. Daniel A. Williams III. One of the research areas I collaborated with Dr. Donaldson on was the linear shallow water theory. Our research around shallow water waves led to many expected and unexpected ideas and techniques. The solitary wave was one idea derived from our research that was so fascinating, I made the sensitive request to change the problem. In pure master mentor fashion, Professor Donaldson’s response was easy and continuous. Not only did he ask can he help out, but we also stepped





**Figure 6.** During a student-faculty party James Donaldson gives Neil Hindman (left) a merry side eye. David James is seated at the right.

away from water waves altogether and started research on optical solitons where we published our first and only article together [BMD04]. Another Donaldson graduate student, Dr. Amatalelah A. Hummel Al-hoori, also wrote a dissertation related to the same subject.

How do you mourn this multidimensional giant? We pay homage to him by honoring his best examples. When I see a researcher who makes a special effort to be an exemplary teacher; when I see a busy administrator who takes the time to mentor and see a person through a rough spot using their time, resources, and efforts; when I see a master teacher who is a first-rate researcher, I am reminded of Dr. James A. Donaldson and feel his presence.

## James Donaldson as Activist and Administrator

*Fern Y. Hunt*

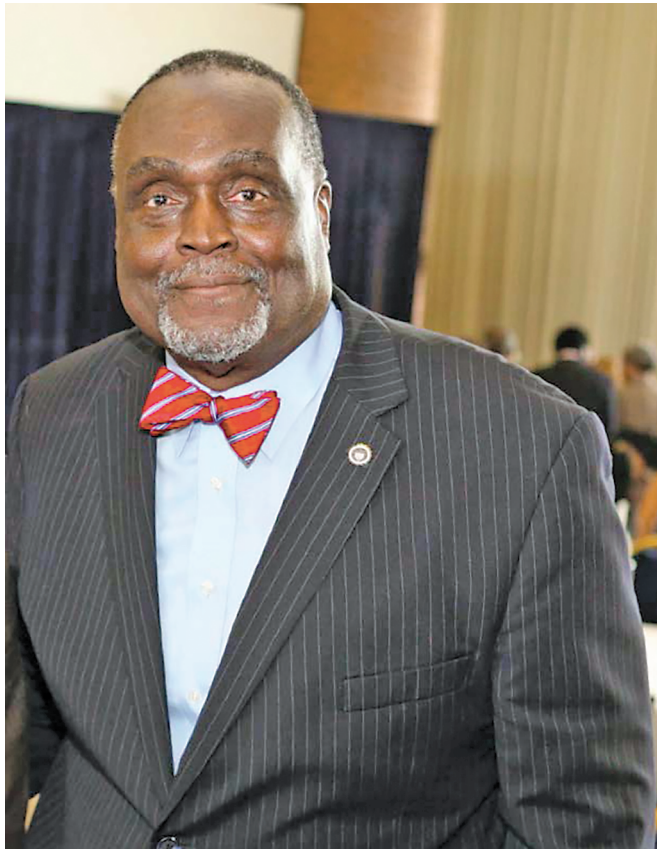
In 1974, while I was a graduate student at the Courant Institute, I attended the International Congress of Mathematicians held in Vancouver, British Columbia. The trip was a reward to myself for having passed the oral preliminary examination the previous year. While there, I met Jim Donaldson, the first African American mathematician I met in person. But I was puzzled. His conversational style, slow and deliberate, was disconcerting at first to my New York City ear. He would often begin with a sly observation, then grab your attention as his voice rolled forward to a witty and/or hilarious ending punctuated by loud laughter. His tone was wry and ironic, but always humane. Puzzlement soon turned to admiration and respect. As it turned out, the next year Donaldson was a visiting

professor at Courant but we rarely met. However, near the end of his term, he talked to me about the new PhD program at Howard and that if I was looking for an academic position after finishing my degree, I should look him up. At the time, I felt too far away from finishing to do more than put this idea aside. But as fate would have it, it came in handy.

By the time I arrived at Howard in the fall semester of 1978, Donaldson had hired a lively and diverse group of younger faculty eager to do research and innovative teaching. This was greatly encouraging to me. Although my early work involved singular perturbations of differential and parabolic partial differential equations, our mathematical interests could not be farther apart. I was then very interested in applications to population genetics and ecology and there were very few connections with his interests aside from the theory of semigroups. Moreover, I was a new researcher in mathematical biology, a field that was still relatively new with few conferences, journals, and colleagues. Despite this, Donaldson was extremely supportive of my work and of the future of the field itself. Indeed, there are several researchers at Howard working in that area today. His greatest impact as a mentor however was a consequence of his superb administrative skills. Whether it was threading NSF proposals through the university bureaucracy or helping me secure funds allowing time off to do research at NIH and later at the National Institutes of Standards and Technology (NIST), he was calm, sagacious, and inventive. I illustrate with one of many examples. In 1980, I got a visiting research position at NIH for a year. The costs would be paid largely by NIH. A month before I was scheduled to begin, NIH announced that due to budget cuts only six-months salary would be paid. By then, Howard had approved an entire academic year of leave. Because of the change in NIH policy we were forced to start the application process all over again with less time to complete it. Jim helped me secure a cost-matching agreement between NIH and Howard that allowed me to take the position as planned.

Donaldson's work with the National Association of Mathematicians and the MAA are mentioned elsewhere but during the 1980s he was also a board member of TransAfrica, the premier US organization engaged in the global movement against apartheid in South Africa. This work was a reflection of his deep connection and concern for mathematicians and mathematics in the African diaspora.

I remember sitting with Jim during lunchtime in the Howard faculty lounge, sometime around 1986. South Africa had declared (yet another) state of emergency, and Nelson Mandela was still in jail. I commented on how hopeless the situation was as the South African



**Figure 7.** James Ashley Donaldson at his retirement as Dean of the College of Liberal Arts and Sciences.

government and its Afrikaner supporters would never yield. To my shock, he said he thought that in fact things might turn out much better than commonly believed and then said, "I think you might like the result." To this day, I am not sure what he knew or how he came to this conclusion, but perhaps his work with TransAfrica offers a clue.

Jim Donaldson was not my teacher and mentor in a traditional sense. Here was a man of towering intellect and personality who held power and made mistakes yes, but who committed acts of kindness, humility, and on occasion self-sacrifice. He did these with humor and sympathy over and over again. He inspired me to show up at my best as a mathematician and as a human being. Thank you, Jim.

James Donaldson died on October 18, 2019, of heart disease.

**ACKNOWLEDGMENTS.** The authors thank Maria Wall, Professor Johnny Houston, Marvin T. Jones, Professor Bourama Toni, and Professor Abdul-Aziz Yakubu for their assistance in the preparation of this article.

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Robin T. Wilson



Daniel A. Williams

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# From Segregation to Department Head

*Nathaniel Whitaker*

As we stood at Fort Monroe at 6:00 a.m., gazing into the sea, I could imagine the ships approaching, bringing the first Africans to what would become this country, bringing my own ancestors. My wife and I had traveled to Fort Monroe in Hampton, Virginia, on August 24, 2019, to observe the 400th anniversary of this arrival. Amazingly, this site was only three miles from where I grew up. It is also the city where the Black women mathematicians in the book and movie *Hidden Figures* lived and worked. Their stories were hidden as were the stories of the triumphs of many African Americans. During much of my time navigating the White world as a mathematician, I myself have felt hidden and invisible. This invisibility is what the narrator in Ralph Ellison's landmark book *Invisible Man* feels. He is a Black man who feels invisible to Whites as he maneuvers through their world. This invisibility in the White world can also be a handicap when trying to further one's education or academic career, since the support from your fellow students and faculty can make a huge difference in your success. The combination of the recent death of George Floyd and the COVID-19 pandemic has somewhat uncloaked or made visible the plight of African Americans and sparked interest in their hidden history. This is the story of my journey from segregation to becoming the Head of the mathematics and statistics department at a major R1 university.

My parents were born in North Carolina in the 1920s in the midst of the Jim Crow era. My grandparents were sharecroppers and my parents began their life together as sharecroppers. They rented land from a White farmer and

paid the owner with the goods that they produced. The system was designed so that the sharecropper would always be in debt. In 1953, when I was two years old, my father took a job in the Newport News Shipyard in Newport News, Virginia. We moved into a segregated housing complex called Newsome Park where most of the shipyard workers lived. Newsome Park provided housing for African Americans in all professions which were accessible then, including shipyard workers, teachers, and computers. These were human computers who worked at the National Advisory Committee for Aeronautics (NACA), which became NASA, performing computations for space flight. From 1943 to 1958, Dorothy Vaughan headed the colored computing section. Mary Jackson and Katherine Johnson were both initially part of Vaughan's group. Mary Jackson became NASA's first Black engineer and Katherine Johnson was a well-respected mathematician who John Glenn and other astronauts relied on for flight calculations. The inspiring stories of these women were told by Margot Lee Shetterly in the book *Hidden Figures: The American Dream and the Untold Story of the Black Women Mathematicians Who Helped Win the Space Race* and later in the associated movie *Hidden Figures*. In Newsome Park, I lived one block away from Dorothy Vaughan. As in North Carolina, segregation was everywhere in Virginia. Our family went to a drive-in theater on Sundays with a fence down the middle, separating Whites and Coloreds (Blacks), with separate concession stands. When my parents bought food from White restaurants, they had to go to the back door to order and pick up the food.

My parents were not highly educated but they have always served as my role models. My father was very good with numbers. He was a self-taught mechanic who could take a car apart and put it back together again. In the south, the game of checkers was very popular among African Americans in the 1950s and 1960s. Men would gather in Black-owned stores with 20 men watching while two players competed in a game of checkers with a long

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queue waiting to play the winner. There were no official tournaments but my father was considered one of the best in the Hampton and Newport News area. He helped me with math while my mother helped me with reading and writing early in elementary school. However, after my first few years in school, I was on my own. My parents instilled in me the need to give back to the African American community. A person that excited my interests in math was my Uncle Bobby, my mother's youngest brother. He would visit us and would challenge me with math questions. He has served as one of the major role models in my life.

Needless to say, I attended a segregated school in Newport News. This was the time of Virginia's massive resistance to desegregation. Before 1954, Virginia tried to maintain its segregated schools, which was allowed, as long as the schools were equal in quality, known as "separate but equal." The facilities at the Black schools were either not equal to those at the White schools or did not exist at all. The Brown versus Board of Education Supreme Court decision in 1954 overturned "separate but equal" and required schools to desegregate. In 1958, Virginia Governor Lindsay Almond closed public schools in Charlottesville, Norfolk, and Warren County rather than comply with desegregation ordered by local judges. Over 10,000 students were left without schools. The schools reopened in winter 1959. From 1959–1964, Prince Edward County closed all of its public schools, refusing to appropriate any of its public funds to schools rather than desegregate. White students were given tuition grants using public funds to attend a private educational academy, while Black students had to fend for themselves. Newport News schools were not forced to desegregate until the late 1960s. Virginia and other southern states resisted desegregation through a wide array of tactics, especially through "Freedom of Choice" plans.

I started my formal schooling in 1956 at the segregated Newsome Park elementary school. When I was in the second grade, I had rheumatic fever and physically attended school for only a month of the entire academic year. My second grade teacher, Mrs. Granderson, brought my school work to me at my house every day, so that I could complete the school year. The African American teachers in our school genuinely cared about us and tried to prepare us for a world that would not be very welcoming to us. When we were given our books at the beginning of each school year and had to put our name in the book, I would see as many as ten to fifteen names of students who had used the book before me, since the books were hand-me-downs from White schools when they acquired newer books. Back then, I was an average student in math, but I improved, especially when the concepts became more abstract.

In 1963, my family moved from Newport News to Hampton, just a month into my seventh-grade year. In Hampton, I attended George Wythe Junior High School. Katherine Johnson, the mathematician portrayed in the

film *Hidden Figures*, lived a half mile from our new home in Hampton and across the street from my new school. My sister Vanessa later became her sorority sister. She loved to play cards and my sister and her other sorority sisters visited her frequently when she moved into a nursing home. They played games with her, but they really came to look in on her. My sister, Katherine Johnson, and a math teacher from Hampton High School are shown in Figure 1.



**Figure 1.** Photo taken in 2013 in Hampton, Virginia. From left to right is Vanessa Whitaker, Katherine Johnson, and Joyce Weeks (a former math teacher at Hampton High School).

The mothers of some of my classmates were also computers working at NASA. One of my classmates at George Wythe was Stephen Smith. His mother, Willianna Smith, worked as a computer under Dorothy Vaughan. In Hampton, as in Newport News, the schools were segregated. The Commonwealth of Virginia was under extreme pressure to integrate their schools and implemented the "Freedom of Choice" plan during my eighth-grade year in 1965. This meant that I could choose to attend any school in the city for my ninth-grade year, but I would have to be bussed there. A friend of mine, Sidney Ricks, asked me if I would attend Thorpe Junior High School with him, an all-White junior high school, since his mother was making him go. I had never really interacted with Whites since our communities and schools were segregated. Thorpe Junior High School was the old Hampton High School, and the very school that Mary Jackson, the engineer in *Hidden Figures*, had petitioned the City of Hampton in order to take classes there, just a few years earlier, in the 1950s. I told Sidney that I would go with him to Thorpe, but when I asked my parents, they said no. My parents had lots of negative experiences in their interactions with Whites. The Whites had perpetrated all types of violence and terror on them and their communities with little fear of justice. In one instance, my mother's uncle helped a White lady who was falling out of a carriage. She told other Whites that he

had behaved towards her in an inappropriate manner. A group of White men came looking for him but he fled to Baltimore and never returned. Lynchings of Blacks were common then as Emmett Till was lynched because of a similar accusation from a White woman. Given their experiences, my parents were rightfully cautious and distrustful with respect to interactions with Whites and were reluctant to allow me to attend this school. My father had a sixth-grade education, and my mother finished the tenth grade. My father had to quit school to work for his family. My mother was unable to finish high school because there was no high school for Blacks in her hometown. Because of educational constraints, they were never able to pursue their dreams but worked hard so that their children could pursue theirs. After a tremendous amount of pleading, my parents finally agreed to let me attend Thorpe Junior High School. Hardly any students chose to leave George Wythe and go to Thorpe that year. In retrospect, I know that Sidney's mother, who was college educated, wanted the best education for her son, and knew that things were not separate and equal at George Wythe.

At Thorpe, we were both put into the more advanced classes, having done well at George Wythe. The students in our homeroom were the children of doctors, lawyers, and other occupations which were usually not accessible to African Americans. I am sure that we were a shock to many of the White students who probably had not had many interactions with African Americans. Early on, one day at school, I saw that our homeroom teacher, Mrs. Bernstein, and some students at the school were very upset. Someone had drawn a swastika on a wall somewhere in the school. Maybe I had learned about this in my previous schooling but at that time I had no idea about it or what it meant. Sidney and I were isolated since most of the White students kept their distance from us. I was very fortunate that I had my friend Sidney in my classes and that I felt the support of my neighborhood every day when I came home from school. We were effectively invisible in the way Ralph Ellison describes the narrator in his book, "I am an invisible man. No I am not a spook like those who haunted Edgar Allan Poe: Nor am I one of your Hollywood ectoplasms. I am a man of substance, of flesh and bone, fiber and liquids, and I might even possess a mind. I am invisible, simply because people refuse to see me." It felt like we did not exist to the White students. This was also the time when boys and girls begin to notice each other. White girls were considered off limits for African American males for many reasons and there were very few Black girls at Thorpe. Academically, I had a hard time adjusting to all the new things and initially I struggled academically. In the ninth grade, we took French II, which was an advanced class. I felt very much behind academically in the class. I remember that Mrs. Bernstein (my homeroom and French teacher) would dictate to us a dialogue in French that we

had to write down. Initially I had no idea what she was saying. I eventually started memorizing what she was going to read in advance in order to do well. Geometry was also a struggle at the beginning. The geometry teacher, Mr. Riddle, would hand back exams from the highest scores to the lowest scores. I was always one of the last names called, near the bottom. In fact, a couple of months or so into the academic year I met with Mr. Riddle and the guidance counselor. They asked me if I wanted to go back into Algebra I. I asked to be given more time. I decided to work harder which included reading the textbook myself. This was a skill that I found out that I had and could use in the future. On the next exam, I made one of the highest scores. It was a little embarrassing because Mr. Riddle told my story in front of the entire class after handing back the exams. This accomplishment made me think that if I worked hard enough, then I could accomplish any goal. Thorpe was much better equipped than George Wythe; for example, in French class, we had audio equipment built-in at our desks where we could listen to native speakers speaking French. In geometry class, we had compasses and protractors for drawing angles and other figures on the board. Nevertheless, even though my previous schools lacked in facilities and equipment, I was happy that for my first eight years I had been nurtured in a Black environment. This provided me with an inner strength that I used and would continue to use to navigate through this foreign world.



**N. Whittaker**

**Sidney Ricks**

**Figure 2.** Photo taken in 1965 in Hampton, Virginia. Ninth-grade graduation photos of Nathaniel Whitaker and Sidney Ricks from H. Wilson Thorpe Junior High School taken from the 1965 *Deacon Dispatch*, H. Wilson Thorpe Junior High School's newsletter.

In the ninth grade at Thorpe, I had to choose what high school I would attend under the "Freedom of Choice" program. The decision now was to choose between attending the White high school, Hampton High School, or the Black high school, Phenix High School. These schools were separate but not equal. Mary Jackson could not take the courses that she needed at Phenix High School to become an engineer. Sidney and I had become very good friends

and we decided together to go to Hampton High School. We had gone through so much together (see Figure 2). At Hampton High, the situation was very similar to Thorpe. I did not like the environment and isolation but at least I knew what to expect. There were a few students that I had known from George Wythe Junior High School that had decided to choose Hampton High School for the tenth grade, as well as a few African American students from other junior high schools. Yet this was far from being a critical mass. The high school was much larger so the percentage of African Americans was probably about the same as at Thorpe. I was typically the only Black student in my classes. In the tenth grade, I did fairly well in my Algebra II class but not quite good enough to get on the track to take calculus in the twelfth grade. Nevertheless, during my time at Hampton High School, I did well in my math classes and enjoyed them. I was a little introverted, did not speak out or create any waves. However, I had some very talented African American male friends that did have some issues, in particular, those who felt comfortable challenging some teachers' ideas about society which related to race and fairness. My friends felt that they were penalized for their opinions. One of my more talented male friends was given a lower grade after speaking out, and as a result did not graduate on time. He is still bitter about that today. I graduated from the overwhelmingly White Hampton High School in 1969. During my senior year, I went to see my guidance counselor, who was White, to help me decide which colleges to apply to. I was, on one hand, disappointed that she only suggested HBCUs (Historically Black Colleges and Universities) but on the other hand felt that it would be a good thing for me. I had missed a lot of social development, being an "invisible man" at Thorpe and Hampton High.

For college, I attended Hampton Institute, Mary Jackson's alma mater and a small historically Black college in my hometown. Hampton Institute (now Hampton University) was and still is now a highly regarded institution of higher learning. My good friend, Sidney Ricks, also attended Hampton Institute. I had not given much thought to my major, but at the last moment I decided to choose mathematics, since that had been an area of strength for me. I did not excel in math in college. My calculus professor had a strong Russian accent. I had him for all semesters of calculus and had a difficult time getting much out of the courses. I did have some excellent African American professors after calculus who served as great role models, including Dr. Geraldine Darden and Mrs. Rosaline Exum, but the lack of a stronger calculus background made higher math courses more difficult. Eventually I changed my major and graduated in economics. I am pretty sure that my lack of success in mathematics was because of not having the correct work ethic, due to an underdeveloped frontal lobe, a problem that a lot of young males suffer from! To be honest, race was not a factor anymore and I was no longer

an invisible man and did enjoy myself a little too much socially. There was also a lot happening on campus. Quite a few African American leaders came to speak on our campus, including Muhammed Ali and Arthur Ashe. They along with the Black faculty at Hampton served as role models to the Black students in a world where Black accomplishments were typically hidden. This exposure was a source of pride and self-esteem. There were lots of famous Black musical groups coming to campus. This was also the time of protests and unrest against the Vietnam war and racism. Our academic year was cut short at least twice due to this unrest. One year, a student actually planted a bomb and blew up the famous Wigwam Building on campus. During my last year, I worked for Vehicle Services, transporting students to activities off campus, such as student teaching. In one instance, I was assigned to transport members of the board of trustees from the airport to campus. I remember distinctly transporting one board member, an older White lady who was a little rude to me, from the airport. Amazingly, the main character in *Invisible Man* drove for Vehicle Services at his Black college and had a similar experience transporting an important White person. My supervisor told me afterwards that the rude woman I had driven was the famous Margaret Mead. At the time, I had no idea who she was.

After graduating from Hampton Institute, I worked for the Army performing cost-benefit analyses on weapon systems at Fort Monroe for three years. Unbeknownst to anyone at that time, this was the site where the first Africans arrived, three miles from where I was raised. After three years, I transferred to Fort Lee in Petersburg, Virginia. Petersburg was a very rural area but not far from Richmond. I was not very happy in my job and I missed academics and mathematics. There was a project that I worked on where I used integrals from calculus to solve a problem. My supervisor did not completely understand but was impressed. Those projects were few and far between. I wanted to give math another chance since I believed that I could be good at it if I really committed myself to it. I started taking evening classes at Virginia Commonwealth University in Richmond in math, physics, astronomy, etc. Instead of relying totally on the instructor, I decided to read the books. I did very well and decided to apply to some graduate schools in mathematics.

I was only accepted at the University of Cincinnati with a teaching assistantship and the University of Connecticut with no support. The decision to leave my good job was not an easy one, but I was not happy there, so I began graduate school at Cincinnati in 1979. I started studying the summer before, reading books and doing problems associated with courses that I would take. I was afraid of failure and worked harder than I had ever worked in my life with great results. In my analysis course, a well-meaning professor told me that I really surprised him. He said,



"in his classes the Asians usually were the best students, the Whites next, and the Blacks were the worst." I guess that he felt that he was giving me a compliment. I was the only African American student in the graduate program in mathematics at Cincinnati. My favorite class was numerical analysis, taught by a young energetic assistant professor, Diego Murio. I loved the subject and Murio's passion. After two years and receiving a Master's degree in mathematics at Cincinnati, I was planning to look for a job where I could use my mathematics. Professor Murio encouraged me to apply to a PhD program, in particular, to the University of California at Berkeley, where he had just done his doctorate. He told me that it was a supportive environment for minorities. Several faculty at Cincinnati wanted me to complete my PhD at Cincinnati. They thought very highly of Berkeley and felt that Berkeley might be a reach for me, but I was not motivated to stay at Cincinnati since I felt very isolated there. I applied to Berkeley but did not tell a lot of people. After I was accepted, I told Professor Murio and he spread the news to the other faculty. I perceived that the analysis professor and many of the other professors were apprehensive about me going to Berkeley. However, I still had the inner belief that I could do anything if I worked hard enough. Maybe I was fooling myself, but I truly believed that. After all, I had conquered Mr. Riddle, VCU, and Cincinnati! I graduated from Cincinnati in May 1981 and got married in June 1981. My wife and I agreed that we would give Berkeley a try.

At that time and even now, there are very few African Americans who earn PhDs in mathematics. From 1982 to 1986, there were 6, 3, 4, 7, and 6, respectively. Most places at that time had no African American graduate students, and even now, not much has changed. Nevertheless, two places have been noteworthy: the University of Michigan and the University of California at Berkeley. At Michigan, during the 1970s, the graduate program averaged approximately six African American graduate students each year. The key figure at Michigan was a White professor, Maxwell Reade (1916–2016). He had the authority to offer fellowships to African American students that he recruited from HBCUs. These students resulted in a critical mass where they supported each other academically and socially. They formalized their group and called it the Ishango Math Society. They encouraged cooperation among African American students (both graduate and undergraduate) through lending books, tutoring, and collaborative studying for courses. Between 1970 and 1976, over 40 African Americans were admitted to the graduate program and six received their PhDs. They have been referred to as the Michigan 6. Maxwell Reade's counterpart at Berkeley was Leon Henkin (1921–2006) with strong support from George Bergman through the Mathematics Opportunity Committee. Professor Henkin negotiated with the department to reserve 10% of the admissions and financial

support for women and underrepresented minorities. From 1978 to 2016, Berkeley awarded approximately 27 PhDs to African Americans in mathematics and I am sure to quite a few women. I made this count using information from George Bergman. Recently, several other schools have had some success, including Maryland with Professor Raymond Johnson.

I had studied the structure of Berkeley's program and it seemed to me that the big hurdle was the Preliminary exam that one must pass by the end of the first year. Berkeley provided me some support to come during the summer and start preparing for the exam. I attended some classes and a Prelim workshop. When I got to Berkeley, I met two more advanced African American graduate students, Richard Baker and Darry Andrews. I was quite surprised since I had seen no African American graduate students in math at Cincinnati. Richard, who is now a professor at the University of Iowa, told me that he was committed to helping me pass the Prelim exam. He met with me quite frequently to go over old problems. Darry, who is at Ohio State now, educated me about negotiating the Berkeley math department as an African American. Some of the faculty were not very accepting of Black graduate students. He noted several African American students who had been victims of bias and unfair treatment from some faculty. Taking a class or having to depend on one of them for your success could be dangerous. A very appealing aspect of the first exam (Prelim exam) was that you received a number to put on your Prelim exam rather than your name. Hence the grader did not know whose paper they were grading. I liked this very much as a way of insuring fairness. The year that I was admitted, the department admitted around 100 new graduate students. Based on previous results, somewhere between 33%–50% would be successful. With such a large class, the passing percentage on the Prelim exam might be at the lower end. I passed the Prelim exam on my first try with a very respectable score. The day that I saw my score beside my given number 12 and the word "pass" was one of the happiest days that I can remember.

The next step was to organize an oral exam (the second exam) where a member of the exam committee had to agree to supervise your thesis. I knew that I wanted to do numerical analysis but more importantly needed a good professor for a thesis advisor. It was important to me to choose someone who believed that I would finish and would be supportive. I had to use my intuition and gut here. At the department's social hour, I remember meeting Professor Alexandre Chorin, who in 2012 would be awarded the National Medal of Science. He was thought of very highly, yet there was a calmness and genuineness about him. I was a little apprehensive about asking him to be my thesis advisor since he already had 10 students, but I did. There were no African American students in the research area that I was interested in, but I got some great support from quite



a few White and international students in preparation for the oral exam. Two of Professor Chorin's students, Claude Greengard (now at Two Sigma) and Robert Krazny (now a professor at Michigan), gave me several mock oral exams. Besides Professor Chorin, the other members of my examining committee were Professors Andy Majda, Ole Hald, and Harry Morrison. I had taken numerical analysis with Professor Majda, who would become one of the leading applied mathematicians of his time. He had very high standards but was very supportive and fair. Professor Morrison was an African American professor from physics who had served as a mentor for myself and many Black doctoral students in STEM at Berkeley. There were very few African American professors in STEM at Berkeley so Professor Morrison was the go-to guy. I had met David Blackwell, a very famous statistician, who is African American, but he was not in the same research area. I took the oral exam and did very well and Chorin became my thesis advisor.

Around this time, several other more junior African American graduate students had entered the PhD program. Two that became very good friends with me were Duane Cooper (now a professor and chair of the math department at Morehouse College) and Janis Oldham (now a professor at North Carolina A & T). Duane and I would have a beer and play Ms Pacman every Friday as a ceremony to end our week. Being more senior, I tried to support Duane and Janis as much as possible as Richard and Darry had supported me. We had exam review sessions and potlucks, similar to the Ishango group at Michigan. That critical mass made a huge difference. Duane and Janis did the same for students entering after them. Berkeley was a good place to be with this number of supportive Black graduate students. Berkeley also recruited a lot of international students and many were very friendly to the Black students. In general, things worked out well for me at Berkeley except for the language exams. Ironically, I had never taken German but passed the German exam on my first try. Nevertheless, I could never pass the French exam after many tries. During my last semester, the department gave a special French exam only for those who were graduating. Voila, I passed. The only negative that I remember about my time there was when two White graduate students tried to physically intimidate me, but I ignored them. I was used to much more, being from the South. Some of my fellow African American graduate students also had negative encounters. However, we supported each other and had backup from Professors Henkin and Bergman. At Berkeley, towards the end of my time, my analysis professor from Cincinnati visited Berkeley. He stopped by my office and congratulated me. He admitted that he had not been confident that I would be successful at Berkeley but he was proud of me. Richard Baker and I finished Berkeley together in 1987. My father and sister flew to California for my graduation.

As I have grown older, I have come to realize more and more how much my parents had been shackled by racism. There were many esteemed, educated people at the graduation ceremony, but my parents are my heroes. For me, receiving my PhD showed them what they could have accomplished without those shackles of Jim Crow holding them down. I stand on their shoulders. People in the community where I grew up did not know what a person with a PhD in mathematics does. One of my mother's best friends said she was very proud of me and now she had someone that could help her with her taxes.

After Berkeley, I decided to try the professorate. I got quite a few relatively good offers for tenure track assistant professorships. In making a decision, certain things were important to me. Most importantly, I wanted to go back east. I even interviewed and got a job offer from my old school, the University of Cincinnati, which I decided to decline. In the end, it came down to deciding between the Applied Math Department at the University of Virginia and the Math & Statistics Department at the University of Massachusetts at Amherst. UVA should win hands down. That is my home state and for me anywhere north of Washington DC would have been too cold. Why would I even visit UMass. During my interview at UVA, I talked to faculty in the department during the day as is typically done. Usually a group of math faculty take you to dinner, but a Black administrator that I had never met, from equal opportunity, took me to dinner. This seemed odd to me. At UMass, someone cooked dinner for me at their house, another person had a reception at their home, and another night we all went to dinner. The department was also very interested in my mathematics and felt that I would fit in. Even then it was hard to choose the "Arctic" over my state university, but I did. One additional fact that influenced me is that there were already two African American Full Professors (Donald StMary and Floyd Williams) at UMass. This was astounding since almost all majority institutions had zero and I would make the number three!

I am now a Full Professor in my 33rd year at UMass. I have published over 40 refereed mathematical papers. In my thesis at Berkeley, I began my research developing numerical methods for evolving fluid interfaces. At UMass, I continued this work but also began work with a colleague, Bruce Turkington, to produce a series of papers on 2-dimensional fluid turbulence. Most recently, I have worked with Panos Kevrekidis in math biology. I also published five other papers in mathematical physics with Panos and our good friend Rudy Horne, who was a professor at Morehouse College. Rudy was African American and was the mathematical consultant for the movie *Hidden Figures*. This is another one of my connections to *Hidden Figures*. Unfortunately, Rudy passed unexpectedly in 2018. I have traveled quite a bit because of my research. I spent a sabbatical year at Ecole Normale Supérieure de Lyon; one

semester in the math department and one semester in the physics department. Still, because of the scarcity of African American researchers in mathematics, it is difficult not to feel like an invisible man. At research conferences, one feels alone and isolated. People form research groups with those that they are most comfortable with and Blacks are typically excluded. In conversations, some people avoid making eye contact, as if you do not exist and your statements are given less value. Part of this is the culture in mathematics but this can be impactful for those, like African Americans, who are already not quite sure if they belong.

Growing up in a supportive African American community and having been taught by my parents the importance of giving back, I have been slightly disappointed to not have been able to impact more African Americans in mathematics as some of my friends that are employed at HBCUs have done. Nevertheless, I have mentored quite a few students and influenced their choices as Professor Murio influenced me. I love numerical analysis and have communicated to others the beauty in it such that they decided to pursue the area. I advised and supervised the thesis of the first African American (Idris Stovall) to receive a PhD in mathematics at UMass Amherst. I have supervised or co-supervised seven PhD theses in total at UMass, four of them women. I have mentored quite a few undergraduate students including a very talented young lady, Heather Harrington, starting from her freshman year at UMass. She now has a permanent faculty position in math biology at the University of Oxford. I played the role of Harry Morrison in mentoring many other Blacks at UMass in STEM.

In the Amherst school system, it was known that Black kids did not do well, in general. This was true regardless of the education or economic status of their parents. When my son was in elementary school, there were very few Blacks in the honors or AP classes in the high school. Trying to counter this, I started tutoring my son early in math. My wife and two other ladies, based on my efforts, started an academic program (called AIMS) for African American children in the Amherst school system. The core of this program was a two-hour math class every Saturday morning. The program built a community and exposed the students to a broad range of activities. At one point, we had over forty kids participating. I taught the upper level math classes with another parent who was an engineer. Other parents taught the lower level classes, starting with the times tables. These were not remedial classes. Our objective was to push the kids farther than their schools did, because we believed that success in math would overflow into other subjects. This program lasted almost nine years and was held on the UMass campus in the math and statistics building. It was very successful in that almost all of the students took AP calculus and other honors courses in the high school. In addition, most went on to very prestigious colleges and have spoken and written about how impactful AIMS was

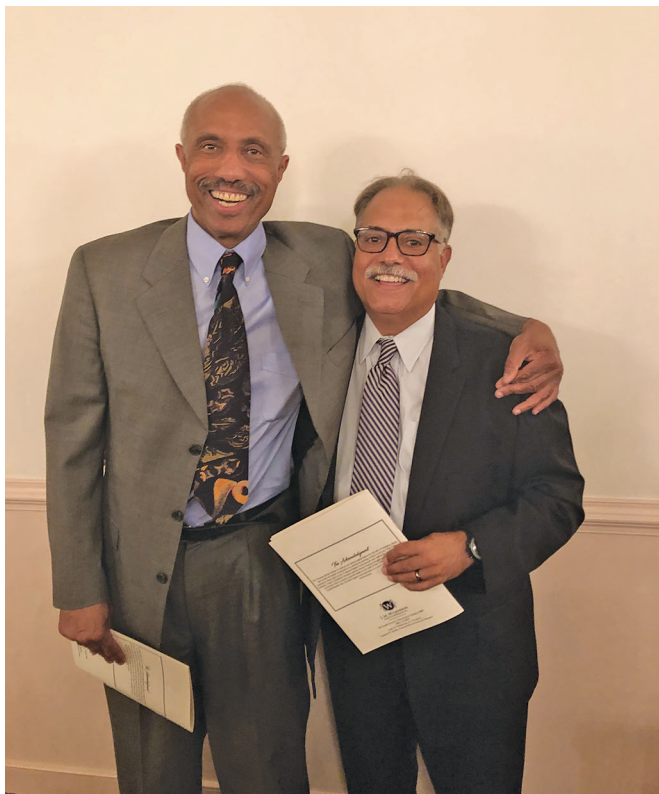
in their lives. I won the University Distinguished Outreach award as well as the UMass system President's Award for my efforts but I was just a part of the efforts of some super people, especially my wife. This is one of my proudest achievements.

I had been thinking about retiring, but in January 2018, after 30 years at UMass, I was offered the Department Head's position. Many of my colleagues asked why would I want it. It is known everywhere as a thankless job. Nevertheless, I decided to take it. It is a huge job, running a department with 43 Tenure Stream Faculty, 17 Visiting Assistant Professors, 20 Lecturers, and a staff of 12. The Department has about 100 graduate students and 1,000 undergraduate majors and teaches over 15,000 students each year. This position gives me a platform to make an impact in many ways. This includes increasing the representation and success of underrepresented groups in the mathematical sciences. I can also serve as a role model. I have been amazed at how excited some African American students have been to meet me. Since becoming Head, I have been very proactive in the recruitment of minorities and women. I have made eight tenure track hires. Three of the eight were women and one was an African American male, which is extremely rare in math and statistics. I have also hired three permanent lecturers which includes an African American female and an African American male. Due to the urging of Assistant Professor Annie Raymond, the department now teaches a 3-credit class once a year at the Hampshire County Jail. Because of the size of our department, the department head does not usually teach but I wanted to be involved so I co-taught the inaugural course in finite math at the jail in Spring 2019 with Professor Raymond. There were five inmates in the class who were very appreciative.

Amy Harmon wrote several articles in the *New York Times* about the difficulties in being a Black mathematician. She counted 1,769 tenured mathematicians in math departments at the top 50 US universities. There were 13 Black tenured professors in these departments or 0.7% of the total. Blacks make up 13% of the population. How many Black department heads are there in the top 50 or top 100 math departments in the US? I know of one other African American department head.

About 20 years ago, I became very interested in genealogy. I especially wanted to know who my ascendants were. Because of a lack of information about individual enslaved people, there were many dead ends. Nevertheless, through the combination of censuses and DNA, I was able to put names to my great- and great-great-grandparents and this caused me to think about their difficult lives. Slavery, Jim Crow, and systemic oppression have limited their dreams for their future and the future of their descendants.

I have come a long way in my life, from a kid on the colored side of a drive-in theater to the Head of a Research



**Figure 3.** Photo taken in 2018 of Nathaniel Whitaker and Sidney Ricks in Petersburg, Virginia.

I mathematics and statistics department. I stand on the shoulders of so many “hidden figures.” In taking the Head’s position, I am fulfilling the wildest dreams of my ancestors. I am inspired by Maya Angelou’s words in her poem *Still I Rise*. As she states, “Bringing the gifts that my ancestors gave, I am the dream and hope of the slave.”

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Nathaniel Whitaker

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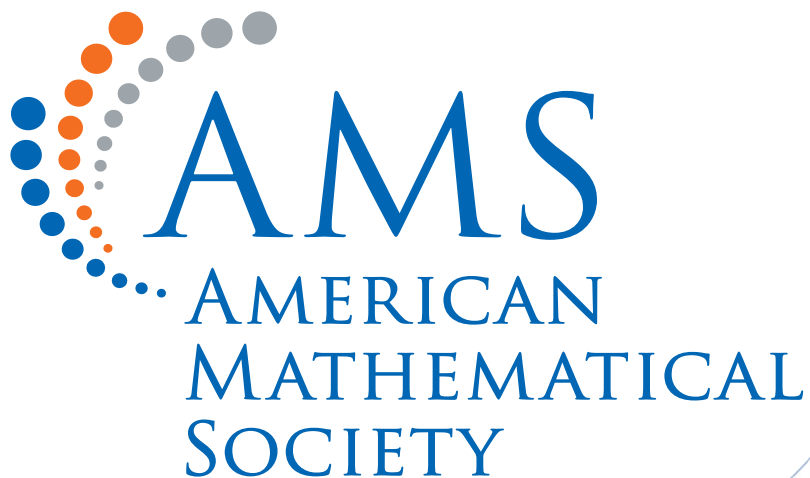
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# Considerations for Increasing Participation of Minoritized Ethnic and Racial Groups in Mathematics

*James A. Mendoza Álvarez and Minerva Cordero*

*Note: The opinions expressed here are not necessarily those of Notices.*

When addressing a group of students and faculty, a colleague says, “There are opportunities for summer mathematics programs; *unfortunately*, these are only for minority students.” During a discussion about ways to increase access for minoritized<sup>1</sup> groups, a colleague offers, “Oh, but *you’re* different,” effectively disregarding suggestions made by a faculty member from a minoritized ethnic group. When discussing the academic job market and negotiations, a colleague remarks, “Well, you’re the good kind of Hispanic because you don’t *look* Hispanic.” Surprisingly, this is a sampling of comments made in professional settings in mathematics that can discourage mathematicians of color or relegate them to an inferior status that neither recognizes their contributions as professionals nor values

their unique perspectives in addressing issues of disenfranchisement of students of color in mathematics. Regrettably, microaggressions such as these, experienced by mathematics faculty from minoritized ethnic and racial groups, are far too common. Why is it *unfortunate* that only minority students can apply for certain mathematics programs when their access and participation has been historically abysmal in many mathematics programs? Judgments and stereotypes about how one should look or act when one identifies with an ethnic group compartmentalize members of minoritized groups and do not recognize the wide diversity in ethnic communities. What are the differences between students or colleagues from ethnic or racial groups historically marginalized in mathematics and those who have not been?

In this article, we share our perspective on increasing participation of minoritized ethnic and racial groups in mathematics, informed by our experiences as Hispanic mathematicians<sup>2</sup> who have successfully navigated the world of academia. James grew up in rural south Texas and attended East Texas State University;<sup>3</sup> he received his PhD in percolation theory from the University of Texas at Austin. During his final years at Austin he became interested in

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<sup>1</sup>We choose the term *minoritized* instead of *minority* to connote the marginalization imposed by individuals or institutions on people of color.

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<sup>2</sup>For full transparency, the authors are also husband and wife.

<sup>3</sup>East Texas State University became Texas A&M Commerce in 1996.

research in mathematics education. Minerva grew up in the countryside in Bayamón, Puerto Rico, and attended the University of Puerto Rico. She received her PhD in finite geometries at the University of Iowa. To situate and reflect on our experiences in mathematics as students, teachers, and researchers, we use Gutiérrez's [Gut07, Gut08] framework to conceptualize equity in mathematics education and Ladson-Billings's [Lad95] propositions that distinguish culturally relevant pedagogy.

Gutiérrez [Gut07] describes four dimensions of equity in mathematics education: access, achievement, identity, and power. To achieve equity, students need access to meaningful mathematics and mathematics resources; in turn, this access must translate into *achievement* outcomes that enable advancement. Moreover, Gutiérrez [Gut08] points out that many studies about differences in performance between minoritized ethnic and racial groups in mathematics and nonminoritized groups focus solely on access or achievement with little attention given to the dimensions of *power* and *identity*. Hence, focusing on power imbalances between minoritized and nonminoritized communities may provide insight into complex issues affecting efforts to increase the number of students from historically marginalized ethnic or racial groups in science, technology, engineering, and mathematics (STEM). Additionally, we must consider the effects of minoritized individuals' perceived pressure to minimize familial or cultural ties to pursue their career choices or adapt to the profession. Gutiérrez emphasizes that studies that narrowly focus on issues of access and achievement have limited ability to address issues of power and identity.

We begin exploring the equity dimension of achievement. Our personal *achievement* in mathematics in our early years perhaps is not so different from our nonminoritized peers. We would both be described by our former teachers as high achievers, having each graduated at the top of our high school classes. The rural schools in south-central Texas and Puerto Rico we attended certainly do not rise to the level of the best schools by almost all metrics, but our experiences and self-perception as academically capable, especially in mathematics, shaped our ideas about persistence and achievement. National Medal of Science awardee Professor Richard Tapia often states that he did not go to a top-notch high school or community college, but in each of those environments he was "the best," and encourages students to strive to "be the best" in whatever environment in which they find themselves [NSTMF19]. In Treisman's [Tre92] research, which led to the creation of the widely successful Emerging Scholars Programs (Treisman-style), he specifically addresses capitalizing on STEM-intending minoritized students' perception of themselves as academically capable (see also [HMT07, Epp99]). Asera [Ase01] describes this as Treisman-style programs' emphasis on "knowing who your students are and recognizing their strengths" [Ase01, p. 11]. Recognizing

the significant achievement in mathematics of minoritized students, given their limited exposure to common experiences afforded to students in high-profile schools or from affluent backgrounds, provides a foundation for valuing the contributions of students from underserved communities. This recognition also presents opportunities for finding ways to provide experiences that validate their past achievement in mathematics and for helping them build appropriate pathways that prepare them for a successful career as a mathematician.

A focus on validating past achievement in mathematics, while creating bridges to critical mathematical thinking that positions students for further study of mathematics, underscores the need for access to teachers, programs, and other experiences that build upon students' ways of knowing. That is, their culture, current family life, neighborhood, and school environment affect their perspectives in ways that many teachers, who do not come from similar backgrounds, may not understand [Civ17]. Thus, the first step toward increasing access should be understanding the human beings involved and their environment; when omitted, well-intentioned efforts promoting "the right mathematics program" for underserved schools often backfire. Putting rich mathematical problems within a student's grasp entails providing scaffolding and other strategies for helping the student see the relevance of the mathematics involved. Some ways of doing this are to provide a historical, cultural, or social context; to situate the knowledge in future career plans; or to be more explicit about how mathematical ideas build from foundational knowledge. From 2009–2015 we had funding from the National Science Foundation Graduate Teaching Fellows in K–12 Education (GK–12) Fellowship Program (NSF DGE #0841400) to work with students and teachers from middle and high schools. We selected schools with the highest percentage of students from historically marginalized ethnic or racial groups in STEM and where the majority were socio-economically disadvantaged (over 90% of the students were from these populations). The goal of the *Mathematically Aligned Vertical Strands Connecting Mathematics Research Pedagogy, and Outreach for GK–12 Fellows and Teachers* (MAVS) project was to develop mathematicians who could communicate research in mathematics in a meaningful way to a broad audience. The underlying principle of the program encompassed building an understanding of the students, teachers, and school environment to better situate a partnership between teachers, research mathematicians, and graduate students [see CEJ14]. The graduate students partnered with teachers to explore the mathematics curriculum in their courses and how they could build a roadmap between their research in mathematics and the mathematics students were learning. The graduate students spent at least ten hours per week in the classroom with the teacher getting to know the school environment and the students. We held steadfast to the idea that we would work

with the highest need schools. Moreover, we insisted that professional development for the graduate students and teachers emphasize high-quality mathematics that started from where the students were. Our commitment to this, even when facing challenges, remained unwavering, in part because of our own personal and professional understanding of underserved environments. The program integrated our firm belief that students be provided experiences that help them make connections between the mathematics they are learning, the mathematics they will learn, and the ways in which it is relevant to their lives. For example, in a seventh-grade class when students were working with unit rates, the graduate student connected this to building insight into the fact that rates are not always constant and into how understanding cell population growth rates was important to her research modeling the inflammatory response during the healing process such as after eye surgery or receiving dental implants.

Achievement and access, as Gutiérrez [Gut08] points out, are often the focus of discourse for addressing issues related to increasing the pipeline of minoritized communities in mathematics and mathematics-based disciplines, but the other two dimensions of equity, those of identity and power, receive less attention. What exactly does identity mean for two Hispanic (one Chicano and one Puerto Rican) full professors in a mathematics department? We navigate a bicultural world: one our professional life and the other our familial connections and cultural traditions. We maintain strong familial connections grounded in our cultural values and may at times forgo making progress on a research paper, attending a research conference, or taking on another project in favor of caring for a loved one, meaningfully engaging in a family event, or spending quality time with our family. The claim here is not that our colleagues would not make similar choices, but that those are considerations that weigh heavily on our minds, more often than not, when we make decisions that might differ from some of our colleagues. The conflict is the worry that the reasons for making the culturally-grounded choices may be misunderstood or be perceived as not being serious about our careers. Although elders in our families emphasized “cuide su trabajo” or “take care of your job” (because in their experience and reality, if you don’t work you don’t eat), sometimes meeting our professional responsibilities entailed making tough choices that run contrary to typical practices in our culture such as living far away from the extended family, not being able to be at the hospital when a loved one was admitted, missing family social events, etc.

The other culture we navigate is that of researchers and faculty members at a research-intensive university. Cribbs, Hazari, Sonnett, and Sadler [CHSS15] indicate that a positive sense of affiliation for mathematics or the development of mathematical identity is rooted in “opportunities...both inside and outside of the classroom that include experiences where students are *recognized* in

mathematics” [CHSS15, p. 1060]. We were fortunate in that our mathematical identity developed from early successes and recognition in mathematics, albeit in underserved school environments. In addition, while we both pride ourselves in our ability to be resourceful and independent, we recognize the importance of mentors at key moments to promote talent in research or teaching, offer useful advice on essential aspects of navigating the tenure and promotion process, or provide guidance in grantsmanship. Without appropriate mentors and persistence, the friction between familial or cultural identity and professional mathematical identity may impede the necessary adaptations for navigating the mathematics profession. Additionally, balancing the tension created by maintaining focus on research expectations while simultaneously giving sufficient attention to broader outreach in the mathematical community can be overwhelming without a network of colleagues providing guidance about how to manage these complementary, but sometimes competing, goals. Our mentors and our own awareness helped us understand that our agency to effect change rested in doing solid work and focusing on key areas relevant to the evaluation of our performance and quality of our work. If, at times, our familial identity or origins might have put us in seemingly disadvantageous situations, having established a firm research identity helped overcome any powerlessness and made it possible to effect change.

What can mathematicians do to help students who perceive themselves as academically capable, had limited access to mathematical experiences that advance mathematical achievement, lack mentors and role models for “adding on” a mathematics identity to their cultural identity, and may feel powerless or invisible? This can be addressed by culturally relevant pedagogy which is distinguishable by “three broad propositions or conceptions regarding self and others, social relations, and knowledge” [Lad95, p. 483]. Some of these ideas were intuitive for us personally and largely derived from our own trajectories. Perhaps this contributed to our successes in the classroom as attested by the numerous teaching awards we each have earned, which include the highest honors in teaching at our institution and the University of Texas System.

The most salient aspects of Ladson-Billings’s [Lad95] work regarding conceptions of self and others that are reflected in our own teaching are believing all students are capable of academic success in mathematics, seeing ourselves in our students and as part of a community, believing that teaching is a way to give back to the community, and embracing the idea that teaching is a process of extracting student knowledge and building upon it [Lad95, pp. 478–479]. Partly due to our own backgrounds and mathematical trajectories, we see broad benchmarks for where a student needs to be in terms of their mathematical development, say by a certain age or certain level. For example, James’s high school did not offer calculus, and Minerva’s high school did not offer a mathematics course

beyond Algebra II. Some might think that a student with this level of exposure may be too far behind and catching up would be insurmountable. Nevertheless, we do recognize that a student needs a plan or a roadmap to eventually obtain the mathematical foundations and mathematical habits of mind to be successful. When we embrace our historically marginalized ethnic and racial groups in our mathematical community, we are welcoming all learners as contributors in the learning environment. We are confident that our students already have important knowledge based in their experiences and backgrounds. Our job is to capitalize on this and help them construct their understandings. Genuine encouragement for solid mathematical ideas and conclusions is common in our classrooms because we put ourselves in the place of the student. We also strive to find ways to validate their assertions, given the status of their knowledge at the time, by practicing an asset-based approach to mathematics education (see also [CPBea18]). An alternate way to gain insight about the common mathematical experiences of STEM-bound students from minoritized groups is to spend time in ethnically and racially diverse public-school classrooms and connect with local schoolteachers who teach students from diverse populations. Also, involving students in classroom tasks that require active engagement and cooperative learning provide opportunities for understanding students' thinking and prior knowledge. This enables the instructor to determine questioning sequences, develop subsequent tasks, or find other ways to draw out insightful student thinking or comments. In their study exploring successful transitions into a university STEM program, Ulriksen, Holmegaard, and Madsen [UHM17] found that construction of disciplinary identity relies upon students' engagement with the curriculum as well as their prior knowledge and experiences.

With respect to social interactions, Ladson-Billings [Lad95] refers to teacher-student relationships that are equitable and reciprocal. We have a heightened awareness of the need to provide students the opportunities to collaborate, explain, and participate in their learning. In the classroom, this can be achieved by reflecting on: Who is being excluded in discussions? Who is isolating themselves during collaborative work? How can we connect students' mathematical understandings in productive ways? Who is struggling with the mathematics and in what way? Establishing a good rapport with students helps facilitate interactions that are equitable and reciprocal. Moreover, it enables instructors to ask students to expand, elaborate, and explore more mathematics in a safe and supportive way.

Student contributions and participation are a central part of our own teaching. This relates to the idea that knowledge is "shared, recycled, and constructed" [Lad95, p. 481]. Having students participate in the construction of a proof or sharing their ideas about subsequent steps to take or using strategies from previous proofs emphasizes the value

placed on participation and engagement in the classroom. If everyone is expected to contribute and participate, this would certainly include students of color. However, developing an awareness for enabling diverse voices to be heard in the classroom is critical for ensuring that minoritized students participate at equitable rates. Also, providing multiple ways to assess students' knowledge and various ways of recognizing excellence empowers students to persist and experience satisfaction in their mathematics learning.

Identifying with our minoritized students (which, by the way, does not require one to be a member of a minoritized group) provides a powerful perspective for advocating for their success in mathematics, their inclusion in the scientific community, and their access to careers in STEM. Part of the process for identifying with them involves informal individual and small group meetings to learn about their views about your class and the university environment and to learn about their past experiences. Some questions we can ask: What am I doing in class that is working for you? What am I doing in class that is not working for you? What events on campus are important to you? How do you feel about recent events on campus? Tell me about your past experiences in mathematics? What strategies did your teachers use that worked well for you? What strategies did not work well for you? Both student confidence and persistence correlate with positive informal contact with faculty (see [KMB10]). Thus, we call for a commitment to finding ways to understand minoritized students' social and mathematical experiences and use these ways as an asset to support students in acquiring the tools to compete in academia or industry at a level that we would want for our own children.

Perhaps the "difference" between a mathematics major from a historically marginalized ethnic or racial group and other mathematics majors derives from the vastly different messages each has received about who can and cannot do mathematics? It is our collective responsibility to dismantle these messages through genuine encouragement, excellent teaching and mentoring, and providing opportunities for success. We call for *authentic* experiences for minoritized students of color that provide many opportunities to experience positive recognition in mathematics that legitimately fosters their mathematical identity.

While we acknowledge that there is no magic bullet or formula for increasing access and participation for students from minoritized ethnic and racial groups, some considerations that arise from our experiences coupled with interpretations from research-based frameworks in mathematics education are:

1. Recognize students' academic accomplishments. For example, acknowledge insightful ideas they offer in class or send them an email about it, or congratulate them on excellent performance on a homework task or an exam.



2. Interact with students in class and encourage them to attend office hours. For example, informal conversations that help instructors get to know their students are much more likely to take place during office hours. Praise their questions and take them seriously.
3. Adopt teaching strategies that encourage active engagement in the classroom and learn how to facilitate productive mathematics discussions and ways to monitor equitable participation by minoritized students.
4. Assign tasks that allow multiple entry points and look for ways to welcome contributions and build upon them.

Both mathematics majors from historically marginalized ethnic or racial groups and other mathematics majors share a love of mathematics and see themselves as “math people.” Recognizing this, we can all capitalize on this common ground. We should treat students equitably by delving into their desire to move forward with mathematics but affirming that their experiences before and during our class may be quite different than their nonmarginalized peers and have nothing to do with the mathematics being taught. We all must give minoritized students equitable, rich mathematical opportunities and provide the support they may need for advancement.

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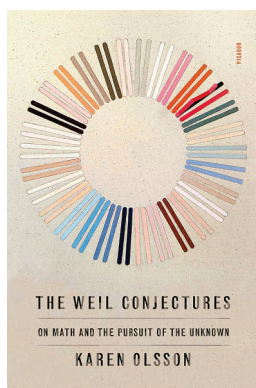
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# The Weil Conjectures

*Reviewed by Brian Hayes*



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***The Weil Conjectures***  
*On Math and the Pursuit of the Unknown*  
by Karen Olsson

In the winter of 1940 Simone Weil was urging her brother André to write an expository account of his mathematical research, for her own benefit and for the world at large. He had plenty of time, she pointed out. He was confined to a French prison, awaiting trial on charges of failing to report for mil-

itary service. André replied:

Telling nonspecialists of my research or of any other mathematical research, it seems to me, is like explaining a symphony to a deaf person. It could be attempted, you could talk of images and themes, of sad harmonies or triumphant dissonances, but in the end what would you have? A kind of poem, good or bad, unrelated to the thing it pretends to describe.

André's dismissive response strikes me as surly and arrogant, and yet there's surely truth in it. The latest ideas from the frontiers of research are seldom fit for armchair consumption. It's the nature of a frontier that you have to do some bushwhacking to get there. On the other hand, if the discoveries of a research community are so abstruse that they can never be understood outside a small coterie of initiates, what's the point of discovering them? Somehow, the explorers of new territory have to send the occasional

dispatch back to civilization, to let us know what they've found.

André did eventually write the account that Simone had asked for, but it was highly technical, "a treatise in the form of a letter," perhaps meant more to intimidate than to inform. (An English translation was published in the *Notices* in 2005 [1].) Simone wrote back, "I understood nothing." She continued to worry that mathematics was becoming too remote from ordinary life. With a fierce sense of social responsibility, she wanted her brother's work to serve human needs, or at least to reveal something about the world we all live in.

Karen Olsson, who tells this story in *The Weil Conjectures: On Math and the Pursuit of the Unknown*, seems sympathetic to both sides of the dispute. Like André, she believes that mathematics should speak for itself. She wants to understand it in mechanistic detail; she won't settle for gauzy metaphors or analogies. But she also shares Simone's concerns. Why should I care about these forbidding abstractions? she asks herself. What do they have to do with my life as a writer, a parent, a citizen? No clear answers are



Figure 1. André and Simone, 1922.

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forthcoming, and yet still she yearns to learn, as if echoing Hilbert's "We must know! We will know!"

Before going further I should make clear that *The Weil Conjectures* is not a textbook or a scholarly monograph. It is not addressed to an audience of mathematicians. But it raises questions about relations between mathematics and society that may well be of interest to the mathematical community. This issue is commonly discussed in terms of *outrreach*—the challenge of communicating research-level mathematics to the public. In Olsson's case it also becomes a question of *inreach*: how can we help someone who feels a powerful attraction to mathematical ideas but cannot negotiate the rugged terrain of prerequisite knowledge?

The heart of Olsson's book is a personal essay, in which she describes her own intense and turbulent encounters with the world of mathematics. That narrative is braided into the stories of the Weil siblings—whose lives were also marked by intensity and turbulence.

André Weil (1906–1998) was a child wonder who grew into an enfant terrible, swaggering across the landscape of French mathematics while still in his twenties. He was a founder and ringleader of the Bourbaki collective. During his jail time in 1940 he proved a variant of the Riemann hypothesis for curves over finite fields. The conjectures referred to in Olsson's title extend this result from curves to varieties (the higher-dimensional analogs of curves) and forge an unexpected link between two distant realms—number theory and topology. By now the conjectures are all theorems (proved by Alexander Grothendieck and Pierre Deligne, among others), yet they are still widely known as the Weil conjectures.

Simone Weil (1909–1943), equally brilliant and precocious, was her older brother's first pupil and devoted childhood companion. Olsson sets the scene: "Simone and André memorize long sections of verse by Corneille and Racine, and they recite pieces of them in turn, staring bug-eyed at each other. It's a contest: although they smirk as they call out the lines, every time one of them misses a word or mangles a phrase, the other delivers a hard slap to the face."

Brother and sister both went on to study at the École Normale Supérieure, but Simone drifted into philosophy rather than mathematics. Later she moved on to social and political activism, and then mystical theology. She could not endure a comfortable life while others suffered, so she sought out work in factories and mines—jobs for which she was utterly unsuited. She volunteered on the Republican side in the Spanish Civil War, but her career at the front ended when she stepped into a cooking pot and scalded her leg.

André's reluctance to take up arms a few years later was not based on pacifist convictions; rather, he felt that his *dharma*—his fate and his duty—was to make mathematics, not war. To get out of jail he ultimately agreed to report

for military service, but a few weeks later the French forces surrendered.

The Weil family escaped Europe at the last possible moment before the Nazis closed the exits. In the New World, André had several vagabond years before finding a home at the University of Chicago; later he moved to the Institute for Advanced Study. In 1942 Simone insisted on returning to Europe, with a plan to parachute into battle and nurse the wounded. She died before that fantasy could be fulfilled, succumbing to tuberculosis and self-induced malnutrition at age 34.

The Weils are fascinating, larger-than-life figures, but their stories have been told elsewhere [2, 3]. In the rest of this review I want to focus on the more personal part of Olsson's narrative. As an adolescent on the lookout for role models, she was drawn to the writings and the life story of Simone, but she also developed an early enthusiasm for mathematics, which eventually led her to an interest in André's work.

"There's a certain kind of young kid for whom the word *algebra* has a magical shimmer," she writes, "portending the enigmas of grades not yet reached, all the unimaginable revelations of junior high and high school." She couldn't wait to get to the  $x$ 's and  $y$ 's. And the romance did not end when those first secrets were revealed. In her sophomore year at Harvard, in the 1990s, she weighed her options and interests, and chose a mathematics major. She remembers late-night walks across a snowy campus. "I experienced then ... a kind of pleasure that (for me) came only after having thought hard about math, the mental equivalent of having gone for a long run. A gentle euphoria."

The pleasures were not all solitary. "We were a small band of students giddily, exhaustedly trekking through an abstract moonscape, helping one another across patches of ice or fighting over which directions to head next. The egos, the insecurities, the unabashed nerdiness! I miss it still .... Then there was the fact that I had a serious boyfriend for the first time .... Part of loving math, for me, was loving a person who also loved math."

But there's more to the story of how Olsson wound up a math major, then *didn't* wind up a mathematician. There was an attack of impostor syndrome (though she never uses that phrase):

From those exceptional kids I detected (or at least imagined) some mix of scorn and pity for someone like me, smart enough to get by, but just the ordinary type of smart. Much as mathematics came with a democratic ideology, according to which it was a realm of rarefied delights open to anyone who wished to work her way along its paths, there also seemed to be an unstated but obvious hierarchy. If math to me was a dark place where I went groping around on my hands and knees, here were these other



people with (it seemed to me) killer night vision who could see everything at once, go prancing from one topic to the next."

Besides, her real ambition was to be a novelist. Upon graduating, she tucked away her mathematics degree and went off in another direction. She became a newspaper reporter and editor in Austin, Texas, and eventually she published a couple of novels (*Waterloo* and *All the Houses*, both set in a political milieu). It was not until 20 years later that the mathematical itch came back, provoked in part by her young son's awakening interest. ("Give me an algebra problem, he begs.")

For reasons that aren't made clear, the Weil conjectures became a focal point of Olsson's mathematical revival. "Though I didn't go far enough in math to really understand the Weil conjectures, nevertheless I wonder, to what extent could I appreciate more about them? A bee in my bonnet, a dubious goal: maybe I could try to apprehend something of their flavor, I think, but at the same time I don't know what that would mean."

She turns to YouTube, where she finds a series of lectures in abstract algebra recorded in 2003 by Benedict Gross [4]. It's a course she took at Harvard a decade earlier, though with a different instructor. She still has the textbook. (Based on her description of the cover, it's Michael Artin's *Algebra*.) As she works her way through the lectures, her comments emphasize blackboard mannerisms, historical digressions, and classroom interruptions. Gross presents a theorem "like a magician announcing his next trick," she says. She tells us less about the mathematics itself. And she doesn't report any progress in penetrating the mysteries of the Weil conjectures.

Later she discovers that a classmate from her Harvard years is now a professor at her hometown university. After weeks of hesitation she sends him a carefully composed email, asking if they might chat sometime about the Weil conjectures. Weeks pass; there's no reply.

It's back to YouTube, but that doesn't go well either.

In the middle of watching the twenty-first online algebra lecture, I hit a wall .... While Professor Gross was elaborating on the Sylow theorems, as he was saying that "any two  $p$ -Sylow subgroups  $H$  and  $H'$  are conjugate," I became instantly tetchy, I couldn't take it any longer. Who cares? I am a midlife mother of two, I thought morosely, and this is the most pointless thing I could possibly be doing.

In the neighborhood supermarket, she spots the professor who never answered her email, and chases him through the aisles until she corners him with her shopping cart in the tortilla section. He apologizes for not responding. They speak about getting together sometime to talk math, but they don't set a date.

As Olsson relates these disheartening developments in her own mathematical journey, she is also wrapping up the narrative thread on the life of André Weil. She passes along some poignant stories of André's last years told by his daughter Sylvie [5], and she admires an elegy written by Goro Shimura [6]. "Is that what I am writing, I wonder, some sort of elegy for math, or for my own entanglement with math? At times it feels that way, but I don't think that's what this is. As it turns out, one stilted encounter in the supermarket is enough to send me back to the algebra lectures, which for no good reason I still want to finish. And so it's on to ring theory, which is of course nothing I need to remember, nothing I need to know."

Much as I enjoyed tagging along with Olsson on her mathematical ramble, I am mildly disappointed with the way the journey ends. I had looked forward to finally reading her account of what the Weil conjectures are all about, and what they mean to her—however fuzzy and fragmentary the appraisal might have to be. Earlier in the book she demonstrates an impressive talent for mathematical exposition. For example, she gives a deft and sure-footed explanation of a fixed point of a continuous function, in terms intelligible to readers who don't know what a function is, or why continuity would matter. I strongly suspect Olsson knows more than she lets on. Nevertheless, when it comes to the big challenge, which I had thought would be the climax to the story, she ducks.

I am also annoyed at her failure to exploit all of the resources available to her. Tuning in to the Gross lectures was surely a good idea, but what about that textbook she mentions? Apparently she never opens it; the book comes off the shelf only as a plaything for her six-year-old. Giving up on the search for a mentor after a single failed attempt also seems pretty feckless. I say this as someone who has often needed help of the same kind, and the mathematical community has always responded generously. If Olsson had persisted, I'm sure she would have found someone eager to offer guidance.

As far as I can tell, Olsson also never consulted the primary literature. If she tried to read the brief paper [7] in which Weil introduced the conjectures, she does not tell us about it. Nor does she mention any of the various commentaries, tutorials, and review articles [8–12] that endeavor to explain the conjectures and the subsequent proofs. None of these publications are easy reading. They assume familiarity with a whole catalog of ideas from abstract algebra, number theory, and topology: finite fields, varieties, rings, zeta functions, fixed-point theorems. That's a heavy cargo of conceptual baggage to tote around. Nevertheless, mastering some of this material seems like a necessary step if you want to write about the subject. There is no royal road to algebraic geometry.

Although Olsson's story can be exasperating at times, on the whole I find her quest inspiring. It's not every day you meet a journalist and novelist who longs for a deeper



understanding of varieties over finite fields and their zeta functions. She is not doing it for grades or for glory, but simply because something about mathematics calls out to her. I hope she will continue, and eventually find fulfillment rather than frustration.

For the mathematics community, Olsson's experience raises the question of how research-level mathematics can be made intelligible to those outside the field. Is it possible? Is it worth the effort? We already have André Weil's answer. He seemed quite comfortable with the idea of mathematics as an elite guild, open only to those of exceptional talent; the rest of the world is deaf to the symphony. Perhaps he was right, but if so the situation is rather sad. Beautiful music is played in an empty concert hall, with no one but the composer and the orchestra able to appreciate it. And there's a practical concern: in general it's the audience that provides material support to the musicians.

Weil's inward-looking view is certainly not universal. For many others, mathematics is something worth sharing—a thing of beauty, a useful tool for understanding the world we live in, a window onto an unexpected universe. They work to engage the public through teaching, lectures, expository writing, mentoring. Olsson's story offers a bit of cheerful news to these evangelists: it's proof that someone out there is listening, keen to hear the message. But it also underlines how much hard work is needed to open a line of communication between research mathematicians and the general public.

The task is not merely translation or interpretation—making the vocabulary of mathematics comprehensible. The crucial challenge is motivation: conveying a sense of why a mathematical idea is worth the trouble of understanding. After giving a brief explanation of Fermat's Little Theorem, Olsson remarks:

Even Fermat's relatively simple theorem starts to grow hair when I try to lay it out in ordinary language, I realize, and it's hard to articulate why it's interesting without invoking more math. At the end of the day, why should a nonmathematician care that André Weil figured out how to count solutions to polynomial equations in finite number fields? In one sense, I myself don't care. I don't understand it well enough to care. But there's a kind of ... not quite gratification but the prospect of it, a door that cracks open just a sliver when I learn about these constructed realms and the relations within and among them ....

Can we open that door just a little wider?

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Brian Hayes

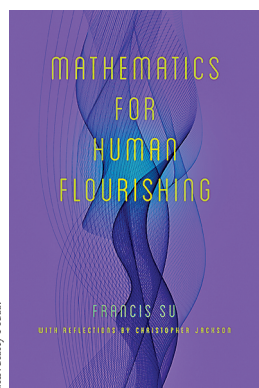
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Figure 1 appeared in *My Father André Weil*, Notices Amer. Math. Soc. 65 (2018), no. 1, 54–57, and is courtesy of Sylvie Weil.

Author photo is courtesy of Rosalind Reid.

Yale University Press, 2020, 288 pages. Cover courtesy of Yale University Press.



***Mathematics for Human Flourishing***  
 by Francis Su

This unconventional book grew out of Su's highly regarded 2017 speech as outgoing president of the Mathematical Association of America. In the speech, which garnered great acclaim among mathematicians and a significant amount of media coverage, Su proposed a moral argument that

mathematics belongs to everyone and that the practice of mathematics can and should shape us as human beings. "This book is not about how great mathematics is," Su writes in the preface, "this is a book that grounds mathematics in what it means to be a human being and to live a more fully human life." This book is suitable for a broad audience, especially for those who do not see themselves as "math people."

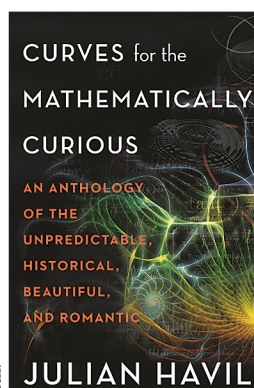
*Mathematics for Human Flourishing* builds upon the themes of Su's speech and explains how mathematics satisfies basic human desires, among which Su includes play, beauty, permanence, truth, justice, freedom, community, and love. With a combination of emotionally charged anecdotes, mind-bending puzzles, and deeply personal reflections, Su explains how mathematics relates to these ideals, each central to human flourishing. Interwoven throughout the book are excerpts from the letters of Christopher Jackson, an incarcerated individual who has corresponded with Su about mathematics for many years. The book ends with questions for further discussion and solutions to the various puzzles interspersed throughout the book.

*The Bookshelf is prepared monthly by Notices Associate Editor Stephan Ramon Garcia.*

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Princeton, 2019, 280 pages. Cover courtesy of Princeton University Press.



***Curves for the Mathematically Curious***  
*An Anthology of the Unpredictable, Historical, Beautiful, and Romantic*  
 by Julian Havil

Havil, the author of several other mathematics titles published by Princeton, has assembled a collection of ten historical and technical profiles of important curves that have arisen throughout the history of mathematics. Each receives roughly twenty pages of treatment,

albeit with a large standard deviation. The selections are not meant to be universal, rather they are reflections of Havil's tastes and inclinations. It is hard to argue with the final list: the Euler spiral, the Weierstrass continuous but nowhere-differentiable function, Bézier curves, the rectangular hyperbola (from which logarithms derive), the quadratrix of Hippias, classical space-filling curves (of Hilbert and Peano), curves of constant width, the normal curve, the catenary, and elliptic curves. These cover a wide range of topics from geometry, calculus, probability, and analysis. Of course, selecting ten curves for this anthology required some hard choices. For example, there is no chapter devoted to conic sections (as Havil cleverly argues by analogy, "not every anthology of poems contains works by Shakespeare").

Mathematicians will appreciate this book as a friendly survey of the historical progression behind some old favorites. *Curves for the Mathematically Curious* is not a simple read for those with short attention spans. Some of the original (often geometric) arguments are reproduced along with the occasionally verbose and somewhat cryptic passages from the original discoverers. Not only does this keep the historical discussion honest it also informs the reader of the trials and errors, the sparks of ingenuity, and the cumbersome notation from the initial development of the topic.

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## *Common Sense Mathematics, Second Edition* by Ethan Bolker and Maura Mast

A headline in *The Boston Globe* on May 1, 2018, claimed, "There were nearly 100,000 Uber and Lyft rides in Boston last year." Should you believe that? The question comes from the first chapter of the new second edition of *Common Sense Mathematics* where the newspaper story is

carefully analyzed, so carefully that the analysis unearths and explains an error in the article. Ethan Bolker and Maura Mast asked themselves several years ago what they expected their students to remember a decade or so after completing their Quantitative Literacy course. They describe their subsequent reflections as "sobering." So they scrapped the whole thing, recast their goals, and redesigned the course. The first edition of this text was the result. They, modestly, claimed that with it they were trying to "change the way our students' minds work." That's all.

Numeracy, Lynn Steen explained in *Mathematics and Democracy*, is not about deep mathematical abstractions, it is about applying relatively elementary mathematical ideas in "subtle and sophisticated contexts." Bolker and Mast decided to meet his challenge by arming their students with habits of mind, and less importantly tools, to confront everyday quantitative information. The book contains literally hundreds of news stories from the mass media invoking numbers or graphs or statistics and it requires the students to dive into those numbers and figure out if they are sensible and believable. The new edition was motivated by a need to update the examples to be more recent. The stories are organized into themes: estimation, averaging, graphical presentation of data, linear functions and models, exponential growth, etc. The presentation via news stories means your students will never ask, "Why do I need to know this?" The answer will be obvious: you don't, unless you want to be able to read the newspaper and to function as a citizen.

The AMS Bookshelf is prepared bimonthly by AMS Acquisitions Specialist for MAA Press titles Stephen Kennedy. His email address is [skennedy@amsbooks.org](mailto:skennedy@amsbooks.org).



## *A Mathematician's Practical Guide to Mentoring Undergraduate Research* by Michael Dorff, Allison Henrich, and Lara Pudwell

Undergraduate research in the sciences got a big boost in 1959 when the NSF dedicated funding to its Undergraduate Research Participation (URP) program. It's safe to say that most mathematicians at that time imagined that undergraduate research in

mathematics was simply infeasible, the frontiers of research were impossibly far from the undergraduate curriculum. Nevertheless a few dozen visionaries created URP programs in mathematics. When the NSF instituted the REU program in the late 1980s mathematics did participate, but our conception hadn't really shifted. We still imagined that research was out of reach of most undergraduates but we recognized that it would be beneficial to a few, elite, super-advanced undergraduates. Over the intervening thirty years we have gradually come to recognize that there are opportunities for undergraduate participation in research and there are real benefits both for the students and the mathematicians involved. It is now not at all remarkable and, indeed, many colleges have incorporated it into their curriculum for all students.

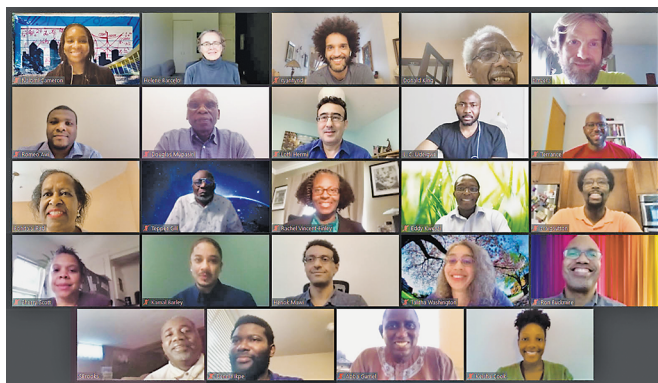
Even with our community's increasing recognition of its value, it can still be daunting to imagine getting started. How do you formulate an accessible, interesting question? Often such programs involve small groups of students; how do you manage the inevitable complicated dynamics of the group? How much is it training and how much is it really research? Do I have to teach them to write a paper, too? How do I find support, financial and departmental?

*A Mathematician's Practical Guide to Mentoring Undergraduate Research* answers all these questions and more. The authors, collectively, have decades of experience and have mentored scores of students. Their guide is packed with nuts-and-bolts implementation advice and inspiring anecdotal evidence of the value of undergraduate research in mathematics.



# MSRI's ADJOINT: African Diaspora Joint Mathematics Workshop

*Hélène Barcelo and Edray Herber Goins*



**Figure 1.** Participants in ADJOINT 2020 on Friday, June 26, 2020, together with Hélène Barcelo, MSRI's Deputy Director. Top row: Naomi Cameron, Hélène Barcelo, Ryan Hynd, Donald King, Timothy Myers. Second row: Romeo Awi, Douglas Mupasiri, Lofti Hermi, Lawrence Udeigwe, Terrance Pendleton. Third row: Bonita Saunders, Tepper Gill, Rachel Vincent-Finley, Eddy Kwessi, Craig Sutton. Fourth row: Sherry Scott, Kamal Barley, Henok Mawi, Talitha Washington, Ron Buckmire. Bottom row: Sean Brooks, Dennis Ikpe, Abba Gumel, Keisha Cook.

ADJOINT is a new program intended primarily for African-American mathematicians and statisticians. It is designed to catalyze research collaborations and provide support for

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*Communicated by Notices Associate Editor Robin T. Wilson.*

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conference participation; its specific goals include increasing the visibility of its participants and developing a sense of community among African-American researchers. The program is structured with an introductory two-week meeting at MSRI which provides opportunities for in-person research work in small groups with project leaders on various topics. The program continues throughout the following year via both online meetings and visits to team members' institutions. ADJOINT is funded by the National Science Foundation (DMS-1915954 and DMS-2016406), the National Security Agency (H98230-20-1-0015), and the Sloan Foundation (G-2020-12602).

## Why ADJOINT?

In February 2019, the *New York Times* published two articles focused on the dearth of Black faculty in the mathematical sciences [3, 4]. As observed in the second of the two articles: "According to the American Mathematical Society, there are 1,769 tenured mathematicians at the math departments of the 50 United States universities that produce the most math Ph.D.s. No one tallies the number of black mathematicians in those departments, but as best I can tell, there are 13. That comes to seven-tenths of 1 percent of the total—perhaps as far as any job classification gets from accurately reflecting the share of black Americans in the general adult population, which stands at 13 percent."

Indeed, these low numbers are well known to the mathematical community. The Conference Board of Mathematical Sciences (CBMS) Fall 2015 Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States [1] reported that African-Americans make up 1%, 3%, and 3%, respectively, of all mathematical sciences faculty in doctoral-level, masters-level, and



bachelors-level four-year mathematics departments. Even with the low numbers of African-American faculty in mathematics, there is concern over how to keep these faculty actively engaged in research. A report on higher education from Hanover Research [5, pg. 10] observes that “Bland, et al. [2] conclude that, ‘when individual faculty’s research productivity is the goal, nothing substitutes for’ four factors in particular: (1) recruiting faculty with a passion for research; (2) providing them with formal mentoring programs; (3) facilitating their networks; and (4) providing time for them to do research.”

The underlying goal of the ADJOINT program is to increase the number, productivity, and visibility of African-American mathematical and statistical researchers. ADJOINT aims to address this issue of underrepresentation by increasing individual faculty research productivity through organized collaborative research groups and formal mentorship opportunities. The ADJOINT program takes its inspiration from MSRI’s very successful Summer Research in Mathematics (SRiM) and MSRI-UP programs. The SRiM program (formerly the Summer Research Program for Women in Mathematics, or SWiM) supports groups of researchers, predominantly women, to advance and complete already existing projects. MSRI’s Undergraduate Program (MSRI-UP) is a six-week REU program whose goal is to identify talented students (18 per summer), especially those from groups underrepresented in math, and introduce them to research in mathematics.

The initial idea for ADJOINT grew out of conversations between four individuals in 2018. That year, Hélène Barcelo was the Acting Director of MSRI, Edray Goins was President of the National Association of Mathematicians (NAM), Michael Singer (North Carolina State University) was Assistant Director of MSRI, and Robin Wilson (California State Polytechnic University at Pomona) was the chair of MSRI’s Human Resources Advisory Committee (HRAC). The four of us worked together to formulate the program and write the initial NSF grant proposal. We chose the title “African Diaspora Joint Mathematics Workshop” in recognition of the inhomogeneity of the Black experience: the term “African Diaspora” describes the dispersion of people from Africa during the Transatlantic Slave Trades.

## Structure of ADJOINT

There are three groups of individuals involved with ADJOINT: program participants, research leaders, and program directors.

**Program participants.** Each year, up to twenty researchers participate in the program. The participants are mainly junior mathematicians that initially spend two weeks in residence at MSRI. During that period, participants spend most of their time engaged in research with team members

under the guidance of the research leaders. The teams are formed prior to the MSRI residency. In addition, professional development workshops and a weekly colloquium series are offered. An early milestone in the program is the research project presentations on Friday afternoons. After this initial two weeks, the researchers remain active during the academic year via virtual meetings as well as visits to team members’ institutions; members are also encouraged to attend and present their work at national and international conferences. The research leaders and program directors continue to support the participants throughout the years.

Participants must be either US citizens or permanent residents, must possess a PhD in the mathematical or statistical sciences, and must be employed at a US institution. Applicants are asked to submit: (1) a cover letter specifying which of the offered research projects the applicant wishes to be part of; (2) a curriculum vitae; (3) a personal statement, no longer than one page, addressing how participation will contribute to the goals of the program; and (4) a research statement, no longer than two pages, describing current research interests, and relevant past research activities, and how they relate to the project of greatest interest.

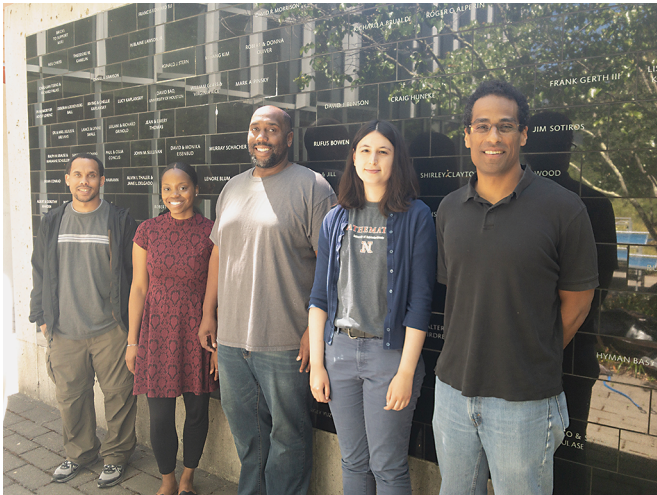
While the participants do not need to be African-American, an applicant’s potential positive impact on the careers of African-Americans in the mathematical sciences is an important factor in the final selection. We consider whether participation would help with promotion and tenure; provide networking opportunities; and encourage the participant to remain active in research. We are especially interested in applicants who are employed at Historically Black Colleges and Universities (HBCUs).

**Research leaders.** Each year, the ADJOINT directors select five research leaders who are respected African-American mathematicians with well-established research programs. Each will lead a group of up to five participants during the two-week part of the program held at MSRI, as well as afterward.

The research leaders and participants will each receive funding for two weeks of lodging, round trip travel to MSRI, and funding to travel to conferences or to team members’ institutions to continue research collaborations.

**Program directors.** ADJOINT is administered by five program directors: Edray Goins (Pomona College) as the lead director, Caleb Ashley (University of Michigan at Ann Arbor), Naiomi Cameron (Spelman College), Jacqueline Hughes-Oliver (North Carolina State University), and Anisah Nu’Man (Spelman College).

The program directors recruit the five research leaders and select the twenty participants. Each summer only one of the directors will be in residence at MSRI during the two



**Figure 2.** Derek Young (left), Chassidy Bozeman, Michael Young, Rachel Kirsch, and Theodore Molla.

intense weeks of research.

The onsite program directors follow and support their cohort for the next five years. Additionally, they oversee the program, provide support for the research teams, and lead participants in professional enhancement activities that include discussions on best practices for job applications, pursuing promotion and tenure, disseminating research, grant writing, and more.

The onsite director for 2020 was Naomi Cameron, and the onsite director for 2021 will be Jacqueline Hughes-Oliver.

### ADJOINT 2019 – Pilot Program

We held a pilot program onsite at MSRI in the summer of 2019; it was funded by the NSF (DMS-1915954). We began small, working with a total of fifteen researchers divided into three working groups. All teams were predominantly comprised of African-American mathematicians at various stages in their careers. We were not able to get all three groups to come to MSRI at the same time, so these groups met at different times for varying lengths.

**Graph theory.** From June 10–21, 2019, Michael Young (Iowa State University) led a research project in graph theory titled *Problems on Tournaments*. The participants were Chassidy Bozeman (Mount Holyoke College), Rachel Kirsch (London School of Economics), Theodore Molla (University of South Florida), and Derek Young (Mount Holyoke College). (See Figure 2.)

Young described the project as follows. A tournament  $T$  is a rebel if the set of domination numbers of all tournaments that do not contain  $T$  is bounded from above. Chudnovsky, Kim, Liu, Seymour, and Thomasse asked if the seven-vertex Paley tournament is a rebel and, more generally, if a similar statement holds for every poset



**Figure 3.** Caleb Ashley (left), Karoline Pershell, Naomi Cameron, Emille Lawrence, and Edray Goins.

tournament. They proved that every tournament that avoids a different tournament on seven vertices has bounded domination number. This research team has been working on determining what other tournaments are rebels and extending the definitions and results to determine what directed graphs are (or are not) rebels. The team is also exploring a conjecture of Mycroft and Naia who ask if, for sufficiently large  $d$ , every outbranching balanced binary tree on  $n = 2^{d+1} - 1$  vertices is contained in every tournament on  $n$  vertices. The group seeks to explore and prove results for specific trees and values of  $n$ .

**Number theory.** From July 8–19, 2019, Edray Herber Goins (Pomona College) led a research project in number theory titled *Compositions of Belyi Maps and their Monodromy Groups*. The participants were Caleb Ashley (University of Michigan), Naomi Cameron (Spelman College), Emille Davie Lawrence (University of San Francisco), Theo McKenzie (University of California at Berkeley), and Karoline Pershell (Association for Women in Mathematics<sup>1</sup> / Service Robotics and Technologies). (See Figure 3.)

Goins described the project as follows. This group's goal was to understand the extended monodromy group  $\text{ExtMon}(\beta)$  by two different methods: (i) geometrically, via surjections  $\Delta(\beta \circ \gamma) \twoheadrightarrow \Delta(\beta)$  on the dessin d'enfants, and (ii) topologically, via surjections  $\text{Mon}(\beta \circ \gamma) \twoheadrightarrow \text{Mon}(\beta)$  on the monodromy groups. The group spent the first week of their visit reviewing Belyi maps, dessin d'enfants, monodromy groups, Naomi Cameron's previous work, and Jacob Bond's previous work. They spent the second week attempting to understand the extending patterns for the specific Belyi map  $\beta(z) = -(z-1)^2/(4z)$  by considering the family of dynamical Belyi maps  $\gamma(z) = z^n$  for

<sup>1</sup>Affiliation as of July 2019.





**Figure 4.** S. James Gates (left), Caroline Klivans, Kevin Iga, and Vincent Rodgers.

natural numbers  $n$ . The group had a follow-up meeting at Pomona College from December 14–17, 2019, but further plans to meet in person are on hold due to the COVID-19 pandemic.

**Physics and combinatorics.** From July 22–26, 2019, Sylvester James Gates, Jr. (Brown University) led a research project in mathematical physics titled *Mathematical Adinkra Symbols: From Physics to Mathematics Investigations across Algebraic Topology and Graph Theory*. The participants were Kevin Iga (Pepperdine University), Caroline Klivans (Brown University), and Vincent Rodgers (University of Iowa). (See Figure 4.)

Gates described the project as follows. In 2005, two theoretical physicists, Michael Faux and S. James Gates, Jr., proposed a type of graph (now known as “Adinkras”) that had apparently not appeared previously in the mathematical literature. Faux and Gates hoped these graphs could be useful in formulating new pathways to solving problems in theoretical/mathematical physics that lie at the foundation of string theory and that certain problems would be illuminated by this new approach. It was viewed as a possible new tool to unravel long-unsolved representation theory problems in the area of supersymmetry. Instead, Adinkras were discovered to be related to a raft of mathematical subjects: (1) Coxeter groups, (2) error-correcting codes, (3) Grothendieck’s “dessin d’enfant,” (4) Beylī pairs, (5) Cimasoni-Reshetikhin dimer models on Riemann surfaces, (6) Donagi-Witten parabolic structure/ramified coverings of super Riemann surface, (7) Morse divisors, (8) Fuchsian uniformization, (9) elliptic curves, and (10) Schur functions. An apparent breakthrough happened at MSRI during the ADJOINT program as the first glimmer of connection was made between the works of Prof. Caroline Klivans and the

“permutahedron” concept she studies in Coxeter groups and the valise Adinkra concept which Prof. Gates investigates. A main deliverable from this study should be a book that Prof. Kevin Iga of Pepperdine will publish to make the development of this young topic accessible to the mathematical community.

**Workshop feedback.** We had a good balance of participants for this pilot program. Of the 15 participants, nine were men while six were women; one identified as Asian, three as White, and 11 as African-American.

The exit surveys of the pilot program were quite positive. One of the participants commented: “It was great working with other African-Americans. However, for me the best part about the program is that more African-Americans will be writing papers together and presenting to the math community. [ADJOINT] can make a direct impact on the number of research papers authored and conference presentations given by African-Americans. That is a huge positive.” Another said: “I felt more comfortable working with this group more than any other research group that I have worked with.” A third remarked: “I felt like my voice was respected. This almost never happens in other research collaborations. In fact, it’s hard to have research collaborations because people don’t take me seriously.”

## ADJOINT 2020

We were inspired by the positive feedback, and decided to expand the program. The first official ADJOINT took place from June 15–26, 2020; it was funded by NSF (DMS-2016406), NSA (H98230-20-1-0015), and the Sloan Foundation (G-2020-12602).

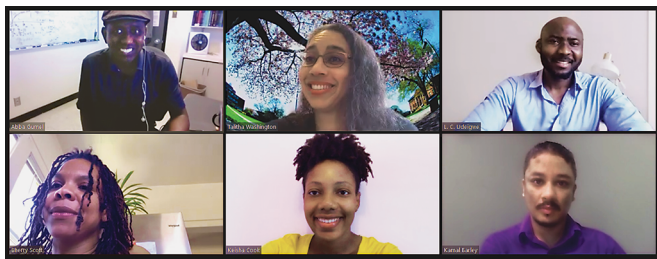
We had five research leaders and eighteen program participants. The onsite program director was Naiomi Cameron (Spelman College). Due to the outbreak of COVID-19, we decided to hold the program virtually using Zoom. Each of the research leaders and program participants was sent an iPad Pro 12.9” along with Smart Keyboard Folio and an Apple Pencil. We used Google’s Jamboard as a primary way to communicate via a virtual whiteboard.

Unfortunately our balance of participants was not quite what we had hoped for. Of the 23 participants, 18 were men but only five were women. Still, 22 identified as African-American and one as White. Seven were faculty at HBCUs (Coppin State University, Hampton University, Howard University, and Southern University and A&M College).

**Mathematical physics.** Tepper Gill (Howard University) led a research project in mathematical physics titled *Analysis, Partial Differential Equations, and Mathematical Physics*. The participants were Eddy Kwessi (Trinity University), Douglas Mupasiri (University of Northern Iowa), and



**Figure 5.** Tepper Gill (top left), Eddy Kwessi, Douglas Mupasiri (bottom left), and Timothy Myers.



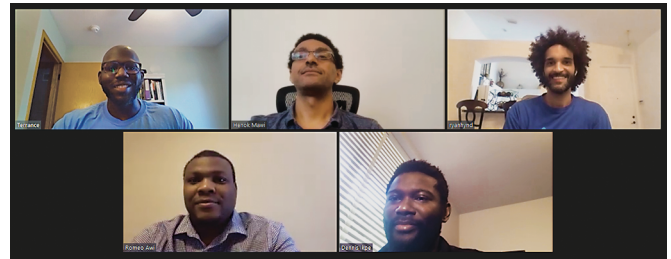
**Figure 6.** Abba Gumel (top left), Talitha Washington, Lawrence Udeigwe; Sherry Scott (bottom left), Keisha Cook, and Kamal Barley.

Timothy Myers (Howard University). (See Figure 5.)

Gill described the project as follows. The center of mass for this research is a new class of separable Banach spaces  $KS^p$ ,  $1 \leq p \leq \infty$ , which contains each corresponding  $L^p$  space as a dense continuous embedding. These spaces are interesting because they contain the HK-integrable functions and the space of distributions. The HK-integral is easy to understand, extends the Lebesgue integral, and integrates nonabsolutely integrable functions. The spaces generate an arena for a number of interesting research topics in analysis, partial differential equations, and mathematical physics.

**Mathematical biology.** Abba Gumel (Arizona State University) led a research project in mathematical biology titled *Mathematics of the Transmission Dynamics and Control of the 2019 Novel Coronavirus*. The participants were Kamal Barley (Stony Brook University), Keisha Cook (Tulane University), Sherry Scott (Milwaukee School of Engineering), Lawrence Udeigwe (Manhattan College), and Talitha Washington (Howard University / National Science Foundation). (See Figure 6.)

Gumel described the project as follows. The world is currently facing a devastating pandemic of a novel Coronavirus (COVID-19), which started as an outbreak of pneumonia of unknown cause in Wuhan, China, in



**Figure 7.** Terrance Pendleton (top left), Henok Mawi, Ryan Hynd; Romeo Awi (bottom left) and Dennis Ikpe.

December of 2019. As of April 26, 2020, COVID-19 (caused by the novel SARS-CoV-2 coronavirus) has spread to over 210 countries and territories, causing about three million infections and 207,000 deaths. In the absence of a safe and effective vaccine against COVID-19, and safe and approved antivirals, control and mitigation efforts are focused on the use of nonpharmaceutical interventions, such as social distancing, quarantine, aggressive containment (i.e., rapid detection, isolation, and contact-tracing of confirmed cases), use of face masks in public, etc. The purpose of this project is to use mathematical modeling approaches and rigorous analysis, coupled with data analytics, to assess the population-level impact of various nonpharmaceutical interventions on controlling and mitigating the burden of the pandemic. In particular, the models we are working on take the form of deterministic (autonomous and nonautonomous) systems of nonlinear differential equations. The project involves the rigorous analysis of the models (using theories and techniques from nonlinear dynamical systems, particularly bifurcation theory) to gain insight into their dynamical features and obtain epidemiological thresholds, in parameter space, that govern the persistence or effective control (elimination) of the pandemic. In addition, statistical tools are being used to estimate the parameters of the models, as well as to fit the models with available data and carry out uncertainty and sensitivity analyses.

**Analysis.** Ryan Hynd (University of Pennsylvania) led two research projects in analysis titled *Approximating Nash Equilibria* and *A Time Optimal Control for Compartmental Models in Epidemiology*. The participants were Romeo Awi (Hampden University), Dennis Ikpe (Michigan State University), Henok Mawi (Howard University), and Terrance Pendleton (Drake University). (See Figure 7.)

Hynd described the first of his two projects as follows. The notion of a Nash equilibrium is an important concept in the theory of noncooperative games. It is informally described as a collective strategy assumed by several players in which no player can decrease her cost by changing her individual strategy. In terms of mathematics, the collective strategy set can be modeled as a



Cartesian product  $X = X_1 \times \cdots \times X_N$ , where each  $X_i$  represents the possible strategies of the  $i$ th player. If  $f_1, \dots, f_N : X \rightarrow \mathbb{R}$  represent the cost functions of the respective players,  $x \in X$  is a Nash equilibrium provided  $f_i(x) \leq f_i(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N)$  for all  $y_i \in X_i$  and  $i = 1, \dots, N$ . Under appropriate continuity and compactness assumptions, it is known that there is a Nash equilibrium. This is typically proved with an analog of Brouwer's fixed point theorem. As a result, this existence result is non-constructive. Nevertheless, we are interested in identifying conditions on  $f_1, \dots, f_N$  and  $X$  such that there is a Nash equilibrium which can be approximated by a *constructive* method.

Hynd described the second of his two projects as follows. The SIR model is perhaps the best-known epidemiology model. It predicts how three compartments of the total population evolve in time. We consider a variant of this model which involves a vaccination rate  $r : [0, \infty) \rightarrow [0, 1]$ . Namely, we study the system of differential equations

$$\dot{S} = -\beta SI - rS,$$

$$\dot{I} = \beta SI - \gamma I,$$

$$\dot{R} = \gamma I,$$

where  $S, I, R : [0, \infty) \rightarrow \mathbb{R}$  represent the susceptible, infected, and recovered compartments of a total population of finite size. Here  $\beta$  and  $\gamma$  are the respective infected and recovery rates per unit time and  $r$  represents a controlled vaccination rate. The problem we have been studying is to characterize the vaccination rate which minimizes the time in which the infected population falls below a given threshold  $\mu > 0$ , that is, we seek to characterize  $r$  such that it minimizes the eradication time  $\min\{t \geq 0 : I(t) \leq \mu\}$ .

**Computational mathematics.** Bonita V. Saunders (National Institute of Standards and Technology) led a research project in computational mathematics titled *Validated Numerical Computations of Mathematical Functions*. The participants were Sean Brooks (Coppin State University), Ron Buckmire (Occidental College), Opel Jones (Towson University), and Rachel Vincent-Finley (Southern University and A&M College). (See Figure 8.)

Saunders described the project as follows. During the late 1930s, 40s, and 50s accurate tables of function values were calculated by human “computers” to facilitate the evaluation of functions by interpolation. In addition to logarithmic and trigonometric functions, these reference tables included values for Gamma, Legendre, Jacobian, Bessel, Airy, and other high-level or “special” functions important for applied and physical applications. The advent of reliable computing machines, computer algebra systems, and computational packages diminished the need for such reference tables, but today’s researchers and



Figure 8. Sean Brooks (top left), Bonita Saunders; Rachel Vincent-Finley (bottom left), and Ron Buckmire.



Figure 9. Craig Sutton (top left), Lotfi Hermi, and Donald King (bottom).

software developers still need a way to confirm the accuracy of numerical codes that compute mathematical function values. Project participants were introduced to the field of validated computations of special mathematical functions, which is the development of codes that compute certifiably accurate function values that can be used to test the accuracy of values produced by personal, commercial, or publicly available codes.

**Differential geometry.** Craig Sutton (Dartmouth College) led a research project in differential geometry titled *Explorations in Inverse Spectral Geometry*. The participants were Lotfi Hermi (Florida International University) and Donald King (Northeastern University). (See Figure 9.)

Sutton described the project as follows. Inverse spectral geometry is the study of the relationship between the *spectrum* of a closed Riemannian manifold—i.e., the sequence of eigenvalues (counting multiplicities) of the associated Laplace-Beltrami operator—and its underlying geometry. Two manifolds are said to be *isospectral* if their spectra agree and a geometric property is said to be *audible* or *spectrally determined* if it is encoded in the spectrum. The spectrum is known to encode the dimension, volume,

and total scalar curvature of a manifold; however, the literature contains numerous examples which demonstrate the spectrum does not completely determine the geometry of the underlying space. Nevertheless, it is expected that certain natural classes of Riemannian manifolds are characterized by their spectra. For example, it is widely believed that the round  $n$ -sphere is uniquely characterized by its spectrum; however, to date, this has only been proven for round spheres of dimension at most six. Our research group is working on problems motivated by the following questions. (1) To what extent is the length spectrum of a manifold encoded in its Laplace spectrum? (2) Is the local geometry of a low-dimensional manifold encoded in its spectrum? (3) How are the covering spectrum of a Riemann surface and its Laplace spectrum related?

**Community building activities.** Each weekday for the two weeks, the research groups were encouraged to work together over Zoom. The entire program came together as a whole several times each week to help build community. We had lunch together every day from 1:00 p.m.–2:00 p.m. EST; there were lunch discussions on “Social Media for Work Productivity” on Wednesday, June 17, 2020, on “Managing Work/Life/Service Loads” on Monday, June 22, 2020, and on “Institutional Impact of COVID” on Thursday, June 25, 2020. We also had three workshops in the afternoon from 2:00–3:00 p.m. EST: a Grant Workshop on Tuesday, June 16, 2020, a Tenure/Promotion Workshop on Tuesday, June 23, 2020, and a Marketability/Leadership Workshop on Wednesday, June 24, 2020.

## ADJOINT 2021

The next program will take place at MSRI from June 21 through July 2, 2021; the onsite program director will be Jacqueline Hughes-Oliver (North Carolina State University). We have four research leaders lined up so far.

**Algebraic geometry.** Danny Krashen (Rutgers University) will lead a research project in algebraic geometry titled *Adventures in Constructive Galois Theory*. He describes the project as follows. Understanding Galois extensions of fields is a central problem in algebra, with a number of open questions, accessible at many levels. In this project we will explore “explicit inverse Galois theory,” which tries to understand which groups arise as Galois groups for a given field, and how. Our goal will be to use these constructive approaches to understand richer algebraic structures and properties that collections of Galois extensions exhibit as a whole.

**Geometric group theory.** Nathan Broaddus (Ohio State University) will lead a research project in geometric group theory titled *Steinberg Modules of Braid Groups*. He describes the project as follows. Many important groups of

interest in topology are duality groups. As such they have an associated group cohomological object which we call the “Steinberg module” of the group. We will begin with an introduction to the braid group and discuss a number of elementary descriptions of its Steinberg module. Our first research goal will be to unify as many of these disparate descriptions as possible.

**Biostatistics.** Emma K. T. Benn (Mount Sinai University) will lead a project in biostatistics titled *Racial/Ethnic Disparities in Health: Applying a More Nuanced Inferential Framework*. She describes the project as follows. Reducing and eliminating health disparities is of utmost concern for many public health and biomedical researchers and has been a stated goal for Healthy People 2000, 2010, and 2020. However, when it comes to racial disparities in health, researchers have done well at describing differences, but have often struggled to identify mutable targets for intervention. This problem exists for a host of reasons, including the complex contextual factors surrounding racial disparities; however, this may also stem from the way in which we operationalize race in research. For the proposed project, we will first explore the operationalization of race as a “cause” when examining racial disparities in health based on multidisciplinary discourse around this topic from statisticians informed by the potential outcomes framework, epidemiologists, clinical investigators, and others. Subsequently, we will critically scrutinize the traditional approaches to investigating disparities in health and apply a more nuanced inferential, rather than descriptive, approach to the statistical analysis of real-world biomedical data with an underlying objective to find efficacious interventions for eradicating health disparities.

**Operations research.** Julie Ivy (North Carolina State University) will lead a project in operations research titled *Using Decision Modeling to Personalize Policy in Complex Human-Centered Problems*. She describes the project as follows. The COVID-19 pandemic highlights the importance of sequential decision making under conditions of uncertainty, learning as the future evolves, and effectively using data to inform decision making. The pandemic further highlights the significant role that mathematical modeling can and should play in addressing complex human-centered problems. This research project will consider these types of problems from a systems modeling perspective. The focus of this project will be decision making under conditions of uncertainty with the goal of modeling complex interactions and quantitatively capturing the impact of different factors, objectives, system dynamics, intervention options, and policies on outcomes with the goal of improving decision quality.



## What's Next?

We plan to continue with ADJOINT on a variety of fronts in 2021. In particular, there was a special session on Saturday, January 9, 2021, at the Joint Mathematics Meetings titled "ADJOINT (African Diaspora Joint Mathematics Workshop) Research Showcase." This gave past participants of ADJOINT the opportunity to present their research, and exposed prospective participants to a preview of future workshops.

For more information about ADJOINT, please visit <https://www.msri.org/web/msri/scientific/adjoint>.

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Hélène Barcelo



Edray Herber Goins

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# FAN CHINA EXCHANGE PROGRAM

- Gives eminent mathematicians from the US and Canada an opportunity to travel to China and interact with fellow researchers in the mathematical sciences community.
- Allows Chinese scientists in the early stages of their careers to come to the US and Canada for collaborative opportunities.

Applications received before **March 15** will be considered for the following academic year.

For more information on the Fan China Exchange Program and application process see [www.ams.org/china-exchange](http://www.ams.org/china-exchange) or contact the AMS Programs Department:

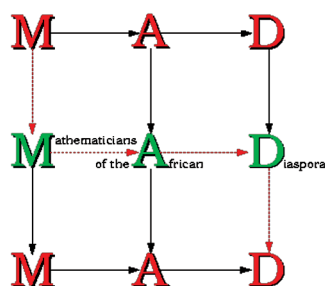
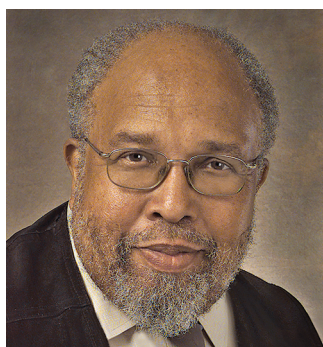
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# Updated “MAD Pages” Website Unveiled October 9, 2020

*Edray Herber Goins*



**Figure 1.** Scott Williams and the Original MAD Pages.

In 1997, Scott Williams (SUNY Buffalo) founded the website “Mathematicians of the African Diaspora” [2], which has since become widely known as the MAD Pages. According to a 2019 blog [1] written by Scott Williams for the American Mathematical Society, “As a child I was struck by the emphasis, within the general American culture, upon achievements in the Sports/Entertainment Industry as indications of success. Within the African American subculture, those indications are even stronger—just consider the winners of the NAACP Image Awards among other celebrations. On the rare occasion a scientist has won an award, there has been a limitation to the medical field. In addition, where it concerns successes in mathematics and the sciences, I discovered much incorrect or misconstrued information available in texts and especially on the web.”

Williams built the site over the course of 11 years, creating over 1,000 pages by himself as a personal labor of

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*Communicated by Notices Associate Editor Robin T. Wilson.*

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**Figure 2.** Donald King, Asamoah Nkwanta, and John Weaver.

love. The site features more than 700 Black mathematicians, computer scientists, and physicists as a way to showcase the intellectual prowess of those from the African Diaspora. Williams provided profiles of these individuals, detailing their education, their journey within mathematics thus far, and their accomplishments. He also created numerous pages discussing Black history within the mathematical sciences, and presented data on the demographics of Black people in the mathematical sciences at the time. Since its creation, the MAD Pages have received more than 20 million visitors, and provided immeasurable inspiration and validation to many Black mathematicians and students.

Scott Williams retired in 2008. After an initial town hall meeting about the future of the MAD Pages, which took place at a Conference for African American Researchers in the Mathematical Sciences (CAARMS), an informal group of mathematicians decided to work together to preserve Williams’ work. In 2015, the National Association of Mathematicians (NAM) formed an ad hoc committee to update the MAD Pages, consisting of NAM President Edray Goins (Pomona College), Committee Co-Chairs Don King (Northeastern University) and Asamoah Nkwanta (Morgan State University), and web developer John Weaver (Varsity Software). The MAD Pages Update Project was funded in part by Temple University, the Educational Advancement Foundation (especially Albert



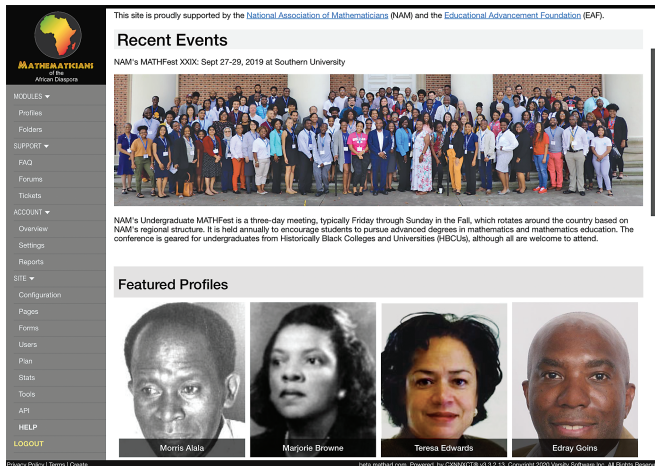


Figure 3. Updated MAD Pages.

Lewis and Harry Lucas), the Mathematical Sciences Research Institute (MSRI), the National Science Foundation (DMS-1560394), Northeastern University, Pomona College, and Washington and Lee University. We employed nearly four dozen undergraduate students from across the country to assist with this project; they greatly contributed to the database's depth and accuracy.

The updated MAD Pages [3] were unveiled to the public on October 9, 2020. "This unveiling date intentionally coincides with the death of Benjamin Banneker, arguably the first African American mathematician," states committee member Edray Goins. "His collected works were lost in a mysterious fire which occurred on the day of his funeral. The MAD Pages is dedicated to the quest of preserving the memory of African American mathematicians, lest they be lost forever." The new pages consist of a database containing biographical information of more than 700 mathematical scientists from the African Diaspora. The old site was a loose collection of profiles and pages with many stories; the navigation system felt dated and lacked any search capabilities. The new site uses a robust database: it is searchable by gender, degree-granting institution, and year of completion. This site also employs a wiki model, allowing users to create their own profiles and update any incorrect information immediately. The new site can be found at <https://www.mathad.com>.

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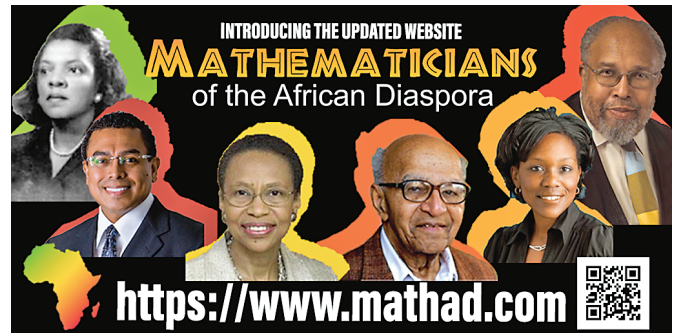


Figure 4.

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## How do AMS Graduate Student Chapters support the mathematical community and beyond?

### AMS Chapter at University of Texas-Arlington

On a Friday in February, after 2 years of planning and discussions, the AMS Chapter at UTA welcomed Dr. Talitha Washington of Howard University to their campus for a day filled with presentations, a lively Q&A and refreshments (sponsored by the student chapter of the Society for Industrial and Applied Mathematics and the College of Science Black Graduate Student Association). Dr. Washington first spoke to an audience primarily comprising graduate students about her own journey from childhood to her time at the National Science Foundation, including challenges related to her identity and where she found opportunities. We invited the undergrads to join us for the colloquium on the life and work of Dr. Elbert Frank Cox, famously known as the first Black PhD in mathematics. Dr. Washington tied landmarks of cultural, societal, and mathematical significance to the development of Dr. Cox as a pioneering mathematician.

Dr. Washington has been an inspiration for many. During the meet and greet, an undergraduate Calculus III student shared with Dr. Washington her interest in engineering and her excitement at being able to meet her, a professional mathematician who looks more like her than the majority of working mathematicians. A special moment indeed and a very inspiring afternoon. The 2 year wait for this event was so worth it for all involved!

### UTA Grad Students "Mav Up" During COVID

We graduate students are doing pretty well within UTA's math program considering the predicaments many are facing. Thankfully, we entered 2020 already having a very active communication line via GroupMe.

The memes do us good!

Certainly, we have faced many challenges with classes, candidacy exams, dissertations, along with teaching activities. Having each other, with our various backgrounds and graduate experiences, has proved to be one of the best guides in a confusing time. Sometimes the appropriate assist has been the willingness to both vent and welcome venting sessions. An important feature of our community is spreading accurate and pertinent information in a timely manner that may turn a venting session into a mathematical or administrative problem-solving journey!

Fortunately, COVID has not stopped us from celebrating each other—I'm especially appreciative of being a 2020 PhD graduate with major support from my peers!

Looking ahead, there are already summer study sessions in the works. A Zoom Pro license is in the pipeline, too, and maybe we will host other graduate chapters in order to unite, to collaborate, and, as we do at UTA, to Mav Up!

### AMS Chapter at Stony Brook University, New York

Despite the shutdown last spring because of COVID-19, Stony Brook's AMS Chapter had a full year of activities. The events provided a friendly atmosphere for graduate students to get together and interact with each other. The chapter invited postdocs as well so that more people interacted and shared their lives at Stony Brook, experiences, research, and career tips, building-up friendships.

The annual and traditional events, Fall Get-Together, Pi-Day, End of Year (canceled), help us keep communicated, related, and connected. Secondly, we wanted to provide graduate students something refreshed both in math and in life. The location of Stony Brook is quite isolated and the life of a graduate students is busy and focused on research during the semester, so we planned a new event this year, a trip to the National Museum of Mathematics in NYC. The members who went really enjoyed the museum. We look forward to more day trips off campus as a group. It is beneficial and helpful for graduate students to refresh, to broaden their perspective, and to have social interaction. We look forward to that and to developing new activities and our traditional ones throughout the year.

This past June, the graduate students in the math department at Stony Brook had a forum for diversity and racial equity on the 10th of June: #ShutDownMath Day. It's not an AMS Student Chapter event, but we think that it is a quite important activity of graduate students willing to support and build up a healthy and respectful environment in the future.





## AMS Chapter at Southern Illinois University Carbondale

The AMS Graduate Student Chapter at Southern Illinois University Carbondale completed several outreach activities directed at undergraduate and younger students in the 2019-2020 academic year before the shutdown (due to COVID) of in-person group activities in late Spring.

Hands-on mathematical activities were a feature of many of the events of the chapter, including game nights, their annual integration bee, and demonstrations for younger groups. The chapter members were entirely responsible for the development and presentation of these activities.

The largest project undertaken by the chapter was the 3rd Annual Integration Bee. Participants included high school students and undergrads. Planning and preparing by the chapter included: developing advertising and outreach, preparing the integration questions and answer keys, and marking the qualifying papers of participants. They saw a group of highly enthusiastic students come on a chilly Saturday morning to tackle day-long integration problems. The highlight of the event was when one of the semi-final rounds went on so long! The judges had

a difficult time deciding who should go to the finals among the highly competitive semi-finalists! After the competition, the participants stayed back to discuss math problems encountered during the day, play a game of set, handle cool mathematical objects, origami, and more. Most of the chapter was involved in some activity on the day of the event itself. The Integration Bee has grown in participation and the competition was keener than in previous years. Pizza and snacks provided a fun social environment that led to interaction among the contestants and chapter members.

The chapter hosted a guest lecture, "Math Colloquium: 3D - Design and Printing for Mathematic Education." In the context of mathematics education, the speaker reflected on his experience with 3D design and printing, using design examples, and discussing pedagogical possibilities. The event was well appreciated by faculty, as well as students, of the Math Department.

In the Fall 2019, 12 middle school girls came with their abundant energy and enthusiasm to a workshop entitled: "Expanding Your Horizons: Mathematics of Origami" conducted by AMS Secretary William Holt, who was assisted by other AMS Chapter members. The girls were taught construction of various model and geometric objects. The event was as enjoyable to the chapter members as it was to the middle schoolers.

## AMS Chapter at Central Michigan University

The AMS Chapter's Integration Bee was a huge success again this year. It was the most enthusiastic and cheerful mathematics event of the year.

We had about 60 people attend the event. About 35 undergraduate students and 14 graduate students took part in the competition. In the undergraduate bracket, the grand integrator was Austin Konkel and the runner up was Shashwat Maharjan. In the graduate bracket, the grand integrator was Mohyedden Sweidan and the runner up was Arkabrata Ghosh.



Photos courtesy of their respective universities.

For information about starting an AMS Graduate Student Chapter, please visit:  
[www.ams.org/studentchapters](http://www.ams.org/studentchapters)

# Overtoun Jenda: A STEM Mentor Extraordinaire

*Ash Abebe, Suzanne Lenhart,  
and Brittany McCullough*

Professor Overtoun Jenda has been recognized with the 2020 Presidential Award for Excellence in Science, Mathematics, and Engineering Mentoring for the substantial STEM mentoring, outreach programs, and networks that he developed and implemented. According to many who know him, he is able to recruit participants and volunteers through his optimistic worldview and genuine interest in connecting groups for mentorship and collaboration. He is said to have a contagious attitude of “We can achieve much with your help.” As some of us know, it is hard to say “no” to him because of his commitment, creative ideas, and infectious energy.

Dr. Jenda is currently Professor of Mathematics and Assistant Provost for Special Projects and Initiatives at Auburn University. He previously served as Associate Provost for Diversity and Multicultural Affairs and Associate Dean of the College of Sciences and Mathematics. He received a PhD in Mathematics from the University of Kentucky after obtaining a BS in Mathematics from the University of Malawi. Prior to coming to Auburn, Dr. Jenda had been a faculty member at the University of Kentucky, University of Botswana, and University of Malawi. Despite heavy administrative duties, Dr. Jenda has managed to maintain an active research program. He published three graduate level books and many research articles in

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Figure 1. Professor Jenda teaching.

homological algebra, as well as several articles in STEM Education.

While many of the programs that Dr. Jenda has implemented provide students with financial support and access to academic resources, mentoring is the common thread in all of his endeavors. He designed and built programs that provide mentoring for students with disabilities, students historically underrepresented in STEM, and first-generation and low-income students. In his programs, students receive mentoring not only from STEM faculty members and administrators, but also from their peers. Dr. Jenda began mentoring students in an unofficial capacity while he was an Associate Lecturer in his home country of Malawi, simply because he wanted to help students succeed. He continued to do so after coming to the US as an Assistant Professor and while working toward promotions, tenure, and entering into university administration. His willingness to provide support for students in need led to a natural progression into leadership positions in the



areas of diversity and STEM education, where he continues to provide mentorship and guidance to students at all levels.

Dr. Jenda has been the Project Director for multiple US National Science Foundation and Alabama State Department of Education grants focusing on STEM education and student success, some of which will be discussed in detail below.

### Reaching out to Students at Auburn

In 1988, during his early days at Auburn University, Dr. Jenda served as academic advisor for 40 students in the department. Since he was the only Black faculty member in the department, many Black students enrolled in math courses would stop by his office for tutoring and advising. He was glad to provide support and encouragement to these students, and it was the relationships formed in this capacity that served as a springboard for several mentoring programs that he would go on to develop over the years. In 1994, he submitted a proposal to the university to fund graduate students to provide tutoring for entry courses in mathematics, chemistry, and physics. His proposal was successful and the funding allowed the mentoring and tutoring of minority students that was previously performed by him alone to be formalized and expanded into what eventually became the College of Sciences and Mathematics Minority Drop-In Center. The Center continues to serve large numbers of students today as the home of a comprehensive academic support program.

Dr. Jenda believes in holding his students to high standards, as a stepping stone to their success. In 1994, he started a summer internship program for minority students, but found few qualifying students. By 1997, after students began participating in the mentoring and tutoring provided by the newly created Drop-In Center, there was a drastic increase in the number of minority students who qualified for his summer internship program.

Dr. Jenda enjoys problem solving, both as a mathematician and as an administrator. Realizing that many talented African-American students come from underserved high schools, in 1997 he created a Summer Bridge Program for 35 minority students, and obtained funding from the university. The goal of this four-week residential program was to give students a head start on their college career by emphasizing academic preparedness, time management skills, and providing a trial run at math and science courses, career awareness activities, and networking opportunities. Dr. Jenda ran this program and served as a mentor for all participants from 1997 until 2006, when he became the university's chief diversity officer. The program still takes place every summer, and he continues to mentor



**Figure 2.** Professor Jenda and first-generation college students on a pre-freshman study-abroad trip to the Virgin Islands.

staff members and graduate assistants in the program and serves as a guest speaker for participants.

From 2006–2010, Dr. Jenda served as the Principal Investigator for the graduate GK–12 Fellows in Science and Mathematics program for East Alabama Schools. This program placed graduate students in STEM disciplines in 9th–12th grade classrooms to serve as resource persons. Dr. Jenda served as a mentor to the fellows, providing guidance on teaching and how to communicate with K–12 teachers and students. The fellows provided overwhelmingly positive feedback about this program, and many of them have gone on to careers in academia.

In 2007, Dr. Jenda designed and established the Provost Leadership Undergraduate Scholarship Program to support underrepresented students through scholarships and mentoring. He hired a full-time staff member to administer the program, while continuing to mentor students from 2007 to 2015. Students were provided financial support with peer mentoring, leadership training, study sessions, and counseling services. This program continues to be one of Auburn University's flagship diversity programs and has provided scholarships and mentoring for over 500 underrepresented minority students.

He has also given opportunities to many undergraduate students in mathematics, through an NSF Summer REU program in algebra and discrete mathematics on Auburn University's campus. Each summer since 1999, at least eight students from around the country participate in daily seminars, problem sessions, and publication efforts under his leadership. He is the PI and one of two faculty mentors (with Dr. Peter Johnson) for the REU program.

## Networking beyond Auburn

Since 1994, Dr. Jenda has been actively involved in mentoring students through the NSF's Louis Stokes Alliances for Minority Participation (LSAMP) program. The LSAMP program aims to assist universities and colleges in diversifying the nation's workforce by increasing the number of STEM baccalaureate and graduate degrees awarded to students from ethnic and racial populations historically underrepresented in these disciplines. From 1994 to 2015, he served as the campus coordinator as part of the Alabama LSAMP. This program provides students with not only scholarship funding, but mentoring from faculty and student peers, free tutoring, research internships, and funding for travel to research conferences. Since 2017, Dr. Jenda has overseen the LSAMP activities for all eight institutions in the Greater Alabama Black Belt Region LSAMP Alliance.

From 2003–2005 and 2008–2010, he awarded Bridge to the Doctorate grants, as extensions of the LSAMP programs. The goal of this program was to increase the production of minority PhDs and facilitate their beginning faculty positions or research careers. Since the LSAMP program funded the first two years of graduate school, this program assisted students in obtaining additional funding for their remaining years of graduate school.

In 2018, in an effort to improve college readiness among students, Dr. Jenda developed the Greater Alabama Black Belt Region STEM Initiative, in partnership with the Alabama State Department of Education and the Cooperative Extension System. The Initiative hosts summer academies each year at high schools. The goal of the program is to help students be better prepared for college by expanding their content knowledge in core subject areas, allowing them to progress toward college admission and successful studies in STEM. Dr. Jenda serves as a mentor for the graduate students who teach in the program, as well as being a resource for the school districts' teachers and administrators. The program involved students from four counties in 2020 and is expected to expand to more counties in 2021.

Dr. Jenda has also created programs for students with disabilities. In 2009, he worked with Auburn faculty and four partner institutions to establish the Alabama Alliance for Students with Disabilities in STEM. This program provided peer mentoring and faculty mentoring for undergraduate and graduate students with disabilities, leading to rates of graduate school entrance higher than those of the general student population and higher levels of persistence in STEM fields and research programs.

"Seeing other individuals with disabilities working toward, and succeeding in, scientific fields at my own university, and at other institutions during the annual conferences, helped show me I was on the right path and not alone," said an Alliance participant, Danielle Tadych.

This alliance was expanded in 2016 through an NSF INCLUDES pilot project award, becoming the South East Alliance for Persons with Disabilities in STEM. This expansion included 21 institutions throughout the Southeastern United States. Dr. Jenda provided the leadership for this alliance, and efforts are underway to expand it to a national alliance to provide support for STEM students, postdoctoral fellows, and faculty members with disabilities nationwide.

## Collaborations in Southern Africa

In 2011, Dr. Jenda launched the Masamu Program (masamu means mathematics in the Bantu language), which promotes US-Africa research collaboration in mathematical sciences and related areas. He mentors faculty and students in the program and serves as the leader of the program's Collaborative Research Network in Mathematical Sciences, consisting of over 80 research faculty members from Africa, the US, Canada, China, and Europe, from 46 colleges and universities. The program provides undergraduate and graduate students with connections to senior research faculty mentors, giving them opportunities for networking, professional development, and publication opportunities [EJO17].

The idea of developing a sustainable US-Africa collaboration in mathematical sciences research was first conceived at the 2009 Southern Africa Mathematical Sciences Association (SAMSA) Conference held in Dar es Salaam, Tanzania, during a strategy session. This led to the creation of the Masamu Program in 2011 with initial financial support from Auburn University, the NSF, and partial funding from the British Council. This program continues to be funded by the NSF with cooperation from partner universities. Each year, the participants attend the SAMSA Conference and work together on research projects in small groups a few days before and after the conference. The collaborative work continues via online meetings throughout the year [LEG<sup>+</sup>17, DAC<sup>+</sup>18, ELA<sup>+</sup>19].

Dr. Edward Lungu, an international leader in Mathematical Biology and Dean of Faculty of Science, Botswana International University of Science and Technology, is co-chair of the Masamu Steering Committee and said about the success of this program: "The achievements of Masamu in Africa are invaluable. Masamu meets every year to train American, European, and African students. The opportunities Masamu has brought have led to improvement in research and teaching in Southern Africa. The number of publications in the region has increased tremendously thanks to the exposure Dr. Jenda has offered to our students who are assisted by their colleagues in developed and well-resourced institutions. There are many collaborative projects between Africa and





**Figure 3.** Masamu Group in Victoria Falls, Zimbabwe, 2014.

other continents—however, very few projects are as successful as the Masamu/SAMSA collaboration.”

Dr. K. A. Jane White, Head of Natural Sciences at the University of Bath in the UK, said, “As soon as I became aware of the Masamu Program, I was keen to be involved for one simple reason—its philosophy. Masamu expects participants to develop and maintain collaborative research activity between academics and PhD students from the sub-Saharan countries and people like me, academics from the UK, Europe, and the USA. That expectation comes from the clear vision that Dr. Jenda has for the Masamu Program and his commitment to the vision of training students and faculty in Africa to deliver high quality mathematical research and teaching.”

Over the years, Dr. Jenda’s mentoring efforts have impacted hundreds of students, especially underrepresented minorities and students with disabilities. His mentees have gone on to enjoy successful careers. He continues to be an exemplary role model and mentor for students, faculty, and staff. The three of us, Abebe, Lenhart, and McCullough, are glad to be a part of TEAM JENDA!

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# Masamu Program, a US-Africa Partnership

*Overtoun Jenda and Edward Lungu*

## Introduction to Masamu

The primary goal of the Masamu (note that “masamu” means mathematics in Southern Africa) Program is to enhance research in the mathematical sciences and related areas within the Southern Africa Mathematical Sciences Association (SAMSA) institutions and beyond through promotion of international research collaboration. The Masamu Program was established in 2010 with the assistance of the National Science Foundation through an Auburn University-SAMSA Memorandum of Understanding (MOU) with particular emphasis on increasing and sustaining research collaborations between mathematicians in the US and the Southern Africa region (Angola, Botswana, Eswatini, Lesotho, Malawi, Mozambique, Namibia, South Africa, Tanzania, Zambia, and Zimbabwe). The strong collaborative support from African and US universities has made the Masamu Program viable and able to expand its reach.

A key component of the Masamu Program is the Masamu Advanced Study Institute (MASI) and Workshop Series in mathematical sciences and related areas, which provides a platform for such collaboration. This is done through the US-Africa Collaborative Research Network (CRN), consisting of US and African senior researchers in several areas of the mathematical sciences and related areas together with additional collaborators from Canada and Europe. The primary goal of the CRN is to improve the

human infrastructure in the mathematical sciences in both the US and Sub-Saharan Africa, especially among women and underrepresented minorities. The target audiences of MASI are graduate and undergraduate students, postdocs, and early-career faculty (those whose rank is less than associate professor), while the workshops are also open to more senior faculty and other researchers in the mathematical sciences and related areas. The Masamu Program especially encourages applications from undergraduate students who have participated in Research Experiences for Undergraduates (REU) programs.

The first institute and workshops were held in 2011 in Livingstone, Zambia. The primary focus of the inaugural institute was on epidemiological modeling. Additional connections were then developed in other areas where there were mutual research strengths, resulting in an expanded CRN that now consists of 82 research mathematicians from Sub-Saharan Africa, US, Canada, China, and Europe from 46 collaborating colleges and universities and research laboratories/centers/institutes. Each research group is co-led by US and African senior research faculty and is subdivided into research teams consisting of senior mathematicians, early-career faculty, students, and postdoctoral researchers.

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**Figure 1.** Michael Washington of CDC presenting to the Mathematical Biology Research Group in Arusha, Tanzania, in 2017.

Each research team includes both US and African participants. African and American researchers work side by side but with different techniques and different questions. This diversity of perspective provides an opportunity for CRN to broaden the spectrum of questions to be studied and the methods/approaches for exploring these questions, resulting in increased research activity and productivity for both African and US participants.

## Programming

SAMSA holds a 4-day international research conference during Thanksgiving week each year, and the Masamu Program runs a 10-day MASI and Research Workshop that overlaps with the conference. US-Africa CRN members take advantage of this gathering and participate in the conference, work on problems, plan research activities for the subsequent year, and present their results at the conference. The conference enhances the institute by providing Masamu participants a platform to make presentations and meet potential research collaborators, and in turn the institute enhances the conference by the lectures presented by CRN senior researchers. Indeed, the conference helps MASI participants learn about new research problems and techniques, since not all SAMSA participants are able to

attend the institute. For example, a Tanzanian mathematician gave a talk at SAMSA 2012 on “connected dominating sets of vertices in graphs” which had some distant connections to US work in this area. The mathematician was then invited to MASI 2013, and this area is now a thriving research area for US students and faculty together with their African counterparts. These activities enable the network to produce an increased number of new PhDs in both the US and Sub-Saharan Africa, more joint research publications, and long-term US-Africa partnerships consisting of researchers from diverse backgrounds. Indeed, the focus is on building long-term US-Africa collaborations rather than a one-off participation, and thus long-term involvement is expected of all participants.

The Kovalevskia Research Grants program was established in 2014 as an initiative of the Kovalevskia Fund, within the SAMSA Masamu Program. These grants are awarded every two years to two upcoming female mathematicians (one in pure and another in applied mathematics) from the Sub-Saharan Africa region who have demonstrated outstanding research and publications in pure or applied mathematics.

One of the 2016 Kovalevskia Research Grants recipients was Dr. Winifred Nduku Mutuku, a senior lecturer



**Figure 2.** 2019 MASI participants in Malawi.



**Figure 3.** Dr. Neal Koblitz and Dr. Winifred Nduku Mutuku at the Kovalevskia Research Awards ceremony in Pretoria in 2016.



**Figure 4.** Kovalevskia Research Awards in Palapye, Botswana, in 2018.



in the Department of Mathematics and Actuarial Science at Kenyatta University, Kenya. Dr. Mutuku received her BS in Education and MS in Applied Mathematics at Kenyatta University and her PhD in Applied Mathematics at Cape Peninsula University of Technology, South Africa, specializing in mathematical modeling and computation of fluid mechanics. She is active in research and currently serves as Chair of the Departmental Board of Postgraduate Studies and Departmental Research Coordinator.

The Masamu Program also sponsors a Research Workshop each year as part of the Masamu Advanced Study Institute (MASI) and the following workshops as part of the SAMSA conference: the Career Development Workshop, the High Performance Computing Workshop, the STEM Education Workshop, and the Department Heads, Chairs and Senior Research Scientists Workshop. The Masamu Program also sponsors colloquia, faculty exchanges, lecture series, and study abroad programs. One of the most sought-after workshops is the STEM Education Workshop in which Masamu runs partnerships with local authorities in host countries during the SAMSA conference. The workshop examines and discusses current evidence-based models, best practices, and strategies for increasing the quantity and quality of undergraduate and graduate degrees in the mathematical sciences. Of particular interest is the opportunity the annual workshop provides to US research faculty and postdocs to collaborate with local teachers in the design of new educational enrichment activities for motivating and exciting high school students to major in mathematics. This has motivated US participants to be more involved in K–12 STEM education back home.

Finally, in 2021 SAMSA will celebrate its 40-year anniversary together with the postponed (due to COVID-19) 10-year anniversary of MASI. To commemorate these special occasions, the US-Africa CRN will host a 2-day international research symposium in lieu of the standard research workshop held at the end of MASI. The theme of the symposium is “Mathematics Connecting to One Health” focusing on the interconnection of human, animal, and environmental health, which are all research focus areas of the US-Africa CRN. At the symposium, we will highlight the work done by the network and work on future collaborative research areas and knowledge transfer to inform policy and practices.

## Accomplishments

The Masamu Program has had nine memorable face-to-face annual Institutes and Workshop Series held in seven different Southern African countries from 2011 to 2019 with 340 participants (including repeats) from 19 countries, with 141 (139 from the US) from North America. The 10th MASI, which took place in 2020, was held virtually due to the pandemic. Eight additional workshops and lectures

have been held outside the annual MASI and Workshop Series over the years.

MASI is a true retreat where you can focus 100% on your research, far away from home and with zero distractions. This has proven to be extremely helpful for US graduate students and faculty, especially those that spend a lot of time teaching. Indeed, most claim that they do more research at the institute in those 10 days than they do during the rest of the academic year. This is a testament to the intense focus on research and being surrounded by top students and faculty from around the world bringing together their research methods and ideas. Productivity is also enhanced by the flexibility that allows participants to work and collaborate on multiple projects and continuously learn new techniques, which is very hard for them to match when they are back in the US. Those that have participated in other institutes in the US say that their experience at MASI is comparable to their visits to other research institutes. What has made MASI successful is the absence of the mentality of the colonial era that espoused the belief that European mathematicians were better than their African counterparts and that Western mathematicians had nothing to learn from their African colleagues. CRN participants are all on equal footing, and this has helped build solid, lifelong collaborations that have strengthened research activity in all participating African and US research programs.

Solutions to the research questions being studied by CRN research teams have important applications in pure and applied mathematics, industry, government, and in society in general. In particular, solutions to these questions have applications in national security, network coverage, surveillance, transportation, finance, and statistics. They also advance understanding of the spread of disease and biodiversity, which have important policy implications as demonstrated by the COVID-19 crisis. The study of tumor



**Figure 5.** Participants visiting Ngorongoro Crater in Tanzania in 2017.



growth, cell motility, and DNA structure have significant medical applications, and mathematical models involving climate data provide important predictions of the impacts of climate change on future human and animal behavior and their interactions.

CRN has grown by 100% since 2011 (from 41 to 82), and we expect the number of participants to continue to grow as we increase the size of each research group and add more research areas. Senior research faculty that are interested in attending the SAMSA conference or would like to be a member of the CRN should contact Paul Horn at the University of Denver and Peter Johnson at Auburn University (for discrete mathematics), or Suzanne Lenhart at the University of Tennessee Knoxville (for mathematical biology), or email [masamu@auburn.edu](mailto:masamu@auburn.edu) for all other areas. Members of the network lead or participate in research teams and should be prepared to attend MASI whenever their schedule allows.

Early-career faculty, graduate and undergraduate students, and postdoctoral researchers can apply to the Masamu Advanced Study Institute (MASI) at [www.masamu.auburn.edu](http://www.masamu.auburn.edu). Participants are selected by the Masamu Program Steering Committee. Selection criteria includes (1) research potential or experience as verified by support letters, and (2) diversity considerations (disability, gender, underrepresented minority, type of college or university). The number of MASI participants is capped at 34 (17 from US) with 10 slots reserved for senior researchers. Thus the selection for MASI is competitive.

### Financial Support

The MASI sustainability model necessitates that faculty and postdocs seek their own funds for attending the institute. The Masamu Program uses NSF funding to support local expenses at the institute and the SAMSA conference depending on financial need. However, faculty and



**Figure 6.** Graph Theory Research Group in Victoria Falls, Zimbabwe, in 2014.

postdocs are urged to find funds to cover their airfare. Selected students will be fully supported. Note that NSF support is only available to US citizens and permanent residents. Sub-Saharan participants, including faculty and students, may also apply for funds from the Simons Foundation “Research and Graduate Studies in Mathematics and its Applications (RGSMA): A Network Approach” project.



**Figure 7.** Edward Lungu and Overtoun Jenda at the 2018 SAMSA conference opening ceremony in Palapye, Botswana.

### Masamu Program Management

The Masamu Program is managed by the Masamu Steering Committee consisting of five representatives from Sub-Saharan Africa, five from the US, and one from Europe. The Auburn University Office of Special Projects and Initiatives administers the US-Africa Collaborative Research Network and runs MASI and workshops on site in Africa. For general inquiries about the program, please contact Dr. Brittany McCullough, Program Analyst, at [masamu@auburn.edu](mailto:masamu@auburn.edu).

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Overtoun Jenda



Edward Lungu

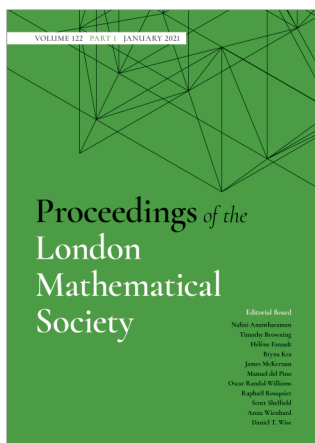
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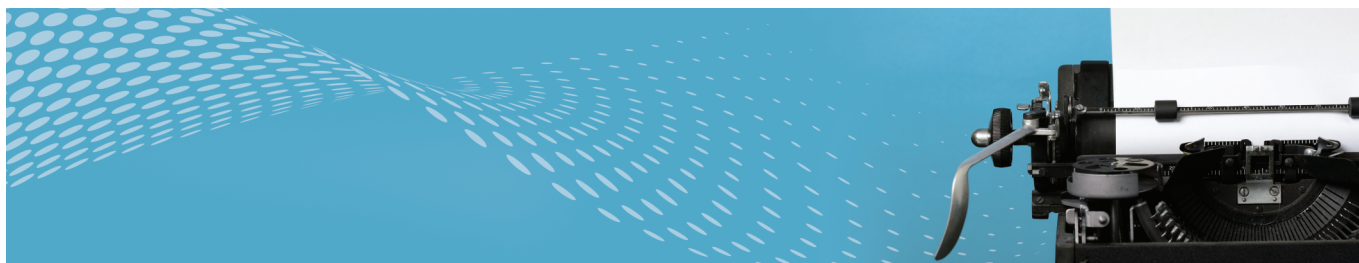
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## Six Surfaces (Almost) Surely

*Danny Calegari*

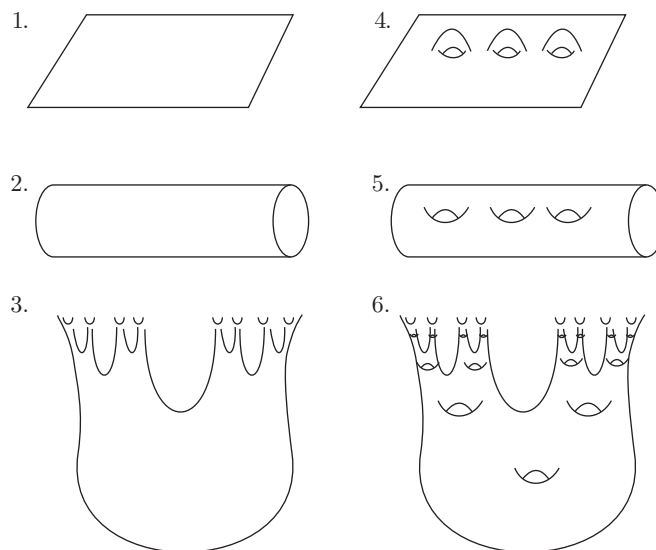
In the television show “Star Trek,” whenever Spock and Kirk and an anonymous ensign marked for death beam down to a strange alien planet, chances are they’ll encounter a landscape oddly reminiscent of Southern California, populated by a race of English-speaking bipedal humanoids whose otherness is certified by one or two facial prosthetics and the unusual color of their polyester jumpsuits.

This uniformity, given the diversity of biological life-forms and habitats on even our own planet, is an artefact of the limited budget and special effects capabilities of Paramount studios in the 1960s. But even in mathematical environments unimaginably rich in possibility, the dice can be loaded in surprisingly frugal ways.

One of my favorite examples is an astonishing theorem of Étienne Ghys, on the topology of “typical” leaves of foliations. A foliation is a manifold clothed in a stripy fabric: the space is filled up with submanifolds (the “leaves” of the foliation) of lower dimension, which locally are arranged in products, but globally can exhibit all kinds of interesting behavior.

Any reasonable nowhere zero vector field integrates to give a 1-dimensional foliation by integral curves. Likewise, a holomorphic vector field on a complex manifold integrates to a foliation by leaves of real dimension two. Though there are some restrictions in higher

dimensions, John Cantwell and Larry Conlon proved that *every* 2-dimensional manifold—i.e., every surface—arises as a leaf of some foliation of some compact manifold.



**Figure 1.** The usual suspects.

Compact (oriented) surfaces are classified by genus—the number of handles—which can be any nonnegative integer. Noncompact (oriented) surfaces are more numerous. There are uncountably many of them. They are classified by two invariants. First there is the space of *ends*. The “ends” of a topological space are—roughly speaking—the “connected components at infinity.” Technically, the path-connected components (i.e.,  $\pi_0$ ) of the complements

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of bigger and bigger compact subsets of a space form an inverse system, and the ends are the points in the inverse limit of this system. The space of ends of a surface is always compact and totally disconnected, so that it embeds in a Cantor set. Every possible homeomorphism type of a (nonempty) compact subset of a Cantor set can arise as the space of ends of some noncompact surface. Second, one considers genus (i.e., number of “handles”). If the genus is finite, it can be any nonnegative integer. If it is infinite, one must consider how the handles interact with the ends—some ends have a neighborhood which is planar, the others (a closed subset of them) are accumulated by handles. This subset of nonplanar ends may again be any closed subset of the space of all ends, and the pair of spaces (ends, nonplanar ends) up to homeomorphism is a complete invariant.

For example: there is exactly one oriented surface (up to homeomorphism) whose space of ends is homeomorphic to the ordinal  $\omega^\omega + 1$ , and for which the only ends accumulated by handles are those whose ordinal is divisible by  $\omega^{666}$ . But out of this crazy proliferation of possibilities, Ghys’s theorem says that a *typical noncompact leaf* is almost surely homeomorphic to one of only *six* (!) possibilities, illustrated in Figure 1.

Three of the surfaces are planar: the plane (one end), the cylinder (two ends), and the sphere minus a Cantor set (sometimes called the *Cantor tree* surface); the points in the missing Cantor set are precisely the ends of this surface. The other three surfaces are obtained from these three by plumbing with infinitely many “handles,” accumulating to every end. Colloquial names for the latter three are the *Loch Ness monster*, the *ladder* surface, and the *blooming Cantor tree* surface.

To understand Ghys’s theorem we must first understand what is meant by the expression “a typical leaf.” The word “typical” refers to a certain kind of probability measure on the foliated manifold called a *harmonic measure*, and a “typical point” is a point chosen at random with respect to this measure. Finally, the leaf containing this typical point is a “typical leaf.” It’s important to get the order right: Dennis Sullivan famously observed that “there is no measurable way to pick a point on a leaf.”

On a compact Riemannian manifold, any initial heat distribution becomes equidistributed over time under the heat flow. Another way of saying this is that the uniform (volume) measure is the unique probability measure invariant under heat flow. Equivalently, it is preserved (on average) by *Brownian motion*, the random process obtained as a parabolically scaled limit of random walk. On a (compact, Riemannian) foliated manifold we want to look for probability measures that are invariant under a restricted sort of heat flow, where heat is only allowed to diffuse along the leaves. Any such probability measure is

called a *harmonic measure* for the foliation. Lucy Garnett proved that every foliation of a compact manifold admits at least one harmonic measure. There might be more than one; for example, the uniform volume measure concentrated on any compact leaf is a harmonic measure. With respect to the local product structure of a foliation, any harmonic measure disintegrates leafwise into the leafwise volume measure times a leafwise harmonic function. If some leaves are noncompact, they might admit many non-constant positive harmonic functions, and in general harmonic measures can be quite interesting.

Ghys’s theorem arises from the fact that we can think about leafwise Brownian motion in two quite distinct ways: as a process on a compact manifold, or as a process on a (typically) noncompact leaf. Leafwise Brownian motion preserves harmonic measure. If we think of it as a measure-preserving random process on a compact manifold, then Poincaré recurrence means that for every Borel set  $B$  of positive harmonic measure, leafwise Brownian motion starting at (almost) any  $x \in B$  will return to  $B$  infinitely often, and in fact the expected time of first return is finite. On the other hand, if we think of Brownian motion as a random process on any particular noncompact leaf, it will turn out that the expected time of return to any compact subset of the leaf is *infinite*.

The simplest case to think about is an end with the geometry of a half-infinite cylinder. Brownian motion in this end is roughly like simple random walk on the positive integers. As is well known, a simple random walk started at any fixed positive integer  $n$  will return to 0 almost surely. However, the expected time before it returns is infinite! Let’s see why. Let  $R(n)$  be the expected time for a random walk beginning at  $n$  to return to 0. Evidently  $R(n)$  is monotone increasing; we will show in fact that  $R(1)$  (and hence  $R(n)$  for all positive  $n$ ) is infinite.

From the definition,  $R(0) = 0$  and

$$R(n) = 1 + \frac{1}{2}(R(n-1) + R(n+1))$$

for every positive  $n$ . Summing the latter formula for  $n$  ranging from 1 to some big  $N$  and rearranging, we see that  $R(N) = R(N+1) - R(1) + 2N$ . If  $R(1)$  were finite, then for sufficiently big  $N$  we would have  $R(N) > R(N+1)$ , which is absurd. So  $R(1)$  (and therefore also  $R(n)$  for all positive  $n$ ) is infinite.

By comparing leafwise Brownian motion in  $M$  and in a leaf  $\lambda$ , we deduce that for any Borel set  $B \subset M$  of positive harmonic measure and for almost every  $x \in B$ , every end of the leaf  $\lambda(x)$  containing  $x$  intersects  $B$ . How might we apply this, focusing for concreteness on the case of a compact manifold foliated by surfaces? We want to show that a leaf with more than two ends has an entire Cantor set of ends; and furthermore, that a leaf of positive genus has every end accumulated by handles. A surface  $S$  has more than two ends if and only if it contains a compact

subset  $P$  whose complement has at least three unbounded components. Let's let  $B(n)$  be the union of all such compact subsets of (leafwise) diameter at most  $n$  over all leaves, and let  $B = \bigcup_n B(n)$ . It turns out that each  $B(n)$  is Borel. To see this, let  $B(n, N)$  denote the set of leafwise compact subsets  $P$  of diameter at most  $n$  whose complement in a ball of radius  $N + n$  has at least three components of diameter at least  $N$ . The property of being some  $P$  in  $B(n, N)$  depends only on the geometry of a ball of radius  $N + n$  in some leaf, and therefore the set  $B(n, N)$  is closed. Since  $B(n) = \bigcap_N B(n, N)$ , it follows that  $B(n)$  (and also  $B$ ) is Borel.

A leaf disjoint from  $B$  has at most two ends. Any other leaf must intersect some  $B(n)$ , and therefore (almost surely), every end intersects  $B(n)$ . This means that every end contains a subset  $P$  of diameter at most  $n$  with at least three unbounded components. This implies that no end is isolated, so that the space of ends is therefore perfect, and thus homeomorphic to a Cantor set.

In a similar way, a surface  $S$  has positive genus if and only if it contains a handle—i.e., a subset  $H$  homeomorphic to a once-punctured torus. Let  $C(n)$  denote the union of all leafwise handles of diameter at most  $n$ , and let  $C = \bigcup_n C(n)$ . Then each  $C(n)$  is closed, and therefore Borel. A leaf disjoint from  $C$  is planar. Any other leaf must intersect some  $C(n)$ , and therefore (almost surely) every end intersects  $C(n)$ , so that every end is accumulated by handles. From this Ghys's theorem follows immediately.

**AUTHOR'S NOTE.** Ghys's theorem is proved in "Topologie des feuilles génériques," *Ann. Math.* 141 (1995), 387–422.



Danny Calegari

#### Credits

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# 2020 *Election*

## Election Results

In the elections of 2020, the Society elected a vice president, a trustee, five members at large of the Council, three members of the Nominating Committee, and two members of the Editorial Boards Committee.

### Vice President



Photo courtesy of Hee Oh

**Hee Oh**

*Yale University*

Term is three years  
(February 1, 2021–January 31, 2024)

### Board of Trustees



Photo courtesy of David R. Morrison

**David R. Morrison**

*University of California,  
Santa Barbara*

Term is five years  
(February 1, 2021–January 31, 2026)

### Members at Large of the Council

Terms are three years (February 1, 2021–January 31, 2024)



Photo courtesy of Alina Carmen Cojocaru

**Alina Carmen  
Cojocaru**

*University of Illinois  
at Chicago*



Photo courtesy of Duane Cooper

**Duane Cooper**  
*Morehouse College*



Photo courtesy of Chase S. Reynolds

**Sarah J.  
Greenwald**  
*Appalachian State  
University*



Photo courtesy of Laurie DeVitt

**Kiran S. Kedlaya**  
*University of  
California, San Diego*



Photo courtesy of Michael Kellert

**Anne Joyce Shiu**  
*Texas A&M University*



## Nominating Committee

Terms are three years (*January 1, 2021–December 31, 2023*)



**Alex Eskin**

*University of Chicago*



**Patricia Hersh**

*University of Oregon*



**Ezra Miller**

*Duke University*

## Editorial Boards Committee

Terms are three years (*February 1, 2021–January 31, 2024*)



**Barbara Lee  
Keyfitz**

*The Ohio State  
University*



**Anna Mazzucato**

*Pennsylvania State  
University*



# 2021 Election

## Nominations by Petition

### Vice President or Member at Large

One position of vice president and member of the Council *ex officio* for a term of three years is to be filled in the election of 2021. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and procedures, which are described below.

### Editorial Boards Committee

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The President will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

### Nominating Committee

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The President will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate's assent and petitions bearing at least 100 valid signatures are required for a name to be placed on

the ballot. In addition, several other rules and procedures, described below, should be followed.

### Rules and Procedures

Use separate copies of the form for each candidate for vice president, member at large, member of the Nominating or Editorial Boards Committees.

1. To be considered, petitions must be addressed to Secretary, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2213 USA, and must arrive by 24 February 2021.
2. The name of the candidate must be given as it appears in the American Mathematical Society's membership records and must be accompanied by the member code. If the member code is not known by the candidate, it may be obtained by the candidate contacting the AMS headquarters in Providence ([amsmem@ams.org](mailto:amsmem@ams.org)).
3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.
4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.
5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.
6. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.

# Nominations by Petition

The undersigned members of the American Mathematical Society propose the name of

\_\_\_\_\_ as a candidate for the position of (check one):

- ☐ **Vice President** (*term beginning 02/01/2022*)
- ☐ **Member at Large of the Council** (*term beginning 02/01/2022*)
- ☐ **Member of the Nominating Committee** (*term beginning 01/01/2022*)
- ☐ **Member of the Editorial Boards Committee** (*term beginning 02/01/2022*)

of the American Mathematical Society.

Return petitions by February 24, 2021 to:  
Secretary, AMS, 201 Charles Street, Providence, RI 02904-2213 USA

## Name and address (printed or typed)

	Signature
	Signature
	Signature
	Signature
	Signature
	Signature





## AMS Prizes & Awards

### Leroy P. Steele Prize for Lifetime Achievement

#### About this Prize

The *Steele Prize for Lifetime Achievement* is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students. The amount of this prize is US\$10,000.

**Next Prize:** January 2022

**Nomination Deadline:** March 31, 2021

**Nomination Procedure:** <https://www.ams.org/steele-prize>

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prizes for Lifetime Achievement should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

### Leroy P. Steele Prize for Mathematical Exposition

#### About this Prize

The *Steele Prize for Mathematical Exposition* is awarded for a book or substantial survey or expository research paper. The amount of this prize is US\$5,000.

**Next Prize:** January 2022

**Nomination Deadline:** March 31, 2021

**Nomination Procedure:** <https://www.ams.org/steele-prize>

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prizes for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

### Leroy P. Steele Prize for Seminal Contribution to Research

#### About this Prize

The *Steele Prize for Seminal Contribution to Research* is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

1. Open (2025)
2. Analysis/Probability (2026)
3. Algebra/Number Theory (2021)
4. **Applied Mathematics (2022)**
5. Geometry/Topology (2023)
6. Discrete Mathematics/Logic (2024)

**Next Prize:** January 2022

**Nomination Deadline:** March 31, 2021

**Nomination Procedure:** <https://www.ams.org/steele-prize>

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prizes for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

## Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of 12 members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at <https://www.ams.org/ams-fellows>.

## Joint Prizes & Awards

### 2021 MOS–AMS Fulkerson Prize

The Fulkerson Prize Committee invites nominations for the Delbert Ray Fulkerson Prize, sponsored jointly by the Mathematical Optimization Society (MOS) and the American Mathematical Society (AMS). Up to three awards of US\$1,500 each are presented at each (triennial) International Symposium of the MOS. The Fulkerson Prize is for outstanding papers in the area of discrete mathematics. The prize will be awarded at the 24th International Symposium on Mathematical Programming to be held in Beijing, China on August 15–20, 2021.

Eligible papers should represent the final publication of the main result(s) and should have been published in a recognized journal or in a comparable, well-refereed volume intended to publish final publications only, during the six calendar years preceding the year of the Symposium (thus, from January 2015 through December 2020). The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided.

The term “discrete mathematics” is interpreted broadly and is intended to include graph theory, networks, mathematical programming, applied combinatorics, applications of discrete mathematics to computer science, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will be based only on their mathematical quality and significance.

Previous winners of the Fulkerson Prize are listed here: [www.mathopt.org/?nav=fulkerson#winners](http://www.mathopt.org/?nav=fulkerson#winners). Further information about the Fulkerson Prize can be found at [www.mathopt.org/?nav=fulkerson](http://www.mathopt.org/?nav=fulkerson) and [https://www.ams.org/prizes-awards/paview.cgi?parent\\_id=17](https://www.ams.org/prizes-awards/paview.cgi?parent_id=17).

The Fulkerson Prize Committee consists of

- Gerard Cornuejols (Carnegie Mellon University)
- Eva Czabarka (University of South Carolina)
- Daniel Spielman (Yale University)

Please send your nominations (including reference to the nominated article and an evaluation of the work) by **February 15th, 2021** to the chair of the committee. Electronic submissions to [gc0v@andrew.cmu.edu](mailto:gc0v@andrew.cmu.edu) are preferred.

Gerard Cornuejols  
Tepper School of Business  
Carnegie Mellon University  
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e-mail: [gc0v@andrew.cmu.edu](mailto:gc0v@andrew.cmu.edu)

# An Interview with Ruth Charney

*Evelyn Lamb*

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. What follows is an edited version of an interview with Ruth Charney, whose two-year term as president began on February 1, 2021. Charney is the Theodore and Evelyn Berenson Professor of Mathematics at Brandeis University. The interview was conducted in fall 2020 by freelance writer Evelyn Lamb.



**Figure 1.** AMS president Ruth Charney.

*Evelyn Lamb is a freelance writer in Salt Lake City, UT. Her email address is [rootsofunityblog@gmail.com](mailto:rootsofunityblog@gmail.com).*

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DOI: <https://dx.doi.org/10.1090/noti2228>

**Notices:** *Let's start with your academic history and how you got involved with the AMS.*

**Charney:** Those are two slightly different things. I had a pretty standard academic history, I think. I was actually an undergraduate at Brandeis. It was a small school and only a few of the math majors planned to go on to graduate school. But it was very friendly, and I enjoyed my undergraduate years very much. After that, I took a gap year to study modern dance, and I had a great time, but the following year I went back to graduate school at Princeton. Then I had postdocs at Berkeley and Yale and eventually took a tenure-track job at Ohio State University. I spent the next 18 or 19 years of my career at Ohio State. Then in 2003, I moved to Brandeis. I have to say, I was very happy at Ohio State. The move had more to do with family issues: my husband's job and my mother, who was getting older and was alone in Vermont. I ended up moving back to my alma mater, Brandeis, and I've been there ever since. I think it's a pretty standard career path, nothing unusual about it, except that there were very few women in math at the time I started.

**Notices:** *Can we go back to modern dance? That is a little bit of an unusual gap year for a mathematician.*

**Charney:** I like to tell that story because it was a big part of me for a long time, and it isn't a part that most of the math community knows about. I started dancing in college; it's not something I did as a kid. I'd been in school all my life, and I planned to go on to graduate school. It wasn't that I didn't know what I wanted to do next; I just decided I needed a year off to do something different, and the one non-math thing I'd really been enjoying was dance. My



sister was living in New York at the time, so I moved to New York to study dance for a year.

I was having such an amazing time dancing that I almost put off graduate school. I applied in the fall to graduate schools, and I got into a couple of schools, including Princeton, my first choice. I thought really hard about whether I could put this off for a year or two, because I was so enjoying myself. But I was afraid that if I didn't accept the offer to go to Princeton, I would regret it later. I think in the end, I realized that you can have a career in math and do dance as a hobby, while the other way around doesn't work very well. And that's totally what I did for many years. Eventually, after I had kids and then moved to Boston, the dance started to slide away. I'm not doing it anymore, although I keep saying I'm going to sign up for dance lessons again! But for a long, long time, it was my main passion outside of work.

*Notices:* And how did you get involved in the AMS?

**Charney:** I think my first serious involvement in professional societies was in 1990. I was asked to be on the AWM (Association for Women in Mathematics) Executive Committee. Two years later, I was elected as a member-at-large of the AMS Council. So it was right in the early '90s that I got involved in both organizations, and frankly, since that time, I've always been at least on some professional committee. From 2006 to 2009, I was vice president of the AMS, and then I was on the Board of Trustees—I was very, very heavily involved. During that period, I also became president of the AWM. Then I took a brief break, and now here I am again. I can't seem to stay away from it!

I've thought a little bit about what got me involved—why I did it to begin with and why I continue to do it. And the obvious answer, which is certainly a factor, is that I felt that these organizations were doing something useful, and that I could contribute something. But I realize also that I was getting something out of it that I really liked. I liked connecting with a broader swath of the math community, hearing how different people saw different issues. I felt that I got a much broader picture of what was going on and a chance to meet interesting people.

*Notices:* And what skills and experience do you think you bring to the job of president?

**Charney:** In one sense, it's exactly what I said, a breadth of points of view. I've seen things from a lot of different sides and hopefully can bring that to the table. I've also been very involved with AMS itself, so I understand how AMS operates. Hopefully I can combine this experience with an awareness of some of the current issues that matter to people in the math community in a productive way.

*Notices:* And what are your priorities as you start serving your term as president?

**Charney:** I've been saying for years—and at this moment in time, it is particularly relevant—we made some real progress in bringing more women into mathematics and encouraging women already in the field. We're not at equality yet, but we've come a long way compared to what things were like when I started. Could we not refocus some of these ideas on other types of diversity and inclusion? Isn't it time to do that? The issues aren't exactly the same, but some of what we've learned from helping women get involved in mathematics and supporting them through their careers could be brought to bear on other types of diversity. If I had to choose one top priority, that would be it. I'm hoping my experience might prove useful, though I realize I'm going to need to talk to a lot of other people to get a better perspective and more ideas on how to address this problem.

We need to look at what the AMS can do. Issues of equality and inclusion have to be addressed at all different levels. You hear people say, "There aren't enough people from underrepresented groups to draw on. They're not even majoring in math. They're not even in our undergraduate classes. How are we supposed to fix this?" And then they blame the problem on K–12. And you know, it's easy to push the blame down. Absolutely, there have to be programs to address these problems in preschool and K–12, but I think the AMS needs to look at what we are in a position to do. We are in a position to address the undergraduate to graduate to postdoc levels, moving people into careers in mathematics. That's where we can actually have



**Figure 2.** Ruth Charney in Boston in 2018.

an effect. I don't have the answers to that, but I have some ideas. I look forward to discussing this with the new AMS Committee on Equity, Diversity and Inclusion.

*Notices:* It seems like you're taking over during a crisis moment in every area of life. COVID-19, of course, has sort of framed this whole year. How do you think you and the AMS will be responding to COVID-19 and the related crises that have come up since it started?

**Charney:** This is my opportunity to give a shout-out to Catherine Roberts, the executive director, and the whole AMS staff. They have been absolutely amazing. The pandemic has had major effects on meetings and on the functioning of the AMS. It's caused complications for the publishing business and for MathSciNet. I was pretty panicky at first, but I'm already feeling like things are under control. Yes, there are still challenges. Things are going to be different. There's not going to be an in-person JMM this year. I've always enjoyed the AMS meetings, especially the social part where you could just hang out, and there's not going to be as much of that for a while. But I feel like the "keeping things functioning part" is going well—the staff is amazing! Which means I get to focus on the issues that I was hoping to focus on. I won't be spending all my time trying to keep the ship from sinking.

*Notices:* Another aspect of the crisis, of course, is employment prospects for people who are graduating with PhDs or finishing their postdocs now. Are there ways the AMS can address that problem?

**Charney:** You're absolutely right. I do think this is a huge problem, and we do need to think about what, if anything, we can do. I have one postdoc and one graduate student both going on the job market this year, and it's very scary. There was discussion about this back in the days of the recession when there were similar sorts of problems. What could we do? One idea was to create more postdoctoral fellowships. The problem is that you can redirect something like the Centennial Fellowship to younger people or create a new fellowship or two, but that's only a fellowship or two. That's nice, but it's not broad enough. How can we affect a larger class? We have an office in Washington that does advocacy for us, to increase funding for NSF and so on. NSF could potentially create many new fellowships, while AMS could at best create one or two. Universities are all suffering right now. It's hard to know exactly where the effects are going to be felt most strongly. This is a huge problem. Various AMS committees are thinking about what else AMS could do in addition to advocacy. I wish I had answers to that.

*Notices:* As you said, part of the problem is that we don't know exactly what effects this crisis will have when all is said and done.

**Charney:** There's a lot of talk about whether this will change things in the long term, even after the virus is gone, and I think for some businesses it will. People are going to do less traveling, and there are definitely going to be changes, but I feel that people go to universities to get away from home, to change their world, not just to go to classes. I don't think that's going to disappear. I think that's going to come back to normal. Maybe there'll be a little more online teaching. When I taught at Ohio State University, I had some students who had children and a job, and they probably would have had an easier time if they could have done it remotely. But the students who come straight out of high school don't want to sit in their bedrooms and attend college by Zoom. I think universities are going to come back, and therefore faculty positions are going to come back. Maybe I'm being naive, but I hope that's correct.

*Notices:* And personally, did you have to scramble to start teaching online in the spring?

**Charney:** Yes, in the middle of the semester, we suddenly went online. That wasn't so hard for me. I was already through a large portion of the course material, so I recorded a few lectures, and then I switched over to having student presentations. I was teaching an undergraduate topology class, and I needed to cover the basic material. Then normally, I would have done something else fun, like the classification surfaces, perhaps. I decided instead to have the students give presentations on various applications of topology to make the class more interactive. This term, I'm teaching our first-year graduate course in algebraic topology, and it is totally online. The graduate students are all first-years, and they don't know each other. So even though I've taught the course a number of times before, I needed to think hard about how to make it work and how to get students to know each other. It's a challenge, and we'll get through it, but I don't know anybody who likes this style of teaching.

*Notices:* In the October issue of the Notices, there was a letter signed by at least 1,500 mathematicians encouraging their colleagues to boycott working with the police, which in the case of mathematicians often means creating predictive policing algorithms. The mathematical community is wrestling with the ethics of how mathematics is used in several different ways. Do you have any thoughts about that, both specifically the algorithms that are used in policing and then more broadly, the ethics of using mathematics in various application areas?

**Charney:** I'm hesitant to give my personal opinion, partly because I haven't thoroughly thought this through, and





**Figure 3.** Ruth Charney on the Brandeis University campus in 2015.

also because I don't want to be speaking in the name of the AMS. You have to be careful when you're president, or president-elect, that there are your personal opinions and then there are the official opinions of the AMS. As far as the role of the AMS, what I'm happy to see is that there's a discussion going on because I think when there are issues that involve mathematics and the mathematical community that are controversial, that the AMS needs to facilitate a discussion, to let everybody speak out and see both sides of the story. As you say, there was this letter, but there have also been a number of responses that said, "Yes, but they're just going to use these algorithms anyway, and the results will get worse and worse if we don't tell them how to fix them." I actually see merit in both sides of the argument right now. Sometimes it is appropriate for the AMS to make a statement about something, but when that happens, it goes through discussion, it gets voted on by the Council, and then it becomes an official statement of the AMS. I think this is what the process has to be because no matter who you are, no matter how strongly you feel, this is the whole mathematical community talking.

I think it's really important that we recognize that these things are happening, and that we make people in the community aware of it, whether that means they stop working on these types of algorithms, or whether it means they get more involved, but in a way that helps improve them. I don't know which is the right answer, and maybe for some people it's one and for some the other. But I'm happy to see that this has become a topic that people are thinking about and talking about.

*Notices:* And you kind of addressed this, but maybe a better way I could have asked that question would be to ask when does it become appropriate for the AMS to make a statement about a broader issue in the country? I remember in July, in response to changes in F-1 visa regulations that would affect many international students, the AMS very quickly issued a statement and

provided information about how members could contact their representatives and oppose those changes.

**Charney:** There is a way to bypass the full process, where it doesn't have to go through the whole Council. But only for things that are extremely timely and where there is a clear consensus in the math community, as there was with the visa regulations. Then you could short circuit the process. But I don't think this business on policing algorithms falls into that category.

*Notices:* Is there anything else you wanted to add?

**Charney:** There is something that concerns me—this certainly isn't a result of COVID, but maybe COVID made it worse—I don't really know. All of the professional societies for years now have been seeing a decrease in memberships. And I think that's because people can access anything online now. It used to be if you weren't a member, you didn't get this, or you didn't get that, and you felt out of touch. I hope people realize now that membership is about supporting the AMS, not just about going to a meeting for a slightly lower fee, or paying a little less for books. I would like people to recognize that what AMS does is really important to the mathematics community, and that they should support it because they believe in it.



Evelyn Lamb

### Credits

Figures 1 and 3 are by Mike Lovett for Brandeis University. Figure 2 is courtesy of Ruth Charney. Author photo is courtesy of Evelyn Lamb.



# MCA 2021

Mathematical Congress of the Americas

## Grants for Virtual MCA 2021

Buenos Aires, Argentina

July 19–24, 2021

With funding from the NSF, the AMS is administering the selection and award process for partial support for US based mathematicians attending the Virtual Mathematical Congress of the Americas (MCA), scheduled for July 19 – 24, 2021, in Buenos Aires, Argentina.

Instructions on how to apply are available on the AMS website at [www.ams.org/mca](http://www.ams.org/mca).

This grant program is being run by the AMS Programs Department, 201 Charles Street, Providence, RI 02904-2213.

For questions or more information, contact Kim Kuda by email at [mca-info@ams.org](mailto:mca-info@ams.org) or by phone at **800.321.4267, ext. 4096** or **401.455.4096**.

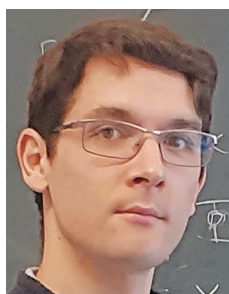


All information currently available about the Virtual MCA 2021 program, organization, and registration procedure is located on the MCA 2021 website:

[www.mca2021.org](http://www.mca2021.org)

# Mathematics People

## Moreira Awarded Brin Prize



Joel Moreira

**Joel Moreira** of the University of Warwick has been selected the recipient of the second Michael Brin Prize for Young Mathematicians in recognition of his “outstanding work on ergodic Ramsey theory and his joint proof of the Erdős sumset conjecture.” The prize was awarded for the papers “Monochromatic sums and products,” *Annals of Mathematics* (2) **185** (2017), no. 3, and “A proof

of a sumset conjecture of Erdős” (with F. K. Richter and D. Robertson), *Annals of Mathematics* (2) **189** (2019), no. 2. Moreira received his PhD degree from Ohio State University in 2016 under the supervision of Vitaly Bergelson. He was Boas Assistant Professor at Northwestern University (2016–2019) before joining the faculty of the University of Warwick.

The Brin Prize for Young Mathematicians recognizes outstanding contributions to dynamical systems made by researchers within four years of the PhD. It carries a cash award of US\$4,000.

—Giovanni Forni  
Chair, Prize Selection Committee

## Lebowitz Awarded Heineman Prize



Joel Lebowitz

**Joel Lebowitz** of Rutgers University has been awarded the Dannie Heineman Prize for Mathematical Physics for his “seminal contributions to nonequilibrium and equilibrium statistical mechanics, in particular, studies of large deviations in nonequilibrium steady states and rigorous analysis of Gibbs equilibrium ensembles.” Working with Elliott Lieb, he proved “the existence of

thermodynamics for ordinary matter with Coulomb interactions,’ the force between two electrically charged particles.” Lebowitz received his PhD from Syracuse University in 1956, after which he was a National Science Foundation Postdoctoral Fellow at Yale University (1956–1957). He has held positions at the Stevens Institute of Technology (1957–1959) and Yeshiva University (1959–1977) before joining the faculty at Rutgers in 1977. His awards and honors include a Guggenheim Fellowship (1976–1977); the Boltzmann Medal (1992); the Max Planck Research Award (1993); the Henri Poincaré Prize (2000); the Vito Volterra Medal of the Accademia Nazionale dei Lincei (2001); the Max Planck Medal of the German Physical Society (2007); and the Grande Médaille from the French Academy of Sciences (2014). He is also involved in human rights work and has been the recipient of the Heinz R. Pagels Human Rights of Scientists Award of the New York Academy of Sciences (1996) and the AAAS Scientific Freedom and Responsibility Award (1999). He is a member of the National Academy of Sciences and a Fellow of the American Physical Society, the New York Academy of Sciences, and the American Association for the Advancement of Science.

—From an American Physical Society announcement

## Haslhofer and Shelukhin Receive Aisenstadt Prize



Robert Haslhofer

**Robert Haslhofer** of the University of Toronto and **Egor Shelukhin** of the University of Montreal have been awarded the 2020 André Aisenstadt Prize in Mathematics of the Centre de Recherches Mathématiques (CRM). Haslhofer was recognized for his work in geometric analysis, differential geometry, partial differential equations, calculus of variations, stochastic analysis, and general relativity. He received his PhD from ETH Zurich in 2012. He served as Courant Instructor at the Courant Institute of Mathematical Sciences from 2012 to 2015 before joining the University of Toronto. He has been the recipient of an

NSF Grant (2014–2017), the Connaught New Researcher Award (2016–2018), a Sloan Research Fellowship (2018–2020), and an NSERC Discovery Grant (2016–2021). He tells the *Notices*: “I’m a cycling enthusiast, and my snowboard and I love powder snow.”



Egor Shelukhin

Shelukhin was honored for work in symplectic topology, contact topology, and geometric analysis. He received his PhD in 2012 from Tel Aviv University under the direction of Leonid Polterovich. He has held positions at the Hebrew University of Jerusalem, Université Lyon 1 Claude Bernard, and Institut Mittag Leffler. He was a CRM-ISM Postdoctoral Research Fellow in Mathematics at the University of Montreal and a Member at the School of Mathematics of the Institute for Advanced Study, Princeton University.

The Aisenstadt Prize recognizes outstanding research by young Canadian mathematicians.

—From CRM announcements

## Cohen Awarded Pascal Medal



Albert Cohen

**Albert Cohen** of Sorbonne Université has been named the recipient of the 2020 Blaise Pascal Medal in Mathematics of the European Academy of Sciences for his work on problems that involve a very large number of variables and whose efficient numerical treatment is therefore challenged by the so-called curse of dimensionality. His research interests include approximation theory, numerical analysis, computational harmonic analysis, signal-image-data processing, and statistics.

Cohen received his PhD in 1990 from the Université Paris IX-Dauphine under the supervision of Yves Meyer. He has held positions at Bell-ATT Labs (1990–1991) and at ENSTA, Paris (1993–1995). Since 1995 he has held the position of professor at Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris. His honors include the V. A. Popov Prize in Approximation Theory (1995), the J. Herbrand Prize of the Académie des Sciences (2000), and the Blaise Pascal Prize of the French Applied and Industrial Mathematical Society (SMAI) and the Académie des Sciences (2004). He has been an invited speaker at ICM 2002 and plenary speaker at ICIAM 2006 and is a senior member

of the Institut Universitaire de France. He enjoys traveling, scuba diving, skiing, and “tentatively playing a couple of musical instruments.”

—Elaine Kehoe

## Bhatnagar Prizes Awarded

The 2020 Shanti Swarup Bhatnagar Prizes for Science and Technology in Mathematical Sciences have been awarded to **Rajat Subhra Hazra** of the Indian Statistical Institute and **U. K. Anandavardhanan** of the Indian Institute of Technology Bombay. Hazra was honored for “outstanding contributions to multiple frontier areas of probability theory such as random matrices (in particular, with dependent entries), his use therein of free probability to which he also contributed, random fields, random graphs, etc., touching upon themes of interest to physics such as sandpile models and membranes.” Anandavardhanan was recognized for “significant contributions in the field of distinguished representations in the Langlands program, especially for his contributions relating base change, distinction, root numbers, and the Asai  $L$ -function for the linear group.” The award recognizes outstanding Indian work in science and technology.

—Bhatnagar Prize announcement

## Krieg Receives Traub Award

**David Krieg** of Johannes Kepler University has been selected the recipient of the 2020 Joseph F. Traub Information-Based Complexity Young Researcher Award. The award is given for significant contributions to information-based complexity by a young researcher who has not reached his or her thirty-fifth birthday by September 30 of the year of the award. The award consists of US\$1,000 and a plaque.

—Erich Novak, Editor  
Journal of Complexity

## International Mathematical Olympiad

The 61st International Mathematical Olympiad (IMO) was held remotely from St. Petersburg, Russian Federation, in September 2020. The team from the People’s Republic of China finished first with a total of 215 points. The Russian Federation took second place with 185 points. The United States team finished in third place with 183 total points. The US team consisted of **Luke Robitaille** (gold medal),



Quanlin Chen (gold medal), William Wang (gold medal), Tianze Jiang (silver medal), Gopal Goel (silver medal), and Jeffrey Kwan (silver medal). Po-Shen Loh was the team leader; Oleksandr Rudenko was deputy team leader.

—IMO announcement

## 2021 AWM Fellows

The Executive Committee of the Association for Women in Mathematics (AWM) established the AWM Fellows Program to recognize individuals who have demonstrated a sustained commitment to the support and advancement of women in the mathematical sciences, consistent with the AWM mission: “to encourage women and girls to study and to have active careers in the mathematical sciences, and to promote equal opportunity and the equal treatment of women and girls in the mathematical sciences.”

The 2021 class of AWM Fellows are researchers, mentors, and educators who are recognized by their peers and students for their commitment to supporting women in the mathematical sciences.

Following are the names and institutions of the 2021 AWM Fellows.

- **Alina Bucur**, University of California, San Diego
- **Sigal Gottlieb**, University of Massachusetts, Dartmouth
- **Eugenie Hunsicker**, Loughborough University
- **Patricia Clark Kenschaft**, Montclair State University
- **Gail Letzter**, National Security Agency
- **Dawn Alisha Lott**, Delaware State University
- **Gretchen L. Matthews**, Virginia Institute of Technology
- **Susan Morey**, Texas State University
- **Bozenna Pasik-Duncan**, University of Kansas
- **Ami Radunskaya**, Pomona College
- **Catherine A. Roberts**, American Mathematical Society
- **Katherine E. Stange**, University of Colorado, Boulder
- **Talitha M. Washington**, Clark Atlanta University and Atlanta University Center
- **Carol S. Woodward**, Lawrence Livermore National Laboratory

—AWM announcement

### Credits

Photo of Joel Lebowitz is courtesy of Joel Lebowitz.

Photo of Egor Shelukhin is courtesy of Egor Shelukhin.

*Introducing*  
an AMS Member




Photo courtesy of Atlanta Convention Photography

**AMS** AMERICAN MATHEMATICAL SOCIETY  
*Advancing research. Creating connections.*

## James J. Madden

Patricia Hewlett Bodin Distinguished Professor  
of Mathematics at Louisiana State University

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AMS Member since: 1980	Favorite Number: 3
Primary Field of Research: Algebra	Favorite Color: Green
Dissertations Advisor: A. W. Hager	Favorite Food: Spinach
Undergrad Institution: Reed College	Erdős Number: 2
PhD Institution: Wesleyan University	Favorite Hobby: Cooking

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**What does the AMS mean to you?**  
Belonging to a community of people devoted to meaningful fun.

**What do you think is the most important thing the AMS offers?**  
Publications and meetings.

**What is your favorite memory from an AMS event?**  
My first contributed talk in 1981.

**Describe the situation when you first fell for math.**  
It was a long process. Not love at first sight. Complicated.

**Were you inspired by a mathematician?**  
No.

**How would you describe math to a non-math person?**  
About exploring an unseen, unheard, unfelt universe with your only vehicle: your mind.

# Community Updates

## Gumel to Give Einstein Lecture



Abba Gumel

**Abba Gumel** of Arizona State University will deliver the 2020 Einstein Public Lecture in Mathematics on the subject “Mathematics of Infectious Diseases.” His work is primarily focused on the use of mathematical approaches to gain insight into the transmission dynamics and control of emerging and reemerging diseases of public health importance.

In conjunction with the AMS Spring Eastern Sectional Meeting (see [https://www.ams.org/meetings/sectional/2284\\_program.html](https://www.ams.org/meetings/sectional/2284_program.html)), the lecture will take place online on Saturday, March 20, 2021, from 1:30 to 2:30 EDT (10:30–11:30 p.m. PDT). The event is free and open to the public. For more information, including how to register, see the website [www.ams.org/meetings/lectures/meet-einstein-lect](http://www.ams.org/meetings/lectures/meet-einstein-lect).

—AMS announcement

## Mathematical Congress of the Americas Grants

With funding from the National Science Foundation (NSF), the AMS is administering the selection, award, and reimbursement process of a grant program to provide support for US-based mathematicians to attend the Mathematical Congress of the Americas (MCA 2021). This year, MCA 2021, scheduled for July 19–24, 2021, is being held virtually. Priority will be given to those whose participation in MCA 2021 is not supported with other funding. Funding will cover the MCA registration fee plus a stipend of up to US\$250. This stipend is intended to assist in offsetting conference-related expenses that will allow the awardee a more focused meeting participation, which otherwise would not be possible. For more information, see [www.ams.org/programs/travel-grants/mca](http://www.ams.org/programs/travel-grants/mca).

Applications are being accepted now through **April 30, 2021**, on MathPrograms. All information currently available about the MCA 2021 program and organization is located on the MCA 2021 website: [www.mca2021.org](http://www.mca2021.org). For questions about the grant program, contact Kim Kuda in the AMS Programs Department at [mca-info@ams.org](mailto:mca-info@ams.org).

—AMS Programs Department

## Arnold Ross Lecture

The Arnold Ross Lectures are intended for talented high school mathematics students. The 2021 Arnold Ross Lecturer is **Noam D. Elkies** of Harvard University. The lecture, rescheduled from 2020, will be given on June 4, 2021, at the 45th American Regions Mathematics League at Pennsylvania State University. See <https://www.ams.org/programs/students/ross-lectures/ross-lectures>.

—AMS announcement

## Maryam Mirzakhani Lectures

The AMS Council established this lecture in 2018 to honor Maryam Mirzakhani (1977–2017), the first woman and the first Iranian to win a Fields Medal. The 2021 lecture was presented virtually in January 2021 by Ciprian Manolescu of Stanford University. His topic was “Khovanov Homology and Surfaces in Four-Manifolds.” The 2020 lecture was given in Denver, Colorado, in January 2020 by Tatiana Toro of the University of Washington on the topic “Differential Operators and the Geometry of Domains in Euclidean Space.”

—AMS announcement

## AMS Opportunities Web Service

The AMS offers an “Opportunities” web service, a free self-service posting site to publicize opportunities for fellowships and grants, nominations for prizes and awards, and proposals for meetings and workshops and to post information about Math Camps, Math Circles, REUs, scholarships, and internships. The Opportunities site has a built-in reminder for annual programs that will help you keep your listing up to date. The service is available at [www.ams.org/opportunities](http://www.ams.org/opportunities). For questions, contact AMS staff at [opportunities@ams.org](mailto:opportunities@ams.org).

—AMS announcement

## Deaths of AMS Members

JACQUES BAZINET, professor, University of Sherbrooke, died on December 12, 2017. Born on April 9, 1931, he was a member of the Society for 53 years.

THOMAS F. BICKEL, of Lebanon, New Hampshire, died on October 27, 2017. Born on November 20, 1937, he was a member of the Society for 53 years.

BEVERLEY M. BROWN, of Ocean Ridge, Florida, died on October 25, 2017. Born on October 4, 1924, he was a member of the Society for 67 years.

FRANK H. BROWNELL, of Bainbridge Island, Washington, died on October 21, 2017. Born on September 20, 1922, he was a member of the Society for 68 years.

RAGNAR-OLAF BUCHWEITZ, professor, University of Toronto Scarborough, died on November 11, 2017. Born on March 18, 1952, he was a member of the Society for 27 years.

GERALD M. BURIOK, of Indiana, Pennsylvania, died on November 8, 2017. Born on November 6, 1943, he was a member of the Society for 46 years.

RICHARD J. DRISCOLL, of Chicago, Illinois, died on October 21, 2017. Born on August 14, 1928, he was a member of the Society for 65 years.

JOEL N. FRANKLIN, professor, California Institute of Technology, died on November 18, 2017. Born on April 4, 1930, he was a member of the Society for 65 years.

TORD H. GANELIUS, of Sweden, died on March 14, 2016. Born on May 23, 1925, he was a member of the Society for 58 years.

RONALD K. GETOOR, of La Jolla, California, died on October 28, 2017. Born on February 9, 1929, he was a member of the Society for 64 years.

RUDOLF GORENFLO, professor, Free University of Berlin, died on October 20, 2017. Born on July 31, 1930, he was a member of the Society for 39 years.

ROBERT D. GULLIVER II, of Minneapolis, Minnesota, died on October 15, 2017. Born on July 24, 1945, he was a member of the Society for 45 years.

RONALD HARROP, of Canada, died on October 22, 2017. Born on May 3, 1926, he was a member of the Society for 60 years.

THOMAS J. HEAD, of Vestal, New York, died on November 10, 2017. Born on January 6, 1934, he was a member of the Society for 59 years.

VIKTOR IVANOV, of St. Petersburg, Florida, died on November 13, 2017. Born on June 8, 1929, he was a member of the Society for 42 years.

JERRY L. LINNSTAEDTER, of Jonesboro, Arizona, died on November 22, 2017. Born on July 25, 1937, he was a member of the Society for 38 years.

ULO LUMISTE, of Estonia, died on November 20, 2017. Born on June 30, 1929, he was a member of the Society for 24 years.

BYRON LEON McALLISTER, of Bozeman, Montana, died on December 9, 2017. Born on April 29, 1929, he was a member of the Society for 62 years.

JOYCE R. McLAUGHLIN, professor, Rensselaer Polytechnic Institute, died on October 23, 2017. Born on October 8, 1939, she was a member of the Society for 65 years.

ADELBERT W. RANSOM, of Rochester, New York, died on November 20, 2017. Born on April 13, 1929, he was a member of the Society for 55 years.

MICHEALE SUSAN RODDY, professor, Brandon University, died on October 5, 2017. Born on August 28, 1955, she was a member of the Society for 27 years.

JAN-ERIK I. ROOS, of Sweden, died on December 15, 2017. Born on October 16, 1935, he was a member of the Society for 51 years.

R. A. ROSENBAUM, of Middletown, Connecticut, died on December 3, 2017. Born on November 14, 1915, he was a member of the Society for 79 years.

CHARLES C. SIMS, of St. Petersburg, Florida, died on October 23, 2017. Born on April 14, 1937, he was a member of the Society for 56 years.

JAMES L. THOMPSON, of Grosse Pointe Woods, Michigan, died on October 17, 2017. Born on October 5, 1940, he was a member of the Society for 46 years.

MYLES TIERNEY, of Montreal, Canada, died on October 6, 2017. Born on September 3, 1937, he was a member of the Society for 56 years.

JOHN W. TOOLE, of Orono, Maine, died on December 18, 2017. Born on April 24, 1924, he was a member of the Society for 67 years.

### Credits

Photo of Abba Gumel is courtesy of Rhonda Olson, Marketing and Communications, Arizona State University.



# Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices may be submitted to [notices@ams.org](mailto:notices@ams.org).

## Call for Nominations for Aisenstadt Prize

The Centre de Recherches Mathématiques (CRM) solicits nominations for the André Aisenstadt Mathematics Prize for an outstanding research achievement by a young Canadian mathematician. The deadline for nominations is **March 1, 2021**. See [www.crm.umontreal.ca/prix/prixAndreAisenstadt/prix\\_attributionAA\\_an.shtml](http://www.crm.umontreal.ca/prix/prixAndreAisenstadt/prix_attributionAA_an.shtml).

—From a CRM announcement

### Early Career Opportunity

## NRC Research Associateship Programs

The National Research Council (NRC) Research Associateship Programs promote excellence in scientific and technological research conducted by the US government through the administration of programs offering graduate-, postdoctoral-, and senior-level research opportunities at sponsoring federal laboratories and affiliated institutions. Application deadlines are **February 1, May 1, August 1, and November 1, 2021**. For details, see sites [nationalacademies.org/pga/rap/](http://nationalacademies.org/pga/rap/).

—National Research Council announcement

### Early Career Opportunity

## NSF-AWM Travel Grants for Women

The National Science Foundation (NSF) and the Association for Women in Mathematics offer the NSF-AWM Mathematics Travel Grants to provide full or partial support for travel and subsistence for a meeting or conference in the applicant's field of specialization. Mathematics Mentoring Travel

Grants help junior women to develop long-term working and mentoring relationships with senior mathematicians. Each grant funds travel, accommodations, and other required expenses for an untenured woman mathematician to travel to an institute or a department to do research with a specified individual for one month. The applicant's and mentor's research must be in a field which is supported by the Division of Mathematical Sciences of the National Science Foundation. Deadline dates for Travel Grants are **February 1, May 30, and October 1, 2021**. The deadline for Mentoring Travel Grants is **February 1, 2021**. See <https://awm-math.org/awards/awm-grants/travel-grants>.

—From an AWM announcement

## Call for Nominations for Traub Prizes

The Joseph F. Traub Prize is given for outstanding achievement in information-based complexity. The deadline for nominations is **March 31, 2021**. Nominations should be sent to [erich.novak@uni-jena.de](mailto:erich.novak@uni-jena.de). See <https://www.journals.elsevier.com/journal-of-complexity/awards/joseph-f-traub-prize>.

The Joseph F. Traub Information-Based Complexity Young Researcher Award is given for significant contributions to information-based complexity by a young researcher who has not reached his or her thirty-fifth birthday by September 30, 2021. The award may be made for work done in a single year or a number of years, published in any journal, number of journals, or monographs. The deadline for nominations is **September 30, 2021**. Nominations may be submitted to [erich.novak@uni-jena.de](mailto:erich.novak@uni-jena.de).

—Erich Novak

The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at [www.nsf.gov/dms](http://www.nsf.gov/dms) and for the Directorate of Education and Human Resources at [www.nsf.gov/dir/index.jsp?org=ehf](http://www.nsf.gov/dir/index.jsp?org=ehf). To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at [www.nsf.gov/mps/dms/about.jsp](http://www.nsf.gov/mps/dms/about.jsp).

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# Classified Advertising

## *Employment Opportunities*

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### MASSACHUSETTS

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**Northeastern University**  
**Department of Mathematics**  
**Tenured/Tenure-Track Positions, Open Level**

The Department of Mathematics, in the College of Science, and the Khoury College of Computer Sciences, at Northeastern University invite applications for an open tenure-track/tenured faculty position at all levels in the area of Mathematics and Machine Learning, beginning in Fall 2021.

Appointments will be based on research contributions at the interface between Mathematics and Computer Science, combined with a strong commitment and demonstrated success in teaching. The appointment will be joint between the Department of Mathematics in the College of Science and the Khoury College of Computer Sciences.

Candidates will be considered from all areas in Data Science, Machine Learning, Topology, Discrete and Computational Mathematics, and Robotics.

In the Northeastern University College of Science, we embrace a culture of respect, where each person is valued for their contribution and is treated fairly. We oppose all forms of racism. We support a culture that does not tolerate any form of discrimination and where each person may belong. As a College, we strive to have a diverse membership, one where each person is trained and mentored to promote their success.

Responsibilities will include teaching undergraduate and graduate courses, mentoring student and conducting an independent research program.

A PhD in Computer Science, Mathematics or a closely related field to one of the above-listed areas of expertise by the state date is required. Successful candidates are expected to have or to develop an independently funded research program of international caliber and teaching excellence in undergraduate and graduate courses. Qualified candidates should be committed to fostering diverse and inclusive environments as well as to promoting experiential learning, which are central to a Northeastern University education.

Review of applications will begin immediately. Complete applications received by December 31, 2020 will be guaranteed full consideration. Additional applications will be considered until the position is filled.

To apply, please submit the documentation requested on the [mathjobs.org/jobs](https://mathjobs.org/jobs). Applicants invited to interviews will be asked to complete a Northeastern University application on the appropriate website.

Northeastern University is an equal opportunity employer, seeking to recruit and support a broadly diverse community of faculty and staff. Northeastern values and celebrates diversity in all its forms and strives to foster an inclusive culture built on respect that affirms inter-group relations and builds cohesion.

All qualified applicants are encouraged to apply and will receive consideration for employment without regard to race, religion, color, national origin, age, sex, sexual orientation, disability status, or any other characteristic protected by applicable law.

To learn more about Northeastern University's commitment and support of diversity and inclusion, please see [www.northeastern.edu/diversity](https://www.northeastern.edu/diversity).

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The *Notices Classified Advertising* section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

The 2021 rate is \$3.65 per word. Advertisements will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted. There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: March 2021—December 23, 2020; April 2021—January 22, 2021; May 2021—February 24, 2021; June/July 2021—April 26, 2021; August 2021—May 28, 2021; September 2021—June 28, 2021; October 2021—July 23, 2021; November 2021—August 24, 2021; December 2021—September 24, 2021.

US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to [classifieds@ams.org](mailto:classifieds@ams.org).

## CHINA

**Tianjin University, China**  
**Tenured/Tenure-Track/Postdoctoral Positions at**  
**the Center for Applied Mathematics**

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.


The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to [mathjobs@tju.edu.cn](mailto:mathjobs@tju.edu.cn).

For more information, please visit [cam.tju.edu.cn](http://cam.tju.edu.cn) or contact Ms. Erica Liu at [mathjobs@tju.edu.cn](mailto:mathjobs@tju.edu.cn), telephone: 86-22-2740-6039.


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# The AMS Mathematics Calendar is an online-only product

You can submit an entry to the Mathematics Calendar at [www.ams.org/cgi-bin/mathcal/mathcal-submit.pl](http://www.ams.org/cgi-bin/mathcal/mathcal-submit.pl)

Questions and answers regarding this page can be sent to [mathcal@ams.org](mailto:mathcal@ams.org).

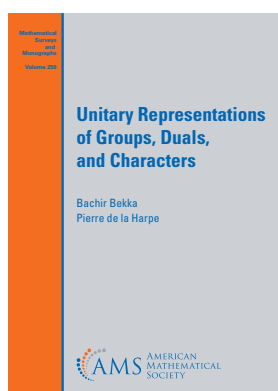


**AMERICAN MATHEMATICAL SOCIETY**  
*Advancing research. Creating connections.*



# New Books Offered by the AMS

## Algebra and Algebraic Geometry



### Unitary Representations of Groups, Duals, and Characters

Bachir Bekka, *IRMAR, Université de Rennes 1, France* and Pierre de la Harpe, *Université de Genève, Switzerland*

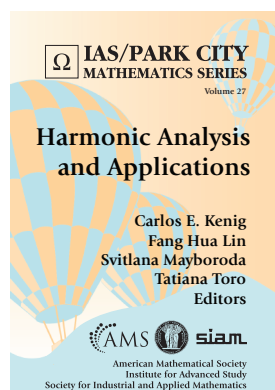
This monograph focuses on dual spaces associated to a group, which are spaces of building blocks of general unitary representations.

*This item will also be of interest to those working in geometry and topology.*

**Mathematical Surveys and Monographs, Volume 250**  
January 2021, 484 pages, Softcover, ISBN: 978-1-4704-5627-6, LC 2020027691, 2010 *Mathematics Subject Classification*: 22D10, 22D25, 22D40, 46L55, List US\$140, **AMS members US\$112, MAA members US\$126**, Order code SURV/250

[bookstore.ams.org/surv-250](http://bookstore.ams.org/surv-250)

## Analysis



### Harmonic Analysis and Applications

Carlos E. Kenig, *University of Chicago, IL*, Fang Hua Lin, *New York University, Courant Institute*, Svitlana Mayboroda, *University of Minnesota, Minneapolis*, and Tatiana Toro, *University of Washington, Seattle*, Editors

These notes give fresh, concise, and high-level introductions to recent developments in the field, often with new arguments not found elsewhere. The volume will be of use both to graduate students seeking to enter the field and to senior researchers wishing to keep up with current developments.

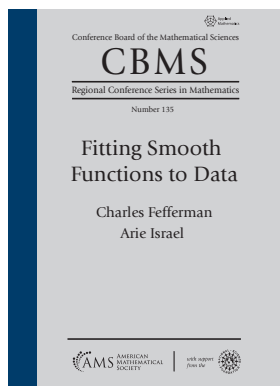
*This item will also be of interest to those working in applications, mathematical physics, and probability and statistics.*

Titles in this series are co-published with the Institute for Advanced Study/Park City Mathematics Institute.

**IAS/Park City Mathematics Series, Volume 27**  
December 2020, 345 pages, Hardcover, ISBN: 978-1-4704-6127-0, LC 2020025415, 2010 *Mathematics Subject Classification*: 42-06, 53-06, 35-06, 28-06, List US\$110, **AMS members US\$88, MAA members US\$99**, Order code PCMS/27

[bookstore.ams.org/pcms-27](http://bookstore.ams.org/pcms-27)

## NEW BOOKS



### Fitting Smooth Functions to Data

Charles Fefferman, *Princeton University, NJ*, and Arie Israel, *University of Texas at Austin*

This book is an introductory text that charts the recent developments in the area of Whitney-type extension problems and the mathematical aspects of interpolation of data.

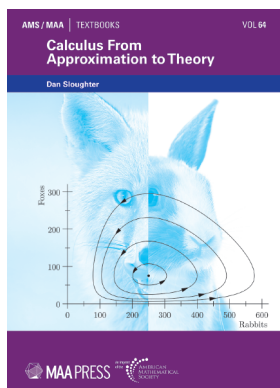
*This item will also be of interest to those working in applications and geometry and topology.*

**CBMS Regional Conference Series in Mathematics, Number 135**

October 2020, 160 pages, Softcover, ISBN: 978-1-4704-6130-0, LC 2020024082, 2010 *Mathematics Subject Classification*: 41A05, 41A29, 42C99, 52A35, 65D05, 65D10, 65D17, List US\$59, **AMS members US\$47.20**, **MAA members US\$53.10**, Order code CBMS/135

[bookstore.ams.org/cbms-135](http://bookstore.ams.org/cbms-135)

## Calculus



### Calculus From Approximation to Theory

Dan Sloughter, *Furman University, Greenville, SC*

*Calculus from Approximation to Theory* takes a fresh and innovative look at the teaching and learning of calculus. One way to describe calculus might be to say it is a suite of techniques that approximate curved things by flat things and through a limiting

process applied to those approximations arrive at an exact answer. Standard approaches to calculus focus on that limiting process as the heart of the matter. This text places its emphasis on the approximating processes and thus illuminates the motivating ideas and makes clearer the scientific usefulness, indeed centrality, of the subject while paying careful attention to the theoretical foundations. Limits are defined in terms of sequences, the derivative is defined from the best affine approximation, and greater attention than usual is paid to numerical techniques and the order of an approximation. Access to modern computational

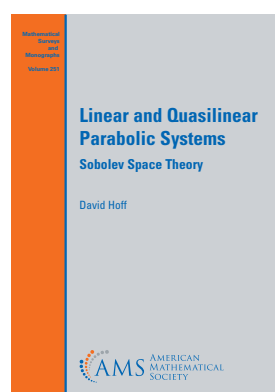
tools is presumed throughout and the use of these tools is woven seamlessly into the exposition and problems. All of the central topics of a yearlong calculus course are covered, with the addition of treatment of difference equations, a chapter on the complex plane as the arena for motion in two dimensions, and a much more thorough and modern treatment of differential equations than is standard.

**AMS/MAA Textbooks, Volume 64**

November 2020, 571 pages, Softcover, ISBN: 978-1-4704-5588-0, LC 2020031331, 2010 *Mathematics Subject Classification*: 26-01, 26A06, 00-01, List US\$99, **AMS Individual member US\$74.25**, **AMS Institutional member US\$79.20**, **MAA members US\$74.25**, Order code TEXT/64

[bookstore.ams.org/text-64](http://bookstore.ams.org/text-64)

## Differential Equations



### Linear and Quasilinear Parabolic Systems

**Sobolev Space Theory**

David Hoff, *Indiana University, Bloomington*

This monograph presents a systematic theory of weak solutions in Hilbert-Sobolev spaces of initial-boundary value problems for parabolic systems of partial differential equations with general essential and natural

boundary conditions and minimal hypotheses on coefficients. Applications to quasilinear systems are given, including local existence for large data, global existence near an attractor, the Leray and Hopf theorems for the Navier-Stokes equations and results concerning invariant regions. Supplementary material is provided, including a self-contained treatment of the calculus of Sobolev functions on the boundaries of Lipschitz domains and a thorough discussion of measurability considerations for elements of Bochner-Sobolev spaces.

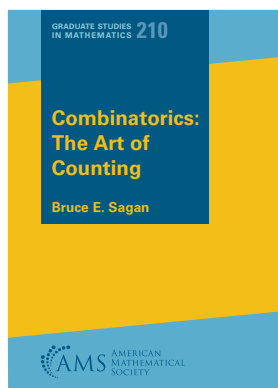
This book will be particularly useful both for researchers requiring accessible and broadly applicable formulations of standard results as well as for students preparing for research in applied analysis. Readers should be familiar with the basic facts of measure theory and functional analysis, including weak derivatives and Sobolev spaces. Prior work in partial differential equations is helpful but not required.

*This item will also be of interest to those working in analysis.*

**Mathematical Surveys and Monographs, Volume 251**  
February 2021, 226 pages, Softcover, ISBN: 978-1-4704-6161-4, LC 2020036612, 2010 *Mathematics Subject Classification*: 35K51, 35K61, List US\$140, **AMS members US\$112**, **MAA members US\$126**, Order code SURV/251

[bookstore.ams.org/surv-251](http://bookstore.ams.org/surv-251)

## Discrete Mathematics and Combinatorics



### Combinatorics: The Art of Counting

**Bruce E. Sagan**, *Michigan State University, East Lansing*

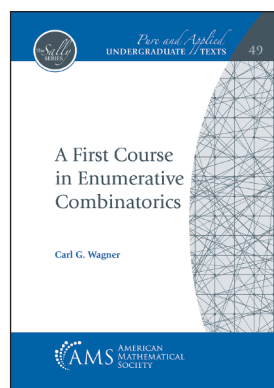
This book is a gentle introduction to the enumerative part of combinatorics suitable for study at the advanced undergraduate or beginning graduate level. In addition to covering all the standard techniques for counting combinatorial objects, the text

contains material from the research literature which has never before appeared in print, such as the use of quotient posets to study the Möbius function and characteristic polynomial of a partially ordered set, or the connection between quasisymmetric functions and pattern avoidance.

The book assumes minimal background, and a first course in abstract algebra should suffice. The exposition is very reader friendly: keeping a moderate pace, using lots of examples, emphasizing recurring themes, and frankly expressing the delight the author takes in mathematics in general and combinatorics in particular.

**Graduate Studies in Mathematics, Volume 210**  
October 2020, 304 pages, Softcover, ISBN: 978-1-4704-6032-7, LC 2020025345, 2010 *Mathematics Subject Classification*: 05–01; 06–01, List US\$89, **AMS members US\$71.20**, **MAA members US\$80.10**, Order code GSM/210

[bookstore.ams.org/gsm-210](http://bookstore.ams.org/gsm-210)



### A First Course in Enumerative Combinatorics

**Carl G. Wagner**, *University of Tennessee, Knoxville*

*A First Course in Enumerative Combinatorics* provides an introduction to the fundamentals of enumeration for advanced undergraduates and beginning graduate students in the mathematical sciences. The book offers

a careful and comprehensive account of the standard tools of enumeration—recursion, generating functions, sieve and inversion formulas, enumeration under group actions—and their application to counting problems for the fundamental structures of discrete mathematics, including sets and multisets, words and permutations, partitions of sets and integers, and graphs and trees. The author's exposition has been strongly influenced by the work of Rota and Stanley, highlighting bijective proofs, partially ordered sets, and an emphasis on organizing the subject under various unifying themes, including the theory of incidence algebras. In addition, there are distinctive chapters on the combinatorics of finite vector spaces, a detailed account of formal power series, and combinatorial number theory.

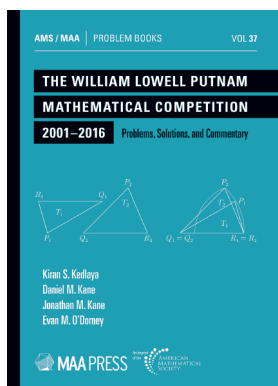
The reader is assumed to have a knowledge of basic linear algebra and some familiarity with power series. There are over 200 well-designed exercises ranging in difficulty from straightforward to challenging. There are also sixteen large-scale honors projects on special topics appearing throughout the text.

**Pure and Applied Undergraduate Texts, Volume 49**  
October 2020, 272 pages, Softcover, ISBN: 978-1-4704-5995-6, LC 2020030053, 2010 *Mathematics Subject Classification*: 05–01, 05A05, 05A15, 05A17, 05A18, 05A19, 05A30, 06A07, 11A25, 11B65, List US\$85, **AMS members US\$68**, **MAA members US\$76.50**, Order code AMSTEXT/49

[bookstore.ams.org/amstext-49](http://bookstore.ams.org/amstext-49)



## General Interest



### The William Lowell Putnam Mathematical Competition 2001–2016

Problems, Solutions, and Commentary

Kiran S. Kedlaya, Daniel M. Kane, both of University of California, San Diego, La Jolla, Jonathan M. Kane, University of Wisconsin-Madison, and Evan M. O'Dorney, Princeton University, NJ

The William Lowell Putnam Mathematics Competition is the most prestigious undergraduate mathematics problem-solving contest in North America, with thousands of students taking part every year. This volume presents the contest problems for the years 2001–2016. The heart of the book is the solutions; these include multiple approaches, drawn from many sources, plus insights into navigating from the problem statement to a solution. There is also a section of hints, to encourage readers to engage deeply with the problems before consulting the solutions.

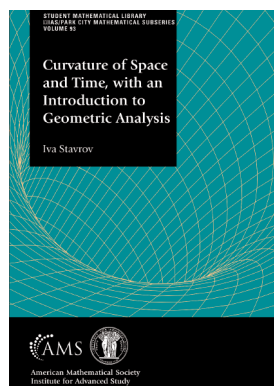
The authors have a distinguished history of engagement with, and preparation of students for, the Putnam and other mathematical competitions. Collectively they have been named Putnam Fellow (top five finisher) ten times. Kiran Kedlaya also maintains the online Putnam Archive.

#### Problem Books, Volume 37

November 2020, 348 pages, Softcover, ISBN: 978-1-4704-5427-2, LC 2020023499, 2010 *Mathematics Subject Classification*: 97U40, 97D50, List US\$65, **AMS Individual member US\$48.75**, AMS Institutional member US\$52, **MAA members US\$48.75**, Order code PRB/37

[bookstore.ams.org/prb-37](http://bookstore.ams.org/prb-37)

## Geometry and Topology



### Curvature of Space and Time, with an Introduction to Geometric Analysis

Iva Stavrov, Lewis & Clark College, Portland, OR

This book introduces advanced undergraduates to Riemannian geometry and mathematical general relativity. The overall strategy of the book is to explain the concept of curvature via the Jacobi equation which, through

discussion of tidal forces, further helps motivate the Einstein field equations.

After addressing concepts in geometry such as metrics, covariant differentiation, tensor calculus and curvature, the book explains the mathematical framework for both special and general relativity. Relativistic concepts discussed include (initial value formulation of) the Einstein equations, stress-energy tensor, Schwarzschild space-time, ADM mass and geodesic incompleteness. The concluding chapters of the book introduce the reader to geometric analysis: original results of the author and her undergraduate student collaborators illustrate how methods of analysis and differential equations are used in addressing questions from geometry and relativity. The book is mostly self-contained and the reader is only expected to have a solid foundation in multivariable and vector calculus and linear algebra.

*This item will also be of interest to those working in mathematical physics.*

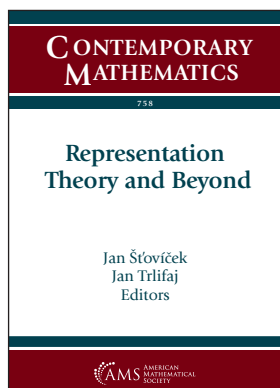
#### Student Mathematical Library, Volume 93

November 2020, 243 pages, Softcover, ISBN: 978-1-4704-5628-3, LC 2020035487, 2010 *Mathematics Subject Classification*: 83–01, 53–01, List US\$59, AMS Institutional member US\$47.20, **All Individuals US\$47.20**, Order code STML/93

[bookstore.ams.org/stml-93](http://bookstore.ams.org/stml-93)

## New in Contemporary Mathematics

### Algebra and Algebraic Geometry



#### Representation Theory and Beyond

Jan Šťovíček and Jan Trlifaj,  
*both of Charles University, Prague, Czech Republic, Editors*

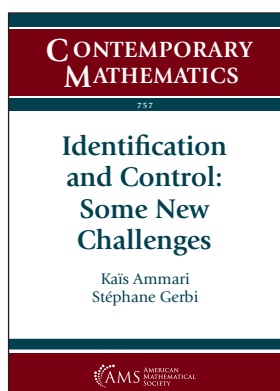
This volume presents several themes of contemporary representation theory together with some new tools, such as stable  $\infty$ -categories, stable derivators, and contramodules.

**Contemporary Mathematics, Volume 758**

January 2021, 298 pages, Softcover, ISBN: 978-1-4704-5131-8, LC 2020002000, 2010 *Mathematics Subject Classification*: 16Gxx, 18Dxx, 18Gxx, List US\$120, **AMS members US\$96**, **MAA members US\$108**, Order code CONM/758

[bookstore.ams.org/conm-758](http://bookstore.ams.org/conm-758)

### Applications



#### Identification and Control: Some New Challenges

Kaïs Ammari, *University of Monastir, Tunisia*, and Stéphane Gerbi, *CNRS & Université Savoie Mont Blanc, Bourget du Lac, France, Editors*

The articles cover new developments in control theory and inverse problems.

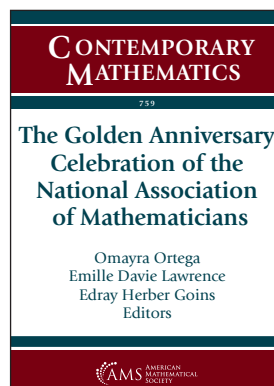
**Contemporary Mathematics, Volume 757**

November 2020, 185 pages, Softcover, ISBN: 978-1-4704-5547-7, LC 2020001921, 2010 *Mathematics Subject Classification*: 35R30, 35Q41, 35K65, 35P20, 35R02, List US\$120, **AMS members US\$96**, **MAA members US\$108**, Order code CONM/757

[bookstore.ams.org/conm-757](http://bookstore.ams.org/conm-757)

FEBRUARY 2021

## General Interest



#### The Golden Anniversary Celebration of the National Association of Mathematicians

Omayra Ortega, *Sonoma State University, Rohnert Park, CA*, Emille Davie Lawrence, *University of San Francisco, CA*, and Edray Herber Goins, *Pomona College, Claremont, CA, Editors*

This volume is put together by the National Association of Mathematicians to commemorate its 50th anniversary.

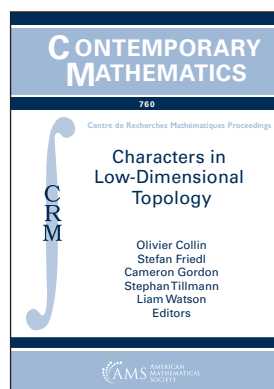
The articles in the book are based on lectures presented at several events at the Joint Mathematics Meeting held from January 16–19, 2019, in Baltimore, Maryland, including the Claytor-Woodard Lecture as well as the NAM David Harold Blackwell Lecture, which was held on August 2, 2019, in Cincinnati, Ohio.

**Contemporary Mathematics, Volume 759**

January 2021, 196 pages, Softcover, ISBN: 978-1-4704-5130-1, LC 2020014035, 2010 *Mathematics Subject Classification*: 01–06, 11B83, 97I50, 65D17, 11G05, 05C69, 34G10, 97D40, 05D10, 35J96, List US\$120, **AMS members US\$96**, **MAA members US\$108**, Order code CONM/759

[bookstore.ams.org/conm-759](http://bookstore.ams.org/conm-759)

## Geometry and Topology



#### Characters in Low-Dimensional Topology

Olivier Collin, *Université du Québec à Montréal, Canada*, Stefan Friedl, *Universität Regensburg, Germany*, Cameron Gordon, *The University of Texas at Austin*, Stephan Tillmann, *The University of Sydney, NSW, Australia*, and Liam Watson, *University of British Columbia, Vancouver, Canada, Editors*

This volume contains the proceedings of a conference celebrating the work of Steven Boyer, held from June 2–8,

2018, at Université du Québec à Montréal, Montréal, Québec, Canada.

Boyer's contributions to research in low-dimensional geometry and topology, and to the Canadian mathematical community, were recognized during the conference.

The articles cover a broad range of topics related, but not limited, to the topology and geometry of 3-manifolds, properties of their fundamental groups and associated representation varieties.

#### Contemporary Mathematics, Volume 760

January 2021, approximately 358 pages, Softcover, ISBN: 978-1-4704-5209-4, 2010 *Mathematics Subject Classification*: 57M25, 57M27, 57M50, 57N10, 53C25, 11F06, 20F06, 20E08, 20F65, 20F67, List US\$120, **AMS members US\$96, MAA members US\$108**, Order code CONM/760

[bookstore.ams.org/conm-760](http://bookstore.ams.org/conm-760)

## New in Memoirs of the AMS

### Algebra and Algebraic Geometry

#### Dualizable Tensor Categories

**Christopher L. Douglas**, *University of Oxford, UK*, **Christopher Schommer-Pries**, *Max Planck Institute for Mathematics, Bonn, Germany*, and **Noah Snyder**, *Indiana University, Bloomington*

**Memoirs of the American Mathematical Society**, Volume 268, Number 1308

November 2020, 88 pages, Softcover, ISBN: 978-1-4704-4361-0, 2010 *Mathematics Subject Classification*: 57R56, 18D10, 55U30, 16D90; 57M27, 17B37, 18E10, List US\$85, **AMS members US\$68, MAA members US\$76.50**, Order code MEMO/268/1308

[bookstore.ams.org/memo-268-1308](http://bookstore.ams.org/memo-268-1308)

### Hecke Operators and Systems of Eigenvalues on Siegel Cusp Forms

**Kazuyuki Hatada**, *Gifu University, Gifu City, Japan*

*This item will also be of interest to those working in number theory.*

**Memoirs of the American Mathematical Society**, Volume 268, Number 1306

November 2020, 165 pages, Softcover, ISBN: 978-1-4704-4334-4, 2010 *Mathematics Subject Classification*: 11F46, 11F60, 11F32, 11F11; 11G25, 14G22, 14F30, 14F20, 11F70, List US\$85, **AMS members US\$68, MAA members US\$76.50**, Order code MEMO/268/1306

[bookstore.ams.org/memo-268-1306](http://bookstore.ams.org/memo-268-1306)

### Double Affine Hecke Algebras and Congruence Groups

**Bogdan Ion**, *University of Pittsburgh, PA* and **Siddhartha Sahi**, *Rutgers University, New Brunswick, NJ*

**Memoirs of the American Mathematical Society**, Volume 268, Number 1305

November 2020, 90 pages, Softcover, ISBN: 978-1-4704-4326-9, 2010 *Mathematics Subject Classification*: 20C08, 17B67, 20F36, 20F34, List US\$85, **AMS members US\$68, MAA members US\$76.50**, Order code MEMO/268/1305

[bookstore.ams.org/memo-268-1305](http://bookstore.ams.org/memo-268-1305)

## Discrete Mathematics and Combinatorics

### Weakly Modular Graphs and Nonpositive Curvature

**Jérémie Chalopin**, **Victor Chepoi**, *both of Aix-Marseille Université and CNRS, Marseille, France*, **Hiroshi Hirai**, *The University of Tokyo, Japan*, and **Damian Osajda**, *Uniwersytet Wrocławski, Poland*

*This item will also be of interest to those working in geometry and topology.*

**Memoirs of the American Mathematical Society**, Volume 268, Number 1309

November 2020, 159 pages, Softcover, ISBN: 978-1-4704-4362-7, 2010 *Mathematics Subject Classification*: 05C12, 51K05, 20F67, 90C27, List US\$85, **AMS members US\$68, MAA members US\$76.50**, Order code MEMO/268/1309

[bookstore.ams.org/memo-268-1309](http://bookstore.ams.org/memo-268-1309)



## Differential Equations

### The Irreducible Subgroups of Exceptional Algebraic Groups

Adam R. Thomas, *University of Bristol, UK and Heilbronn Institute for Mathematical Research, Bristol, UK*

**Memoirs of the American Mathematical Society**, Volume 268, Number 1307

November 2020, 197 pages, Softcover, ISBN: 978-1-4704-4337-5, 2010 *Mathematics Subject Classification*: 20G05, 20G15, 20G41, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/268/1307

[bookstore.ams.org/memo-268-1307](http://bookstore.ams.org/memo-268-1307)

## Mathematical Physics

### Conformal Symmetry Breaking Differential Operators on Differential Forms

Matthias Fischmann, *Aarhus University, Denmark*, Andreas Juhl, *Humboldt-Universität, Berlin, Germany*, and Petr Somberg, *Charles University, Praha, Czech Republic*

*This item will also be of interest to those working in algebra and algebraic geometry.*

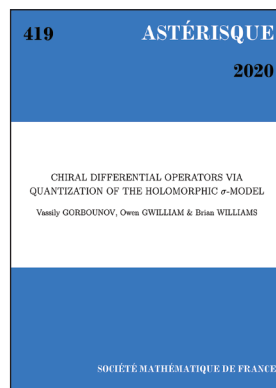
**Memoirs of the American Mathematical Society**, Volume 268, Number 1304

November 2020, 116 pages, Softcover, ISBN: 978-1-4704-4324-5, 2010 *Mathematics Subject Classification*: 22E46, 35J30, 53A30; 22E47, 33C45, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/268/1304

[bookstore.ams.org/memo-268-1304](http://bookstore.ams.org/memo-268-1304)

## New AMS-Distributed Publications

### Algebra and Algebraic Geometry



### Chiral Differential Operators via Quantization of the Holomorphic $\sigma$ -Model

Vassily Gorbounov, *University of Aberdeen, Higher School of Economics, The Laboratory of Algebraic Geometry and Homological Algebra and Moscow Institute of Physics and Technology*, Owen Gwilliam, *University of Massachusetts, Amherst*, and Brian Williams, *School of Mathematics, University of Edinburgh, Scotland*, Editors

*inburgh, Scotland, Editors*

The curved  $\beta\gamma$  system is a nonlinear  $\sigma$ -model with a Riemann surface as the source and a complex manifold  $X$  as the target. Its classical solutions pick out the holomorphic maps from the Riemann surface into  $X$ . Physical arguments identify its algebra of operators with a vertex algebra known as the chiral differential operators (CDO) of  $X$ . We verify these claims mathematically by constructing and quantizing rigorously this system using machinery developed by Kevin Costello and the second author, which combine renormalization, the Batalin-Vilkovisky formalism, and factorization algebras. Furthermore, we find that the factorization algebra of quantum observables of the curved  $\beta\gamma$  system encodes the sheaf of chiral differential operators. In this sense our approach provides deformation quantization for vertex algebras. As in many approaches to deformation quantization, a key role is played by Gelfand-Kazhdan formal geometry.

*This item will also be of interest to those working in mathematical physics and geometry and topology.*

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Astérisque**, Number 419

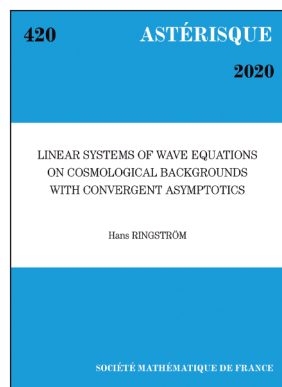
October 2020, 210 pages, Softcover, ISBN: 978-2-85629-920-3, 2010 *Mathematics Subject Classification*: 81T15,

## NEW BOOKS

81T40, 81T70, 17B69, 55N34, List US\$68, **AMS members US\$54.40**, Order code AST/419

[bookstore.ams.org/ast-419](http://bookstore.ams.org/ast-419)

## Differential Equations



### Linear Systems of Wave Equations on Cosmological Backgrounds with Convergent Asymptotics

**Hans Ringström**, *KTH Royal Institute of Technology, Sweden and Institut Mittag-Leffler, Dept. of Mathematics, Sweden*, Editor

The subject of the article is linear systems of wave equations on cosmological backgrounds with convergent asymptotics. The condition of convergence corresponds to the requirement that the second fundamental form, when suitably normalized, converges. The

model examples are the Kasner solutions. The main result of the article is optimal energy estimates. However, we also derive asymptotics and demonstrate that the leading order asymptotics can be specified (also in situations where the asymptotics are not convergent).

*This item will also be of interest to those working in geometry and topology and mathematical physics.*

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Astérisque**, Number 420

October 2020, 512 pages, Softcover, ISBN: 978-2-85629-926-5, 2010 *Mathematics Subject Classification*: 35L10, 35L15, 35Q75, 53B30, 53C50, 58J45, 83F05, List US\$105, **AMS members US\$84**, Order code AST/420

[bookstore.ams.org/ast-420](http://bookstore.ams.org/ast-420)

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# Meetings & Conferences of the AMS

## February Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at [www.ams.org/meetings](http://www.ams.org/meetings).

**Important Information About AMS Meetings:** Potential organizers, speakers, and hosts should refer to <https://www.ams.org/meetings/meetings-general> for general information regarding participation in AMS meetings and conferences.

**Abstracts:** Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of  $\text{\LaTeX}$  is necessary to submit an electronic form, although those who use  $\text{\LaTeX}$  may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in  $\text{\LaTeX}$ . Visit [www.ams.org/cgi-bin/abstracts/abstract.pl](http://www.ams.org/cgi-bin/abstracts/abstract.pl). Questions about abstracts may be sent to [abs-info@ams.org](mailto:abs-info@ams.org). Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

### Associate Secretaries of the AMS

**Central Section:** Georgia Benkart, University of Wisconsin–Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: [benkart@math.wisc.edu](mailto:benkart@math.wisc.edu); telephone: 608-263-4283.

**Eastern Section:** Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: [steve.weintraub@lehigh.edu](mailto:steve.weintraub@lehigh.edu); telephone: 610-758-3717.

**Southeastern Section:** Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: [brian@math.uga.edu](mailto:brian@math.uga.edu); telephone: 706-542-2547.

**Western Section:** Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: [lapidus@math.ucr.edu](mailto:lapidus@math.ucr.edu); telephone: 951-827-5910.

## Meetings in this Issue

### 2021

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*See [www.ams.org/meetings](http://www.ams.org/meetings) for the most up-to-date information on the meetings and conferences that we offer.*

*The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at <https://www.ams.org/welcoming-environment-policy>.*



# Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See <https://www.ams.org/meetings>.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

**New: Sectional Meetings Require Registration to Submit Abstracts.** In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

## Spring Southeastern Virtual Sectional Meeting

*now meeting virtually, EST (hosted by the American Mathematical Society)*

**March 13–14, 2021**

Saturday – Sunday

**Meeting #1164**

Southeastern Section

Associate secretary: Brian D. Boe

Program first available on AMS website: January 28, 2021

Issue of *Abstracts*: Volume 42, Issue 2

### Deadlines

For organizers: Expired

For abstracts: January 19, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/sectional.html>.*

### Invited Addresses

**Yaiza Canzani**, University of North Carolina-Chapel Hill, *Eigenfunction concentration via geodesic beams*.

**Jiongmin Yong**, University of Central Florida, *Time-Inconsistency — A Mathematical Perspective*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.*

*Advanced Topics in Graph Theory and Combinatorics* (Code: SS 2A), **Songling Shan**, Illinois State University.

*Advances in Computational Dynamics* (Code: SS 6A), **Jorge L Gonzalez**, Georgia Institute of Technology, **Andrey Shilnikov**, Georgia State University, and **J.D. Mireles James**, Florida Atlantic University.

*Celestial Mechanics and Applied Astrodynamics* (Code: SS 10A), **Bhanu Kumar** and **Molei Tao**, Georgia Institute of Technology.

*Commutative Algebra and its Interaction with Algebraic Geometry and Combinatorics* (Code: SS 11A), **Justin Chen**, Georgia Institute of Technology, and **Youngsu Kim**, California State University, San Bernardino.

*Differential Graded Methods in Commutative Algebra* (Code: SS 1A), **Saeed Nasseh**, Georgia Southern University, and **Adela Vraciu**, University of South Carolina, Columbia.

*Functional Differential Equations, Theory and Applications* (Code: SS 8A), **Joan Gimeno**, University of Rome Tor Vergata, and **Rachel Kuske** and **Jiaqi Yang**, Georgia Institute of Technology.

*Graphs in Data Science* (Code: SS 9A), **Nicolas Fraiman**, University of North Carolina, Chapel Hill, and **Soledad Villar**, Johns Hopkins University.

*Groups, Geometry, and Topology* (Code: SS 4A), **Tara Brendle** and **Maxime Fortier-Bourque**, University of Glasgow, and **Dan Margalit** and **Yvon Verberne**, Georgia Institute of Technology.

*Integrable Nonlocal Systems* (Code: SS 14A), **Matthew Russo**, Florida State University.

*Optimization and Real Algebraic Geometry* (Code: SS 15A), **Saugata Basu** and **Ali Mohammad Nezhad**, Purdue University.

*Recent Developments on Analysis and Computation for Inverse Problems for PDEs* (Code: SS 3A), **Dinh-Liem Nguyen**, Kansas State University, and **Loc Nguyen** and **Khoa Vo**, University of North Carolina at Charlotte.

*Stochastic Control and Related Topics* (Code: SS 7A), **Andrzej Swiech**, Georgia Institute of Technology, and **Jiongmin Yong**, University of Central Florida.

*Superalgebras, Quantum Groups, and Related Topics* (Code: SS 13A), **Jonas Hartwig** and **Dwight A. Williams, II**, Iowa State University.

*Topology and Geometry of 3- and 4-Manifolds* (Code: SS 5A), **Siddhi Krishna**, Georgia Institute of Technology and Columbia University, **Miriam Kuzbary**, Georgia Institute of Technology, **Beibei Liu**, Max Planck Institute for Mathematics and Georgia Institute of Technology, and **JungHwan Park**, Georgia Institute of Technology.

*Tropical Geometry, F1-connections and Matroids* (Code: SS 12A), **Kalina Mincheva**, Tulane University, and **Jaiung Jun**, SUNY at New Paltz.

## Spring Eastern Virtual Sectional Meeting

*now meeting virtually, EDT (hosted by the American Mathematical Society)*

**March 20–21, 2021**

Saturday – Sunday

**Meeting #1165**

Eastern Section

Associate secretary: Steven H. Weintraub

Program first available on AMS website: January 28, 2021

Issue of *Abstracts*: Volume 42, Issue 2

### Deadlines

For organizers: Expired

For abstracts: January 19, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/sectional.html>.*

### Invited Addresses

**Abba Gumel**, Arizona State University, *Mathematics of Infectious Diseases* (Einstein Public Lecture in Mathematics).

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.*

*Algebraic Geometry in Dynamics* (Code: SS 9A), **Nguyen-Bac Dang**, Stony Brook University, and **Nicole Looper**, **Rohini Ramadas**, and **Joseph H. Silverman**, Brown University.

*Applications and Asymptotic Properties of Discrete Dynamical Systems: A Session in Honor of the Retirement of Orlando Merino* (Code: SS 15A), **Elliott Bertrand**, Sacred Heart University, **Zachary Kudlak**, United States Coast Guard Academy, **Mustafa Kulenovic**, University of Rhode Island, and **David McArdle**, University of Connecticut.

*Applied Combinatorics* (Code: SS 3A), **Carina Curto**, Pennsylvania State University, and **Pedro Felzenszwalb** and **Caroline Klivans**, Brown University.

*Commutative Algebra* (Code: SS 1A), **Laura Ghezzi**, Department of Mathematics, New York City College of Technology-CUNY, **Saeed Nasseh**, Georgia Southern University, and **Oana Veliche**, Northeastern University.

## MEETINGS & CONFERENCES

*Current Trends in Combinatorial Commutative Algebra* (Code: SS 5A), **Kuei-Nuan Lin**, Pennsylvania State University, Greater Allegheny, and **Augustine O’Keefe**, Connecticut College.

*Fractional Calculus and Fractional Differential/Difference Equations* (Code: SS 19A), **Lyubomir Boyadjiev**, City University of New York, **Pavel Dubovski**, Stevens Institute of Technology, and **Mark Edelman**, Yeshiva University and New York University.

*Gauge Theory, Geometry, and Low-Dimensional Topology* (Code: SS 17A), **Paul Feehan**, Rutgers University, and **Daniel Ruberman**, Brandeis University.

*Geometric and Functional Inequalities and Nonlinear Partial Differential Equations* (Code: SS 16A), **Joshua Flynn**, University of Connecticut, **Nguyen Lam**, Memorial University of Newfoundland Grenfell Campus, **Jungang Li**, Brown University, and **Guozhen Lu**, University of Connecticut.

*Hopf Algebras, Tensor Categories, and Related Homological Methods* (Code: SS 13A), **Pablo S. Ocal**, Texas A&M University, and **Julia Plavnik**, Indiana University Bloomington.

*Metric techniques in Analysis* (Code: SS 7A), **Vasileios Chousionis** and **Sean Li**, University of Connecticut.

*Mirror Symmetry and Enumerative Geometry* (Code: SS 4A), **Mandy Cheung**, Harvard University, and **Siu-Cheong Lau** and **Yu-Shen Lin**, Boston University.

*Moduli of Curves, Hilbert Schemes, and Tropical Geometry* (Code: SS 10A), **Ignacio Barros**, Northeastern University, **Noah Giansiracusa**, Bentley University, and **Rob Silversmith**, Northeastern University.

*New Applications and Methods in Financial Mathematics* (Code: SS 14A), **Gu Wang**, Worcester Polytechnic Institute, and **Bin Zou**, University of Connecticut.

*Nonlinear Wave Equations, General Relativity, and Connections to Fluid Dynamics* (Code: SS 12A), **Stefanos Aretakis**, University of Toronto, **Aynur Bulut**, Louisiana State University, and **Sung-Jin Oh**, University of California, Berkeley.

*Probability and Combinatorics* (Code: SS 18A), **Zhongyang Li**, University of Connecticut, and **Mei Yin**, University of Denver.

*Recent Advances in Schubert Calculus and Related Topics* (Code: SS 2A), **Cristian Lenart** and **Changlong Zhong**, State University of New York at Albany.

*Recent Developments in Automorphic Representations* (Code: SS 8A), **Spencer Leslie**, Duke University, and **Tian An Wong**, University of Michigan-Dearborn.

*Recent Developments in Differential Geometry* (Code: SS 11A), **Megan Kerr**, Wellesley College, and **Catherine Searle**, Wichita State University.

*Stochastic Analysis* (Code: SS 6A), **Parisa Fatheddin** and **Aurel Stan**, Ohio State University, Marion.

## Spring Central Virtual Sectional Meeting

*now meeting virtually, CDT (hosted by the American Mathematical Society)*

**April 17–18, 2021**

Saturday – Sunday

**Meeting #1166**

Central Section

Associate secretary: Georgia Benkart

Program first available on AMS website: To be announced  
Issue of *Abstracts*: Volume 42, Issue 2

### Deadlines

For organizers: Expired  
For abstracts: February 16, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/sectional.html>.*

### Invited Addresses

**Fabrice Baudoin**, University of Connecticut, *Title to be announced.*

**Malabika Pramanik**, University of British Columbia and BIRS, *Title to be announced.*

**Maksym Radziwill**, Caltech, *Title to be announced.*



## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Commutative Algebra* (Code: SS 7A), **Ayah Almousa**, Cornell University, and **Sean Sather-Wagstaff**, Clemson University.  
*Graph Theory and Applications* (Code: SS 9A), **Katherine F. Benson**, University of Wisconsin-Stout, **Christine A. Kelley**, University of Nebraska-Lincoln, **Esmeralda L. Năstase**, Xavier University, and **JD Nir**, University of Manitoba.

*Interactions between Representation Theory, Poisson Geometry, and Noncommutative Algebra* (Code: SS 5A), **Jason Gaddis**, Miami University, **Padmini Veerapen**, Tennessee Technological University, and **Xingting Wang**, Howard University.

*Legendrian Knots and Surfaces* (Code: SS 3A), **Honghao Gao**, Michigan State University, and **Dan Rutherford**, Ball State University.

*Nonsmooth Analysis and Geometry* (Code: SS 1A), **Luca Capogna**, Worcester Polytechnic Institute, and **Gareth Speight** and **Nageswari Shanmugalingam**, University of Cincinnati.

*Numerical Linear Algebra* (Code: SS 8A), **Lothar Reichel**, Kent State University, **Jianlin Xia**, Purdue University, and **Qiang Ye**, University of Kentucky.

*Probabilistic and Diffusion Methods in Analysis and Geometry* (Code: SS 4A), **Rodrigo Bañuelos** and **Jing Wang**, Purdue University, and **Ju-Yi Yen**, University of Cincinnati.

*Recent Progress in Analytic Number Theory* (Code: SS 6A), **Seungki Kim**, University of Cincinnati, **Xiannan Li**, Kansas State University, and **Xuancheng Fernando Shao**, University of Kentucky.

*Sharp Estimates in Harmonic Analysis* (Code: SS 2A), **Kabe Moen**, University of Alabama, **Leonid Slavin**, University of Cincinnati, and **Alex Stokolos**, Georgia Southern University.

## Spring Western Virtual Sectional Meeting

now meeting virtually, PDT (hosted by the American Mathematical Society)

**May 1–2, 2021**

Saturday – Sunday

Program first available on AMS website: February 25, 2021

Issue of *Abstracts*: Volume 42, Issue 2

### Meeting #1167

Western Section

Associate secretary: Michel L. Lapidus

### Deadlines

For organizers: Expired

For abstracts: February 16, 2021

The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtg/sectional.html>.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Advances in Functional Analysis and Operator Theory* (Code: SS 9A), **Michel L. Lapidus**, University of California, Riverside, **Marat V. Markin**, California State University, Fresno, and **Igor Nikolaev**, St. John's University.

*Algebraic and Combinatorial Aspects of Polytopes* (Code: SS 14A), **Federico Ardila**, San Francisco State University and Los Andes University, **Laura Escobar**, Washington University in St. Louis, and **Raul Penaguião**, University of Zurich.

*Algebraic K-theory, Motivic Homotopy Theory, and Perfectoid Spaces* (Code: SS 12A), **Shanna Dobson**, California State University, Los Angeles.

*Analysis, Combinatorics, and Geometry of Fractals* (Code: SS 4A), **Kyle Hambrook**, San Jose State University, and **Chun-Kit Lai**, San Francisco State University.

*Categorical and Combinatorial Methods in Representation Theory, and Related Topics* (Code: SS 19A), **Mee Seong Im**, U.S. Naval Academy, **Bach Nguyen**, Xavier University of Louisiana, and **Arik Wilbert**, University of Georgia.

*Commutative Algebra* (Code: SS 21A), **Juliette Bruce**, Mathematical Sciences Research Institute, Berkeley, **Monica Lewis**, University of Michigan, and **Sean Sather-Wagstaff**, Clemson.

## MEETINGS & CONFERENCES

*Connections between homotopical algebra and geometry* (Code: SS 6A), **Ryan Grady**, Montana State University, and **Chris Rogers**, University of Nevada, Reno.

*Diagrammatic and Combinatorial Methods in Representation Theory* (Code: SS 17A), **Robert Muth**, Washington & Jefferson College, **Nick Davidson**, Reed College, **Peter Tingley**, Loyola University Chicago, and **Tianyuan Xu**, University of Colorado Boulder.

*Differential Geometry and Geometric PDE* (Code: SS 1A), **Alfonso Agnew**, **Nicholas Brubaker**, **Thomas Murphy**, **Shoo Seto**, and **Bogdan Suceavă**, California State University, Fullerton.

*Geodesics in Hyperbolic 2- and 3-Manifolds* (Code: SS 10A), **Maria Trnkova**, University of California, Davis, and **Andrew Yarmola**, Princeton University.

*Geometric Analysis* (Code: SS 22A), **Ovidiu Munteanu**, University of Connecticut, and **David Bao**, San Francisco State University.

*Geometric and Categorical Methods in Representation Theory* (Code: SS 16A), **Ana Balibanu**, Harvard University, **Daniele Rosso**, Indiana University Northwest, and **Jonathan Wang**, Massachusetts Institute of Technology.

*How do Industry Professionals Use Big Data?* (Code: SS 15A), **Luella Fu**, San Francisco State University.

*Localization and delocalization in ergodic quantum systems* (Code: SS 2A), **Ilya Kachkovskiy**, Michigan State University, and **Wencai Liu** and **Rodrigo Matos**, Texas A&M University.

*Nonlinear PDEs and fluid dynamics* (Code: SS 13A), **Igor Kukavica**, **Juhi Jang**, and **Wojciech Ozanski**, University of Southern California.

*Quivers, Tensors, and Their Applications* (Code: SS 20A), **Visu Makam**, Institute for Advanced Study, **Francesca Gandini**, Kalamazoo College, and **Alana Huszar** and **Robert Cochrane**, University of Michigan.

*Regularity Theory for Linear and Nonlinear PDEs* (Code: SS 11A), **Zongyuan Li**, Rutgers University, **Weinan Wang**, University of Arizona, and **Xueying Yu**, Massachusetts Institute of Technology.

*Research in Mathematics by Early Career Graduate Students* (Code: SS 8A), **Michael Bishop**, **Marat V. Markin**, and **Khang Tran**, California State University, Fresno.

*Social Change through Mathematics and Education* (Code: SS 18A), **Federico Ardila** and **Shandy Hauk**, San Francisco State University, **Ashia Wilson**, Massachusetts Institute of Technology, and **Robin Wilson**, California State Polytechnic University, Pomona.

*Topological Perspectives in Graph Theory, Classical and Recent* (Code: SS 3A), **Jonathan L. Gross**, Columbia University, **Timothy Sun**, San Francisco State University, and **Thomas W. Tucker**, Colgate University.

*Women in Commutative Algebra - One hundred years of Idealtheorie in Ringbereichen* (Code: SS 7A), **Eloísa Grifo** and **Alessandra Costantini**, University of California, Riverside.

## Grenoble, France

Université de Grenoble-Alpes

July 5–9, 2021

Monday – Friday

### Meeting #1168

Associate secretary: Michel L. Lapidus

Program first available on AMS website: Not applicable

Issue of Abstracts: Not applicable

### Deadlines

For organizers: Expired

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/internmtgs.html>.

### Special Sessions

*Advances in Functional Analysis and Operator Theory*, **Marat V. Markin**, California State University, Fresno, USA, **Igor Nikolaev**, St. John's University, USA, **Jean Renault**, Université d'Orleans, France, and **Carsten Trunk**, Technische Universität Ilmenau, Germany.

*Algebraic Geometry (Associated with Plenary Speaker Claire Voisin)*, **Radu Laza**, Stony Brook University, USA, **Catriona Maclean**, Grenoble, France, and **Claire Voisin**, Paris, France.

*Automorphic Forms, Moduli Spaces, and Representation Theory (Associated with Plenary Speaker Vincent Lafforgue)*, **Jean-François Dat**, Sorbonne Université, France, and **Bao-Chau Ngo**, University of Chicago, USA.

*Classical and Quantum Fields on Lorentzian Manifolds*, **Dietrich Häfner**, Université Grenoble Alpes, France, and **Andras Vasy**, Stanford University, USA.

*Combinatorial and Computational Aspects in Topology*, **Eric Samperton**, University of Illinois, USA, **Saul Schleimer**, University of Warwick, United Kingdom, and **Greg McShane**, Université Grenoble-Alpes, France.

*Contact Geometry*, **David E. Blair**, Michigan State University, USA, **Gianluca Bande**, Università degli Studi di Cagliari, Italy, and **Eric Loubeau**, Université de Bretagne Occidentale, France.

*Deformation of Artinian algebras and Jordan type*, **Anthony Iarrobino**, Northeastern University, USA, **Pedro Macias Marques**, Universidade de Evora, Portugal, **Maria Evelina Rossi**, Università degli Studi di Genova, Italy, and **Jean Valles**, Université de Pau et des Pays de l'Adour, France.

*Deformation Spaces of Geometric Structures*, **Sara Maloni**, University of Virginia, USA, **Andrea Seppi**, Université Grenoble Alpes, France, and **Nicolas Tholozan**, Ecole Normale Supérieure de Paris, France.

*Derived Categories and Rationality*, **Matthew Ballard**, University of South Carolina, USA, **Emanuele Macrì**, Université Paris-Saclay, France, and **Patrick McFaddin**, Fordham University, USA.

*Differential Geometry in the Tradition of Élie Cartan (1869 - 1959)*, **Vincent Borelli**, Université Claude Bernard, **Bogdan Suceavă**, California State University, Fullerton, USA, **Mihaela B. Vajiac**, Chapman University, USA, **Joeri Van der Veken**, KU Leuven, Belgium, **Marina Ville**, Université de Tours, France, and **Luc Vrancken**, Université Polytechnique Hauts-de-France, Valenciennes, France.

*Drinfeld Modules, Modular Varieties and Arithmetic Applications*, **Tuan Ngo Dac**, CNRS Université Claude Bernard Lyon 1, France, **Matthew Papanikolas**, Texas A&M University, USA, **Mihran Papikian**, Pennsylvania State University, USA, and **Federico Pellarin**, Université Jean Monnet, France.

*Fractal Geometry in Pure and Applied Mathematics*, **Hafedh Herichi**, Santa Monica College, USA, **Maria Rosaria Lancia**, Sapienza Università di Roma, Italy, **Therese-Marie Landry**, University of California, Riverside, USA, **Anna Rozanova-Pierat**, CentralSuplec, Université Paris-Saclay, France, and **Steffen Winter**, Karlsruhe Institute of Technology, Germany.

*Functional Equations and Their Interactions*, **Guy Casale**, IRMAR, Université de Rennes 1, France, **Thomas Dreyfus**, IRMA, Université de Strasbourg, France, **Charlotte Hardouin**, IRMAR, Université de Toulouse 3, France, **Joel Nagloo**, CUNY, New York, USA, **Julien Roques**, Institut Camille Jordan, Université de Lyon 1, France, and **Michael Singer**, North Carolina State University, Raleigh, USA.

*Graph and Matroid Polynomials: Towards a Comparative Theory*, **Emeric Gioan**, LIRMM, France, **Johann A. Makowsky**, Israel Institute of Technology- IIT, Israel, and **James Oxley**, Louisiana State University, USA.

*Groups and Topological Dynamics*, **Nicolas Matte Bon**, University of Lyon, France, **Constantine Medynets**, United States Naval Academy, USA, **Volodymyr Nekrashevych**, Texas A&M University, USA, and **Dmytro Savchuk**, University of South Florida, USA.

*Group Theory, Algorithms and Applications*, **Indira Chatterji**, Université de Nice, France, **Francois Dahmani** and **Martin Deraux**, Institut Fourier, Université Grenoble Alpes, France, and **Delaram Kahrobaei**, CUNY and NYU, USA.

*History of Mathematics Beyond Case-Studies*, **Catherine Goldstein**, CNRS, IMJ-PRG, France, and **Jemma Lorenat**, Pitzer College, USA.

*Integrability, Geometry, and Mathematical Physics*, **Luen-Chau Li**, Pennsylvania State University, USA, and **Serge Parnen-tier**, Université Claude Bernard Lyon 1, France.

*Inverse Problems*, **Hanna Makaruk**, Los Alamos National Laboratory (LANL), USA, **Robert Owczarek**, University of New Mexico, Albuquerque and Los Alamos, USA, **Tomasz Lipniacki**, Polish Academy of Sciences, Poland, and **Piotr Stachura**, Warsaw University of Life Sciences-SGGW, Poland.

*Low-Dimensional Topology*, **Paul Kirk**, University of Bloomington, USA, **Christine Lescop**, CNRS, Institut Fourier, Université Grenoble Alpes, France, and **Jean-Baptiste Meilhan**, Institut Fourier, Université Grenoble Alpes, France.

*Mathematical Challenges in Complex Quantum Systems (Associated with Plenary Speaker Simone Warzel)*, **Alain Joye**, Institut Fourier, Université Grenoble Alpes, France, **Jeffrey Schenker**, Michigan State University, USA, **Nicolas Rougerie**, Université Grenoble-Alpes and CNRS, France, and **Simone Warzel**, Zentrum Mathematik, TU München, Germany.

*Mathematical Knowledge Management in the Digital Age of Science*, **Patrick Ion**, University of Michigan, Ann Arbor, USA, **Thierry Bouche**, Université Grenoble-Alpes, France, and **Stephen Watt**, University of Waterloo, Canada.

*Mathematical Physics of Gravity, Geometry, QFTs, Feynman and Stochastic Integrals, Quantum/Classical Number Theory, Algebra, and Topology*, **Michael Maroun**, AMS-MRC Boston, USA, and **Pierre Vanhove**, EMS/SMF CEA Paris Saclay, France.

*Modular Representation Theory*, **Pramod N. Achar**, Louisiana State University, USA, **Simon Riche**, Université Clermont Auvergne, France, and **Britta Spath**, Bergische Universität Wuppertal, Germany.

*Percolation and Loop Models (Associated with Plenary Speaker Hugo Duminil-Copin)*, **Ioan Manolescu**, University of Fribourg, Switzerland.



*Quantitative Geometry of Transportation Metrics*, **Florent Baudier**, Texas A&M University, USA, **Dario Cordero-Erausquin**, Sorbonne Université, France, **Alexandros Eskenazis**, University of Cambridge, United Kingdom, and **Eva Pernecka**, Czech Technical University in Prague, Czech Republic.

*Recent Advances in Diffeology and their Applications*, **Jean-Pierre Magnot**, Université d'Angers, France, and **Jordan Watts**, Central Michigan University, USA.

*Rough Path and Malliavin Calculus*, **Fabrice Baudoin**, University of Connecticut, USA, **Antoine Lejay**, University of Lorraine, France, and **Cheng Ouyang**, University of Illinois at Chicago, USA.

*Spectral Optimization*, **Richard S. Laugesen**, University of Illinois at Urbana Champaign, USA, **Enea Parini**, Aix Marseille University, France, and **Emmanuel Russ**, Grenoble Alpes University, France.

*Statistical Learning (Associated with Plenary speaker Peter Bühlmann)*, **Christophe Giraud**, Paris Saclay University, France, **Cun-Hui Zhang**, Rutgers University, USA, and **Peter Bühlmann**, ETH Zürich, Switzerland.

*Sub-Riemannian Geometry and Interactions*, **Luca Rizzi**, CNRS, Institut Fourier, Grenoble, France, and **Fabrice Baudoin**, University of Connecticut, USA.

## Buenos Aires, Argentina

*The University of Buenos Aires*

**July 19–23, 2021**

*Monday – Friday*

### Meeting #1169

Associate secretary: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of *Abstracts*: Not applicable

### Deadlines

For organizers: Expired

For abstracts: To be announced

## Buffalo, New York

*University at Buffalo (SUNY)*

**September 18–19, 2021**

*Saturday – Sunday*

### Meeting #1170

Eastern Section

Associate secretary: Steven H. Weintraub

Program first available on AMS website: August 5, 2021

Issue of *Abstracts*: Volume 42, Issue 3

### Deadlines

For organizers: February 18, 2021

For abstracts: July 27, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/sectional.html>.*

### Invited Addresses

**Kirstin Eisentraeger**, Pennsylvania State University, *Title to be announced.*

**Jason Manning**, Cornell University, *Title to be announced.*

**Jennifer Mueller**, Colorado State University, *Title to be announced.*

# Omaha, Nebraska

Creighton University

**October 9–10, 2021**

Saturday – Sunday

## Meeting #1171

Central Section

Associate secretary: Georgia Benkart

Program first available on AMS website: August 19, 2021

Issue of *Abstracts*: Volume 42, Issue 3

## Deadlines

For organizers: March 9, 2021

For abstracts: August 10, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgsectional.html>.*

## Invited Addresses

**Daniel Erman**, University of Wisconsin-Madison, *Title to be announced.*

**Jasmine Foo**, University of Minnesota-Twin Cities, *Title to be announced.*

**Kay Kirkpatrick**, University of Illinois Urbana-Champaign, *Title to be announced.*

# Albuquerque, New Mexico

University of New Mexico

**October 23–24, 2021**

Saturday – Sunday

## Meeting #1172

Western Section

Associate secretary: Michel L. Lapidus

Program first available on AMS website: September 2, 2021

Issue of *Abstracts*: Volume 42, Issue 4

## Deadlines

For organizers: March 23, 2021

For abstracts: August 24, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgsectional.html>.*

## Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.*

*Dissipative Systems and Their Applications* (Code: SS 3A), **Mingji Zhang** and **Bixiang Wang**, New Mexico Institute of Mining and Technology.

*Inverse Problems: In Memory of Professor Zbigniew Oziewicz* (Code: SS 1A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico.

*Recent Advances in Studies of Electrodifffusion Phenomena* (Code: SS 2A), **Weishi Liu**, University of Kansas, **Hamid Mofidi**, University of Iowa, and **Mingji Zhang**, New Mexico Institute of Mining and Technology.

## Mobile, Alabama

*University of South Alabama*

**November 20–21, 2021**

*Saturday – Sunday*

### Meeting #1173

Southeastern Section

Associate secretary: Brian D. Boe

Program first available on AMS website: September 30, 2021

Issue of *Abstracts*: Volume 42, Issue 4

### Deadlines

For organizers: April 20, 2021

For abstracts: September 21, 2021

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtg/sectional.html>.*

### Invited Addresses

Sara Del Valle, Los Alamos National Laboratory, *Title to be announced*.

## Seattle, Washington

*Washington State Convention Center and the Sheraton Seattle Hotel*

**January 5–8, 2022**

*Wednesday – Saturday*

### Meeting #1174

Associate secretary: Georgia Benkart

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Charlottesville, Virginia

*University of Virginia*

**March 11–13, 2022**

*Friday – Sunday*

### Meeting #1175

Southeastern Section

Associate secretary: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: August 12, 2021

For abstracts: January 18, 2022

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtg/sectional.html>.*

### Invited Addresses

Moon Duchin, Tufts University, *Title to be announced* (Einstein Public Lecture in Mathematics).

Laura A Miller, University of North Carolina at Chapel Hill, *Title to be announced*.

Betsy Stovall, University of Wisconsin-Madison, *Title to be announced*.

Yusu Wang, The Ohio State University, *Title to be announced*.



## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Advances in Difference, Differential, Fractional Differential and Dynamic Equations with Applications* (Code: SS 6A), **Muhammad Islam** and **Youssef Raffoul**, University of Dayton.

*Advances in Infectious Disease Modeling: From Cells to Populations* (Code: SS 5A), **Lauren Childs**, **Stanca Ciupe**, and **Omar Saucedo**, Virginia Tech.

*Advances in Operator Algebras* (Code: SS 11A), **Ben Hayes** and **David Sherman**, University of Virginia.

*Algebraic Groups: Arithmetic and Geometry* (Code: SS 1A), **Raman Parimala**, Emory University, **Andrei Rapinchuk**, University of Virginia, and **Igor Rapinchuk**, Michigan State University.

*Celebrating Diversity in Mathematics* (Code: SS 7A), **Lauren Childs**, Virginia Tech, **Sara Maloni**, University of Virginia, and **Rebecca R.G.**, George Mason University.

*Combinatorial Methods in Geometric Group Theory* (Code: SS 19A), **Tarik Aougab**, Haverford College, **Marrissa Loving**, Georgia Institute of Technology, and **Priyam Patel**, University of Utah.

*Commutative Algebra* (Code: SS 2A), **Eloísa Grifo**, University of California, Riverside, and **Sean Sather-Wagstaff**, Clemson University.

*Homotopy Theory* (Code: SS 10A), **Julie Bergner** and **Nick Kuhn**, University of Virginia.

*Integrable Probability* (Code: SS 14A), **Leonid Petrov**, University of Virginia, and **Axel Saenz**, Tulane University.

*Knots and Links in Low-Dimensional Topology* (Code: SS 13A), **Thomas Mark**, University of Virginia, and **Allison Moore**, University of California Davis.

*Knot Theory and its Applications* (Code: SS 20A), **Hugh Howards** and **Jason Parsley**, Wake Forrest University, and **Eric Rawdon**, St Thomas University.

*Mathematical String Theory* (Code: SS 8A), **Ilarion Melnikov**, James Madison University, **Eric Sharpe**, Virginia Tech, and **Diana Vaman**, University of Virginia.

*Probabilistic Methods in Geometry and Analysis* (Code: SS 12A), **Fabrice Baudoin** and **Li Chen**, University of Connecticut.

*Recent Advances in Graph Theory and Combinatorics* (Code: SS 17A), **Neal Bushaw**, Virginia Commonwealth University, and **Martin Rolek** and **Gexin Yu**, College of William and Mary.

*Recent Advances in Harmonic Analysis* (Code: SS 3A), **Amalia Culiuc**, Amherst College, **Yen Do**, University of Virginia, and **Eyvindur Ari Palsson**, Virginia Tech.

*Recent Advances in Mathematical Biology* (Code: SS 23A), **Junping Shi**, College of William & Mary, **Xhisheng Shuai**, University of Central Florida, and **Yixiang Wu**, Middle Tennessee State University.

*Recent Progress on Singular and Oscillatory Integrals* (Code: SS 4A), **Betsy Stovall** and **Joris Roos**, University of Wisconsin-Madison.

*Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS)* (Code: SS 18A), **Chun-Ju Lai** and **Daniel K. Nakano**, University of Georgia, and **Weiqliang Wang**, University of Virginia.

*Tensors and Complexity* (Code: SS 16A), **Visu Makam**, Institute for Advanced Study, and **Rafael Oliveira**, University of Waterloo.

*Topics in Convexity and Probability* (Code: SS 22A), **Steven Hoehner**, Longwood University, and **Mark Meckes** and **Elizabeth Werner**, Case Western Reserve University.

*Trends in Teichmüller Theory* (Code: SS 21A), **Thomas Koberda** and **Sara Maloni**, University of Virginia, and **Giuseppe Martone**, University of Michigan.

*Vertex Algebras and Geometry* (Code: SS 9A), **Marco Aldi**, Virginia Commonwealth University, **Michael Penn**, Randolph College, and **Nicola Tarasca** and **Juan Villarreal**, Virginia Commonwealth University.

*Youth and Enthusiasm in Arithmetic Geometry and Number Theory* (Code: SS 15A), **Evangelia Gazaki** and **Ken Ono**, University of Virginia.

# Medford, Massachusetts

Tufts University

**March 19–20, 2022**

Saturday – Sunday

## Meeting #1176

Eastern Section

Associate secretary: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: August 24, 2021

For abstracts: January 18, 2022

The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtg/sectional.html>.

## Invited Addresses

**Daniela De Silva**, Barnard College, Columbia University, *Title to be announced*.

**Enrique R. Pujals**, Graduate Center, CUNY, *Title to be announced*.

**Christopher T Woodward**, Rutgers University, New Brunswick, *Title to be announced*.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Analysis on Homogeneous Spaces* (Code: SS 7A), **Jens Christensen**, Colgate University, **Matthew Dawson**, CIMAT, Mérida, México, and **Fulton Gonzalez**, Tufts University.

*Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups and Related Topics* (Code: SS 3A), **S. Allen Broughton**, Rose-Hulman Institute of Technology, **Jen Paulhus**, Grinnell College, and **Aaron Wootton**, University of Portland.

*Equivariant Cohomology* (Code: SS 4A), **Jeffrey D. Carlson**, The Fields Institute, and **Loring Tu**, Tufts University.

*Homological Methods in Commutative Algebra* (Code: SS 6A), **Janet Striuli**, Fairfield University and National Science Foundation, and **Oana Veliche**, Northeastern University.

*Inverse Problems and Their Applications* (Code: SS 1A), **Youssef Qranfal**, Wentworth Institute of Technology.

*Mathematical Methods for Ecology and Evolution in Structured Populations* (Code: SS 8A), **Olivia Chu**, Princeton University, **Daniel Cooney**, University of Pennsylvania, and **Chadi Saad-Roy**, Princeton University.

*Mathematics of Data Science* (Code: SS 2A), **Vasileios Maroulas**, University of Tennessee Knoxville, and **James M. Murphy**, Tufts University.

*Symmetries of Polytopes, Maps, and Graphs* (Code: SS 5A), **Gabe Cunningham**, University of Massachusetts Boston, and **Mark Mixer**, Wentworth Institute of Technology.

# West Lafayette, Indiana

Purdue University

**March 26–27, 2022**

Saturday – Sunday

## Meeting #1177

Central Section

Associate secretary: Georgia Benkart

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

## Deadlines

For organizers: August 31, 2021

For abstracts: January 25, 2022

The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtg/sectional.html>.

## Invited Addresses

**Christine Berkesch**, University of Minnesota, *Title to be announced*.

**Matthew Hedden**, Michigan State University, *Title to be announced.*

**Brian Street**, University of Wisconsin, *Title to be announced.*

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Analysis and Probability in Sub-Riemannian Geometry* (Code: SS 5A), **Jeremy Tyson**, University of Illinois at Urbana-Champaign, and **Jing Wang**, Purdue University.

*A Women in Analysis Research Network Event* (Code: SS 4A), **Donatella Danielli-Garafalo**, Purdue University, and **Irina Mitrea**, Temple University.

*Combinatorial Algebra and Geometry* (Code: SS 6A), **Christine Berkesch**, University of Minnesota, and **Laura Matusevich** and **Aleksandra Sobieska**, Texas A&M University.

*Harmonic Analysis* (Code: SS 2A), **Shaoming Guo** and **Brian Street**, University of Wisconsin-Madison.

*Quantum Algebra and Quantum Topology* (Code: SS 1A), **Shawn Cui**, Purdue University, **Julia Plavnik**, Indiana University, and **Tian Yang**, Texas A&M University.

*Recent Developments in Commutative Algebra* (Code: SS 7A), **Jennifer Kenkel**, University of Michigan, and **Linquan Ma** and **Uli Walther**, Purdue University.

*The Interface of Harmonic Analysis and Analytic Number Theory* (Code: SS 3A), **Theresa Anderson**, Purdue University, **Robert Lemke Oliver**, Tufts University, and **Eyvindur Palsson**, Virginia Tech University.

## Denver, Colorado

*University of Denver*

**May 14–15, 2022**

*Saturday – Sunday*

**Meeting #1178**

Western Section

Associate secretary: Michel L. Lapidus

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: October 12, 2021

For abstracts: March 15, 2022

## El Paso, Texas

*University of Texas at El Paso*

**September 17–18, 2022**

*Saturday – Sunday*

Central Section

Associate secretary: Georgia Benkart

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Chattanooga, Tennessee

*University of Tennessee at Chattanooga*

**October 15–16, 2022**

*Saturday – Sunday*

Southeastern Section

Associate secretary: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/sectional.html>.*



## MEETINGS & CONFERENCES

### Invited Addresses

**Giulia Saccà**, Columbia University, *Title To Be Announced*.

**Chad Topaz**, Williams College, *Title To Be Announced*.

**Xingxing Yu**, Georgia Institute of Technology, *Title To Be Announced*.

### Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Active Learning Methods and Pedagogical Approaches in Teaching College Level Mathematics* (Code: SS 6A), **Hashim Saber**, University of North Georgia.

*Applied Knot Theory* (Code: SS 1A), **Jason Cantarella**, University of Georgia, **Eleni Panagiotou**, University of Tennessee at Chattanooga, and **Eric Rawdon**, University of St Thomas.

*Boundary Value Problems for Differential, Difference, and Fractional Equations* (Code: SS 9A), **John R Graef** and **Lingju Kong**, University of Tennessee at Chattanooga, and **Min Wang**, Kennesaw State University.

*Geometric and Topological Generalization of Groups* (Code: SS 4A), **Bikash C Das**, University of North Georgia.

*Nonstandard Elliptic and Parabolic Regularity Theory with Applications* (Code: SS 2A), **Hongjie Dong**, Brown University, and **Tuoc Phan**, University of Tennessee, Knoxville.

*Probability and Statistical Models with Applications* (Code: SS 5A), **Sher Chhetri**, University of South Carolina, Sumter, and **Cory Ball**, Florida Atlantic University.

*Quantitative Approaches to Social Justice* (Code: SS 7A), **Chad Topaz**, Williams College.

*Special Session on Combinatorial Commutative Algebra* (Code: SS 8A), **Michael Cowen**, **Hugh Geller**, **Todd Morra**, and **Sean Sather-Wagstaff**, Clemson University.

*Structural and Extremal Graph Theory* (Code: SS 3A), **Hao Huang**, Emory University, and **Xingxing Yu**, Georgia Institute of Technology.

## Salt Lake City, Utah

University of Utah

October 22–23, 2022

Saturday – Sunday

Western Section

Associate secretary: Michel L. Lapidus

Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtg/sectional.html>.

### Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Algebraic combinatorics and applications in harmonic analysis* (Code: SS 3A), **Joseph Iverson** and **Sung Y. Song**, Iowa State University, and **Bangteng Xu**, Eastern Kentucky University.

*Approximation Theory and Numerical Analysis* (Code: SS 2A), **Vira Babenko**, Drake University, and **Akil Narayan**, University of Utah.

*Building bridges between commutative algebra and nearby areas* (Code: SS 5A), **Benjamin Briggs** and **Josh Pollitz**, University of Utah.

*Commutative Algebra* (Code: SS 4A), **Adam Boocher**, University of San Diego, **Eloísa Grifo**, University of California, Riverside, and **Jennifer Kenkel**, University of Michigan.

*Extremal Graph Theory* (Code: SS 1A), **Bernard Lidický**, Iowa State University.

*Fractal Geometry, Dimension Theory, and Recent Advances in Diophantine Approximation* (Code: SS 9A), **Alexander M. Henderson**, University of California, **Machiel van Frankenhuysen**, Utah Valley University, and **Edward K. Voskanian**, The College of New Jersey.

*Free boundary problems arising in applications* (Code: SS 14A), **Mark Allen**, Brigham Young University, **Mariana Smit Vega Garcia**, Western Washington University, and **Braxton Osting**, University of Utah.

*Geometry and Representation Theory of Quantum Algebras and Related Topics* (Code: SS 6A), **Mee Seong Im**, United States Military Academy, West Point, **Bach Nguyen**, Xavier University of Louisiana, and **Arik Wilbert**, University of Georgia.

*Graphs and Matrices* (Code: SS 11A), **Mark Kempton**, **Emily Evans**, and **Ben Webb**, Brigham Young University.

*Higher Topological and Algebraic K-Theories* (Code: SS 18A), **Agnès Beaudry**, University of Colorado Boulder, **Jonathan Campbell**, Duke University, and **John Lind**, California State University, Chico.

*Inverse Problems* (Code: SS 12A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico.

*Knotted surfaces and concordances* (Code: SS 15A), **Mark Hughes**, Brigham Young University, **Jeffrey Meier**, Western Washington University, and **Maggie Miller**, Princeton University.

*Mathematics of Collective Behavior* (Code: SS 10A), **Roman Shvydkoy** and **Daniel Lear**, University of Illinois at Chicago.

*PDEs, data, and inverse problems* (Code: SS 7A), **Jared Whitehead**, Brigham Young University.

*Recent advances in algebraic geometry and commutative algebra in or near characteristic  $p$*  (Code: SS 8A), **Bhargav Bhatt**, University of Michigan, and **Karl Schwede**, University of Utah.

*Recent advances in the theory of fluid dynamics* (Code: SS 17A), **Elaine Cozzi**, Oregon State University, and **Magdalena Czubak**, University of Colorado Boulder.

*Recent Advances of Numerical Methods for Partial Differential Equations with Applications* (Code: SS 16A), **Joe Koebe**, Utah State University, **Yunrong Zhu**, Idaho State University, and **Jia Zhao**, Utah State University.

*Several Complex Variables: Emerging Applications, Connections, And Synergies* (Code: SS 13A), **Jennifer Brooks**, Brigham Young University, and **Dusty Grundmeier**, Harvard University.

*Topics in graphs, hypergraphs and set systems* (Code: SS 19A), **David Galvin**, University of Notre Dame, **John Engbers**, Marquette University, and **Cliff Smyth**, The University of North Carolina at Greensboro.

## Boston, Massachusetts

*John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel*

**January 4–7, 2023**

*Wednesday – Saturday*

Associate secretary: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Atlanta, Georgia

*Georgia Institute of Technology*

**March 18–19, 2023**

*Saturday – Sunday*

Southeastern Section

Associate secretary: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced

For abstracts: To be announced

## Fresno, California

*California State University, Fresno*

**May 6–7, 2023**

*Saturday – Sunday*

Western Section

Associate secretary: Michel L. Lapidus

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

### Deadlines

For organizers: October 4, 2022

For abstracts: March 7, 2023

## MEETINGS & CONFERENCES

The scientific information listed below may be dated. For the latest information, see <https://www.ams.org/amsmtgs/sectional.html>.

### Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <https://www.ams.org/cgi-bin/abstracts/abstract.pl>.

*Advances by Scholars in the Pacific Math Alliance* (Code: SS 22A), **Andrea Arauza Rivera**, California State University, East Bay, **Mario Banuelos**, California State University, Fresno, and **Jessica De Silva**, California State University, Stanislaus.

*Advances in Functional Analysis and Operator Theory* (Code: SS 6A), **Michel L. Lapidus**, University of California, Riverside, **Marat V. Markin**, California State University, Fresno, and **Igor Nikolaev**, St. John's University.

*Algebraic Structures in Knot Theory* (Code: SS 4A), **Carmen Caprau**, California State University, Fresno, and **Sam Nelson**, Claremont McKenna College.

*Algorithms in the study of hyperbolic 3-manifolds* (Code: SS 26A), **Robert Haraway, III** and **Maria Trnkova**, University of California, Davis.

*Analysis of Fractional Differential and Difference Equations with its Application* (Code: SS 20A), **Bhuvaneswari Sambandham**, Dixie State University, and **Aghalaya S. Vatsala**, University of Louisiana at Lafayette.

*Artin-Schelter regular algebras and related topics* (Code: SS 27A), **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington.

*Combinatorics Arising from Representations (associated with the Invited Address by Sami Assaf)* (Code: SS 16A), **Sami Assaf**, University of Southern California, **Nicolle Gonzalez**, University of California, Los Angeles, and **Brendan Pawloski**, University of Southern California.

*Complexity in Low-Dimensional Topology* (Code: SS 14A), **Jennifer Schultens**, University of California, Davis, and **Eric Sedgwick**, DePaul University.

*Data Analysis and Predictive Modeling* (Code: SS 8A), **Earvin Balderama**, California State University, Fresno, and **Adriano Zambom**, California State University, Northridge.

*Inverse Problems* (Code: SS 5A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

*Math Circle Games and Puzzles that Teach Deep Mathematics* (Code: SS 13A), **Maria Nogin** and **Agnes Tuska**, California State University, Fresno.

*Mathematical Biology: Confronting Models with Data* (Code: SS 21A), **Erica Rutter**, University of California, Merced.

*Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova)* (Code: SS 1A), **Natalia Komarova** and **Jesse Kreger**, University of California, Irvine.

*Methods in Non-Semisimple Representation Categories* (Code: SS 11A), **Eric Friedlander**, University of Southern California, Los Angeles, **Julia Pevtsova**, University of Washington, Seattle, and **Paul Sobaje**, Georgia Southern University, Statesboro.

*Recent Advances in Mathematical Biology, Ecology, Epidemiology, and Evolution* (Code: SS 10A), **Lale Asik**, Texas Tech University, **Khanh Phuong Nguyen**, University of Houston, and **Angela Peace**, Texas Tech University.

*Research in Mathematics by Early Career Graduate Students* (Code: SS 7A), **Doreen De Leon**, **Marat Markin**, and **Khang Tran**, California State University, Fresno.

*Scientific Computing* (Code: SS 19A), **Changho Kim**, University of California, Merced, and **Roummel Marcia**.

*The use of computational tools and new augmented methods in networked collective problem solving* (Code: SS 18A), **Mario Banuelos**, California State University, Fresno, **Andrew G. Benedek**, Research Centre for the Humanities, Hungary, and **Agnes Tuska**, California State University, Fresno.

*Women in Mathematics* (Code: SS 12A), **Doreen De Leon**, **Katherine Kelm**, and **Oscar Vega**, California State University, Fresno.

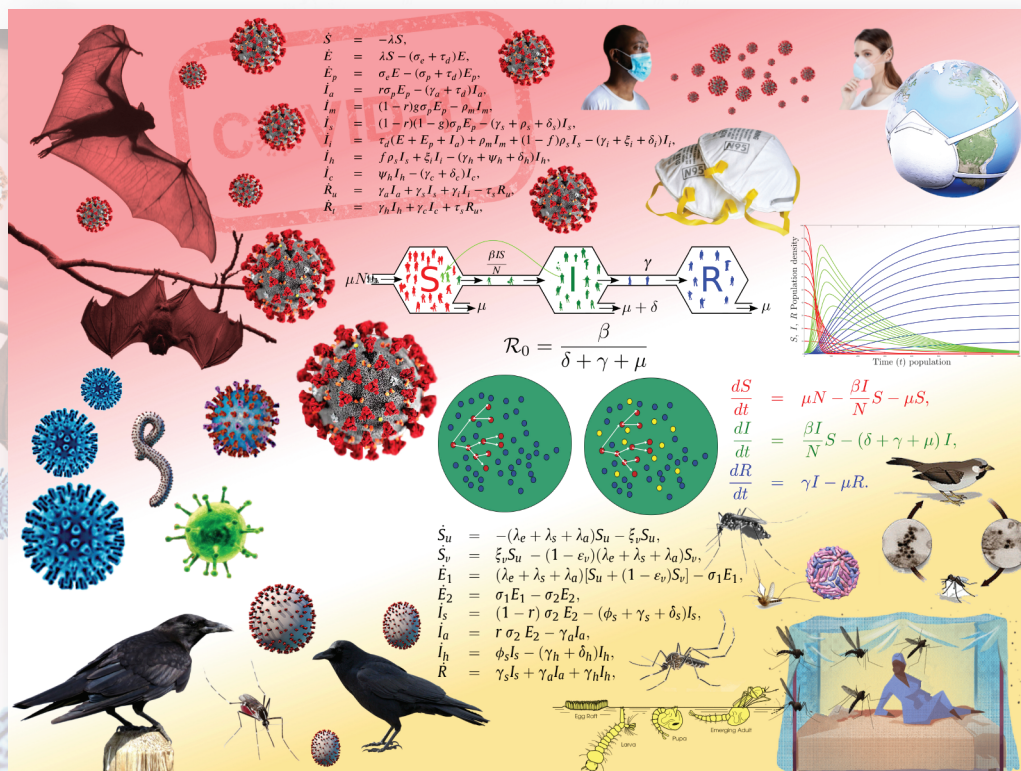
*Zero Distribution of Entire Functions* (Code: SS 9A), **Khang Tran** and **Tamás Forgács**, California State University, Fresno.



The American Mathematical Society Presents  
The 2021 AMS Einstein Public Lecture in Mathematics

**ABBA GUMEL**  
ARIZONA STATE UNIVERSITY

# Mathematics of Infectious Diseases



Credit: Rhonda Olson (Marketing and Communications, Arizona State University)

**SATURDAY,  
MARCH 20**  
1:30–2:30 PM EDT  
(10:30–11:30 AM PDT)

In conjunction with the AMS Spring Eastern Sectional Meeting, the lecture takes place online.

Mathematics has historically been used as a vital tool for providing insight and understanding on the mechanisms of the spread, control, and mitigation of emerging and re-emerging infectious diseases, dating back to the pioneering works of Daniel Bernoulli (on modeling the potential impact of a smallpox vaccine) in the 1760s and the compartmental modeling frameworks of the likes of William Kermack, Anderson McKendrick, and Sir Ronald Ross in the early 1900s. This lecture will address some of the mathematical techniques and theories used to formulate, parametrize, and analyze mathematical models for the transmission dynamics and control of infectious diseases.

In this public lecture, Gumel will emphasize discussion on infectious diseases that continue to inflict major public health and socio-economic challenges to humankind, including the ongoing novel 2019 coronavirus pandemic.

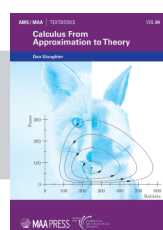
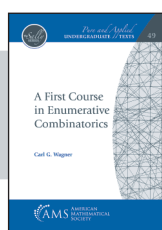
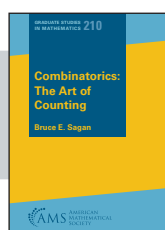
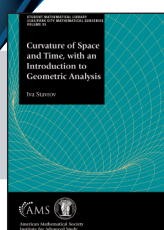
Event details:  
<http://www.ams.org/meet-einstein-lect>


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 = Textbook

## Curvature of Space and Time, with an Introduction to Geometric Analysis

Iva Stavrov, *Lewis & Clark College, Portland, OR*

This book introduces advanced undergraduates to Riemannian geometry and mathematical general relativity. The overall strategy of the book is to explain the concept of curvature via the Jacobi equation which, through discussion of tidal forces, further helps motivate the Einstein field equations.

**Student Mathematical Library**, Volume 93; 2020; 243 pages; Softcover; ISBN: 978-1-4704-5628-3; List US\$59; All individuals US\$47.20; Order code STML/93

## Combinatorics: The Art of Counting

Bruce E. Sagan, *Michigan State University, East Lansing, MI*

This book is a gentle introduction to the enumerative part of combinatorics suitable for study at the advanced undergraduate or beginning graduate level. In addition to covering all the standard techniques for counting combinatorial objects, the text contains material from the research literature which has never before appeared in print.

**Graduate Studies in Mathematics**, Volume 210; 2020; 304 pages; Softcover; ISBN: 978-1-4704-6032-7; List US\$89; AMS members US\$71.20; MAA members US\$80.10; Order code GSM/210

## A First Course in Enumerative Combinatorics

Carl G. Wagner, *University of Tennessee, Knoxville, TN*

*A First Course in Enumerative Combinatorics* provides an introduction to the fundamentals of enumeration for advanced undergraduates and beginning graduate students in the mathematical sciences.

**Pure and Applied Undergraduate Texts**, Volume 49; 2020; 272 pages; Softcover; ISBN: 978-1-4704-5995-6; List US\$85; AMS members US\$68; MAA members US\$76.50; Order code AMSTEXT/49

## Calculus From Approximation to Theory

Dan Sloughter, *Furman University, Greenville, SC*

*Calculus from Approximation to Theory* takes a fresh and innovative look at the teaching and learning of calculus. One way to describe calculus might be to say it is a suite of techniques that approximate curved things by flat things and through a limiting process applied to those approximations arrive at an exact answer. Standard approaches to calculus focus on that limiting process as the heart of the matter. This text places its emphasis on the approximating processes and thus illuminates the motivating ideas and makes clearer the scientific usefulness, indeed centrality, of the subject while paying careful attention to the theoretical foundations.

**AMS/MAA Textbooks**, Volume 64; 2020; 571 pages; Softcover; ISBN: 978-1-4704-5588-0; List US\$99; Individual member US\$74.25; MAA members US\$74.25; Order code TEXT/64

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