The Legacy of Dick Askey (1933–2019)

Howard S. Cohl, Mourad E. H. Ismail, and Hung-Hsi Wu

1. History, by Paul Terwilliger

Richard Allen (Dick) Askey, who devoted his life's work to mathematics and mathematics education, died on October 9, 2019 at the age of 86.

Dick was born on June 4, 1933, in St. Louis, Missouri. In 1955, he earned a BA from Washington University in St. Louis, and in 1956 an MA from Harvard. He then pursued a doctorate at Princeton, and finished his course work in 1958. During 1958–1961, while completing his thesis, Dick was an instructor back at Washington University. In 1961, he earned a PhD from Princeton; his advisor was Salomon Bochner. After a two-year instructorship at the University of Chicago, Dick was appointed assistant professor in the department of mathematics at the University of Wisconsin, where he served for the remainder of his career. He became associate professor (1965–1968), professor (1968–1986), Gábor Szegő Professor (1986–1995), John Bascom Professor (1995–2003), and professor emeritus (2003–2019).

Dick advised 14 PhD students and five postdoctoral fellows, all of whom thrived under his guidance. Later in this article, we will hear from his former PhD student Dennis Stanton (see §2.5), and his informal student/mentee Tom Koornwinder (see §2.8), who tell of their deep bond with Dick and his profound influence over them. Epic work with his former PhD student James Wilson is also featured. Dick's later PhD students Shaun Cooper and Warren Johnson, and his later postdoc Frank Garvan, shared Dick's interest in q-series, number theory, and the mathematics of Srinivasa Ramanujan. Mourad Ismail (see §2.6) was one of Dick's postdocs, and they developed a profound collaboration that continued throughout their careers. Dick also had a strong collaboration with George Gasper (see §2.7), who spent a year (1967–1968) at the University of Wisconsin as a visiting lecturer. Early in his career, Dick's interest in the mathematics of Ramanujan brought him into contact with George Andrews (see §2.1), and they became lifelong collaborators. A shared interest in Ramanujan also brought Dick into contact with Bruce Berndt (see

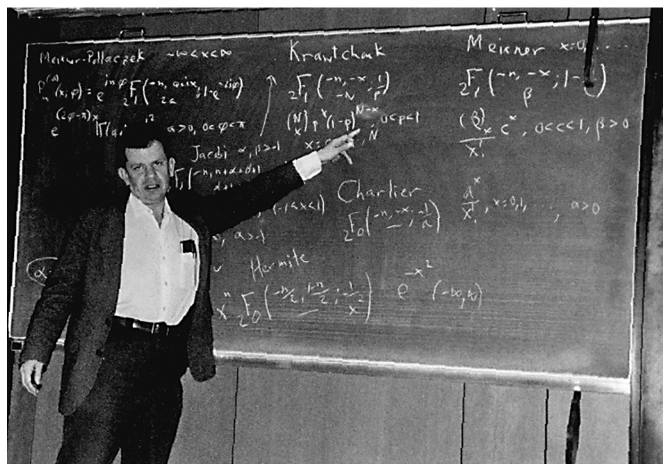

Figure 1. Dick Askey, Oberwolfach, March 1983.

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§2.3) who greatly benefited from Dick's guidance for over four decades. Dick had collaborations with a large number of other mathematicians. For lack of space, we mention only a few: Mizan Rahman, Ranjan Roy, Paul Nevai, Deborah Haimo, Samuel Karlin, Natig Atakishiyev, Sergeĭ Suslov and Stephen Wainger.

Figure 2. Dick Askey, University of Minnesota, March 1988.

Dick was a preeminent mathematician of his generation, as the following awards and distinctions suggest. Dick was a Guggenheim Fellow (1969–1970); invited speaker at the International Congress of Mathematicians (1983); Vice President of the American Mathematical Society (AMS) (1986–1987); Honorary Fellow of the Indian Academy of Sciences (1988); Fellow of the American Academy of Arts and Sciences (1993); Member of the National Academy of Sciences (1999); Fellow of the Society for Industrial and Applied Mathematics (2009); and Fellow of the AMS (2012). Dick received an honorary doctorate from SASTRA University in Kumbakonam, India (2012), and a Distinguished Mathematics Educator Award from the Wisconsin Mathematics Council (2013). Dick won a Lifetime Achievement Award at the International Symposium on Orthogonal Polynomials, Special Functions and Applications in Hagenberg, Austria, on July 24, 2019.

Dick's primary research interest was Special Functions; many of these are extensions of the functions on your scientific calculator. When asked why do research on special functions, Dick emphasized that one studies special functions not for their own sake, but because they are useful. Roughly speaking, special functions are the functions that have acquired a name after repeated use.

It took some courage for Dick to start his research career on the topic of special functions. During the period 1950–1970, it was widely believed that the existence of large, fast computing machines would minimize the value of special functions. This belief was wrong. Taking a broad view of the relationships between special functions and the rest of mathematics and physics, Dick and a small group of like-minded researchers resurrected the field and attracted many young, talented, and ambitious mathematicians to the area.

Dick was an author or coauthor of over 180 research articles. We mention two that had a profound influence. An inequality in his 1976 paper coauthored with George Gasper [AG76] was used by Louis de Branges to prove the Bieberbach conjecture in 1985. In a *Memoir* published by the AMS in 1985 [AW85], Dick and his former doctoral student, James Wilson, introduced the Askey–Wilson polynomials, which have become indispensable in combinatorics, probability, representation theory, and mathematical physics. The importance of these polynomials is suggested by the fact that the previously known families of hypergeometric and basic hypergeometric orthogonal polynomials, 43 families in total, are all special or limiting cases of the Askey–Wilson polynomials. The Askey–Wilson polynomials are viewed by many mathematicians as a sublime gift to their community.

Dick wrote two books, and he edited four more. His book, *Orthogonal Polynomials and Special Functions* [Ask75] focused on classical orthogonal polynomials, related questions about positivity, and inequalities. His book, *Special Functions* [AAR99], coauthored with George Andrews and Ranjan Roy, has become the standard text on the subject.

The elegance of Dick's mathematical writing brings to mind the following quotation of Sun Tzu in *The Art of War: The supreme art of war is to subdue the enemy without fighting*. Many of Dick's proofs have this quality.

Figure 3. First DLMF Editorial Board meeting, NIST, January 2000.

Dick lent his expertise to several projects that produced reference materials on special functions. In one project the National Institute of Standards and Technology

(NIST) created the Digital Library of Mathematical Functions (DLMF); see [NIST:DLMF]. The DLMF is the 21st century successor to a classic text by Abramowitz and Stegun called the *Handbook of Mathematical Functions* (1964, MR167642). Dick served as an associate editor for the DLMF project. In this capacity, Dick gave advice on all aspects of the project, from its conception around 1995 to the initial release in 2010. In addition to his advising work, Dick coauthored the chapters on Algebraic and Analytic Methods, the Gamma Function, and Generalized Hypergeometric Functions & Meijer *G*-Function. In another effort, Dick was involved in updating the Bateman Manuscript Project. The result is the Askey–Bateman Project in the Encyclopedia of Special Functions, edited by Mourad Ismail and Walter Van Assche, and published by Cambridge University Press. Volumes 1 and 2 have recently appeared, and they cover univariate/multivariate orthogonal polynomials along with some multivariate special functions that Dick was interested in (see §§2.2, 2.4, 2.5, 2.6, 2.8).

Dick was passionate about the history of mathematics, and he emphasized this topic in his lecturing and writing. Dick helped to edit *A Century of Mathematics in America* [Ask89]. Dick never tired of bringing to the world's attention the genius of the mathematician Srinivasa Ramanujan (1887–1920). As part of this effort, in 1983 Dick commissioned the sculptor Paul Granlund to create a bronze bust of Ramanujan. Four copies were originally made, one of which is now in London at the headquarters of the Royal Society.

Early in his career, Dick made a commitment to improving K–12 mathematics education (see §§3.1, 3.2, 3.3). He wrote several dozen articles on this subject; for instance "Good intentions are not enough" [Ask01]. Dick was an advocate for the Singapore primary mathematics textbooks, and helped to create some of their Teacher Guides. Dick served on the AMS Education Committee (1998–2001) and the US National Committee for Mathematics (1999–2004). At the state level Dick consistently engaged in reviews and discussions concerning Wisconsin state standards, assessment documents, and professional resources. Dick's mathematical credentials and common sense made him an effective critic of various fads in school mathematics education. Concerning his position, we will give him the last word:

> Like a stool which needs three legs to be stable, mathematics education needs three components: good problems, with many of them being multi-step ones, a lot of technical skill, and then a broader view which contains the abstract nature of mathematics and proofs. One does not get all of these at once, but a good mathematics program has them as goals and makes incremental steps towards them at all levels.

Figure 4. Persi Diaconis and Dick Askey, 80th Birthday Conference, Madison, Wisconsin, December 2013.

2. Askey's Contribution to Mathematics Research

Dick's lifelong devotion to the study of special functions is summed up by this quote from Persi Diaconis:

> Dear Friend Dick, You are one of my heroes. Not just because of your wonderful work but because of your bravery under fire. As we both know, there was a long time when our math world just didn't know what to think about orthogonal polynomials: was it applied math, a corner of representation theory, or numerical analysis? Just what was it?? Anyway, it got "no respect." You kept soldiering on and beat the ... at their own game.

In order to motivate and describe some of Askey's deep mathematical contributions, it will be helpful to delve into the subject of q-analysis and q-series. As we will see, some q-series serve as generating functions for statistics on partitions and also extend classical sums and integrals. The importance of looking at q-series in a new and modern way started with Andrews and Askey's work on orthogonal polynomials in the 1970s and suddenly many mathematicians quickly joined in. The spread of interest in q-series was so fast that some called it "the q-disease" because it was "highly contagious." We are glad to see that the q-disease is here to stay.

As Askey was one of the first to recognize, the subject of q-series started with Fermat's evaluation of $\int_0^a f(x)\,\mathrm{d}x$, where he replaced it by the infinite Riemann sum using the mesh points aq^n, that is,

$$\int_0^a f(x)\,\mathrm{d}_q x = \sum_{n=0}^{\infty} (aq^n - aq^{n+1}) f(aq^n),$$

which is now referred to as a q-integral. When $f(x) = x^m$ the sum is a geometric progression and equals $a^{m+1}(1-q)/(1-q^{m+1})$, which tends to $a^{m+1}/(m+1)$ as $q \to 1$. This suggests considering the $(1-q^n)/(1-q)$, $n \in \mathbb{N}$, as analogues of the natural numbers and we are naturally led to the q-shifted factorials

$$(a;q)_0 := 1, \ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ n \in \mathbb{N} \cup \{\infty\}, \quad (1)$$

and the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

respectively. We assume $0 < q < 1$. One can then write $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$, which then defines $(a;q)_n$ for $n \in \mathbb{C}$.

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has many combinatorial interpretations. If q is a prime power, it is the number of k-dimensional subspaces of an n-dimensional space over a field with q elements. It is a polynomial in q which is unimodal. It is also a generating function with power series variable q of the number of partitions of integers which are of length k and have at most $n-k$ parts. Many partition-theoretic identities have an analytic equivalence of the type "an infinite series involving q-shifted factorials equals an infinite product." For example the analytic forms of the famous Rogers–Ramanujan identities are:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \\ \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \end{aligned} \quad (2)$$

To see the connection with partitions the reader should observe that the powers of q on the right-hand side of the first equation are sums of terms of the form $5k+1$ or $5k+4$, that is, partitions whose parts are $\equiv 1, 4 \pmod 5$. Although the partition-theoretic interpretation of the left-hand side is not obvious, it can be thought of as the number of partitions into parts where any two parts differ by at least 2.

A basic hypergeometric function is a power series whose coefficients are quotients of products of q-shifted factorials. The subject of q-series studies evaluations of certain q-series as quotients of infinite products, transformations connecting different q-series, and orthogonal polynomials which arise as q-series.

There are two divided difference operators associated with q-analysis. The first,

$$(D_q f)(x) = \frac{f(x) - f(qx)}{x - qx},$$

has been used since the 19th century. The second is the Askey–Wilson operator $\mathcal{D}_q$. We write x as $(z+1/z)/2$ and denote $f(x)$ by $\breve{f}(z)$. Then

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})(z - 1/z)/2}.$$

We note that if we think of the numerator as Δf, then the denominator will be exactly Δx.

There are many q-deformed physical models in which the Hamiltonian is a second-order linear operator in $\mathcal{D}_q$. One case is the $U_q(sl_2(\mathbb{C}))$-quantum invariant Heisenberg XXZ model of spin 1/2 of a size $2N$ with the open (Dirichlet) boundary condition. The Bethe ansatz equations of this model are

$$\left(\frac{\sin\left(\lambda_k + \frac{1}{2}\eta\right)}{\sin\left(\lambda_k - \frac{1}{2}\eta\right)} \right)^{2N} = \prod_{j \neq k, j=1}^{n} \frac{\sin\left(\lambda_k + \lambda_j + \eta\right) \sin\left(\lambda_k - \lambda_j + \eta\right)}{\sin\left(\lambda_k + \lambda_j - \eta\right) \sin\left(\lambda_k - \lambda_j - \eta\right)},$$

where $1 \leq k \leq n$. The solution of this system is identified with the zeros of a polynomial solution to a second-order equation in $\mathcal{D}_q$ using a major modification of a technique which Stieltjes initiated to solve a one-dimensional electrostatic equilibrium problem. For details see §16.5 and §3.5 in Ismail (2009, MR2542683).

Although many q-identities become hypergeometric identities as $q \to 1$, there are many q-results which exist only when $q \neq 1$. For example, the analytic form of the Rogers–Ramanujan identities (2) are genuine q-series results and there is no $q \to 1$ limit.

2.1. Askey's monographs, by George E. Andrews.

Dick edited many proceedings of conferences and collaborated with Madge Goldman and others on mathematics books for elementary school. He wrote two books for research mathematicians: *Orthogonal Polynomials and Special Functions* [Ask75] (stemming from his 1974 Conference Board of the Mathematical Sciences (CBMS) lecture series at Virginia Tech), and *Special Functions* [AAR99] (joint with Ranjan Roy and George Andrews). Dick's editing and comments on Gábor Szegő's collected papers in three separate volumes [Sze82] has been immensely valuable.

Orthogonal Polynomials and Special Functions is an elegant introduction to Dick's early achievements, but more importantly, also to the topics that he viewed as most significant. He and I had been in correspondence since 1970, and he invited me as one of the participants in the CBMS

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conference. It was about then that our two quite different fields of interest converged. I recall saying to Dick that before I met him, I didn't know an orthogonal polynomial from a perpendicular one. He responded, "I'd hate to tell you what I thought a partition was!" The confluence of our interests led Dick to invite me to Wisconsin for the 1975–1976 academic year. Thus began the long path leading to the book, *Special Functions*. We decided to run a seminar on our joint interests. As a topic, we chose Wolfgang Hahn's paper, *Ueber Orthogonalpolynome, die q-Differenzengleichungen genuegen* (1949, MR0030647). Here was the world of q and orthogonal polynomials tied neatly together. We decided we should definitely write a book. I set to work to produce three chapters, and Dick was collecting notes from our seminar to supplement his contribution. Each of us got enmeshed in other projects after the glorious year of 1975–1976. Cambridge University Press continued to nudge us over the decades, and we intended, over and over again, to put everything together. As time wore on, it seemed this book would never happen. The late Ranjan Roy was entirely responsible for rescuing it from oblivion. Ranjan attended subsequent lectures by Dick and realized that the long-awaited book would never come about unless someone took over the onerous task of taking the rough notes and ideas from both of us and putting them into a coherent and readable text. It was clear from Ranjan's other singly authored books that he was an excellent mathematical expositor, and hence he was the perfect choice for our book. Both Dick and I were grateful beyond words that Ranjan was able to turn our 1976 dream into a 1999 reality.

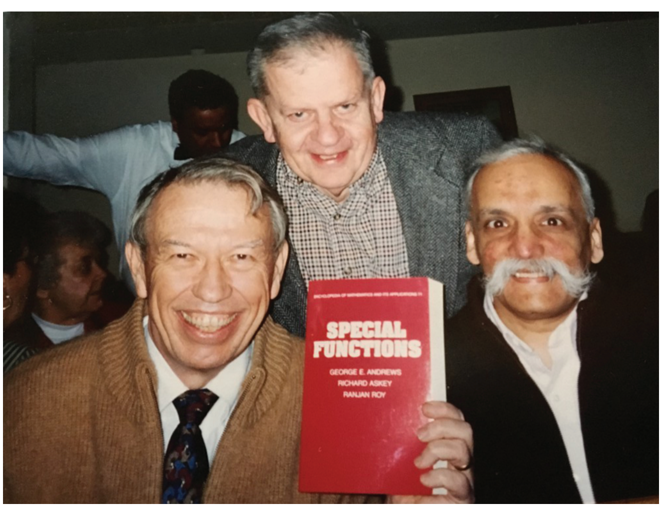

Figure 5. Andrews, Askey, and Roy, Baltimore, 2003.

2.2. Askey and Ramanujan, by Krishnaswami Alladi.

Richard Askey was widely acknowledged as a leader in the field of special functions. He was also a major figure in the world of Ramanujan, for he was instrumental, along with George Andrews and Bruce Berndt, in making the mathematical world aware of the wide-ranging and deep contributions of Ramanujan. Indeed, he, George Andrews, and Bruce Berndt have been jocularly referred to as "the gang of three" in the Ramanujan world. Here I shall share some personal recollections, but in doing so, I shall focus on Askey's role in educating us about the remarkable contributions of Ramanujan pertaining to special functions, and in his efforts to foster the legacy of Ramanujan.

I first met Askey at the Joint Summer Meeting of the AMS and MAA at the University of Michigan, Ann Arbor, in August 1980. At that time, I was a Hildebrandt Research Assistant Professor there just after having completed my PhD. He was giving the J. Sutherland Frame Lecture on "Ramanujan and some extensions of the gamma and beta functions" which I attended. I was charmed by his conversational, yet engaging, lecturing style. A vast panorama of the area of special functions unfolded in his lecture, revealing his encyclopedic knowledge of the subject. He discussed some of Ramanujan's startling discoveries and exhorted everyone in the audience to study the work of the Indian genius. In particular, he emphasized Ramanujan's important q-analog of the beta integral. He also discussed Selberg's multidimensional extension of the beta integral, and spoke about his conjectured q-analog of the Selberg integral, which provided an extension of the integrals of both Ramanujan and Selberg. He concluded his lecture by pointing out that in physics there are incredible formulas in several variables that are being analyzed and that a genius like Ramanujan would be of invaluable help. As Askey put it:

> The great age of formulae may be over, but the age of great formulae is not!

Askey's paper under the same title as the lecture had just appeared in the May 1980 issue of the *American Mathematical Monthly*. I was working in analytic number theory at that time, but before the end of that decade, owing to the lectures of Andrews, Askey, and Berndt that I heard at the Ramanujan Centennial in India in 1987, I entered the world of q, or as Askey would say, I was smitten with the q-disease!

Around the time of that Summer Meeting, Askey sent letters to academicians and persons with an interest in fostering scientific legacies to contribute towards the

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creation of busts of Ramanujan. I received one such letter. Responding to a plea from Janaki, Ramanujan's 80-year-old widow who was living in poverty in Madras, India, Askey had contacted the famous American sculptor Paul Granlund and commissioned him to produce busts based on the famous passport photograph of Ramanujan. The response to Askey's letter of request was overwhelming, and so it was possible for Granlund to produce ten bronze busts, and these were ready by 1983, well in time for the Ramanujan Centennial in 1987.

The Ramanujan Centennial was an occasion when mathematicians around the world gathered in India to pay homage to the Indian genius, and take stock of the influence his work has had and the impact it might have in the future. Askey was one of the stars of the centennial celebrations. There were several conferences in India during December 1987–January 1988, and Askey was a speaker in almost all of them. I organized a one-day session during a conference at Anna University, Madras, in December 1987. He graciously accepted our invitation to inaugurate that conference and to speak in my session. Mrs. Janaki Ramanujan was present at the inauguration, and she thanked Askey profusely for his effort in getting the busts of Ramanujan made. After the inauguration, Askey delivered a magnificent lecture on "Beta integrals before and after Ramanujan" in my session. We were also honored to have him give a public lecture entitled "Thoughts on Ramanujan" at our family home in Madras under the auspices of the Alladi Foundation that my father, the late Prof. Alladi Ramakrishnan, had created in memory of my grandfather Sir Alladi Krishnaswami Iyer.

With my research focused on the theory of partitions and q-series since 1990, we have had a series of conferences at the University of Florida emphasizing this area. Professor Askey has visited Gainesville several times both as a lead speaker at these conferences, and for History Lectures and talks on mathematics education during the regular academic year. I have enjoyed every one of his lectures in Gainesville and at meetings elsewhere. I want to share with you one interesting episode.

In 1995, there was a two-week meeting on special functions, q-series, and related topics at the Fields Institute in Toronto. The first week was an instructional workshop, and the second week was a research conference. I attended the second week. The great I. M. Gel'fand was scheduled to be the Opening Speaker for the research conference. I was looking forward to Gel'fand's lecture since I had heard so much about the Gel'fand Seminar he had conducted in Moscow, and how he would dominate the seminar and cut people down to size. It turned out that Gel'fand could not come to Toronto due to ill health (he was 82 years old). So Askey got up and said that he was the one who had invited Gel'fand, and if the person you had invited is unable to come, then you should give a talk in his place. So in his imitable style, Askey gave a masterly lecture on special functions that I thoroughly enjoyed.

His insight and critical comments have been immensely useful to me in various ways. Starting from the Ramanujan Centennial, I wrote articles annually for Ramanujan's birthday for *The Hindu*, India's National Newspaper, comparing Ramanujan's work with that of various mathematical luminaries in history. I benefited from Askey's comments and (constructive) criticism in preparing these articles. A collection of these articles appeared in a book that I published with Springer in 2012 for Ramanujan's 125th birthday.

By the time the Ramanujan 125 celebrations came around in 2012, I was firmly entrenched in the Ramanujan World, and so was involved with the celebrations in various ways. In particular, owing to my strong association with SASTRA University, I organized a conference at their campus in Kumbakonam, Ramanujan's hometown. We felt that Askey, Andrews, and Berndt had to be recognized in a special way in Ramanujan's hometown for all they had done to help us understand the plethora of identities that Ramanujan had discovered. So *The Trinity* (Askey, Andrews, and Berndt)—as I like to refer to them in comparison with the three main Gods, Brahma, Vishnu, and Shiva of the Hindu religion (!)—were awarded Honorary Doctorates by SASTRA University in a colorful ceremony at the start of which they entered the auditorium with traditional South Indian Carnatic music being played on the Nadaswaram, a powerful wind instrument. Askey enjoyed the ceremony but felt that the music was too loud; in fact, that is how the Nadaswaram is, since it is played in festivals attended by a thousand people or more!

There is much that can be said of Dick Askey. But I will conclude by emphasizing that, in spite of his eminence, he was a very friendly and helpful person. It is rare to find eminence combined with humanity, and Askey had this precious combination which has been beneficial to so many of us. In particular, we owe a lot to him for helping us understand some aspects of Ramanujan's fundamental work on special functions, and for his efforts in fostering the legacy of the Indian genius.

2.3. Askey and Ramanujan's notebooks, by Bruce C. Berndt. Although I was a graduate student at Wisconsin, and Richard Askey was a close friend and strong supporter of my work for over 50 years, one of my life's biggest regrets is that I never took a course from him. My first

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experiences with number theory came in the spring semester of my third year and fall semester of my fourth year at Wisconsin when I enrolled in courses in modular forms, taught, respectively, by Rod Smart and Marvin Knopp. Perhaps surprisingly, modular forms led me to a doctoral dissertation in which Bessel functions played a leading role. My association with Dick and his advocacy of my research for the next 53 years began at this time.

While on my first sabbatical leave at the Institute for Advanced Study in February 1974, I discovered that theorems that I had proved on Eisenstein series at the Institute enabled me to prove some formulas from Ramanujan's notebooks. Starting in May 1977, I began devoting all of my research efforts for the next forty years to proving the claims made by Ramanujan in his (earlier) notebooks and later, with George Andrews, in his lost notebook. It is unfortunate that there is not sufficient space here in which to express my appreciation and indebtedness to my many doctoral students who enormously aided me in this long endeavor. By far, the most important and strongest advocate of not only my work, but also that of my students, during these several decades was Dick Askey.

During my efforts to find proofs of Ramanujan's claims in his notebooks [Ram57], Askey provided many insights, references, and proofs. In my five volumes (abbreviated by Parts I–V), devoted to Ramanujan's claims, I referred to Askey's help a total of 31 times, more than any other mathematician. (For brevity of the present exposition, complete references to Askey's several relevant papers can be found in [Ber85].) We now provide a sampling of Askey's contributions to our editing of Ramanujan's notebooks [Ram57].

Already in Part I, published in 1985, Askey read in detail most of the chapters, and, in particular, supplied many important references. In generalizing a result of Ramanujan [Ber85, p. 302], Askey showed that a more general integral

$$\int_0^\infty \frac{t^{n-1}(-at;q)_\infty}{(-t;q)_\infty}\,\mathrm{d}t, \qquad |a|<q^n,$$

using (1), $n=x$, $a=q^{x+y}$, is a q-analogue of the beta integral (3).

As most friends of Askey are aware, he had a prodigious knowledge of the literature, especially that from the 19th century and earlier. The chapters in Part II are significantly richer because of Askey's historical observations. In Chapter 11, which features hypergeometric series, Askey's influence is most pronounced, as he supplied several proofs and observations. For example, in Entry 29(ii) of Chapter 11 in his second notebook [Ram57, pp. 86–87], Ramanujan offered an identity for hypergeometric functions, namely,

$${}_3F_2\left[\begin{matrix}-2\alpha,-2\beta,-\gamma\\ -\alpha-\beta+\frac12,\delta\end{matrix};1\right]={}_4F_3\left[\begin{matrix}-\alpha,-\beta,-\gamma,\gamma+\delta\\ -\alpha-\beta+\frac12,\frac12\delta,\frac12(\delta+1)\end{matrix};1\right],$$

where α,β, or γ is a nonnegative integer. Askey and Wilson showed that this identity leads to an orthogonal set of polynomials on $(-\infty,\infty)$ with respect to a weight function involving a product of gamma functions.

The Rogers–Ramanujan identities (2) appear in Part III [Ber85, pp. 77–79], where a lengthy discussion of all known proofs up to 1991, including a new proof from Askey, can be found. On page 284 in his second notebook, Ramanujan writes,

> The difference between $\dfrac{\Gamma(\beta-m+1)}{\Gamma(\alpha+\beta-m+1)}$ and
> $$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}+\frac{\alpha m}{1!}\frac{\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+n+2)}$$
> $$+\frac{\alpha(\alpha+1)}{2!}m(m+2n+1)\frac{\Gamma(\beta+2n+1)}{\Gamma(\alpha+\beta+2n+3)}$$
> $$+\cdots$$

but he does not tell us what the difference is. We might guess that it is 0, and it is in some cases. Askey provided the answer, which you can find in [Ber85, Part IV, pp. 344–346].

Two of the most intriguing entries in the 100 pages of unorganized material in Ramanujan's second notebook pertain to Gaussian quadrature. On pages 349 and 352 of [Ram57, Vol. 1] [Ber85, Part V, pp. 549–560], Ramanujan provides theorems on Gaussian quadrature with respect to a discrete measure in which orthogonal polynomials play a central role. We provide a portion of one example. Let

$$S(x,n):=\sum_{k=0}^{n-1}\varphi(x-n+1+2k).$$

(Ramanujan did not provide any hypotheses for φ.) Ramanujan then gives four successive approximations to $S(x,n)$, the first of which is simply $\varphi(x)$, and the second is

$$\frac12\left\{\varphi\left(x+\sqrt{\frac{n^2-1}{3}}\right)+\varphi\left(x-\sqrt{\frac{n^2-1}{3}}\right)\right\}.$$

Askey pointed out that an application leads to Hahn polynomials, which were introduced by Chebyshev in 1875 and are constant multiples of

$${}_3F_2\left[\begin{matrix}-k,k+\alpha+\beta+1,-t\\ \alpha+1,-N\end{matrix};1\right].$$

The proofs are due to Askey and do not apparently appear elsewhere. As Askey pointed out, it was surprising to learn of these theorems, because nowhere else in Ramanujan's work is there any indication that he knew about Gaussian quadrature and orthogonal polynomials.

Although he would not acknowledge such an acclamation during his time, Askey was generally recognized as the world's leading authority on q-series, which flow abundantly throughout Ramanujan's lost notebook [Ram88]. The advice and comments on q-series that Askey kindly gave to George Andrews and this writer in preparing our five volumes on Ramanujan's lost notebook [AB05] cannot be overemphasized.

Ramanujan and Dick Askey would have immensely enjoyed having long conversations with each other.

2.4. Askey and algebra, by Luc Vinet and Alexei Zhedanov. Even though an algebra now bears his name, Askey himself was not very involved with algebraic studies. In fact, he often told us that the nomenclature to which we shall refer should be changed so as to not mention him. It is however a striking manifestation of his legacy that his work has led to constructs which are becoming more and more important in Algebra and Mathematical Physics.

Empirically, it is observed that the possibility of solving physical models rests on the underlying presence of symmetries that can be somewhat hidden. Their mathematical description has led to the identification of structures such as Lie algebras, superalgebras, quantum algebras, and their representation theory. The special functions that appear in the solutions must thus offer a lead as to what are the algebraic entities poised to account for the symmetries of the systems in question. This relates to the long tradition championed by Wigner, Gel'fand, Vilenkin among others of interpreting special functions algebraically.

What is then the algebra encoded in $p_n(x; a, b, c, d|q)$, the Askey–Wilson polynomials [AW85], and their corresponding finite set, the q-Racah polynomials? The answer (to which we have contributed) is rooted in the bispectral properties of the Askey–Wilson polynomials which are eigenfunctions of a q-difference operator $\mathcal{L}_q^{(a,b,c,d)}$ in addition to satisfying, as required for orthogonal polynomials, a three-term recurrence relation where the variable x can be viewed as the eigenvalue of an operator acting on the discrete degree variable. Focusing on these two operators and setting $K_0 = \mathcal{L}_q^{(a,b,c,d)} + (1 + q^{-1}abcd)$ and $K_1 = x$, the following relations are found:

$$[K_0, K_1]_q = K_2,$$
$$[K_1, K_2]_q = \mu K_1 + \nu_0 K_0 + \rho_0,$$
$$[K_2, K_0]_q = \mu K_0 + \nu_1 K_1 + \rho_1,$$

where $[A, B]_q := q^{1/2}AB - q^{-1/2}BA$ and μ, ν, and ρ are related to the parameters a, b, c, d of the polynomials p_n. Since the realization is not affected by the truncation, the algebra is also the one associated to the q-Racah polynomials. Focusing on generators and relations, this can be taken to define the *Askey–Wilson algebra* abstractly. It is remarkable that this algebra encapsulates the properties of the Askey–Wilson polynomials which can indeed be obtained from the construction of representations. An interpretation of the polynomials as overlaps between the eigenbases of K_0 and K_1 follows. This relates to the theory of Leonard pairs and to P- and Q-polynomial association schemes, both already mentioned (see §2.5).

Thus, whenever the Askey–Wilson or q-Racah polynomials are present, the Askey–Wilson algebra is lurking. Now it is known that the q-Racah polynomials are basically the $6j$-coefficients of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}(2))$. Such coefficients arise in the recouplings of three irreducible representations. This suggests, as is the case, that the Askey–Wilson algebra occurs as the centralizer of the diagonal action in representations of the three-fold product of $\mathcal{U}_q(\mathfrak{sl}(2))$. Here the generators are realized as the intermediate Casimir elements and the parameters are related to the values of the initial and total Casimir operators. The Askey–Wilson algebra is ubiquitous: it is a coideal subalgebra of $\mathcal{U}_q(\mathfrak{sl}(2))$, a truncation of the q-Onsager algebra, it is connected to double affine Hecke algebras (DAHA), it identifies with the Kauffmann bracket skein algebra of a four-punctured sphere, offers a framework to extend Schur–Weyl duality, and so on. We are much endebted to Askey for all that.

The bispectral properties of the Racah polynomials can similarly be packaged in an algebra to which the name of Racah has been attached. It can be obtained as the $q \to 1$ limit of the Askey–Wilson algebra after an affine transformation of the generators has been performed to revert to ordinary commutators. This Racah algebra is the centralizer of the diagonal action of $\mathfrak{sl}(2)$ in its three-fold tensor product and is the symmetry algebra of the generic superintegrable model in two dimensions.

In their classification of P- and Q- polynomial association schemes, Bannai and Ito found a case that corresponds to the $q \to -1$ limit of the q-Racah polynomials which are now referred to as the Bannai–Ito polynomials. We observed (with S. Tsujimoto) that these are eigenfunctions of a certain Dunkl shift operator. The corresponding eponymous algebra proved to be the centralizer of three copies of the superalgebra $\mathfrak{osp}(1|2)$. This led to the characterization of a "-1 scheme" complementing the Askey tableau. Askey took an interest in these studies, and on numerous occasions, he expressed the view that it should be possible to extend this to other roots of unity.

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There is much more that Askey has generated in Algebra with the introduction of the Askey–Wilson polynomials. For instance, a noteworthy conduit has been their multivariate generalizations. Two directions have been followed in this respect. One, in the framework of symmetric functions, led to the Macdonald–Koornwinder polynomials corresponding to the BC_n root lattice. DAHAs are in this case the associated algebraic structures. The other took the recoupling path with Tratnik, Gasper, and Rahman providing different generalizations of the Racah and Askey–Wilson orthogonal polynomials in many variables. The extensions of the Askey–Wilson, Racah, and Bannai–Ito algebras that these last classes of polynomials entail are currently being developed.

The influence of Askey on Algebra will endure. To make clear that there are many more dimensions to this impact, we may add that the other Askey polynomials, those defined on the circle which are biorthogonal (see [Sze82, Vol. 1]), have been connected to the Heisenberg group and it is expected that more algebraic advances will arise from the exploration of these functions. This is another illustration that Askey's results will keep cross-fertilizing areas of representation theory, special functions, and mathematical physics in ways that have not yet been fully imagined.

2.5. Askey and combinatorics, by Dennis Stanton.

Askey considered enumerative questions on Laguerre polynomials with Ismail (1976, MR0406808) and also Ismail–Koornwinder (1978, MR0514623). Some integrals could be evaluated by counting certain permutations, and weighted versions with parameters also existed. Even–Gillis (1976, MR0392590) had considered similar questions. One of their methods was the MacMahon Master Theorem in enumeration. At the same time, Foata (1978, MR0498167) had used the exponential formula in enumeration to give a beautiful proof of Mehler's formula for Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(a)H_n(b)\frac{u^n}{n!} = (1-4u)^{-\frac{1}{2}} \exp\left(\frac{4abu-4(a^2+b^2)u^2}{1-4u^2}\right).$$

This began a good deal of work in enumeration and specific orthogonal polynomial systems, orchestrated by Askey and Foata. They were instigators for important conferences in Columbus, Oberwolfach, and Tempe for researchers in both areas. At this time q-analogues of classical polynomials were intensely studied, leading to connections to partition theory and quantum groups. The philosophy of using weighted objects to represent analytic statements carried over to general orthogonal polynomials, and was developed by Flajolet (1980, MR0592851) using continued fractions, and by Viennot (1985, MR0838979) using combinatorial techniques. This combinatorial philosophy was used by Zeilberger–Bressoud (1985, MR0791661) to prove the q-Dyson conjecture.

Askey realized that a finite set of orthogonal polynomials with a finite discrete orthogonality always has a dual orthogonality. He reorganized the orthogonality relation as row or column orthogonalities of an orthogonal matrix. The $6j$ symbols had such orthogonalities. Askey and Wilson (1979, MR0541097) showed these symbols, once rescaled, were orthogonal polynomials in one variable. They are part of the classical scheme of hypergeometric orthogonal polynomials, namely ${}_4F_3(1)$ functions with four free parameters. Both $6j$ orthogonalities could be reformulated for polynomials. Askey made a partially ordered set which organized the classical hypergeometric polynomials (e.g., Hermite, Laguerre, Charlier, Jacobi, Meixner, Krawtchouk, Racah), the *tableau d'Askey*. The Hasse diagram of this poset (partially ordered set), drawn and distributed by J. Labelle (1984, MR0838967), became a focus of study. The q-Racah polynomials, defined by basic hypergeometric series, were the top element of the discrete part of this diagram, and would be key polynomials in algebraic graph theory.

A distance regular graph G (see Bannai–Ito, 1984 MR0882540) has very regular properties. Among them is that the $|G| \times |G|$ indicator matrix A_j for vertices (v_1, v_2) at distance j in the graph is a polynomial of degree j in the distance one matrix A_1. If this polynomial is denoted $p_j(A_1)$, the polynomials $p_j(x)$ have a discrete orthogonality relation using the values of $p_j(\lambda_k)$, where $\lambda_0, \lambda_1, \ldots$ are the eigenvalues of A_1. Delsarte (1973, MR0384310) studied a special case of these graphs called P- and Q-polynomial association schemes, for which there are polynomials q_k and real numbers μ_j satisfying $p_j(\lambda_k) = q_k(\mu_j)$. These have two sets of orthogonalities, one for p_j and one for q_k, just as the Racah polynomials did. Askey knew that all of the known infinite families of such schemes had eigenmatrices given by special or limiting cases of the q-Racah polynomials. Leonard (1982, MR0661597) proved the surprising result that the eigenmatrices are always special or limiting cases of the q-Racah polynomials. Wang (1952, MR0047345) classified the two-point homogeneous spaces, whose spherical functions, the continuous versions of $p_j(\lambda_k)$, are classical orthogonal polynomials. This gives some hope to classify such association

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schemes. Terwilliger (2001, MR1826654) has developed a detailed study of the linear algebra behind the pair of matrices (A_1, A_1^*) for p_j and q_k.

Askey (1975, MR0481145) promoted the study of linearization and connection coefficient problems for orthogonal polynomials. The linearization coefficients had graph- and group-theoretic interpretations, which could be restated as an enumeration problem. Rogers' original proof of the Rogers–Ramanujan identities (2) used a connection relation for the q-Hermite polynomials to the q^{-1}-Hermite polynomials. Rogers had these polynomials, but did not know their orthogonality relation.

Askey considered as crucial the integrals or sums for the total mass of the weight function for orthogonal polynomials. For the Jacobi polynomials, this integral is a beta function. The Askey–Wilson integral (1985, MR0783216) (5) is the evaluation for the continuous weight at the top of the *tableau d'Askey*. Askey was keenly interested in multivariable and root system versions (1980, MR0595822), and organized work in this area. His student Walter Morris (1982, MR2631899) wrote a thesis with many such conjectured integrals for root systems, concurrent with Macdonald (1981, MR0633515). These became important as measures for orthogonal polynomials in several variables; Macdonald (1989, MR1100299) and Koornwinder (1992, MR1199128).

2.6. Askey and beta integrals, by Mourad E. H. Ismail.

One of Dick Askey's contributions is the insight to see that many q-series identities, old and new, are different q-analogies of the beta integral

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{3}$$

We require $\Re(z, a, b) > 0$. One remarkable example is the Ramanujan ${}_1\psi_1$ sum

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(b/a;q)_\infty (q;q)_\infty (q/az;q)_\infty (az;q)_\infty}{(b;q)_\infty (b/az;q)_\infty (q/a;q)_\infty (z;q)_\infty} \tag{4}$$

for $|b/a| < |z| < 1$; see [AAR99, §10.5] for details.

The Selberg integral

$$\int_{[0,1]^n} \prod_{s=1}^{n} t_s^{\alpha-1}(1-t_s)^{\beta-1} \prod_{1\le j<k\le n} |t_j - t_k|^{2\gamma}\,\mathrm{d}t_1 \cdots \mathrm{d}t_n = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}$$

is the n-dimensional version of the beta integral, which corresponds to the case $n = 1$. Dick recognized that the Selberg integral is the key to the development of a deep theory of multivariate special functions. He also formulated q-analogues of this integral and of Aomoto's generalization of the Selberg integral. Askey also promoted the work of I. G. Macdonald and others on root systems. The last forty years saw great progress in this area spearheaded by Dick's tireless promotion and encouragement. This eventually led to the theory of Macdonald and Koornwinder polynomials. The interested reader may consult the beautiful survey of the Selberg integral, its applications, and significance by Peter Forrester and Ole Warnaar (2008, MR2434345).

Another important contribution is the Askey–Wilson integral

$$\int_0^\pi \frac{\left(e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{\prod_{j=1}^{4}\left(t_j e^{i\theta}, t_j e^{-i\theta}; q\right)_\infty}\,\mathrm{d}\theta = \frac{2\pi\,(t_1t_2t_3t_4;q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(t_jt_k;q)_\infty} \tag{5}$$

for $|t_j| < 1$. The orthogonality of the Askey–Wilson polynomials follows from (5) in a standard way. This again can be interpreted as a q-beta integral. After a certain scaling and letting $q \to 1$ it becomes the Wilson integral

$$\int_0^\infty \frac{\prod_{j=1}^4 \Gamma(a_j + i\sqrt{x})}{\Gamma(2i\sqrt{x})}\,\mathrm{d}x = \frac{2\pi \prod_{j=1}^4 \Gamma(a_j)}{\Gamma(a_1+a_2+a_3+a_4)},$$

valid for $a_j \ge 0$.

The Wilson integral and the Wilson polynomials appeared first in Wilson's dissertation written under Askey's supervision. The Askey–Wilson integral is a fundamental result that led to a better understanding of q-special functions, their transformations, and analytic properties.

2.7. Askey, positivity, inequalities, and applications, by George Gasper.

In the spring of 1967, shortly after accepting a visiting lecturer position at the Mathematics Department of the University of Wisconsin in Madison, I received a package from Professor Richard Askey containing several interesting reprints and preprints of his papers (partially joint work with Isidore Hirschman, Jr., Steve Wainger, and Ralph Boas) and a letter encouraging me to attend his graduate-level special functions course at the university during the 1967–1968 academic year. So, in addition to teaching the graduate complex variables course and attending analysis seminars, talks, and Wainger's harmonic analysis course, I also attended Askey's special functions course.

Dick's course covered gamma and beta functions, generalized hypergeometric functions and series, Bessel functions, (mostly classical) orthogonal polynomials (and their three-term recurrence relations, differential/

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difference equations, asymptotic expansions, etc.), summability, fractional integrals, infinite products, etc. You can now study most of the topics covered in the course by reading the corresponding material in the 1999 Andrews, Askey, and Roy *Special Functions* book [AAR99]. Dick was an excellent, knowledgeable lecturer who communicated enthusiastically with his audiences and was able to get them involved in the discussions. With his prodigious memory he could lecture on several mathematical topics and rapidly write complicated formulas on the blackboards without referring to his notes. At the beginning of classes he would frequently pass out stapled piles of blue mimeographed sheets containing the formulas, definitions, theorems, etc., that he was going to discuss. Also, in his classes and talks he would point out related open problems that he and others had tried to solve, and strongly encouraged his audience to try to solve them. It was Askey's encouragements to solve interesting and important open problems that led to many significant papers.

Via the positivity of the generalized translation operator for Jacobi series in Gasper (1971, MR0284628), Askey (1972, MR0340672) proved that if $\alpha, \beta \geq -\frac{1}{2}$ and $\sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x)/P_k^{(\beta,\alpha)}(1) \geq 0$, where $-1 \leq x \leq 1$, $n \in \mathbb{N}_0$, then in order for all of the partial sums of the Poisson kernel in powers of r, $0 \leq r < 1$, for Jacobi series to be nonnegative for $x, y \in [-1, 1]$, it is necessary and sufficient that $0 \leq r \leq 1/(\alpha + \beta + 3)$. He applied Bateman's fractional integral and some identities and inequalities for Jacobi polynomials to show that the inequality displayed above holds when $-1 < \alpha \leq \beta + 1$, $\alpha + \beta > 0$. In (1972, MR0301897) and (1979, MR0539375) he applied this inequality and an inequality for sums of Jacobi polynomials in Gasper (1977, MR0432946) to prove that the Cotes numbers for Jacobi abscissas are positive if $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, or $-1 < \alpha \leq \frac{3}{2}$, $\beta = \alpha + 1$, or if $-1 < \beta \leq \frac{3}{2}$, $\alpha = \beta + 1$. He also stated that it should be possible to fill in the convex hull of these (α, β) points and those in an earlier paper with Fitch (1968, MR0228166) and that it is possible that the Cotes numbers are positive on the rectangle $-1 < \alpha, \beta \leq \frac{3}{2}$, which would be the best possible rectangle, both of which are still open problems.

In a paper [AG76] that was submitted for publication in 1973 and published in 1976, Askey and Gasper used a sum of squares of ultraspherical polynomials and Bateman's fractional integral to prove that the sum of Jacobi polynomials displayed above is nonnegative for $\beta \geq 0$, $x \in [-1, 1]$, $n \in \mathbb{N}_0$, if and only if $\alpha + \beta \geq -2$. Unexpectedly, several years later the special cases $\{(\alpha, \beta) : \beta = 0, \alpha = 2, 4, 6, \ldots\}$ of the above inequality turned out to be the inequalities that de Branges (1985, MR0772434) needed in February 1984, to complete his proof of the Bieberbach conjecture, and of the more general Robertson and Milin conjectures. For additional information, see the Askey and Gasper paper, Askey's personal account, and the other papers and personal accounts in [DDM86].

Askey's papers (1973, MR0315351) and (1974, MR0372518) helped lead to the conjecture in Askey & Gasper [AG76] that if $0 \leq \lambda \leq \alpha + \beta$, $\beta \geq -\frac{1}{2}$, then

$$\sum_{k=0}^{n} \frac{(\lambda+1)_{n-k}(\lambda+1)_k}{(n-k)!\ k!} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\beta,\alpha)}(1)} > 0,$$

where $-1 < x \leq 1$, $n \in \mathbb{N}_0$, except when $\lambda = 0$, $\alpha = -\beta = \frac{1}{2}$, when the sum is nonnegative and there are cases of equality. They proved several special cases of this conjecture (see Theorems 1, 2, 7, 8, and 11 in [AG76]) and showed that a proof of it for the special case $0 \leq \lambda = \alpha + \beta$, $\beta \geq -\frac{1}{2}$, would prove Askey's conjecture in (1973, MR315351) that the Cesàro $(C, \alpha + \beta + 2)$ means of the Poisson kernel for Jacobi series are positive for $\alpha, \beta \geq -\frac{1}{2}$, and hence, equivalently, that the $(C, \alpha + \beta + 2)$ means of the Jacobi series of a nonnegative function are nonnegative when $\alpha, \beta \geq -\frac{1}{2}$. Eventually, Gasper (1977, MR1535644) proved these conjectures and some other related conjectures via a sum of squares of Jacobi polynomials.

See [GIK+00] for detailed comments on more of Dick's work on orthogonal polynomials, orthogonality, inequalities, and other topics.

The last open problem that Dick encouraged me to solve was to prove an inequality involving hypergeometric functions that arose in a paper that Brown and Davies were writing entitled "Financing efficiency of securities-based crowdfunding." In order to make the proofs of some of their main theorems mathematically rigorous, they needed a proof of their conjecture that certain quotients of quotients of hypergeometric functions were bounded above by one. For my subsequent proof and comments, see Addendum 1 in the Supplementary Data for the recently published Brown and Davies paper (2020, in *The Review of Financial Studies* 33(2020), 3975–4023) with the above title.

We will miss Dick, his phone calls, emails, papers, talks, and encouragements to solve interesting and important open problems.

2.8. Very positive memories about Dick Askey, by Tom H. Koornwinder.

I got acquainted with Dick Askey during 1969–1970 when he spent a sabbatical at the Mathematical Centre (now CWI) in Amsterdam. He was young and

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energetic, full of ideas, very stimulating for us young PhD students. The lectures he gave there already essentially contained the material for his NSF Regional Conference Lectures [Ask75] at Virginia Polytechnic Institute. These lecture notes consider four canonical problems for systems $\{p_n\}$ and $\{q_n\}$ of orthogonal polynomials (see also the Introduction in [AG71]):

(i) (product formula) $p_n(x)p_n(y)$ as integral of $p_n(z)$;
(ii) (linearization) $p_m(x)p_n(x)$ as sum of $p_k(x)$;
(iii) (transmutation) $q_n(x)$ as integral of $p_n(z)$;
(iv) (connection formula) $q_n(x)$ as sum of $p_k(x)$.

Note the dualities (i)↔(ii) and (iii)↔(iv). In these problems, Dick was particularly interested in positivity results, i.e., cases with a *nonnegative* integration or summation kernel. For instance in (i), $p_n(x)p_n(y) = \int_a^b p_n(z)K(x,y,z)\,dz$ with $K(x,y,z) \geq 0$. At such positivity results, one might arrive either by an explicit expression of the kernel or by a general theorem. *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$, orthogonal on $(-1,1)$ for weight function $(1-x)^\alpha(1+x)^\beta$ $(\alpha,\beta>-1)$, were always the most prominent examples and objects of study.

We use the normalization

$$R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1).$$

The product formula for *ultraspherical polynomials* $R_n^{(\alpha,\alpha)}(x)$ $(\alpha>-\frac{1}{2})$ is classical:

$$R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y)=c\int_{-1}^{1} R_n^{(\alpha,\alpha)}(z)\,(1-t^2)^{\alpha-\frac{1}{2}}\,dt,$$

where $z = z(x,y,t) = xy + t\sqrt{1-x^2}\sqrt{1-y^2}$ and $c\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\,dt=1$. Passing to the integration variable z gives the kernel form. Expansion of $R_n^{(\alpha,\alpha)}(z(x,y,t))$ in terms of $R_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(t)$ gives the *addition formula*.

For $\alpha = \frac{1}{2}(d-3)$ these formulas can be interpreted in terms of *spherical harmonics* on S^{d-1}; see [AAR99, Ch. 9]. Formulated even more smartly, the ultraspherical polynomials are then *spherical functions* ϕ for the *Gel'fand pair* (see Gel'fand (1950, MR0033832)) $(G,K)=(\mathrm{SO}(d),\mathrm{SO}(d-1))$, and thus satisfy Gel'fand's product formula $\phi(x)\phi(y) = \int_K \phi(xky)\,dk$ $(x,y \in G$; dk normalized Haar measure on $K)$. For any Gel'fand pair (G,K) harmonic analysis for K-bi-invariant functions, including (positive) convolution, can be equivalently described as harmonic analysis on the double coset space $K\backslash G/K$ in terms of special functions coming from the spherical functions. Usually these special functions depend on parameters, and only a few discrete choices of them come from a Gel'fand pair. If positive convolution is still available for other parameter values, for instance by a product formula with nonnegative kernel, and if certain further axioms hold, then things fit into a so-called *hypergroup*, independently developed by Dunkl (1973, MR0320635), Jewett (1975, MR394034), and Spector (1975, MR0447974).

While Askey was in Amsterdam, George Gasper (who had started an intensive collaboration with Dick while he was a visiting lecturer in Madison) found a product formula for Jacobi polynomials $(\alpha > \beta > -\frac{1}{2})$ with explicit nonnegative kernel. Thus the positive convolution structure was settled. Then Askey asked me to find a (different looking) product formula and corresponding addition formula for Jacobi polynomials from a group-theoretic context. I could tackle this by starting with decomposition of spherical harmonics on the unit sphere in $\mathbb{C}^d$ with respect to the unitary group. Slightly earlier and independently the same approach was followed by Vilenkin and Šapiro (1967, MR0219662; 1968, MR0230955).

Concerning item (ii) the linearization coefficients for $p_m(x)p_n(x)$ are nonnegative if $p_n(x)$ can be interpreted as a spherical function; see [AB76, p. 141]. More generally, Gasper showed nonnegativity for Jacobi polynomials if $\alpha \geq \beta > -1$ and $\alpha+\beta>-1$.

In his conference lectures Askey often made the following point. Just as a product formula suggests an expansion called an addition formula, there should be a dual addition formula suggested by an explicit linearization formula. In a 2018 publication (MR3791629), I solved this for ultraspherical polynomials by observing that the linearization coefficients are the orthogonality weights for special Hahn polynomials.

A very useful thing I learnt from Dick is about the occurrence of *fractional integrals* [Ask75, §2.9] in special functions. Transmutation formulas (item (iii)) are often (generalized) fractional integrals. Item (iv) (connection coefficients) gave rise to Dick's most explicit contact with analysis on groups. He considered isometric imbeddings of projective spaces to get positivity of connection coefficients for the spherical functions being Jacobi polynomials (see [Ask68, AB76]).

In 1975, George Andrews raised Dick's interest in q-theory, and through Dick we all became enthusiastic about q. The work of Dick and his student Jim Wilson culminated in the introduction of the *Askey–Wilson polynomials* [AW85], being on top of the *q-Askey scheme*. I have been happy to bring these polynomials to quantum groups (1993, MR1215439) and to several variables (1992, MR1199128), two areas in which Dick did not work himself, but whose importance he always emphasized.

We will miss Dick's wise lessons such as: "Study the old masters"; "Look for interactions with and applications to

Figure 6. Doron Lubinsky, Paul Nevai, Dick Askey, Tom Koornwinder attending OPSFA1 Polynômes Orthogonaux et Applications, Bar-le-Duc, France, October 1984.

other fields"; and Paul Turán's "Special functions are useful functions!"

3. Askey's Contribution to Mathematics Education

3.1. Askey's contributions to school mathematics education, by Hung-Hsi Wu. Most professional mathematicians who get involved in school mathematics education do so late in their careers for some external reason. Dick Askey was an exception. He seemed to have been interested in education all his life. When he was only in his thirties, he was already on record as having voiced his displeasure with some of the New Math's excesses in formalism as manifested in the textbooks.

I got to know Dick entirely by accident. In 1992, I was inadvertently drawn into the Math Wars by innocently writing a critique of one of the new reform curricula. For that, I immediately came under fire from the reform crowd. For a month or two, I was completely at sea. It was Serge Lang who suggested that I write to Dick because Serge had read one of Dick's comments on school mathematics and Serge approved. So I did, and Dick provided the advice and support I had sought. Years later when Dick and I had become close friends, he told me with a chuckle that when he first got my email, he was highly suspicious because he couldn't be sure whether I was a nut or not. Happily, he decided in due course that I was not, and that marked the beginning of a friendship that was to last for the next twenty-seven years until his passing.

As it happened, both Dick and I had the same diagnosis of the most important issue in the ongoing education crisis: our teachers have not been provided the needed mathematical knowledge to discharge their duties. He fought to the very end of his life for ways to improve teachers' content knowledge, and I believe we found some comfort in each other's support in those often lonely battles.

Dick told me that he was proudest of his ability to work behind the scenes to get results in education. I think the reason he could do that was because of his stature as a mathematician; many in education still value a mathematician's input. For example, he had friends in the Madison school district and, because they consulted him often, he managed to have a voice in some of the local educational decisions. He also maintained a good working relationship with many in the reform effort. Not infrequently, the latter turned to Dick for advice on mathematical issues. They sent him book manuscripts for informal inspection, and his suggestions sometimes resulted in major revisions of the manuscripts or even changes in the organization's policies. Understandably, these efforts of Dick's would go unnoticed by the general public. In his last years, Dick was the consultant for the Dimension Math series of `http://singaporemath.com` for grades K–6. The publisher, Jeffery Thomas, told me that he would not have tackled this series without Dick's participation.

Dick wrote some quite influential articles on school mathematics education. Comments on these articles together with his overall contributions to education are the subject of the following articles by Al Cuoco and Roger Howe.

3.2. Remembering Dick Askey, by Al Cuoco. My first encounter with Dick Askey was over 20 years ago, when I called him to complain about some of his public statements tied to the National Council of Teachers of Mathematics (NCTM) *Standards*.

I launched into a tirade about how his criticisms—essentially about missing topics, "mangled" (his word) treatments of other topics, and lack of rigor and high expectations—needed be taken seriously, about how crucial it was for mathematicians to play key roles in mathematics education, and about how his caustic tone would cause people to dismiss his criticism and that of other mathematicians.

Dick listened to all this without saying a word. Then, in a very quiet and almost faltering voice, he said that he wanted young people in this country to have the best education possible, and he wanted to make sure that our children were not subjected to untested ideas about education. In the ensuing hour, I listened to Dick's detailed description of his worries about the sorry state of school algebra, the poor mathematical preparation of teachers, the concentration on low-level details in school geometry and trig, and what he considered the overall mathematical shortchanging of our children. I realized that Dick Askey was

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simply a person who loved mathematics—especially classical mathematics—and who thought that young people can and should develop a passion for it.

Since then, right until he moved to assisted living, I had hundreds of discussions with Dick about mathematics and education. He continued to be an advocate for mathematical integrity in school mathematics. He continued to be skeptical about anything in education he perceived as faddish or untested, he tirelessly criticized texts and materials that he found mathematically flawed, and he continued to make public proclamations about this or that curriculum or educational theory in ways that often made my blood boil.

I've experienced Dick's education criticism first hand. I'd often ask Dick to comment on early drafts of our materials. In addition to pointing out resources and historical references, Dick provided dozens of ideas about how to improve the presentations ("I prefer not to use L'Hôpital's rule when the addition formula for sine will work just as well"), many suggestions for "the right way" to develop a topic, detailed edits, and, of course, heavy doses of personal opinion ("to be brutally frank, I don't like your treatment of this topic"). Much of this work was pro bono, and, if Dick couldn't find time to help us out, he'd always find a colleague ready to pitch in, with just the right expertise.

There are many stories to tell about Dick's work in education. I want to tell two.

Early in 2019, I asked Dick to tell me about the (read "his") important ideas in trigonometry. Shortly after that, I received sixty pages of handwritten notes. I have no idea whether he wrote them for us or had them in his back pocket. But they are brilliant. So typically Askey, they were simple and direct, no fancy detours or applications (Dick loved the word "parsimonious"), full of connections among topics, and plenty of the kind of advice that I came to expect from him:

> The theorem of Pythagoras is so important that you should know more than one proof. "Know" is a word which can mean different things to different people. Some settle for a memorized proof. That may be necessary as a first step, but one wants to go beyond that to understand the idea behind the proof. Once this is done, then it is usually easy to recall the proof. Also, the ideas can frequently be used in other settings.

I showed the notes to a few folks on the MathEd[1] list that Roger describes. Shaun Cooper, one of Dick's PhD students and now at Massey University in New Zealand, looked at them and immediately saw their importance. Shaun agreed to edit them, add solutions, and TeX them up. These turned into a wonderful contribution to the field, especially for teachers at every level who want a blueprint for a trig course that highlights the essence of the subject. The monograph is now available at `http://go.edc.org/askey-trig-2021`.

Dick was not always a harsh critic of efforts to improve K–12 mathematics education. For example, Dick published several articles in the NCTM journal, *Mathematics Teacher*, all aimed at helping teachers see the innards of Fibonacci numbers and recursively defined functions [Ask04]. And Dick's relationship with the *Common Core State Standards for Mathematics* was quite different from his reaction to the NCTM *Standards*. Although he was initially skeptical that they would be useful, he consented to join the feedback group. Dick's work here was characteristically sincere and based on careful reading and reasoning. He would argue strongly for his viewpoint, but did not insist on winning every time. His focus was steadily on the integrity of the mathematics. He ended up being a strong advocate for CCSSM.

My second example is from 2010. Dick and I were invited to a conference at the University of Santiago in Chile, which was devoted to all aspects of mathematics textbooks. Just before he was about to travel, he developed a kidney stone and couldn't go. We talked about alternatives for his talk (someone could read his notes, for example), but he decided that he was going to deliver it himself. So, he taped the presentation and attached it to an email (!). The file was 1.5 GB, and this was 2010, so it took all night and part of the next morning to download. With great help from the tech folks at the university, we got it to load and run on what had to be a wimpy Mac. It still plays, and it's worth viewing at `https://go.edc.org/Askey-Santiago`.

It's a perfect example of Dick's unadorned style, his criticism of textbooks at the time, and his taste in trigonometry. You can also see how he stops every now and then, grimacing in pain from the stone. But he pulled it off in his usual way, and the talk was a big hit.

In the end, it's Dick's love for mathematics that drove his involvement in education. This love shows through in many of his actions and activities, but never so well as in his interview with a BBC radio program devoted to the life of Ramanujan. Dick speaks of beautiful and amazing formulas and of the kind of genius that "you can't imagine yourself being, no matter how much smarter you could be." I've used this tape with my high school classes many times, and my students always leave with a better

[1] *Anyone wishing to join MathEd can send a request to Jim Madden with the subject line "Request to join MathEd." They should include: (1) name; (2) email address; (3) affiliation; (4) a statement of their willingness to be identified to other users as a new subscriber; and (5) an acknowledgment that MathEd is intended to enable patient, well-informed discussion among professional mathematicians and mathematics teachers concerning mathematics education.*

understanding of how mathematics, even very technical mathematics, can be a thing of beauty. Every time *I* hear it, I'm struck by how this very mild mannered Dick Askey, a mathematician so in love with his discipline that when he speaks of it, his voice softens, could cause such a stir in mathematics education.

3.3. Askey and mathematics education, by Roger Howe.

I got to know Dick Askey through our mutual involvement in mathematics education. An important part of our interaction was through MathEd, an email discussion group about mathematics education, which Dick, in cooperation with Hung-Hsi Wu of UC Berkeley and Chi Han Sah of SUNY Stony Brook, founded in the early to mid-1990s. In the more than two decades since it was first created, it has been a rich source of information and food for thought for me, and I believe for many of its several dozen members.

The most important aspect of MathEd was its members, recruited by Dick and Wu and Han, who included many of the US mathematicians most concerned about K–12 education during the 1990s, and several from other countries, also some excellent high school teachers and mathematics educators of various kinds. The range of views and experience represented by the members led to rich and thought-provoking exchanges.

Another important feature of MathEd is one of its ground rules, which can be paraphrased as "What happens on MathEd stays on MathEd." This means that no messages on MathEd should be shared with people outside the group, without permission from the sender. This restriction promotes stimulating exchanges, with members not having to worry that a provocative or partially thought out position might result in the kind of punitive attacks for which the internet has become notorious.

Dick also helped me learn about math education in other ways. He read extensively, and recommended items that he found valuable. He pointed me to the work of Harold Stevenson, studying mathematics education in Asia, especially Japan. This is where I first learned how Japan does a much better job with math education than we do, partly due to their valuable practice of "Lesson Study."

When Han Sah died in 1997, Dick and Hung-Hsi both attended his memorial gathering in Stony Brook. I also went, since I had known Han since the 1970s, when we were both in the Math Department there. I had met Wu when I was a graduate student and he was on the faculty in Berkeley in the 1960s, but this was the first time I met Dick in person.

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After Han Sah's passing, Dick arranged for the MathEd discussion group to be hosted at the University of Wisconsin, and served as its moderator from then almost until he passed away. Now LSU, home of three long-time MathEd contributors, has assumed the hosting task. The longevity of the group, and especially the continuity after the successive passing of hosts, testifies to the value the group finds in the discussions.

Both Dick and I were involved in the "math wars" of the 1990s, when the NCTM *Standards* promoted a new vision of mathematics education, heavily influenced by constructivism, the then dominant paradigm in education. I think we shared similar reservations about constructivism as a complete theory for education, but Dick was much more forthright than I was about expressing his reservations, and more specifically, how they were embodied in the NCTM *Standards*. Quite possibly as a result, he was less sought after for the various committees that were formed to deal with mathematics education. This was not for the best, because his insights and keen analytical skills could have improved the reports of any committees he served on.

Despite this, Dick was able to reach a broad public through several significant articles. One was a review for the *American Educator* [Ask99], of Liping Ma's highly influential book, *Knowing and Teaching Elementary Mathematics*. This was a report on the responses that a group of Chinese elementary math teachers gave to a set of questions designed (by Deborah Ball and colleagues at the National Center for Research on Teacher Education) to investigate teachers' understanding of mathematical issues in the elementary curriculum. It revealed a dramatic gap between the understanding of Chinese teachers and American teachers. Despite the large impact this book made in the US mathematics education community, its lessons were unfortunately not absorbed by policy makers.

Dick's review is incisive, and shows his skill at making points by close analysis and careful discussion. He gives a detailed and insightful discussion of the responses of the Chinese teachers to one of the questions, about constructing a word problem whose answer requires a division of fractions. He then uses this to make larger points. A major one (which indeed is made also in the book), is that "elementary" mathematics in fact has substantial depth. This means that elementary teachers need commensurate opportunities to master it.

However, as Dick pointed out in some detail, our educational system did not afford those opportunities. As evidenced here and elsewhere, Dick was deeply aware of the need to increase the mathematical understanding of the teaching corps, and it was a continuing theme of his writing over several decades. I believe that this perspective was shared by many of the mathematicians who got involved in mathematics education in the 1990s. This message has been echoed strongly by Wu, and I hope by me, and it has been featured in a number of studies of high standing, notably *Adding It Up* (the NRC review of research in math education, commissioned with the hope of ending the math wars of the 1990s), and the report of the National Mathematics Panel [NMAP08] (see findings 17, 18, and especially 19, pages xx, xxi). Unfortunately, these recommendations have had little effect on US educational policy.

Dick's article "Good intentions are not enough" [Ask01] connects the need for greater teacher understanding to the NCTM *Standards*. Released in 1989, and supplemented and revised throughout the 1990s, the *Standards* was the central document of that period for efforts to reform mathematics education. It advocated for novel, highly interactive approaches to teaching. However, it ignored the issue of the mathematical understanding of teachers that has been holding back mathematics learning in the US for many decades. As Dick correctly pointed out in his article, the kind of mathematics teaching envisioned in the *Standards* required considerably more understanding than traditional methods, and so would exacerbate this issue.

One of the most enjoyable activities I participated in with Dick was the workshop, which he and Patsy Wang-Iverson of Research for Better Schools (RBS) cooperated in organizing, in the summer of 2004 at the Wingspread Estate, nestled in the beautiful Wisconsin countryside. Its main building was designed by Frank Lloyd Wright. It is now run by the Johnson Foundation as a conference center, with meals and social events arranged by a full-time staff. Dick was one of the moderators of the introductory session, and an active contributor throughout. Here, and whenever I met him, his careful thinking and somewhat ironic sense of humor always made for interesting and enjoyable conversation.

The primary activity of the Wingspread meeting was to study and discuss the videos of math lessons from Trends in International Mathematics and Science Study, the comparative study of mathematics achievement in countries around the world done in the late 1990s. We viewed videos of 8th grade math lessons from Australia, the Czech Republic, Hong Kong, Japan, Netherlands, Sweden, and the US, and tried to analyze them from the point of view of promoting student thinking, and in particular, of making connections between different aspects of the lesson. Our conclusions were posted on the RBS website.

Another project I worked on with Dick was the report *No Common Denominator*, put out by the National Center for Teacher Quality. This was a survey and evaluation of the mathematical education and training given to students in schools of education across the country. The picture it painted was rather dismal. There was little uniformity in the amount of mathematics covered or the topics, or the entrance or graduation requirements. There was little evidence in most programs that prospective elementary teachers gained a "deep understanding" of the mathematics they would need to teach, or in many programs, even that this was a focus of instruction. Out of 77 studied, only 10 were found to give adequate preparation in mathematics. This was published in 2008, seven years after the Conference Board on the Mathematical Sciences (CBMS) had published *The Mathematical Education of Teachers*, containing its strong recommendations for content courses in teacher preparation programs, and *Adding It Up* had made parallel recommendations. CBMS revisited and republished its recommendations in 2011, after the release of the Common Core Standards. It is too late for Dick, but I will continue to hope that these ideas eventually have an impact on US mathematics education.

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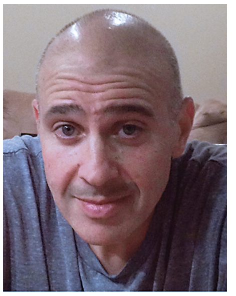

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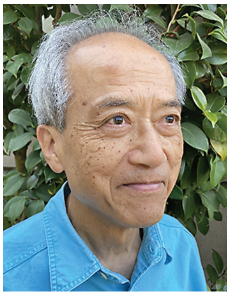

Hung-Hsi Wu

Credits

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