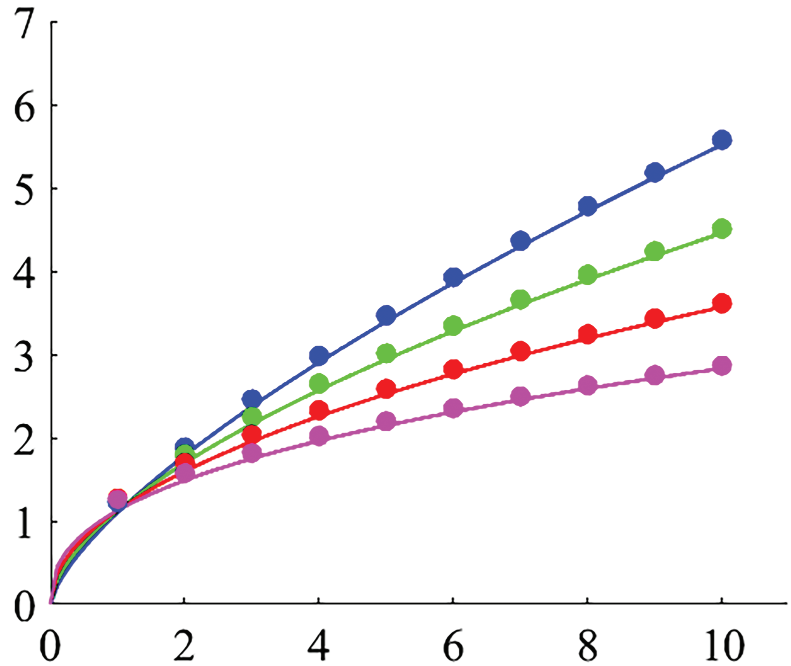
The $\mathbb{N}_0$ Story: Discrete Fractional Calculus



Raegan Higgins and Heidi Berger

1. Introduction

The derivative is at the heart of science, technology, engineering, and mathematics (STEM) and specifically calculus. It seems natural, therefore, to extend this idea. What does it mean to take a half derivative? A $\sqrt{2}$th derivative? These are the questions that sparked the creation of fractional calculus. Indeed, L'Hôpital asked this first question in a 1695 letter to Leibniz. It was not until the 1800s that a firm theoretical foundation for the fractional calculus was provided. S. F. Lacroix made the first significant step to the creation of fractional calculus. In 1812, he defined a fractional derivative by means of an integral, and he first mentioned a derivative of arbitrary order in 1819 [Ros77]. Lacroix casually noted without clear practical intent that for an integer m, the power rule

$$D^m x^p = p(p-1)(p-2)\cdots(p-m+1)x^{p-m}$$

can be written as

$$D^m x^p = \frac{\Gamma(p+1)}{\Gamma(p-m+1)} x^{p-m} \tag{1.1}$$

even when m is *not* an integer. This formulation relies on the gamma function, which is defined as follows.

Definition 1.1. The gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

for all complex numbers z such that the real part of z is positive.

Integration by parts yields

$$\Gamma(z+1) = z\Gamma(z)$$

when the real part of z is positive. This very important property allows the domain of the gamma function to be extended to all complex numbers except zero and the negative integers. A well-known consequence is that the

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gamma function generalizes the factorial as $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$.

Several mathematicians worked on fractional calculus in the 19th century. However, only Riemann and Liouville are known for contributions to both the fractional integral and fractional derivative. For $\nu > 0$ and $a \in \mathbb{R}$, the Riemann-Liouville fractional integral is

$$D_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-\tau)^{\nu-1} f(\tau)\, d\tau \qquad (1.2)$$

provided f is locally integrable. The Riemann-Liouville fractional integral is an area projected with a gamma function, which generalizes the classical definite integral [Pod02]. While it is intuitive to replace $-\nu$ with ν to get the fractional derivative operator $D_a^\nu f(t)$, this approach fails since the integral $\int_a^t (t-\tau)^{-\nu-1} f(\tau)\ d\tau$ is generally divergent. So the Riemann-Liouville fractional derivative can be considered using the secondary construction

$$\begin{aligned} {}_{RL}D_a^\nu f(t) &\\ &= \frac{d^N}{dt^N} D_a^{-(N-\nu)} f(t) \\ &= \frac{1}{\Gamma(N-\nu)} \frac{d^N}{dt^N} \left[\int_a^t (t-\tau)^{N-\nu-1} f(\tau)\ d\tau \right], \quad (1.3) \end{aligned}$$

where N is the unique positive integer satisfying $N-1 < \nu < N$.

An alternative, but not equivalent, formulation of the fractional derivative is the Caputo fractional derivative, which was introduced in 1967 by Michele Caputo. For $\nu \geq 0$, $n \in \mathbb{N}$, and $a \in \mathbb{R}$, the Caputo fractional derivative of order ν for a function f is

$$\begin{aligned} {}_C D_a^\nu f(t) &= D_a^{-(n-\nu)} \frac{d^n}{dt^n} f(t) \\ &= \frac{1}{\Gamma(n-\nu)} \left[\int_a^t (t-\tau)^{n-\nu-1} f^{(n)}(\tau)\ d\tau \right], \end{aligned}$$

where $n-1 < \nu < n$ and f is a real-valued continuous function that has continuous derivatives up to order $n-1$ on $[a,b]$ such that the $(n-1)$st derivative is absolutely continuous on $[a,b]$. Caputo's formulation has the advantage over the Riemann-Liouville fractional derivative in that it is not necessary to define the fractional order initial conditions when solving differential equations using Caputo's definition. While the Riemann-Liouville fractional integral has the semigroup property (for $\alpha,\ \beta > 0$, $D_a^{-\alpha} f(t) \cdot D_a^{-\beta} f(t) = D_a^{-\alpha-\beta} f(t)$), neither the Riemann-Liouville nor the Caputo fractional derivative possess it.

The theory of fractional calculus builds around the Riemann-Liouville definition of the fractional derivative, often following analogously the theory of continuous whole-order calculus.

Although the field of fractional calculus has been studied for over two hundred years, the field of discrete fractional calculus has only gained traction in the past decade [EGJP16, GP15]. In what follows, we present an understanding of monotonicity and convexity through the lens of discrete fractional calculus. In doing so, we see how the nonlocal nature of the fractional difference leads to surprising results that defy the intuition students gain on monotonicity and convexity in an introductory calculus course. The nonlocal nature of the fractional difference also yields interesting applications to biological modeling, which is how our study concludes.

To see the beauty and need of the discrete fractional calculus, we begin with the integer-order calculus. The definitions and theorems presented in this section can be found in [GP15]. Within the study of discrete fractional calculus, we study real-valued functions defined on a shift of the natural numbers, either $\mathbb{N}_a = \mathbb{N}_0 + \{a\} = \{a, a+1, a+2, \dots\}$ or $\mathbb{N}_a^b := \{a, a+1, \dots, b\}$ for fixed $a, b \in \mathbb{R}$ such that $b-a$ is a positive integer. Analogous to a whole-order derivative for real-valued functions, there are two main types of difference functions: the delta, or forward difference, and the nabla, or backward difference. This paper focuses on the delta difference. In doing so, we will see some surprising results that are counterintuitive to what is seen in the continuous case.

Consider a map $f : \mathbb{N}_a \to \mathbb{R}$, where a is a real number. The well-known *forward difference operator* Δ is defined by

$$\Delta f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a. \qquad (1.4)$$

Differences of higher order $N \in \mathbb{N}$ are defined iteratively: $\Delta^N f(t) := \Delta(\Delta^{N-1} f(t)), t \in \mathbb{R}_a$. Just as differences generalize differentiation, so too can summations generalize definite integrals. The delta definite integral of $f : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$\int_c^d f(s)\ \Delta s := \sum_{s=c}^{d-1} f(s),$$

where $c \leq d$ are in $\mathbb{N}_a$ with the convention $\sum_{s=c}^{c-k} f(s) := 0$ whenever $k \in \mathbb{N}_1$. In order to motivate delta fractional sums and differences, we need to first define the whole-order delta sum of a function $f : \mathbb{N}_a \to \mathbb{R}$, which requires the concept of "t to the r falling."

The generalized falling function is built from the gamma function. Defined according to the value of n, the generalized falling function is given as follows:

1. If $n \in \mathbb{N}$, then $t^{\underline{n}} := t(t-1)(t-2)\cdots(t-n+1)$.
2. If $n = 0$, $t^{\underline{n}} := 1$.
3. If $n = -1, -2, -3, \dots,$

$$t^{\underline{n}} := \frac{1}{(t+1)(t+2)\cdots(t-n)}.$$

4. If $n \notin \mathbb{Z}$, then

$$t^{\underline{n}} := \frac{\Gamma(t+1)}{\Gamma(t-n+1)}. \tag{1.5}$$

It is understood that $t^{\underline{n}}$ is given only for the values of t and n for which these formulae are meaningful. Moreover, in the case where $t-n+1$ is a pole of the gamma function but $t+1$ is not, we declare $t^{\underline{n}} := 0$. Using (1.4), we see that $\Delta t^{\underline{n}} = nt^{\underline{n-1}}$. This notation enables us to generalize the power rule in a manner consistent with (1.1).

The generalized falling function is used to define the discrete Taylor monomials. These take the place of the Taylor monomials $\frac{(t-s)^n}{n!}$ in the continuous calculus.

Definition 1.2. We define the discrete Taylor monomials (based at $s \in \mathbb{N}_a$) $h_n(t,s)$, $n \in \mathbb{N}_0$, by

$$h_n(t,s) = \frac{(t-s)^{\underline{n}}}{n!}, \; t \in \mathbb{N}_a.$$

Some important properties of the discrete Taylor monomials are given below.

Theorem 1.3. *Let $t, s \in \mathbb{N}_a$ and $n \in \mathbb{N}_0$. Then*

1. $h_0(t,a) = 1$;
2. $h_n(t,t) = 0$;
3. $\Delta h_{n+1}(t,a) = h_n(t,a)$;
4. $\Delta_s h_{n+1}(t,s) = -h_n(t,s+1)$, *where $\Delta_s h_n(t,s)$ denotes the derivative with respect to s;*
5. $\int h_\nu(t,a)\,\Delta t = h_{\nu+1}(t,a) + C$;
6. $\int h_n(t,s+1)\,\Delta s = -h_{n+1}(t,s) + C$

for a constant C.

The discrete Taylor monomials allow us to establish a rule for repeated summation.

Theorem 1.4. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. Then*

$$\begin{aligned} &\int_a^t \int_a^{\tau_1} \int_a^{\tau_2} f(\tau_n)\Delta\tau_n \cdots \Delta\tau_2\, \Delta\tau_1 \\ &\quad = \int_a^t h_{n-1}(t,s+1) f(s)\Delta s. \end{aligned} \tag{1.6}$$

This is proved using induction on n, integration by parts (Theorem 1.58 of [GP15]), and Theorem 1.3.5. Motivated by (1.6), we express the nth integer sum $\Delta_a^{-n} f(t)$ for $n \in \mathbb{N}$ as

$$\Delta_a^{-n} f(t) := \int_a^t h_{n-1}(t,s+1) f(s)\Delta s.$$

However, since $h_{n-1}(t,s+1) = 0$ whenever $s = t-1, t-2, \ldots, t-n+1$, we obtain

$$\Delta_a^{-n} f(t) := \int_a^{t-n+1} h_{n-1}(t,s+1) f(s)\Delta s, \tag{1.7}$$

which is the definition of the nth integer sum of $f(t)$. With (1.7) in hand, we now define the fractional sum, which parallels (1.2).

Definition 1.5. Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then the νth-order fractional sum (based at a) of f is given by

$$\begin{aligned} \Delta_a^{-\nu} f(t) &:= \int_a^{t-\nu+1} h_{\nu-1}(t,\tau+1) f(\tau)\,\Delta\tau \\ &= \sum_{\tau=a}^{t-\nu} h_{\nu-1}(t,\tau+1) f(\tau) \end{aligned}$$

for each $t \in \mathbb{N}_{a+\nu}$, where

$$\begin{aligned} h_\nu(t,s) &:= \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)} \\ &= \frac{\Gamma(t-s+1)}{\Gamma(t-s-\nu+1)\Gamma(\nu+1)} \end{aligned} \tag{1.8}$$

is the νth fractional Taylor monomial based at s.

The following formulas about the fractional Taylor monomials generalize the integer version given in Theorem 1.3.

Theorem 1.6. *Let $t, s \in \mathbb{N}_a$. Then*

1. $h_\nu(t,t) = 0$;
2. $\Delta h_\nu(t,a) = h_{\nu-1}(t,a)$;
3. $\Delta_s h_\nu(t,s) = -h_{\nu-1}(t,s+1)$;
4. $\int h_\nu(t,a)\,\Delta t = h_{\nu+1}(t,a) + C$;
5. $\int h_\nu(t,s+1)\,\Delta s = -h_{\nu+1}(t,s) + C$,

whenever these expressions make sense.

Properties 2 and 3 generalize the power rule for the derivative of polynomials, and properties 4 and 5 generalize the power rule of integration of polynomials.

As in the continuous case (1.3), this leads to the definition of the νth-order fractional difference. Suppose $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Define $N \in \mathbb{N}$ to be the unique positive integer satisfying $N-1 < \nu < N$. Then the νth fractional difference is

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N-\nu}. \tag{1.9}$$

To illustrate these definitions, we give the following example.

Example 1.7. Find the half-derivative of $f(t) = t$ on both the real numbers and the integers.

Using the Riemann-Liouville definition of the fractional derivative (1.3) with $a \in \mathbb{R}$ and $\nu = 1/2$ ($m = 1/2$, $p = 1$ in (1.1)), we find that

$$\begin{aligned} D_a^{1/2} t &= \frac{1}{\Gamma(1/2)} \frac{d}{dt} \int_a^t \frac{\tau}{(t-\tau)^{1/2}}\, d\tau \\ &= \frac{1}{\sqrt{\pi}} \left[\frac{a}{\sqrt{t-a}} + 2\sqrt{t-a} \right]. \end{aligned}$$

We now calculate the half-derivative of t on $\mathbb{Z}$. Using (1.9) with $\nu = 1/2$, $N = 1$, and any $a \in \mathbb{R}$, we have

$$\Delta_a^{1/2} t = \Delta^1 \Delta_a^{-1/2} t, \quad t \in \mathbb{N}_{a+1/2}.$$

So we need to find $\Delta_a^{-1/2}t$ first. We will do that using the following form of integration by parts (Theorem 1.58 of [GP15]):

$$\int_b^c u(\tau)\Delta v(\tau)\Delta\tau = u(\tau)v(\tau)\Big|_b^c - \int_b^c v(\tau+1)\Delta u(\tau)\Delta\tau.$$

Above, we choose $u(\tau) = \tau$ and $\Delta v(\tau) = h_{-1/2}(t, \tau+1)$. Consequently, using Theorem 1.6, we obtain

$$\begin{aligned} \Delta_a^{-1/2}t &= \int_a^{t+1/2} h_{-1/2}(t,\tau+1)\,\tau\,\Delta\tau \\ &= -\tau\, h_{1/2}(t,\tau)\Big|_a^{t+1/2} \\ &\quad + \int_a^{t+1/2} h_{1/2}(t,\tau+1)\Delta t \\ &:= T_1 + T_2. \end{aligned}$$

Using (1.8), we have

$$\begin{aligned} T_1 &= \frac{1}{\Gamma(\frac{3}{2})}\left[-(t+1/2)\left(-\frac{1}{2}\right)^{\underline{1/2}}\right] \\ &\quad + \frac{1}{\Gamma(\frac{3}{2})}\left[a(t-a)^{\underline{1/2}}\right] \\ &= \frac{2}{\sqrt{\pi}}a(t-a)^{\underline{1/2}} \end{aligned}$$

since $(-1/2)^{\underline{1/2}}$ is zero because $t-n+1=0$ is a nonpositive integer and $t+1=1/2$ is not a nonpositive integer where $t=-1/2$ and $n=1/2$ in (1.5). In a similar fashion, one can show that

$$T_2 = \frac{4}{3\sqrt{\pi}}(t-a)^{\underline{3/2}},$$

and so

$$\begin{aligned} \Delta_a^{1/2}t &= \Delta^1\left[\frac{2}{\sqrt{\pi}}\left[a(t-a)^{\underline{1/2}} + \frac{2}{3}(t-a)^{\underline{3/2}}\right]\right] \\ &= \frac{2}{\sqrt{\pi}}\left[\frac{1}{2}a(t-a)^{\underline{-1/2}} + (t-a)^{\underline{1/2}}\right], \end{aligned}$$

which is analogous to $D_a^{1/2}t$. Figure 1 illustrates the case $a=0$ for multiple values of ν on the integers.

2. Monotonicity and Convexity

Early on in calculus, students learn that given a function $f : X \subseteq \mathbb{R} \to \mathbb{R}$, with X open and $c \in X$ at which f' exists, we have

$$f'(c) := \lim_{x\to c}\frac{f(x)-f(c)}{x-c}.$$

So $f'(c)$ only considers the ***local*** behavior of f near the point $x = c$.

One very important consequence of the local nature of the derivative operator is that the operator possesses a strong connection to the monotonic behavior of f. In particular, as every first-semester calculus student learns, given

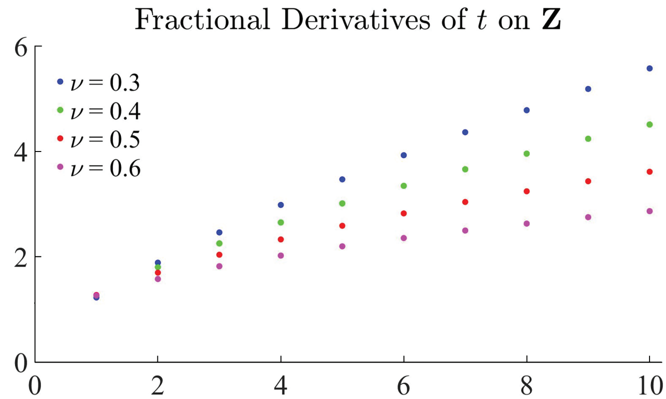

Figure 1. The fractional derivative of t is shown over the integers for various values of ν.

a differentiable function $f : X \subseteq \mathbb{R} \to \mathbb{R}$, if $f'(x) > 0$ for all x in some set $U \Subset X$, that is, U is compactly contained in X, then it follows that f is increasing on $\overline{U}$. The mean value theorem establishes this connection rigorously.

The situation in the discrete case is even more transparent. In particular, (1.4) computes the amount of change in f as we move the time point from t to $t+1$. Thus, only the time points t and $t+1$ are considered, and so, the behavior of f at any other points is ignored and plays no role whatsoever in the computation of $\Delta f(t)$. As in the continuous case, one consequence of the local structure of the difference operator is its connection to the monotonicity of f. In the integer-order setting, a short string of biconditional statements proves the connection between $\Delta f(t) \geq 0$ on $\mathbb{N}_a$ and f being increasing on $\mathbb{N}_a$. A natural follow-up question is whether this result holds for the discrete fractional case.

From the definition of the νth fractional difference as given by (1.9), we see there is a certain domain shift. Notice that the domain of the map $t \mapsto \Delta_a^\nu f(t)$ is different from that of the map $t \mapsto f(t)$. That is, the fractional forward difference shifts the domain of f from $\mathbb{N}_a$ to $\mathbb{N}_{a+N-\nu}$. This happens because the νth fractional difference is built from the νth-order fractional sum.

Here we see the nonlocal structure of the fractional difference. The map $t \mapsto \Delta_a^{-\nu}f(t)$ is a linear combination of the set $\{f(a), f(a+1), \dots, f(t-\nu)\}$. It is clear that the fractional difference of f at fixed time t depends on all values of f at all previous times. The implicit nonlocal structure of the discrete fractional difference makes it interesting and complex.

The first to consider the connection between the sign of $\Delta_a^\nu f$ and the monotonicity f were Dahal and Goodrich, who provided a partial answer in [DG14]. They concluded the following.

Theorem 2.1. *If y is nonnegative on $\mathbb{N}_0$ and $y(0) = 0$, then for fixed $\nu \in (1,2)$ with $\Delta_a^\nu y(t) \geq 0$ on $\mathbb{N}_{2-\nu}$, y is increasing on $\mathbb{N}_0$.*

The presence of the restrictions that y be nonnegative and $y(0) = 0$ is surprising since they are not required in the continuous or integer-order setting. Although not presented here, it is worth noting that the proof is vastly more complicated than that of its integer-order counterpart. This is due to the nonlocal structure of the fractional difference.

We see that $\Delta y(0) \geq 0$, which is an immediate consequence of the assumptions $y(0) = 0$ and y being nonnegative. Jia et al. discovered [JEP15] that the condition $\Delta y(0) \geq 0$ was necessary to generalize the work of Dahal and Goodrich. Their refinement states that if $y : \mathbb{N}_0 \to \mathbb{R}$ is nonnegative such that $\Delta y(0) \geq 0$, then for fixed $\nu \in (1,2)$ with $\Delta_a^\nu y(t) \geq 0$ on $\mathbb{N}_{2-\nu}$, y is increasing on $\mathbb{N}_0$.

After this improved result, Goodrich, Jia, et al. produced another refinement [EGJP17]. The prerequisite that $\Delta y(0)$ be nonnegative is sufficient, but not necessary to the degree that it can be replaced by a somewhat weaker condition. Before giving a more up-to-date monotonicity result, a preliminary result is needed. This was originally proved by Jia et al. [JEP15] and is necessary to improve the original monotonicity result. It is stated as follows.

Lemma 2.2. *Assume that $\Delta_a^\nu y(t) \geq 0$ for each t in $\mathbb{N}_{a+2-\nu}$ with $\nu \in (1,2)$. Then*

$$\Delta y(a+k+1) \geq -h_{-\nu}(a+k+2-\nu, a)y(a) - \sum_{\tau=a}^{a+k} -h_{-\nu}(a+k+2-\nu, \tau+1)\Delta y(\tau)$$

for each $k \in \mathbb{N}_0$, where

$$h_{-\nu}(t, \tau+1) = \frac{(t-\tau)^{\underline{-\nu}}}{(t-\nu-\tau)!\,(a+k+2-\tau)!} < 0$$

for $\mathbb{N}_{a+2-\nu}$, $a \leq \tau+1 \leq t+\nu-1$.

This result establishes a lower bound for $\Delta y(a+k+1)$. Now we give a more general result for monotonicity theorems when using the delta fractional difference.

Theorem 2.3. *If $y : \mathbb{N}_a \to \mathbb{R}$ is such that $\Delta_a^\nu y(t) \geq 0$ for fixed $\nu \in (1,2)$ on $\mathbb{N}_{a+2-\nu}$ and*

$$y(a+1) \geq \frac{\nu}{k+2} y(a)$$

for each $k \in \mathbb{N}_0$, then $\Delta y(t) \geq 0$ on $\mathbb{N}_{a+1}$.

There are two immediate observations to be made. First, y is not required to be a nonnegative map. This is an improvement over Dahal and Goodrich's result. However, if y is nonnegative, one can obtain more information from this more general result. Secondly, we observe that the hypotheses do not require an "initial monotonicity."

The lower bound on $y(a+1)$ in the theorem and the lower bound of $\Delta y(a+k+1)$ in the lemma are used together to conclude that y is monotone increasing. It is worth noting that the lower bound on $y(a+1)$ is sufficient. Its necessity is still an open question.

We now turn our attention to the relationship between the delta fractional difference and the convexity of the map y. As with monotonicity, the relationships are not straightforward and prove to be more complicated than in the integer-order setting. Erbe et al. [EGJP16] state the following proposition.

Proposition 2.4 ([EGJP16, Prop. 3.1]). *Let $y : \mathbb{N}_a \to \mathbb{R}$. Then $\Delta^2 y(t) > 0$ for each $t \in \mathbb{N}_a$ if and only if y is convex. Similarly, $\Delta^2 y(t) < 0$ for each $t \in \mathbb{N}_a$ if and only if y is a concave map on $\mathbb{N}_a$.*

This can be proved rather easily. However, if we move to the fractional-order case, the connection is more complicated. Goodrich [Goo14] initially studied this connection. Later Jia et al. [BEP15] pointed out that Goodrich's result omitted the important hypothesis $\Delta^{N-1}y(a) \geq 0$. Together they proved the following theorem.

Theorem 2.5. *Assume that $\Delta_a^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{N+a-\nu}$ with $N-1 < \nu < N$, $N \in \mathbb{N}_3$. Additionally, suppose that $(-1)^{N-i}\Delta^i y(a) \geq 0$ for $i = 0, 1, \dots, N-2$ and $\Delta^{N-1}y(a) \geq 0$. Then $\Delta^{N-1}y(t) \geq 0$ for each t in $\mathbb{N}_a$.*

The proof relies on and complements Theorem 2.1. It repeatedly applies the result to the map $w : \mathbb{N} \to \mathbb{R}$ given by $w(t) = \Delta^{N-2}y(t)$ to establish that w is increasing. From this result, one has the following corollary.

Corollary 2.6 ([EGJP16]). *Fix $\nu \in (N-1, N)$ with $N \in \mathbb{N}_3$. Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{N-\nu}$. In case N is odd, assume that*

$$\begin{cases} \Delta_0^j y(0) < 0, & j = 0, 2, \dots, N-3, \\ \Delta_0^j y(0) > 0, & j = 1, 3, \dots, N-4, \end{cases}$$

whereas in the case N is even, assume that

$$\begin{cases} \Delta_0^j y(0) > 0, & j = 0, 2, \dots, N-3, \\ \Delta_0^j y(0) < 0, & j = 1, 3, \dots, N-4. \end{cases}$$

If, in addition, $\Delta_0^{N-2}y(0) \geq 0$ and $\Delta_0^{N-1}y(0) \geq 0$, then $\Delta_0^{N-1}y(t) \geq 0$ for each $t \in \mathbb{N}_0$.

In the case $2 < \nu < 3$, that is, $N = 3$, we see the following geometrical interpretation of the above result.

Corollary 2.7 ([BEP15]). *Assume that $y : \mathbb{N}_a \to \mathbb{R}$ satisfies $\Delta_a^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_a$ with $2 < \nu < 3$, and $y(a) \leq 0$, $\Delta y(a) \geq 0$, $\Delta^2 y(a) \geq 0$. Then $\Delta^2 y(t) \geq 0$ for $t \in \mathbb{N}_a$.*

By taking a closer look at the conditions, we see that y must be *initially* nonnegative, increasing, and convex. If all this happens and $\Delta_a^\nu y(t)$ is nonnegative, then we have $\Delta^2 y(t) \geq 0$. As a matter of fact, we can deduce y is convex if $\Delta^2 y(t) > 0$. An improvement on this result follows.

Theorem 2.8 ([Goo16]). *Fix $\nu \in (2,3)$ and suppose that $\Delta_a^\nu f(t) \geq 0$ for each $t \in \mathbb{N}_{3+a-\nu}$. If for each $k \in \mathbb{N}_{-1}$ we have*

$$\begin{aligned} &\frac{1}{-\nu+1} f(a+2) + \frac{\nu+2+k}{(\nu-1)(k+3)} f(a+1) \\ &\quad - \frac{\nu}{(3+k)(4+k)} f(a) \\ &\quad \leq 0, \end{aligned} \tag{2.1}$$

then $\Delta^2 f(t) \geq 0$ for each $t \in \mathbb{N}_{a+1}$.

Define a function f such that $f(a) = 0, f(a+1) = 1$, and $f(a+2) = 1.9$. Then it satisfies the hypotheses of Theorem 2.8. If we fix $\nu = \frac{5}{2}$, then inequality (2.1) holds, but $\Delta^2 f(a) = -\frac{1}{10} < 0$. Thus, Theorem 2.8 does not require any initial convexity. The sharpness of these results is an open question. As with the monotonicity results, it would be interesting to determine the optimal convexity-type result.

3. Biological Modeling

In recent years, discrete fractional calculus has begun to be applied in the biosciences. The appeal of discrete fractional calculus is that the fractional difference of a function depends on its whole time history, and not on its instantaneous behavior. Such a characteristic of the fractional difference operator in biosciences is perfectly suited for the description of materials with memory. Thus, the nonlocal nature of the fractional difference operator that led to surprisingly complex results involving monotonicity and convexity also lends itself to advances in modeling.

In [AAHN15], F. M. Atıcı et al. used discrete fractional calculus to model tumor volume via sigmodial curves. The chosen curves were Gompertz, Logistics, Richards, and Weibull, with each curve having a discrete, continuous fractional, and discrete fractional version. As the continuous version of the curves have the (natural) exponential function, a description of how to replace it with an appropriate representation for each form of each curve was given.

The authors accessed tumor volume data for 28 mice taken over 17 consecutive days. Figure 2 is a plot of the daily mean tumor volume (y-value denoted by a red dot) of 28 mice and of the standard deviations of volume (represented by bars). Using residual sum of squares of the aggregated mean tumor volume, continuous fractional Gompertz (in yellow), continuous fractional Logistic (in gray), continuous Richards (in green), and continuous Weibull (in blue) versions fit the data best and these fits are plotted in Figure 2.

The focus then became how the models fit to the individual mice's data and their predictive performance. To assess these, residual sum of squares (RSS), standard error (SE) of the estimate, adjusted coefficient of determination (ACD), and cross-validation methods were used. The

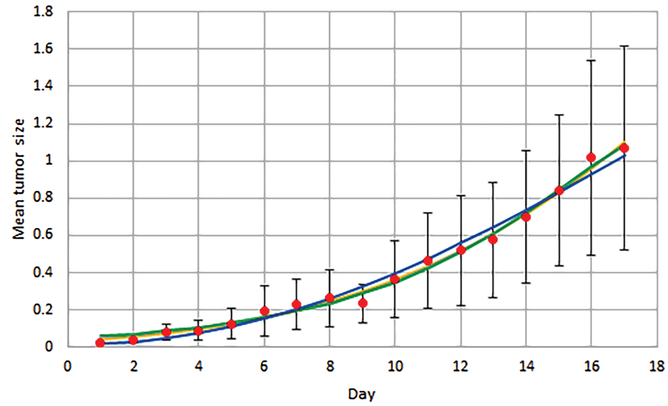

Figure 2. Mean tumor volume with standard deviation by day and fits of curves.

authors found that now in the individual case the discrete versions significantly outperform the continuous ones in terms of data fitting; they produced the most models with minimum RSS. This improvement may be seen because time is measured on a discrete scale. Furthermore, this difference is insightful because individual mice have different growth curves, and that growth can be quite different if the model is applied to mean tumor volume. When model types were compared across all versions, Richards curve gave the best fit for the largest number of mice data sets.

The discrete versions were again better than the continuous ones in producing minimum SE of the estimate and the max ACD. Although the discrete form of the Richards curve had the most (20) mice data that best fit with respect to ACD, the Logistic curve had the most (17) across all versions. This is thought to be so because max ACD penalizes models for using more parameters. Logistic curves have three parameters while Richards and Weibull curves have four.

Better fit does not necessarily mean a good predictive performance. In order to gauge the predictive performance of the models, the authors used the k-fold cross validation method and predicted residual sum of squares (PRESS). A smaller value of PRESS indicates better performance in predicting a future value of tumor volume. The fractional models were better predictors than the traditional ones. The authors observed that there were 32 continuous fractional and 30 discrete fractional models that had the minimum PRESS compared to 24 continuous and 26 discrete.

In summary, the traditional models resulted in better fit, while the fractional models were better at predicting future outcomes than the traditional ones.

In [AABM17], F. M. Atıcı et al. continue modeling using fractional calculus. While the goal remained to model tumor growth in mice, this time only discrete fractional equations were considered. Starting from the Gompertz

Mouse Identifier	*Old method of partial sums in Discrete*	*New method of partial sums in Discrete*	*Improvement with new method in Discrete*	*New method of partial sums in Discrete-Fractional*	*Improvement with new method in Discrete-Fractional*
$1-4$	0.0733707	0.004505497409094128	93.85%	0.0041820751	7.17%
$1-5$	0.0527466	0.04261294839958312	19.21%	0.0256315825	39.85%
$5-1$	0.0120763	0.010400107799865513	13.88%	0.00932411659	10.34%
$5-3$	0.0161821	0.013243640560150597	18.15%	0.01001867981	24.35%
$9-1$	0.0599209	0.017423882860512472	70.92%	0.01389409623	20.25%
$9-2$	0.0114475	0.014405508857738373	N/A	0.00856935321	40.51%
$9-4$	0.00010619	0.000062307725400199	41.32%	0.00005501819	11.7%
$A6-1$	0.013071	0.020630076592876006	N/A	0.01475519747	28.47%
$A6-2$	0.0154899	0.014043018252413534	9.34%	0.01165751860	16.98%
$A6-4$	0.0273169	0.042661876659798564	N/A	0.04050010184	5.06%
$A6-5$	0.00211683	0.0031591615221599366	N/A	0.00199468791	36.86%
$B2-1$	0.00167148	0.002623607499610515	N/A	0.00173688210	33.79%
$B2-2$	0.0295633	0.020772251406296828	29.73%	0.01496914862	27.93%
$B2-3$	0.174809	0.14893675896412792	14.80%	0.13590911970	8.74%
$B2-5$	0.00495535	0.004895335102352389	1.21%	0.00425793938	13.02%

Table 1. Models with minimum RSS.

differential equation

$$y' = (c - b \ln y(t))\, y(t), \tag{3.1}$$

where b and c are parameters, the authors described the process of obtaining a fractional Gompertz difference equation and developing a method to estimate parameters for discrete Gompertz equations. The proposed fractional Gompertz difference equation is

$$\Delta^\nu y = (c - b \ln y(t))\, y(t), \tag{3.2}$$

where $\nu \in (0, 1.5)$.

The authors developed a parameter estimation method which they call the "improved partial sum method." The advantage of this method is that it can be used for *both* solutions and equations. Using their method does not require solving the model for data fitting. There are two reasons for exploring this new approach for data fitting. The first is to avoid using solutions obtained from approximation techniques, and the second is to keep the number of parameters as small as possible. They showed that the improved partial sum method gave better results for least squares fitting with (3.2) (see Table 1). For a certain mouse, column 2 gives the RSS value using the old method of partial sums for the discrete model ((3.2) with $\nu = 1$), and column 4 gives the RSS value under the new method of partial sums for the same discrete model.

When comparing the RSS values in the discrete case for the improved method of partial sums and the original one, the RSS values decreased 10 times out of a possible 15 with the greatest improvement being 93.85% (as seen in column 3). When using the new method for both discrete and discrete fractional models, the RSS values decreased for all 15 mice (when compared to the old method for the discrete model). In all instances, the SE associated with the discrete fractional model ((3.2) with $\nu \in (0, 1.5) \setminus \{1\}$) improved partial sum method was lower than the SE associated with the discrete improved partial sum method. Similarly, the r^2 values for the discrete fractional model were always higher.

To further solidify the fact that the discrete fractional Gompertz equation modeled the data well, the authors compared its RSS using the improved partial sum method to the RSS of the continuous Gompertz equation (3.1) with the Marquardt-Levenberg algorithm. The last column of Table 2 (IMP means improvement) shows data fitting with the new method for discrete fractional is an improvement over data fitting using the Marquardt-Levenberg algorithm for the continuous Gompertz equation. The greatest improvement was 93%.

In short, the authors presented a discrete fractional equation to model tumor growth in mice. For data fitting purposes, they introduced an algorithm for parameter estimations without needing to solve the equations. From the statistical measures presented, the reader can conclude that the discrete fractional model offers improved data fitting when compared to its discrete form.

4. Final Words

Fractional equation theory is a rich branch of mathematics that includes fractional differential equations and discrete fractional difference equations. This area has gained considerable popularity within the past decade, although it was initially studied by L'Hôpital in 1695.

A recent paper by Goodrich and Lizama [GL20a] investigated convolution-type inequalities, i.e., inequalities of the form

$$\big(\Delta^n (a * u)\big)(t) \geq 0$$

for some positive integer n. Here $a * u$ represents the finite convolution of a and u. By choosing a in a particular

Mouse Identifier	*New method (Discrete-Fractional)*	*M-L Alg. (Gompertz-Curve)*	*IMP (%)*
$1-4$	0.00442823	0.00442823	0%
$1-5$	0.0256315825	0.0391794	35%
$5-1$	0.00932411659	0.010066	7%
$5-3$	0.01001867981	0.0154649	35%
$9-1$	0.01389409623	0.0332244	58%
$9-2$	0.00856935321	0.0114397	25%
$9-4$	0.00005501819	0.00008154	33%
$A6-1$	0.01475519747	0.012371	N/A
$A6-2$	0.01165751860	0.1741	93%
$A6-4$	0.04050010184	0.0237391	N/A
$A6-5$	0.00199468791	0.00210743	5%
$B2-1$	0.00173688210	0.00169775	N/A
$B2-2$	0.01496914862	0.0228022	34%
$B2-3$	0.13590911970	0.179615	24%
$B2-5$	0.00425793938	0.00508966	16%

Table 2. Comparing two models.

way we can recover from the abstract convolution inequality certain fractional difference inequalities and thus obtain various monotonicity- and convexity-type results. An interesting unresolved question is to what extent this abstraction to the convolution setting is useful in answering these types of qualitative questions regarding fractional difference operators. The approaches used in [GL20a] are interesting and elegant. But just how far they can be pushed in the effort to fully understand the qualitative properties of discrete fractional operators is an open question.

A second open question is one of multiplicity and optimality. There are many different monotonicity-type (similarly, convexity-type) results in the literature now. But how many undiscovered ones are there? Are there better ones (in some reasonable sense), which have yet to be discovered? For example, comparing the results in Goodrich [Goo14, Goo16] and Baoguo, Erbe, and Peterson [BEP15] we see a variety of convexity-type results. But are there others which have not been discovered and which would be interesting for one reason or another?

A third open question concerns extremality. In particular, it is well known that in calculus a change in the sign of the derivative indicates the existence of a minimum or a maximum. But can anything be said in the case of a fractional difference? This would seem to have a natural connection to monotonicity- and convexity-type results. Nonetheless, the only work of which we are aware is a paper by Dahal, Drvol, and Goodrich [DDG17]. This represents another collection of open problems in the area.

Sequence fractional differences are a final area of current research that is worth investigating. The discrete fractional operator is, in general, noncommutative, i.e., $\Delta^{\mu}\Delta^{\nu} \neq \Delta^{\nu}\Delta^{\mu}$ for noninteger μ and ν. This is in considerable contrast to the integer-order setting, wherein the operator action $f \to (\Delta \circ \Delta)f$ can be unambiguously defined

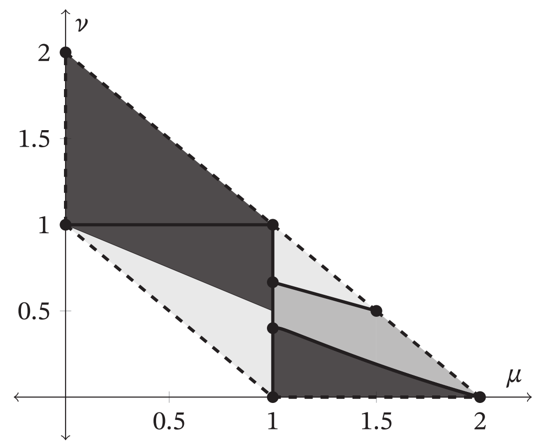

Figure 3. Parameter pair (μ, ν) and qualitative behavior of f. `https://www.springer.com/journal/11117`.

by

$$\begin{aligned}((\Delta \circ \Delta)f)(t) &:= (\Delta^2 f)(t) \\ &:= f(t+2) - 2f(t+1) + f(t).\end{aligned}$$

This leads to the concept of a "sequential fractional difference," which is the composition of two or more fractional difference operators of (possibly) unequal orders—for example,

$$\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f(t).$$

An interesting question then arises: what sort of connections (if any) exist between the sign of $\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f(t)$ and the convexity, concavity, or monotonicity of f itself? This study of the connection between sequential fractional differences and the monotonicity and convexity of functions is very much in its infancy. We refer to results in [GL20b]. The study of the fractional sequential inequality

$$\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a}f(t) \geq 0 \tag{4.1}$$

results in a very delicate interplay between the choice of the parameter pair (μ, ν) and whether there is any connection between (4.1) and the qualitative behavior of f (e.g., monotonicity and convexity). For example, consider Figure 3. In this drawing, where we assume (for simplicity) that the admissible parameter space is restricted to $1 < \mu + \nu < 2$,

1. the ***dark grey*** region is where there is a strong connection between the sign of (4.1) and the monotone behavior of f;
2. the ***light grey*** region is where there is (effectively) no connection between the sign of (4.1) and the monotone behavior of f; and
3. the ***medium grey*** region is where whether there is any kind of connection is, at present, ***unknown***.

Note that in this drawing we see that the case $(\mu, \nu) \in (0,1) \times (0,2)$ is completely dispatched, whereas the case

$(\mu, \nu) \in (1,2) \times (0,1)$, with $\mu + \nu \in (1,2)$, still has some "gaps" in it. In addition, we notice that

1. in case $(\mu, \nu) \in (0,1) \times (0,1)$ the "good" region is *above* the "bad" region; whereas
2. in case $(\mu, \nu) \in (1,2) \times (0,1)$ the "good" region is *below* the "bad" region.

Why should this be? This is indeed a very good open question.

A final open area of research is to extend fractional calculus beyond the real numbers and integers. That is, to define the fractional dynamic calculus and fractional dynamic equations on an arbitrary, nonempty closed subset of the real numbers, known as a time scale $\mathbb{T}$. Georgiev attempts to do this in [Geo18]. However, when the time scale is the integers, the definition of the Riemann-Liouville fractional delta-integral of order $\alpha > 0$ conflicts with the standard definition of the discrete Riemann-Liouville integral of order $\alpha > 0$ as given in [GP15]. Consequently, the unification of the continuous and discrete fractional calculus is not achieved.

The applications of fractional calculus extend to finance and economics. The authors of [TPV20] investigate the economic growth of G20 counties from 1970 to 2018. While integer-order models gave good estimates of gross domestic product, they found that fractional-order more accurately predicted economic growth without increasing the number of parameters or sacrificing the ability to predict the progression of gross domestic product in the short-term.

Furthermore, discrete fractional calculus can be applied to statistics. In [GG17], the authors introduce two generalized random variables. They show the expectation of the first type requires both the delta Riemann left fractional sum and difference of the probability function. Similar results hold between the second type and the nabla fractional sum and difference. These relationships allow for the development of fractional versions of discrete uniform distributions.

Even with these successes on the integers, there are still avenues for future research in discrete fractional calculus, with the eventual hope of more general applications to time scales.

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