Remembrances of Harold Widom

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Estelle Basor

The idea of small Oxford-like colleges, snow free winters, and views of the ocean appealed to Harold Widom when he joined the faculty of the University of California at Santa Cruz in the Fall of 1968. The first course he taught was a two quarter sequence in undergraduate real analysis, and I was very fortunate to be a student in the class. He came to class often with only a tiny piece of note paper. With the note paper, chalk, and the blackboard he would produce the most exquisite and captivating lectures. Harold taught in a heuristic style. He would map out how one might arrive at a result and then fill in the details. He once told me that for him, mathematics was about looking for and finding answers, and in analysis, he could do that better than anyone.

Harold was born in Newark, New Jersey, on September 23, 1932. His brother Benjamin Widom (Ben) was born five years earlier. Harold said very little to me about his early years. I only found out about his childhood through my recent correspondence with Ben and with Harold's daughters Jennifer and Barbara Widom. From Ben, I learned that Harold was only eight when their father, Morris Widom, died, but that neither had seen him in the preceding three years, since their father had been in a tuberculosis sanitarium in Arizona and then in Colorado. Their father was a dentist who contracted tuberculosis while serving in the US army in the First World War, having been drafted soon after immigrating to the US from Russia.

In 1939, Harold, Ben, and their mother moved to Brooklyn. Harold attended Stuyvesant High School in Manhattan where he was captain of the math team. Other

Figure 1. A very young Harold.

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team members included Elias Stein and Paul Cohen. Ben also recalled that Harold told him that meeting Ben's mathematically inclined high-school and college friends, including Herman Zabronsky and Melvin Hausner, helped pave his path to mathematics. After graduating from high school at the age of 16 in 1949, he attended the City College of New York for two years and while there, in 1951 at the age of 18, was named a Putnam Fellow. (If the dates do not seem to make sense, it is because the Putnam exam was given in March of 1951.) Harold then transferred to a special PhD program at the University of Chicago and received his Master's Degree a year later and his PhD under the direction of Irving Kaplansky in 1955 when he was only 22. Despite the pace of his achievements, he once said he wasted a whole year of graduate school playing bridge.

Figure 2. Harold and Ben Widom circa 1938.

In 1955, Harold joined the faculty of Cornell where much of his fundamental research in operator theory began. Ben had previously joined the chemistry faculty and the two brothers lived down the street from each other. Influenced by Mark Kac, he became interested in the properties of Toeplitz matrices and operators. He proved many beautiful results including ones about the invertibility of the operators, index theorems for the operators, the connectedness of the spectrum, asymptotic results for the spectra, and much more. Toeplitz operators are discrete convolution operators and if one traces through his work, they (or close cousins) are always at its heart. His work included the study of Wiener-Hopf operators, the continuous version of Toeplitz operators. A particular Wiener-Hopf operator is the one whose kernel is the sine kernel, $\frac{\sin(x-y)}{x-y}$, closely tied to the groundbreaking work in random matrix theory done jointly with Craig Tracy. For a much more complete description of his work in the area of Toeplitz operators and random matrix theory, we refer the reader to two recent articles [4, 9].

One day at the end of class in the second half of undergraduate real analysis at UC Santa Cruz, Harold told me that the Mathematics Department was starting a graduate program in the Fall and asked me if I was interested in becoming one of the first students. I did not hesitate to agree to this knowing that there was a possibility that I could work with him. There were seven students in the first mathematics graduate student office. Six of us were Santa Cruz students and one other was Lidia Luquet who was actually Harold's student from Cornell, but in residence at Santa Cruz to finish her work. Lidia and I were two of eventually eight students who finished their degrees with Harold. Four of us were women, a remarkable number given the fact that in the 1970s and 80s, only about 14% of PhDs in mathematics were awarded to women. Harold's brilliance as a mathematician was clearly a reason the four of us wanted him as an advisor. But on a deeper level, I think we knew he would always be encouraging, kind, honest, and fair.

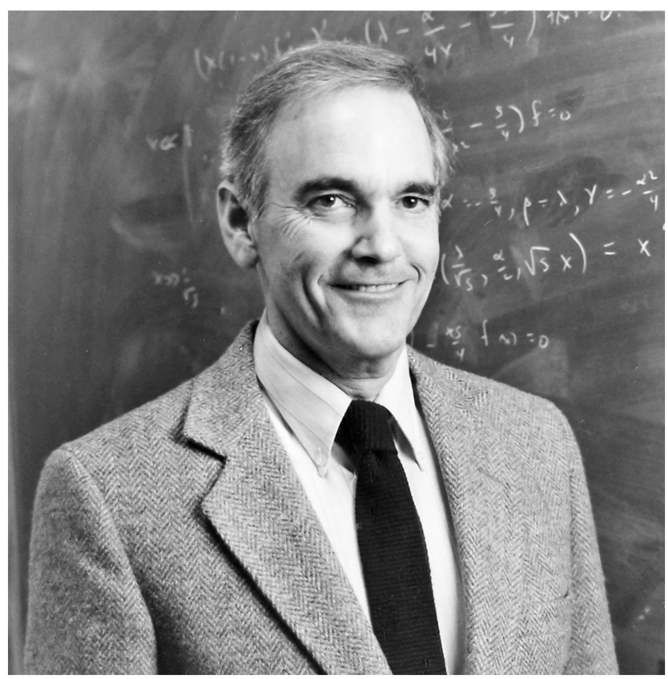

Figure 3. Harold in 1990, in a favorite place—by a blackboard.

As of now, MathSciNet lists 167 publications by Harold. He wrote the volume *Asymptotic expansions for pseudodifferential operators on bounded domains* which was published by Springer-Verlag, in 1985, as part of the Lecture Notes in Mathematics series. He also wrote two beautiful short books for students. These were *Lectures on Integral Equations* and *Lectures on Measure and Integration*, based on lectures he gave at Cornell. The notes were written by David Drazin and Anthony J. Tromba, both students in his classes at the time. They were first published by Van Nostrand and later by Dover. At the time of the original publication, Van Nostrand focused on publishing current mathematics in an informal and heuristic style, and the notes clearly display Harold's ability to convey mathematics in a most natural and compelling way.

One of the most endearing qualities of Harold is that he never drew attention to himself or his work. The work spoke for itself and he received numerous awards. In 2002, he, along with Craig Tracy, shared the George Pólya Prize for their work on random matrices. In 2006, he was elected to the American Academy of Arts and Sciences, and he and Craig received the Norbert Wiener Prize in Applied Mathematics. Then in 2020, Harold and Craig received the American Mathematical Society's Steele Prize for Seminal Research.

Harold had many interests outside of mathematics. He played the violin as a child and was part of the UCSC orchestra for several years. He loved good mysteries, tennis, hiking, and chocolate cake. While I am forever grateful to Harold for being my thesis advisor, I am most grateful for his friendship. He seemed to have a sixth sense about getting in touch when I was stuck by either a mathematics question or some other problem.

In the reminiscences that follow a clear pattern emerges. For all of us who knew him well and collaborated with him, he often answered a question we could not answer ourselves and in the process we became aware of not only his mathematical insight and power but also his goodness, kindness, and generosity as a person.

Estelle Basor

Albrecht Böttcher

I encountered the name Harold Widom for the first time in the late 1970s. I then wrote my diploma paper under

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Figure 4. Hiking in 2014.

the guidance of Bernd Silbermann. Its topic was the extension to block Toeplitz matrices of Silbermann's new approach to Szegő's limit theorem. To approach the problem he gave me a copy of an article on curled up photocopy paper that resisted placing it between two books. This article was Widom's 1976 paper [18]. I read it with excessive appetite, and although I did not understand everything then, it was one of the papers that paved my way. Both my diploma paper and my very first publication, jointly with Silbermann in 1980, are imbued by arguments based on Widom's ideas and techniques.

In those years, German authors gradually began to publish in English. I used that paper by Widom to make a list of idiomatic phrases I could use in my forthcoming papers. One of my (and I think also of his) favorites was the tension-creating "what results is . . . ." Since then I have used this turn of phrase many times, but, unfortunately, with rare success: it has usually not been accepted by the copy-editors and rather been changed into the dry "it results. . . " or "the result is. . . ."

In the 1980s, the conjecture of Fisher and Hartwig was the focus of research on Toeplitz determinants. This conjecture predicts the asymptotic behavior of the determinants generated by a large class of so-called singular symbols, that is, by functions with zeros, poles, and discontinuities, and thus in the case where the assumptions of Szegő's strong limit theorem are violated. When diving into this subject, I read further papers by Widom and learned of the tremendous work he had done in the 1970s on Toeplitz determinants with singular symbols. The article [4] describes his extraordinary achievements in detail. During the 1980s, Harold became one of my true mathematical heroes, and so it was a highlight in my life to meet him in person at an Oberwolfach conference in 1989. At that time, there were many mathematicians I had known by name only, but had never met in person. Things changed with the fall of the Iron Curtain in 1989.

In 1992, I was invited to participate in the conference dedicated to Harold on his 60th birthday, which was taking place in Santa Cruz. This was another unforgettable event in my life. It was my very first trip across the Atlantic Ocean and I met in person many of my other mathematical heroes, including Estelle Basor, Ronald Douglas, and Donald Sarason. I had lots of inspiring discussions with all the participants, in particular with Harold, who also took me on a half-day trip to Carmel Bay in the south of Santa Cruz.

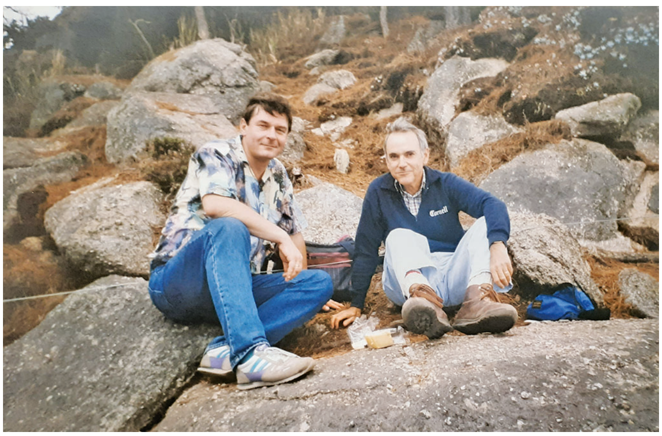

Figure 5. Albrecht Böttcher and Harold Widom at Point Lobos in 1992.

Since those days, I had many encounters and a good deal of email correspondence with Harold. I was in particular happy that he visited me in Chemnitz twice, and each time combined the visit with a trip around Saxony with me. It was always fascinating to learn how he looked at things and to profit from his ingenious insights and ideas. Here is a concrete story. The $n \times n$ Toeplitz matrices $T_n(\alpha)$ generated by the Fourier coefficients of the function $|e^{i\theta} - 1|^{2\alpha}$ with a positive integer α play a distinguished role in several problems; see, e.g., [6]. The matrix $T_n(\alpha)$ is a symmetric banded Toeplitz matrix the entries in the first row of which are the $2\alpha + 1$ numbers

$$(-1)^k \binom{2\alpha}{\alpha - k} \quad (0 \le k \le 2\alpha)$$

followed by zeros. The smallest eigenvalue of $T_n(\alpha)$ has the asymptotic behavior

$$\lambda_{\min}(T_n(\alpha)) \sim \frac{c_\alpha}{n^{2\alpha}} \quad \text{as } n \to \infty,$$

where $a_n \sim b_n$ means that $a_n/b_n \to 1$. Kac, Murdock, and Szegő (1953) proved that $c_1 = \pi^2 \approx 9.87$ and Parter (1961) showed that $c_2 \approx 500.547$, which is the fourth power of the smallest positive number μ satisfying $\cos\mu \cosh\mu = 1$. Around 2006, I found that $c_3 = (2\pi)^6 \approx 61\,529$ and this made me conjecture that the coefficients c_α obey the asymptotics $c_\alpha \sim ((\alpha+1)\pi/2)^{2\alpha}$ as $\alpha \to \infty$. The first three values of the asymptotic expression are $\pi^2, 493, (2\pi)^6$, which is in nearly perfect agreement with the data. I told Harold about this and he immediately replied that according to his intuition this cannot be the right asymptotics. So we sat down at our desks and after some intense correspondence we were able to prove that actually

$$c_\alpha \sim \sqrt{8\pi\alpha}\left(\frac{4\alpha}{e}\right)^{2\alpha} \quad \text{as } \alpha \to \infty;$$

see [6]. This time the first three values of the asymptotic formula are $10.85, 531, 64\,269$. And what about the fourth term? Numerical computations indicate that $c_4 \approx 14\,000\,000$, the right asymptotics indeed delivers $14\,457\,978$, whereas the wrong asymptotics gives $1\,445\,565$. Thus, had I considered the fourth term, I would have seen that my conjecture was wrong. Harold didn't need a fourth term. He felt it immediately!

At the MSRI workshop on random matrix theory in Berkeley in 2002 which was dedicated to Harold on his 70th birthday, I opened my talk with the words "I am very sorry, but although related to Harold's work, the following will be an absolutely deterministic talk. However, I am here to celebrate Harold's birthday and to profess my bond of friendship with and my admiration for him, even at the price that you will rank me as absolutely not belonging to the random matrix community."

Throughout my life, I have had the fortune to learn from and to work with many outstanding mathematicians, those from the generation of my parents, those of my age, and of course those from the younger generations. All of these people have strongly influenced my interests and my way of doing mathematics. Harold Widom is in the top of the colleagues to whom I owe the most. I am extremely glad and thankful for having had him as a partner and friend for decades, for benefitting from his genius and personality. Now he has left us, but I am consoling myself with the certainty that his name and achievements will live on and inspire future generations.

Albrecht Böttcher

Ivan Corwin

Harold Widom set a wonderful example of what it means to be a mathematician. On every occasion that I got to spend time with him, he had a childlike giddiness and excitement about him, and an unmatched fearlessness. Before witnessing this, I had the impression that every senior mathematician knows every answer (or at least pretends to). In watching Harold think (for instance, when he, Craig Tracy, and I spent time together at Craig's house or at Oberwolfach), it was illuminating to see that he was willing to struggle with problems, to try new ideas and areas, and take chances. I had heard stories about Harold as a teenager and I think Percy Deift had once shown me a photo of Harold on the Stuyvesant math team. I never saw Harold as old. He always seemed to have the energy of that high school math competitor that I imagined from Percy's photograph.

I first met Harold in December of 2008 at Joel Lebowitz's 100th Statistical Mechanics conference at Rutgers. I was a second year graduate student then, and I had devoted most of that fall to carefully reading his recent work with Craig Tracy on the asymmetric simple exclusion process (ASEP). At Courant, I was organizing a weekly reading seminar on this material so I was quite excited when I saw Harold and Craig for the first time. The specifics of how I ended up talking to them are hazy for me, but I do distinctly remember making a claim to them about a calculation I had made which was just plain wrong. Despite this, I left our brief yet meaningful interaction there feeling quite positive—two giants of mathematics had been willing to hear me out and help me understand my mistake.

It is difficult to overstate the impact that Harold's work has had on my own mathematical development as well as the whole fields of random matrix theory and integrable probability. My 2013 survey on "Two ways to solve ASEP" started with the following statement: "Exact formulas in probabilistic systems are exceedingly important, and when a new one is discovered, it is worth paying attention. This is a lesson that I first learned in relation to the work of Tracy and Widom on the asymmetric simple exclusion process (ASEP) and through my subsequent work on the Kardar-Parisi-Zhang (KPZ) equation." When I sent a draft of this survey to Harold, he replied (in his characteristically short and to-the-point style) "I liked your first sentence a lot. I liked the rest, too, of course, but that requires more concentration."

Looking back at the quote from my survey, I feel like it missed an important point. It is the hands of a sculptor, not the clay, which forms a work of art. Harold had an amazing ability and tenacity to find new ways to shape and massage formulas until they yielded to him. So that I do not talk purely in metaphors, let me give an example. This is the example that I studied for my entire second year of graduate school.

Start with $k \in \mathbb{N}$ particles inhabiting sites of $\mathbb{Z}$. Sites only have the capacity to hold at most one particle, so the state of this system can be kept track of by an ordered vector $\vec{x} = (x_1 < \cdots < x_k) \in \mathbb{Z}^k$. Particles evolve in the following stochastic manner. Every particle has a propensity to jump left or right by one site, and these jumps occur independently according to exponentially distributed waiting times. The left jumps occur at rate q and the right jumps at rate p. We normalize time so $p + q = 1$. The only interaction between particles comes in the form of an *exclusion* rule which states that any attempted jump which would violate the one-particle-per-site condition, is ignored. The above is an informal description of the *interacting particle systems* known as ASEP.

Despite the simplicity of its definition, ASEP is an incredibly rich mathematical system (with connections to PDEs, algebraic combinatorics, symmetric function theory, Hecke algebras to just name a few) and also a paradigmatic model which reveals universal physical phenomena believed to be present in wide classes of particle systems (as well as related areas such as interface growth). One of the most fundamental problems is to understand how the system behaves when k is large.

The starting point for Tracy and Widom's ASEP work is the following transition probability formula. For $t > 0$ and $\vec{y} = (y_1 < \cdots < y_k) \in \mathbb{Z}^k$, let $P_{\vec{y}}(\vec{x}; t)$ denote the probability that ASEP started at time zero in state $\vec{y}$ is in state $\vec{x}$ at time t. Then, provided that $p \neq 0$,

$$P_{\vec{y}}(\vec{x}; t) = \sum_{\sigma \in S_k} \int A_\sigma \prod_{j=1}^{k} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{\epsilon(\xi_j) t} \frac{d\xi_j}{2\pi i}. \qquad (1)$$

This formula needs some explanation. The sum $\sigma \in S_k$ is over permutations on k elements; there are k complex contour integrals in the variables $\xi_1, \ldots, \xi_k$ which are taken to lie on small circles around the origin with radius small enough so as not to contain any poles of the A_σ term; this A_σ term is given by the formula

$$A_\sigma = \prod \{S_{\alpha\beta} : \{\alpha, \beta\} \text{ is an inversion of } \sigma\},$$
$$S_{\alpha\beta} = -\frac{p + q\xi_\alpha \xi_\beta - \xi_\alpha}{p + q\xi_\alpha \xi_\beta - \xi_\beta},$$

where the product is of terms $S_{\alpha\beta}$ over all pairs $1 \leq \alpha < \beta \leq k$ such that $\sigma(\alpha) > \sigma(\beta)$. Finally, $\epsilon(\xi) = p\xi^{-1} + q\xi - 1$.

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For example, when $k = 2$, this formula reads

$$P_{(y_1,y_2)}(x_1, x_2; t) = \int\int \frac{d\xi_1}{2\pi i}\frac{d\xi_2}{2\pi i} e^{\epsilon(\xi_1)t+\epsilon(\xi_2)t}\Big(\xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} - \frac{p+q\xi_1\xi_2-\xi_1}{p+q\xi_1\xi_2-\xi_2}\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1}\Big).$$

The $k = 2$ formula above was known from 1997 work of Schütz; the general k formula was not proved until Tracy and Widom's 2007 work. The structure of the above formula has some precedent coming from the *coordinate Bethe ansatz* in exactly solvable models in statistical mechanics. Indeed, the formula can be thought of as a sort of Plancherel formula for the Fourier transform associated to the general of ASEP.

What Tracy and Widom were able to do with the above formula is utterly amazing. First, they used the formula to extract a formula for the marginal distribution of the m^{th} particle (for $m \leq k$). Then, they set $y_i = i$ for $i \in \{1, \dots, k\}$ and took $k \to \infty$. This infinite initial configuration corresponds to having every site to the right of the origin occupied at time zero, and every other site empty. This is known as *step initial data* since the density of particles resembles a step function. At this point, Tracy and Widom were able to massage their formula for the distribution of the m^{th} particle to yield the following result: If $N_0(t)$ denotes the number of particles which are to the left of or at the origin at time t, then

$$\text{Probability}\Big(N_0(t) = m\Big) = -(p/q)^m \int \frac{\det(I-\zeta K)_{L^2(C_R)}}{\prod_{j=0}^{m}\Big(1-\zeta\,(p/q)^j\Big)}\frac{d\zeta}{2\pi i} \tag{2}$$

where the integral in ζ is over a contour enclosing $\zeta = q^{-k}$ for $k \in \{0, \dots m-1\}$, and $\det(I-\zeta K)_{L^2(C_R)}$ is the Fredholm determinant of the operator $K : L^2(C_R) \to L^2(C_R)$ whose integral kernel is given by

$$K(\xi, \xi') = q \cdot \frac{e^{\epsilon(\xi)t}}{p + q\xi\xi' - \xi},$$

and where C_R is a sufficiently large circle centered at zero.

In studying probabilistic systems one is often interested in taking scaling limits where the system size and time diverge. This is because, like in the field of statistical mechanics, probabilistic systems are often thought of as microscopic models for real world systems; and the real world is big. Moreover, there is a pervasive and in some cases demonstrated belief that under such scaling limits, many different microscopic systems will behave similarly. This is akin to the ubiquitous occurrence of the bell curve across many domains of math and science.

For the ASEP with step initial data, the natural question is to study the behavior of $N_0(t)$ for large t. This is most interesting when there is a net drift of particles to the left, meaning that we assume $q > p$. Under this condition, Tracy and Widom proved the following asymptotic result:

$$\lim_{t\to\infty} \text{Probability}\Big(\frac{N_0(t/(q-p)) - t/4}{2^{-1/3}t^{1/3}} \geq -s\Big) = F_{\text{GUE}}(s) \tag{3}$$

where $F_{\text{GUE}}(s)$ is the GUE Tracy-Widom distribution function which Tracy and Widom had introduced fifteen years earlier in the seemingly different study of random matrix theory. There are various ways to define this distribution function, though the most relevant for the above result is in terms of a specific Fredholm determinant. The special case of this result where $q = 1$ and $p = 0$ was known about a decade earlier due to work of Johansson. In that special case (known as TASEP where the T stands for "totally"), the transition probability formulas in (1) can be rewritten in terms of a single determinant. This is a major simplification and from that point, the calculation becomes more "standard" in the context of random matrix theory techniques.

The path from (2) to (3) was, in contrast, quite nonstandard, involving a number of ingenious manipulations. For instance, one readily sees that as m and t grow in (2) (as they do in our asymptotic limit), the kernel K varies (due to the factor $e^{\epsilon(\zeta)t}$), the contour for ζ varies (due to the condition on enclosing $\zeta = q^{-k}$ for $k \in \{0, \dots, m-1\}$) and the denominator involves more terms. There exist standard methods in asymptotic analysis for integrals and Fredholm determinants, but none of them apply here. Tracy and Widom had to extensively manipulate their formula by various tricks of analysis such as modifying the kernel without changing the determinant or factoring the determinant and evaluating a resulting resolvent. Even as someone who long studied these calculations, it is hard to know what guided them. My only guess is that Harold could just feel his way through these calculations, much like how a master sculptor just knows how the clay should feel and move.

I will end my foray into mathematical exposition here and just provide a few final reflections. Tracy and Widom's work on ASEP, as with their earlier work in random matrix theory, was trail blazing. Though Tracy and Widom, themselves, did not seek to develop their work into general theory or refined methods, this mantel was taken up by many inspired by their work and the connections that it hinted at. Indeed, it is easy to see Tracy and Widom's work at the start of great epochs of mathematical work in both random matrix theory and interacting particle systems.

Now in the fall semester of 2021, we are running the third semester program at MSRI on random matrix theory and interacting particle systems–the first two having been in 1999 and 2010. My last email from Harold came on November 27, 2017. I had written to let him know about the plan for the MSRI program, hoping that he would allow me to list him as a potential participant. He wrote back "Sure, you can put me down as interested in participating in the proposed program. Hope it goes through." Though Harold is not present to participate in person, his mathematical contributions and intellectual curiosity have an unmistakable impact on the program, as it has had on my whole career. He is sorely missed.

Ivan Corwin

In the Trenches with Harold

Persi Diaconis

About five years ago, I was stuck. I needed to get my hands on the largest eigenvalue of the matrix

$$\frac{1}{4}\begin{bmatrix} 2 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & & \ddots & & & & \\ & \ddots & & \ddots & & & \\ & & 1 & \bullet & 1 & & \\ & & & \ddots & & \ddots & \\ & & & & \ddots & & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 & \end{bmatrix} \tag{4}$$

where $\bullet = 2\cos\{(2\pi j)/n\}$, $0 \le j \le n-1$. The matrix is a sum of a diagonal matrix and a circulant, but the usual Weyl bounds only give that the largest eigenvalue is bounded above by the sum of the two largest eigenvalues. This gives a bound of 1; I needed $1-(c/n)$.

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The matrix in (4) is sort of like a Toeplitz matrix,

$$\begin{bmatrix} c_0 & c_1 & & & & & \\ c_1 & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & c_1 \\ & & & & & c_1 & c_0 \end{bmatrix} \tag{5}$$

(constant on the diagonals). Harold is the world's expert on the eigenvalues of Toeplitz matrices. A famous result, Szegö's strong limit theorem, gives the limiting distribution of the eigenvalues $\{\lambda_j\}_{j=0}^{n}$ in the bulk: $(1/n)\sum_0^n \delta_{\lambda_j}$ (under conditions on the c_j). The extreme eigenvalues are harder.

Harold had written the definitive paper [12] showing that the largest eigenvalue was

$$1-\frac{a_1}{n}+\frac{a_2}{n^2}+O\left(\frac{1}{n^3}\right)$$

(again, under natural conditions on the c_j).

My matrix is of Kac–Murdoch–Szegö type,

$$\begin{bmatrix} f_0(0) & f_1\left(\frac{1}{n}\right) & & & & \\ f_{-1}\left(\frac{1}{n}\right) & f_0\left(\frac{1}{n}\right) & f_1\left(\frac{2}{n}\right) & & & \\ & f_{-1}\left(\frac{2}{n}\right) & f_0\left(\frac{2}{n}\right) & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & f_1(1) \\ & & & & f_{-1}(1) & f_0(1) \end{bmatrix} \tag{6}$$

with "nice" functions on the diagonals. The limiting distribution of the eigenvalues in the bulk was known, from the Kac–Murdoch–Szegö theorem. I hoped Harold could tweak his Toeplitz knowledge to get results for matrices like (4) and (6).

This turned out to be quite a hard problem. Limiting results for the general form (6) are unknown. Harold got interested in my special case. He changed variables, rescaled, threw parts of the matrix away, and announced (for (4)),

$$\lambda_{\max}=1-\frac{\pi}{2n}+o\left(\frac{1}{n}\right).$$

It really was magical. It matched the data, too. However, several trips between Stanford and Santa Cruz didn't enlighten me, so Harold volunteered: "I'll write down a proof." This was a real learning experience for me: seeing a master hard (and soft) analyst at work. He found a topology that allowed convergence of operators to give refined information on convergence of the spectrum. He showed that a scaled version of my matrix converges to the harmonic oscillator

$$L=-\frac{1}{4}\frac{d^2}{dx^2}+\pi^2x^2.$$

He then got that the kth largest eigenvalue behaved as

$$1-\frac{(2k-1)\pi}{2n}+o\left(\frac{1}{n}\right).$$

He got the eigenfunctions, too. All of this delivered in ten or so typeset pages.

And then the fun started. "You know, the result is kind of nice, it deserves a nicer proof. Wait up." Three versions later, a book-quality proof emerged. Our work resulted in two joint papers with Dan Bump, Angela Hicks, and Laurent Miclo [7, 8]. These give the motivation (to random walk on the Heisenberg group) and include many appearances of the matrix (4): Harper's operator, Hofstadter's butterfly, the FFT, ... Various subsets of these coauthors cooked up quite different proofs — i.e., using the uncertainty principle, with a "pure probability" proof among them — but the scholarship and brilliance of Harold's argument is the clear winner.

Persi Diaconis

Torsten Ehrhardt

Before I even met Harold in person, I learned to appreciate his mathematics. When I was a student in the research group on Toeplitz operators in Chemnitz (Germany), the name Harold Widom was highly regarded. It was in the early 1990s, when I pondered his 1973 paper [17] on the asymptotics of Toeplitz determinants as I tried to make some progress on the proof of the Fisher-Hartwig conjecture. I still remember the effort it took me to comprehend this intricate piece of analysis. As Hirschman pointed out in his review of this paper, it "represents a jump of several quanta in depth and sophistication [...]." While the Fisher-Hartwig conjecture was raised in 1968, it took several decades and the work of several mathematicians until the conjecture was proved in quite some generality in 2009 [10]. It was Harold who laid out a method and proved the first important cases of the conjecture.

The work of Harold Widom in the theory of Toeplitz operators goes of course much further. Admittedly, it was probably less appreciated by myself at the time because his results had already entered the textbooks on Toeplitz operators [5]. Harold Widom was among those who developed the foundations for the invertibility and Fredholm theory for Toeplitz operators in the late 1950s and 1960s. To give some idea, he established an invertibility criterium for Toeplitz operators in terms of the Wiener-Hopf factorization of the symbol [13] as well as a criterion of a different kind, which is now known as the Widom-Devinatz Theorem [14]. Furthermore, he provided a complete Fredholm theory for Toeplitz operators on the Hardy space H^p with piecewise continuous symbols, which foreshadowed future developments of the theory in the 1980s. Let me also highlight the beautiful result that the spectrum of a Toeplitz operator is always connected [15], which answered a question posed by Paul Halmos.

Circling back to the asymptotics of Toeplitz determinants, Harold's 1976 paper [18] turned out to be another milestone. There he proved the block case of the strong Szegő limit theorem (now referred to as the Szegő-Widom limit theorem). Most importantly for the future, the main novelty was the introduction of operator theoretic methods to prove the determinant asymptotics. The method is exemplified by what is called Widom's formula, $T_n(ab) = T_n(a)T_n(b) + P_nH(a)H(\tilde{b})P_n + W_nH(\tilde{a})H(b)W_n$, a simple but powerful formula that is the basis for various localization results. In fact, I have used this formula and its variations numerous times.

It was at a conference in Winnipeg in 1994 when I met Harold for the first time, but due to the large setting there was only little time for interaction. Instead, it was mostly during the programs on Random Matrices Theory held at MSRI in Berkeley in 1999 and in 2002 that our professional relationship developed. At that time Harold and Craig Tracy had already made their seminal contribution to this field by identifying the limits of certain random matrix probability distributions, the so-called Tracy-Widom distributions. These distribution functions can be expressed as Fredholm determinants, which are of a similar nature as Toeplitz determinants. This shift in Harold's research interest opened up a whole new world and had a significant impact on me. In fact, certain quantities in Random Matrix Theory can be expressed not only as determinants of Toeplitz operators, but also in terms of other kinds of "structured" operators, such as Wiener-Hopf, Toeplitz+Hankel, and Bessel operators. It was this path of research that I followed in the subsequent years.

Finally, when I joined the UCSC Mathematics department in 2004, he and I became colleagues. Although Harold had already retired in 1994, he continued to be active in many ways. He maintained a blackboard both at his home and at his university office which ought not to be erased and which captured the problems on which he was currently working. While he continued to collaborate with Craig Tracy, he closely followed my research. It was

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a blessing that, for instance, when I was talking with my PhD students I could always refer to Harold as the source of deeper knowledge.

Harold was a remarkably nice person to speak to. He was a person of great integrity, of fair and thoughtful judgement, and an extraordinary character. In my first years at UCSC, I resorted to his advice many times. I remember him always being in good spirits and encouraging. I also relied on him in many personal matters such as housing in Santa Cruz.

Harold kept teaching until he was 79 years old. He was known to the students to enter the classroom with at most a tiny piece of paper and deliver his lecture easily and elegantly on the blackboard. I felt honored to have taken over his last course in 2011. Throughout the years he frequently came to the departmental colloquia, and he often gave me a ride in his old stick-shift (!) car to join the dinner afterwards.

My wife and I also enjoyed the hospitality of Harold and his wife Linda. I know that Harold was a good violin player, but I regret that I never heard him play. He also loved hiking and nature. I must admit that I really 'envied' him that he had made it to the top of the Half Dome in his younger years.

Harold Widom remained mathematically active until his last months. When a colleague of ours approached us in the Fall of 2019 with an intricate asymptotic question, it was Harold who came up with the correct answer first. He seemed in good spirits when I congratulated him on his 88th birthday in September 2020. He had recently broken a hip but had been recovering. Sadly, he fell seriously ill a few months later.

Harold was a wonderful person, and I will greatly miss him.

Torsten Ehrhardt

Al Kelley and Tony Tromba

One of us (Tony) first met Harold in 1962 while an undergraduate taking graduate classes at Cornell. Already a Full Professor, Harold had a reputation as a brilliant lecturer. He was the only Cornell mathematics professor whose graduate student audiences could erupt mid class in spontaneous applause. His John von Neumann-like mathematical quickness was legendary; he could answer your question before you even finished stating it. His Cornell lectures on Analysis and Operator Theory were beautiful and inspiring. No wonder that they are still in print more than 50 years after he gave them.

Harold attended Stuyvesant High School in Manhattan together with two other famous 20th-century mathematicians, Elias Stein of Princeton and Paul Cohen of Stanford, all of whom would ultimately specialize in the field of mathematical analysis. Elias was one year older than Harold and Paul two years younger. For several years Paul, who would go on to win a Fields Medal in Mathematics in 1962 in recognition of his path breaking solution to Hilbert's first problem, was generously tutored in high school by Harold. All three remained together to study analysis under the guidance of Antoni Zygmund and Alberto Calderón in graduate school at the University of Chicago. Reflecting on his life, Paul, in a speech at Stanford in 2001, thanked Harold for the profound influence he had on his early mathematical career.

We were both colleagues of Harold at Santa Cruz. One of us (Al) was already a faculty member when Harold joined in 1968 as a founding member of the department, and the other of us (Tony) joined two years later in 1970. From the time he came, Harold was the most influential member of our department. In 1994, he jumped at the opportunity for early retirement so he could focus his life on research, especially on his work with Craig Tracy. This was truly a huge loss for our department. Tony always joked that when Harold retired, the UCSC mathematics department entered a completely new chapter, Chapter 11. However, for Harold it was the right decision.

His joint and seminal work with Craig Tracy on random matrices and their historic discovery of the Tracy-Widom distribution brought them both enormous fame and wide international recognition. The densities of the Tracy-Widom distributions are on the cover of each issue of the journal *Random Matrices: Theory and Applications*, a rare tribute to someone's work.

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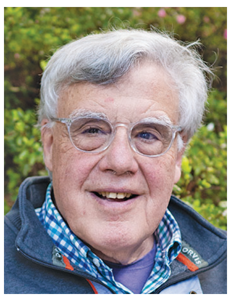
For over 15 years, the three of us hiked almost every week. We thoroughly enjoyed being together and having extended conversations on almost any topic, mathematical, political, or simply campus and departmental issues.

After some time we only hiked every other week or so, and then finally much less often. One favorite (and most spectacular) hike was to go from Twin Gates on Empire Grade down to Wilder Ranch.

Harold was a brilliant mathematician and a truly wonderful person. It was a great blessing for both of us to have had him as a colleague and as a close friend for over 50 years. His profound mathematical discoveries places him among the giants of the University of California.


Al Kelley


Tony Tromba

Barry Simon

I have long admired Harold Widom as one of the most original and talented analysts of the past 50 years. While my personal contact with Harold was limited, I think that it is appropriate to say something about his work on orthogonal and Chebyshev polynomials, especially his seminal 1969 paper in *Advances in Mathematics* [16]. There has been extensive literature motivated by this paper discussing what has become known as Szegő-Widom asyptotics and Widom factors.

While this work is not as extensive as Widom's large opus on Toeplitz operators nor his even more extensive work on asymptotics of certain stochastic processes, he did have another important paper on OPs and the *Advances* paper led him to significant work on the analytic structure of Riemann surfaces leading to the specification of the Parreau-Widom condition.

Widom's work had its roots in the 1919 work of Faber and the 1920 work of Szegő, which looked at asymptotics, respectively, of Chebyshev polynomials and of OPUC (orthogonal polynomials on the unit circle). This work and Widom's relies on the specification of these polynomials via minimization. Given a compact set, $\mathfrak{e} \subset \mathbb{C}$, the Chebyshev polynomial, $T_{n,\mathfrak{e}}(z)$, of degree n is the monic polynomial that minimizes the L^∞ norm over $\mathfrak{e}$ and, given a probability measure, $d\mu$, on $\mathbb{C}$ with all finite moments, the monic OP is the monic polynomial that minimizes the $L^2(d\mu)$ norm.

Faber considered a set, $\mathfrak{e}$, which is a simple, closed analytic curve plus the area that curve surrounds. If $B_\mathfrak{e}(z)$ is the Riemann map of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ to the unit disk with $B_\mathfrak{e}(z) = C(\mathfrak{e})/z + O(z^{-2})$ (where $C(\mathfrak{e}) > 0$) near $z = \infty$, then Faber proved that as $n \to \infty$ for any $z \notin \mathfrak{e}$ one has that

$$\frac{T_{n,\mathfrak{e}}(z)B_\mathfrak{e}(z)^n}{C(\mathfrak{e})^n} - 1 \to 0.$$

Szegő proved a somewhat analogous result for OPUC leading to later extensive work on Szegő asymptotics. Before turning to Widom's great realization, I should mention that the Ukrainian mathematician, Akhiezer, had related ideas although I think Widom only knew of Akhiezer's work on the case where $\mathfrak{e}$ is two bounded real intervals, where exact solutions in terms of elliptic functions obscure the big picture.

Widom considered the situation where $\mathfrak{e}$ is the union of a finite number of sufficiently smooth closed Jordan curves or Jordan arcs. It turns out that there is a natural generalization of $B_\mathfrak{e}$. In the case of connected $\mathfrak{e}$, $-\log|B_\mathfrak{e}(z)|$ is the unique (real) harmonic function which is 0 on $\partial\mathfrak{e}$ and asymptotic to $\log(|z|)$ at ∞. There is such a function (the potential theoretic Green's function, $G_\mathfrak{e}$) even when $\mathfrak{e}$ is not connected so one can attempt to define $B_\mathfrak{e}$ by adding i times the conjugate harmonic function to $G_\mathfrak{e}$ and exponentiating its negative.

The problem is that unlike the case where $\mathfrak{e}$ is connected, the function $B_\mathfrak{e}$ is not single valued—its absolute value is single valued but if one analytically continues around a component of $\mathfrak{e}$, there can be a change of phase. Put more exactly, there is a character of the fundamental group of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$, so that $B_\mathfrak{e}$ is character automorphic with some character, call it $\chi_\mathfrak{e}$.

Widom realized that the asymptotics that Faber found thus couldn't hold because 1 is a single valued function while the ratio is character automorphic with a character, $\chi_\mathfrak{e}^n$, which varies with n. Widom found a suitable replacement for 1—it should be replaced by the character automorphic function that solved a natural minimum problem among the class of all character automorphic functions. This implies the natural asymptotics is not a limit but approach to an almost periodic set.

When $\mathfrak{e}$ consists only of smooth closed curves, Widom proved his ansatz correct for both Chebyshev and orthogonal polynomials. For the important class of finite unions of intervals in $\mathbb{R}$, he solved the OP problem but only proved asymptotics for the norms of the Chebyshev polynomials but he left their asymptotics open.

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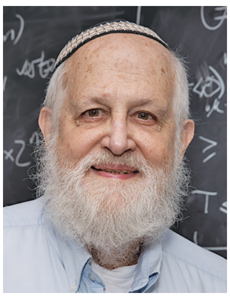
In the 50+ years since, the subject has blossomed in the hands of (with apologies to those left out) Peherstorfer-Sodin-Yuditskii, Totik and Christiansen-Simon-Zinchenko settling, among other things, the above conjecture and extensions to a natural class of infinite gap subsets of $\mathbb{R}$. So Widom was the founder of a subarea of the theory of special functions of subtlety and great beauty.

In my limited dealing with Harold, I was impressed with what a nice guy he was and with his obvious sense of humor. I vividly remember the introduction of his talk at Percy Deift's 60th birthday conference given when Harold was 74. After mentioning some of Percy's work, he said *It's remarkable how much Percy has accomplished*—perfectly timed pause—*for such a young man*. Harold brought the house down.



Barry Simon

How I Came to Work with Harold Widom

Craig A. Tracy

How I came to Toeplitz determinants. My interest in Toeplitz determinants began in graduate school since my thesis, under the supervision of Barry McCoy, dealt with the 2D Ising model and related statistical mechanical models. It was well-known by then that the 2D Ising spin-spin correlation function $\langle\sigma_{00}\sigma_{0,N}\rangle$ could be expressed as a Toeplitz determinant and an explicit formula for the spontaneous magnetization M_0 can be obtained from the strong Szegö limit theorem since $M_0^2 = \lim_{N\to\infty}\langle\sigma_{00}\sigma_{0,N}\rangle$ with $T < T_c$ (T is temperature and T_c is the critical temperature) [11]. At the critical temperature the symbol of the Toeplitz matrix is singular.

Somewhat later I became aware of the work of Harold Widom and Estelle Basor on Toeplitz determinants with singular symbols [1, 9, 17].

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How I came to meet Estelle Basor. In the summer of 1984 I was invited to speak at the University of Wyoming in a summer school organized by Hermann Flaschka. It was my good fortune that Estelle attended and I got to know her and her husband (and fellow mathematician) Kent Morrison. I lectured on Bethe Ansatz and the XXZ quantum spin chain. In the spring of 1991, Estelle was on sabbatical at UC Santa Cruz and she invited me to visit. On this occasion I was introduced to Harold Widom. We spoke for a few minutes but it was clear he wanted to get back to work.

My time at RIMS. In the Fall of 1991 I was on sabbatical at Kyoto University. I got interested in the asymptotics of a certain tau-function, $\tau(t;\theta,\lambda)$, introduced by M. Sato, T. Miwa, and M. Jimbo in their study of holonomic quantum fields. Here t is a distance variable and θ and λ are parameters. The asymptotic problem is to find the short-distance behavior as $t\to 0^+$. Jimbo had proved

$$\tau(t;\theta,\lambda)\sim\tau_0(\theta,\lambda)t^{\frac{1}{2}(\sigma(\theta,\lambda)^2-\theta^2)},$$

but he did not determine the constant $\tau_0(\theta,\lambda)$.

I suspected that the techniques introduced by Widom [17] could be used to solve this asymptotic problem. But I needed someone to collaborate with who *really* understood [17]. The number two world's expert on [17] is Estelle. So I wrote to Estelle suggesting we work together. Again to my good fortune, she accepted. We proved that the *Widom integral operator*[1] determined the τ-function asymptotics and produced an explicit formula for $\tau_0(\theta,\lambda)$ in terms of the Barnes G-function.[2] This same constant arises in Toeplitz determinants with a singular symbol. We concluded the introduction with the statement

> Perhaps the most significant aspect of the present paper is that the short distance asymptotics of the τ-function are also determined by the Widom operator. This indicates that the connections between these various asymptotic formulas take place on a very fundamental level.

While at RIMS I attended lectures by E. Brézin on random matrices. I had learned a little about Dyson's random matrix theory work as a graduate student, but it was Brézin's lectures that brought me up to speed. The question I asked (myself) was to determine asymptotic formulas for the probability of finding exactly n eigenvalues in an interval of length s, for large s and fixed n for the three Gaussian ensembles $\beta = 1, 2, 4$. Again I turned to Estelle for help. We made good progress but there was a crucial constant we could not determine.

[1] *See (1.16) in* [2].
[2] *See (1.9) in* [2].

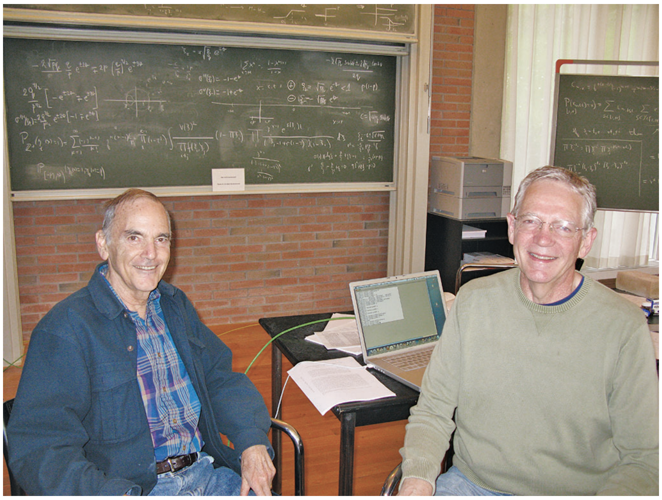
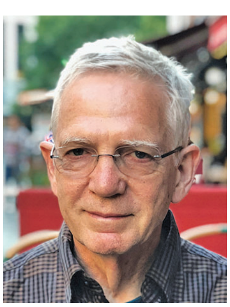


Figure 6. Harold Widom (left) and Craig Tracy (right) in May 2009 as part of MFO's Research in Pairs program.

Enter the Dragon. As related to me by Estelle, she had shown Harold our paper [2]. He inquired about what we were doing now and Estelle said we're working on a problem in random matrix theory. She added that we were stuck with determining a certain constant. Harold asked if he could take a look at the problem. Estelle and I were delighted that Harold wanted to work on this problem. A few days later I received a fax where this constant was evaluated. This resulted in our joint publication [3].

Harold and I went on to collaborate continuously for nearly 30 years. Our major joint work is elegantly summarized in the *Bulletin* article "Harold Widom's Work in Random Matrix Theory" by I. Corwin, P. Deift, and A. Its.

On one of our many visits to Oberwolfach, we were out for a hike. We had both agreed that at a certain time we'd turn around so as not to get lost at night in the Black Forest. As that time approached, we rounded a bend and could see our goal—still in the distance. We both said simutaneously "let's keep going." Then Harold turned to me and said "That's why we work well together!" (We reached our goal and we didn't get lost.)



Craig A. Tracy

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Credits

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