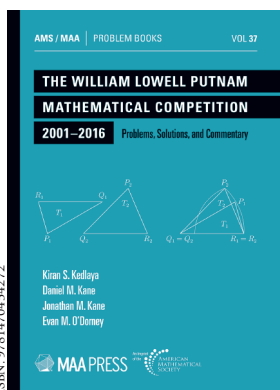




## “THE PUTNAM”

*Reviewed by Daniel H. Ullman*



*The William Lowell Putnam  
Mathematical Competition,  
2001–2016: Problems, Solutions,  
and Commentary*

By Kiran S. Kedlaya,  
Daniel M. Kane, Jonathan M. Kane,  
and Evan M. O’Dorney

*“The moving power of mathematical invention is not reasoning but imagination.”*

—Augustus De Morgan

### William Lowell Putnam

William Lowell Putnam was born on November 22, 1861 in Roxbury, Massachusetts. He grew up in Cambridge, Massachusetts, across the street from Loeb House in Harvard Yard, which for many years was the official residence of Harvard presidents. He went to high school at Cambridge Latin School and then attended Harvard College, where he excelled in his studies. He graduated magna cum laude in 1882 with a degree in mathematics.

After completing law school at Harvard, in 1888 he married Elizabeth Lowell, who, like William, counted John Lowell, a delegate to the Constitutional Convention of 1787, as great-great-grandfather. Although William enjoyed a distinguished career in business and the law, he never ventured far from Harvard or from mathematics. Elizabeth’s brother Abbott Lawrence Lowell became president of Harvard in 1909, a position he held for 24 years. Julian

Lowell Coolidge, another member of the extended family, was a professor of mathematics at Harvard, and he chaired the department from 1927 to 1940. George Birkhoff, first appointed to the Harvard mathematics department in 1912, was a close friend of Putnam.

One passion of Putnam (with curious resonance to the present political time) was to change the date of the presidential inauguration from March 4 to January 20. The long interim between Election Day and Inauguration Day, Putnam argued, delayed a new president’s ability to pursue an agenda and address national emergencies. The March 4 date was enshrined in the Constitution, however, so the crusade would require a constitutional amendment. Putnam championed this issue with the American Bar Association, but he died in 1923, without the chance to see the final success of this movement: The Twentieth Amendment to the Constitution was ratified in 1933, establishing January 20 as inauguration day for incoming presidents.

In 1921, Putnam wrote a piece for the Harvard alumni magazine advocating for an intercollegiate scholarly competition that would capture the sort of team spirit familiar in the realm of athletic competitions. While he did not name mathematics as the disciplinary home for this competition, it seemed natural to establish a competition in mathematics in honor of Putnam’s own academic interest. Putnam’s widow Elizabeth created the William Lowell Putnam Intercollegiate Memorial Fund in 1927 to endow such a competition. After her death in 1935, the competition took roughly its present form under the auspices of the MAA, and the first William Lowell Putnam Mathematical Competition began in 1938.

### The Collected Problems

The legacy of that competition, nowadays known affectionately as “the Putnam,” is chronicled in four volumes of collected problems, the first three of which, (1), (2), and (3), cover the years 1938–1964, 1965–1984, and 1985–2002, respectively. AMS/MAA Press has now published the fourth volume in this series, covering the years 2001–2016. Each

*Daniel H. Ullman is a professor of mathematics at the George Washington University. He is director of the Putnam Competition. His email address is du11man@gwu.edu.*

*Communicated by Notices Associate Editor Katelynn Kochalski.*

*For permission to reprint this article, please contact: reprint-permission@ams.org.*

DOI: <https://dx.doi.org/10.1090/noti2538>

of these volumes reprints the exam questions and provides the solutions, along with information about the results of the competition.

The organization of the present volume is no surprise. The problems are featured in order first, and readers who want to develop their solving skills should attempt the problems without turning further pages. Then comes a section of brief hints, which help a reader to understand the approach to a problem and not just to see a polished solution. After that is the section of solutions (comprising  $\frac{3}{4}$  of the book). There is also a section on results of the competition, displaying some of the same obsessive attention to detail that one sees more typically among sports fans. There is a short section at the end offering strategies for participants. Those who want to learn to solve competition problems at this level would be wise to pay attention to these strategies, but there is no substitute for slow and deep study of the problems, hints, and solutions from the earlier sections of the book. There is also a topic index, so that a reader can easily locate all the problems in a given area of mathematics.

The hints and solutions here are the work of the four authors of the book, who are masters of the art of Putnam problem solving. The hints are short but just right, neither too much nor too little. The solutions are beautifully written. Remarks after many of the solutions provide additional insight.

The problems are the heart of the Competition. The ideal Putnam problem is attractive and natural, is unapproachable via standard classroom techniques, and yet gives way once the magic key is discovered. The joke goes like this: All Putnam problems are easy, but it's usually very hard to see *why* they are easy. One is given a hypothesis  $A$  and a conclusion  $C$ , and one must argue that  $A$  implies  $B$  and  $B$  implies  $C$ . But no one tells you what  $B$  is. Finding  $B$  is the key. That's where imagination and creativity come in.

Taken together, the four volumes show how the competition has evolved over nearly 80 years. The problems from the early years of the competition seem banal and routine compared to problems from more recent years. Here are two of my favorite problems from the current volume, the first an easier problem, the second a harder one:

- 2005, Problem A1: Show that every positive integer is a sum of one or more numbers of the form  $2^r 3^s$ , where  $r$  and  $s$  are nonnegative integers and no summand divides another. (For example  $23 = 9 + 8 + 6$ .)
- 2001, Problem A6: Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Before reading on, take out a pen and some paper and devote some time to solving these two problems.

(If you just followed the instruction in the previous sentence, then this volume is for you.)

Welcome back. The first of these is a gem for the ages and is attributed to Paul Erdős. The proof is via mathematical

induction—no surprise there—but the numbers 2 and 3 each play a delicate role in the argument. You just have to smile when you discover the solution. The second problem has a statement that is a masterpiece of simplicity, yet the answer is difficult to find. (Spoiler alert: Don't read the rest of this paragraph if you want to attempt this problem without any hints.) The problem statement suggests strongly that the answer must be "no," but in fact the answer is "yes"! The arc can have length as long as approximately 4.0027. Knowing that the answer is "yes" makes the problem significantly easier to solve, though there remains work to be done. Only one student earned full marks for this problem during the competition, employing the imagination to make that guess and the power to execute the details of the calculation. That combination of imagination and power is rare. It is what we often call genius.

### Mathematical Competition

The Putnam Competition is now firmly established as the preeminent collegiate mathematical competition in the US and Canada. Counting among the winners (Putnam Fellows) of the Competition are Fields Medalists, Nobel Laureates, and MacArthur Fellows. Many Putnam Fellows go on to leading careers in mathematical research or in other endeavors to which their creative problem-solving skills apply.

But thousands of students participate in the Putnam Competition each year, and few of them will be Putnam Fellows or runners-up. In fact, the vast majority of participants each year do not solve even one of the 12 problems completely. The legacy of the Putnam Competition is rightly measured by how it affects the entirety of the mathematical community, the entirety of the mathematical enterprise.

There is no doubt that the Putnam Competition identifies, rewards, and nurtures elite mathematical talent. But might it also discourage such talent? What of the participant who is not so precocious and yet aspires to further study of mathematics, but who scores zero or near zero on the Putnam? The MAA and the AMS have as part of their mission to promote mathematics across all levels. Does the Putnam work against this goal by telling so many from our diverse community that they are not in possession of mathematical talent? (Such questions are raised, for example, in [4].)

Indeed, there is a possibility of harm, and some students should not attempt the Putnam. But the Putnam can benefit students across the spectrum. As in sports, the enterprise selects a few elite participants who win awards and prizes, yet many others benefit. Exercise, whether physical or mental, is healthy, and it develops such characteristics as strength, stamina, and focus in all who participate, not just in the winners. During the competition, all participants experience an intense period of concentration, rare outside of this setting, during which their creative minds are unlocked. These are the moments when one confronts one's own intellectual powers and stretches them as far as they

are prepared to go in pursuit of solutions. It may happen that these few hours are the time in which one advances intellectually at the greatest rate.

Another argument sometimes levied against the Putnam is that it measures an ability that is neither necessary nor sufficient for success in a mathematical career. There are certainly examples of productive research mathematicians who did not excel on the Putnam, perhaps because they do not thrive under time pressure. And there are certainly examples of Putnam Fellows who do not excel in mathematics research, perhaps because they lack stamina or interest.

Still, the Putnam is a test of creativity and power, and these attributes are highly correlated with excellence in advancing mathematics and in using mathematics to solve problems. The ability to excel in the Putnam Competition translates readily to other domains, such as mathematical research or mathematical careers in business, government, or industry. It is unfortunate that, under the influence of the modern assessment movement in undergraduate education, we are more and more encouraged to teach and test only direct applications of standard techniques. This prepares students very little to be able to tackle the Putnam. The mantra that we must only test students in the classroom on problems that we expect most of them to solve is utterly lacking in ambition and excitement. Our mathematics classrooms should be more, not less, like the Putnam, where we challenge students with problems that are difficult to unlock rather than tasking students to solve routine problems that we train them to master. In this way, we develop the mathematical imagination to which De Morgan refers in the quote at the beginning of this review.

### The Case for Joy

The competition that carries William Lowell Putnam's name was inspired by his belief that a scholarly discipline might be more like an athletic endeavor. Does mathematical competition resemble athletic competition? In at least one respect, they are quite different: Mathematical competitions are not good spectator sports. Moreover, the values of teamwork and cooperation, so critical to the success of athletic teams, are for the most part absent from the modern Putnam Competition.

But in several respects, the analogy is useful. Where athletic competitions promote and encourage physical strength and stamina, so mathematical competitions promote and encourage mental strength and stamina. The preparation one invests to succeed in sports is comparable to the mathematical preparation we offer in the best of our undergraduate classrooms. The proper mission of athletics is not merely to identify victors but to promote values that transfer outside the athletic realm. Similarly, the mission of the Putnam Competition is not merely to identify Putnam Fellows but to promote the development of creative and insightful thinking toward the solution of challenging problems.

It is said that mathematical ability is based in the left hemisphere of the brain while creativity is based in the right hemisphere. That is a view based on the common misconception of mathematics as a discipline rooted in memorization, facts, and routine calculation and opposite from emotion and imagination. Adherents of this view probably do not have any vision of what a mathematics competition is really about. The Putnam is a test of the very human capacity to think flexibly.

Mathematical competition and athletic competition share one more attribute, often overlooked: Students participate in the Putnam competition because it is *joyous*. Like sports, it is meant to be enjoyed. It is exciting. You strive to win, to excel, but you enjoy the effort even if you do not. Never mind that the competition builds your strength, power, concentration, and insight. Never mind that it's good for you. Think of the Putnam Competition as mathematics for fun.

Those who insist that competition has no place in mathematics (or in any human endeavor, see [6]) don't appreciate that competition can make things fun. To solve a Putnam problem is pleasurable. There is no need to be a Putnam Fellow to enjoy the competition. Participants can set their own bar for doing well. Some may aspire to solve just one problem; others may attempt them all. It is the same with marathon running. Many thousands of runners compete, although few can hope to win the race. Still, every runner who competes tests their limits, grows stronger from the experience, and deserves congratulations.

The Putnam Competition is a test of creativity, not facts. In this respect, it belongs to a higher stratum than trivia competitions such as spelling bees or geography contests, the training for which demands prodigious memorization. Students who learn to use the theorems and techniques of the undergraduate curriculum may find themselves at a complete loss when facing Putnam problems unless they have also grown accustomed to tackling challenging problems whose solutions do not involve routine application of standard techniques. Putnam problem solving is also a distinctly human talent. We are not near to a time when a machine can succeed at the Putnam Competition. By contrast, machines easily defeat humans in a test of spelling, geography, chess, or arithmetic.

Certainly, the psychology of our students is a factor in how we teach, but we ought not to assume that challenging our students will lead to low self-esteem. Instead, we should teach them that math is hard and that it requires creativity and imagination that can be developed with practice. We should assure them that the effort exerted toward difficult challenges is beneficial even if those efforts do not succeed every time. This may not resonate with students who have become infected with the popular view that mathematical talent is entirely innate. But for the others, we ought to congratulate them for developing their creativity as they tackle these challenges.

One of the best ways to prepare for the Putnam and to develop mathematical talent is slow, close, careful study of this latest collection of Putnam problems. A student should work on each problem for at least an hour if necessary before consulting the hint, then take at least another hour before consulting the solution. The hints, solutions, and commentary provided by these expert authors will help to advance the next generation of mathematics students in the art of creative problem solving. And this advances the broad mission of both the MAA and the AMS. Success of students on the Putnam today is a leading indicator of the health of the mathematical enterprise tomorrow.

### References

- [1] P. R. Halmos, A. M. Gleason, R. E. Greenwood, and L. M. Kelly, *The William Lowell Putnam Mathematical Competition, Problems and Solutions: 1938–1964*, MAA Press, Washington, DC, 1980. Reissued 2019.
- [2] G. L. Alexanderson, L. F. Klosinski, and L. C. Larson, *The William Lowell Putnam Mathematical Competition, Problems and Solutions 1965–1984*, MAA Press, Washington, DC, 1985.
- [3] K. S. Kedlaya, B. Poonen, and R. Vakil, *The William Lowell Putnam Mathematical Competition, 1985–2000: Problems, Solutions, and Commentary*, MAA Press, Washington, DC, 2004.

- [4] R. Merris, Letter to the Editor: *Low Putnam Scores are Demoralizing*, FOCUS, the newsletter of the MAA, October/November, 2017.
- [5] William Lowell Putnam, *A Suggestion for Increasing the Undergraduate Interest in Studies*, The Harvard Graduates Magazine (1921), 210–212.
- [6] Alfie Kohn, *No Contest: The Case Against Competition*, Boston: Houghton Mifflin, 1992.



Daniel H. Ullman

### Credits

Photo of Daniel H. Ullman is courtesy of Susan Colley.

## Give the Gift of AMS Membership

The gift of connection to the mathematical community in three easy steps. Eligible purchasers will receive a special **AMS blanket**.



For more information:  
[www.ams.org/membership/givemembership](http://www.ams.org/membership/givemembership)