Susan Montgomery: A Journey in Noncommutative Algebra

Yevgenia Kashina

Susan Montgomery was born in 1943 and grew up in Lansing, Michigan. She did her undergraduate studies at the University of Michigan where her advisor was J. E. McLaughlin, who inspired her interest in algebra. Having obtained her undergraduate degree in Mathematics in 1965, Susan received an NSF Graduate Fellowship and started her graduate studies at the University of Chicago. In 1969, she defended her Ph.D. thesis titled "The Lie Structure of Simple Rings with Involution of Characteristic 2" under the supervision of I. N. Herstein. She then spent one year at DePaul University. In 1970, Susan joined the faculty at the University of Southern California, where she currently is a professor.

Susan received multiple awards and recognition, including a John S. Guggenheim Memorial Foundation Fellowship in 1984, an Albert S. Raubenheimer Distinguished Faculty Award from the Division of Natural Sciences and Mathematics at USC in 1985, and a Gabilan Distinguished Professorship in Science and Engineering at USC in 2017–2020. In 2012, Susan was selected as an Inaugural Fellow of the AMS and elected as a Fellow of the American Association for the Advancement of Science.

Susan served on many professional society committees, including the AMS Board of Trustees in 1986–1996, the Board on Mathematical Sciences of the National Research Council in 1995–1998 and its Executive Committee in 1997–1998, and the AWM Scientific Advisory Committee in 2015–2017. Furthermore, she was elected as vice president of the AMS for the 2014–2017 term. Moreover, she served as the chair of the Mathematics Department at USC in 1996–1999.

While in her early work Susan studied rings with involutions and group actions on rings, her current research is focused on Hopf algebras and their representations. She has published two books and more than 100 research papers; in addition, she was a coeditor for seven collections of papers on various topics in algebra. Her monograph *Hopf algebras and their actions on rings* became the most cited book on Hopf algebras and quantum groups. Below we discuss Susan's research accomplishments, focusing on several important topics.

1. Group Algebras and Rings with Involution

The first thesis problem on which Susan worked as a graduate student was to determine whether, in characteristic p, left invertible elements of a group algebra are also right invertible. This property was observed by Kaplansky in characteristic 0; his proof followed from results concerning von Neumann algebras. Montgomery was able to give a shorter proof of the characteristic 0 case, based on the properties of C^*-algebras instead (see [Mon69]), but the characteristic p proof eluded her for almost two years. This question still remains open, even though many mathematicians have tried to solve it in the last fifty years.

Susan's second thesis problem was on Lie simplicity of simple rings with involution of characteristic 2. Given an associative ring R, one can introduce a Lie structure on R by defining the Lie product via $[x, y] = xy - yx$ for all x, y in R. This definition immediately raises the question how the ideal structures of R as an associative ring and R as a Lie ring are connected. In particular, if R is simple as an associative ring, what can we say about its Lie simplicity? One of the motivations for such a study, given by Herstein, was to investigate whether the simplicity of four infinite families of simple, finite-dimensional Lie algebras defined as matrices is in fact a consequence of the simplicity of the associative matrix algebra over a field. In the 1950s, there was a series of papers where these questions were considered; in particular, Herstein and Baxter proved several results about the Lie structure of the skew-symmetric elements of a simple ring with involution of

Yevgenia Kashina is a professor of mathematics at DePaul University. Her email address is ykashina@depaul.edu.

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characteristic not equal to 2. The case of characteristic 2 was still unknown at the time; it then became the main topic of Susan's thesis. We will now describe her results on Lie simplicity of simple rings with involution of characteristic 2, which were later published in [Mon70]. Note that, for many of the results discussed below, the rings were not required to have identity; however, for simplicity, we will always assume that R is a ring with identity.

Recall that, if $A \subseteq B$ are additive subgroups of R, we say that A is a Lie ideal of B provided that $[A, B] \subseteq A$, where $[A, B]$ is an additive subgroup of R generated by the commutators $[a, b]$ for all $a \in A$ and $b \in B$. Let R be a simple ring, that is, R has no proper nontrivial two-sided ideals; then its center Z is a field. Assume, in addition, that R has an involution *, which is, by definition, a self-inverse anti-automorphism of R, and that * fixes every element of Z. Let S and K be respectively the sets of symmetric and skew-symmetric elements of R and let $V = \{x + x^* \mid x \in R\}$ be the set of trace elements. It is easy to see that $V \subseteq S$ are additive subgroups of R and K is a Lie subring of R. Herstein and Baxter obtained the following results on characterization of Lie ideals of K and $[K, K]$ in the case of characteristic different from 2:

Theorem 1. *Let R be a simple ring with involution of characteristic not 2 and assume that $\dim_Z R > 16$. Then*

1. *(Herstein) Every Lie ideal of K either is contained in Z or contains $[K, K]$.*
2. *(Baxter) Every proper Lie ideal of $[K, K]$ is contained in Z.*

Note that, since Z is fixed by the involution, the intersection of K and Z is trivial. Thus every proper Lie ideal of $[K, K]$ is trivial and therefore the latter part of the theorem implies that $[K, K]$ is a simple Lie ring. Moreover, since the 4×4 matrices over a field, with the transposition used for the involution, do not behave well, the condition $\dim_Z R > 16$ is necessary.

In her thesis, Montgomery investigated the case of characteristic 2. It turned out that, in contrast to the case of characteristic not 2, the Lie ideals were now characterized in terms of $[V, V]$, and not $[K, K]$. Moreover, Montgomery showed that in this situation $[V, V] = [[S, S], S]$ and therefore the principal difference is that previously the main object of study was the Lie square of K, whereas in characteristic 2 the main object becomes the Lie cube of S. Montgomery's results can be summarized in the following theorem:

Theorem 2. *Assume that R is a simple ring with involution of characteristic 2 and that $\dim_Z R > 144$. Then*

1. *Every Lie ideal of V either is contained in Z or contains $[V, V]$.*
2. *Every Lie ideal of S either is contained in Z or contains $[V, V]$.*
3. *Every proper Lie ideal of $[V, V]$ is contained in Z.*

In light of the Lie square of V being equal to the Lie cube of S, the latter part of the theorem implies the following Lie simplicity result:

$$\frac{[[S, S], S]}{[[S, S], S] \cap Z} \quad \text{is a simple Lie ring.}$$

The methods used to prove the theorem involved applications of several results on polynomial identities; the degree of one of these identities was 144 which caused the condition on the dimension of R over Z. Then, in joint work with Lanski, Montgomery used similar methods to extend her results to prime rings of characteristic 2.

Montgomery spent most of the next few years studying rings with involution, in particular when properties of the ring were inherited by the symmetric elements. Note that one can introduce a Jordan structure on R by defining a new product via $x \circ y = xy + yx$ for all x, y in R; then the set S of symmetric elements becomes its Jordan subring. Using this observation, Montgomery obtained many important results describing the connection between the ideal structure of R as an associative ring and R as a Jordan ring, as well as the ideal structure of the Jordan ring S.

2. Fixed Rings of Automorphism Groups of a Ring

Figure 1. Susan Montgomery at USC in 1971.

Since the techniques used by Montgomery in her studies of rings with involution relied on polynomial identities, she was led to investigate the question of when having a subring satisfying a polynomial identity implies that the entire ring satisfies a polynomial identity. Recall that a ring satisfies a polynomial identity and is called a PI-ring if there is a polynomial with integral coefficients in noncommuting variables which vanishes under the substitutions from the ring; PI-rings generalize the class of rings which are finitely generated as modules over their center. In [Mon74], Montgomery proved several results for the subrings being centralizers and, as an application, gave an affirmative answer to a question raised by Bjork; namely, she showed that under certain conditions the ring itself satisfies a polynomial identity if the fixed subring of an automorphism group satisfies a polynomial identity. A natural direction from this work was studying fixed rings R^G of automorphism groups G of a ring, in particular, the connection between the structure of the ring and its fixed subring. Another important problem is to investigate the relationship between the fixed ring R^G and the skew group ring $R * G$.

Here, we say that a group G acts via automorphisms on a ring R if there exists a homomorphism from G to Aut(R), the group of automorphisms of R. Often this homomorphism is going to be injective, so that G could be considered as a subgroup of $\mathrm{Aut}(R)$. For any $g \in G$, the action of g on R will be denoted by $r \mapsto g \cdot r$. An automorphism g of R is called inner if there exists a unit in R such that g acts on R via conjugation with this unit; otherwise g is called outer. A subgroup G of $\mathrm{Aut}(R)$ is called inner if every element of G is inner and it is called outer if its only inner element is 1. If G is finite, we say that R has no $|G|$-torsion provided that $|G|r = 0$ implies $r = 0$.

Recall that the fixed subring R^G under the action of G is

$$R^G = \{r \in R \mid g \cdot r = r, \text{ for all } g \in G\}.$$

The fixed ring R^G is guaranteed to be nontrivial under the following sufficient conditions: Either R has no $|G|$-torsion and R is not nilpotent, as proved by Bergman and Isaacs in 1973, or R has no nilpotent elements, as shown by Kharchenko in 1975. For this reason, most of the following results will, implicitly or explicitly, rely on one of these hypotheses.

Another important ring to study in the situation of a group acting on a ring is the skew group ring $R * G$, which extends the semidirect product for groups: a free R-module with basis $\{g \in G\}$ with multiplication defined via $(rg)(sh) = r(g \cdot s)gh$ for r, s in R and g, h in G. Looking closely at this product formula, we can notice that the action of G is responsible for interchanging g and s via $gs = (g \cdot s)g$. This construction can be extended to a more general notion of the crossed product $R *_\sigma G$ with elements of the form $r\bar{g}$, where elements from G and R commute by the same rule as before, but $\bar{g}\bar{h} = \sigma(g, h)\overline{gh}$ for a 2-cocycle $\sigma : G \times G \to U$, where U is a group of units of R. If the 2-cocycle σ is trivial then $R *_\sigma G = R * G$, a skew group ring, and if the group action is trivial then $R *_\sigma G = R^t[G]$, a twisted group ring. Furthermore, if $R = k$ is a field and G acts trivially on k, then $k * G = kG$ is called a group algebra.

Note that in the original definitions one has a right group action, denoted by $r \mapsto r^g$, and the skew group multiplication defined via $(rg)(sh) = rs^{g^{-1}}gh$. We chose the current notation so that it matches the (left) Hopf algebra action and smash product defined in Section 3.

This raises an important question: Considering the three rings, R, R^G, and $R * G$, if one of them is prime, semiprime, semisimple Artinian, primitive, semiprimitive, or satisfies polynomial identities, can we say the same about the other two and under what conditions? And what is the relationship between Jacobson radicals of these rings? Montgomery provided answers to many of these questions and her results and techniques motivated the work of other researchers.

All of the notions mentioned above are basic properties of noncommutative rings that one wants to understand to get an initial picture of the ring's structure. Some of them reduce to well-known properties when the ring is commutative, for example, a commutative ring is primitive if and only if it is a field. Or, when a ring is Artinian (that is, satisfies the descending chain condition on ideals, which generalizes the notion of finite-dimensional algebras), the conditions of being simple, prime, and primitive are equivalent. A ring is called semiprimitive (or Jacobson semisimple) if its Jacobson radical is zero. A ring is semiprimitive if it is semisimple, that is, semisimple as a module over itself; when a ring is Artinian these two notions coincide. A ring is called prime provided that the zero ideal is a prime ideal (that is, if a product of two ideals is zero then one of them is zero) and is called semiprime provided that it has no nontrivial nilpotent ideals (in other words, if a square of an ideal is zero then the ideal itself is zero).

In 1980, Montgomery published a Springer Lecture Notes volume titled *Fixed Rings of Finite Automorphism Groups of Associative Rings* [Mon80], in which she summarized the progress made in the field in the 1970s. We will now discuss some of her most important results in this area.

In 1973, Susan visited Israel, where she started a lifelong collaboration with Miriam Cohen. In their first paper, they proved that, assuming that a ring R has no nilpotent ideals and no $|G|$-torsion, R is semisimple Artinian if R^G is semisimple Artinian. The converse of this result was shown by Levitzki in 1935, under the assumption of $|G|^{-1}$ being an element of R (see [Mon80, Theorem 1.15]). In 1976, Montgomery proved a more general result, completely describing the relation between Jacobson radicals of the ring and its fixed subring: Assuming that $|G|$ is a bijection on R (that is, if $|G|R = R$ and there is no $|G|$-torsion), $J(R^G) = J(R)^G$, that is, the Jacobson radical of R^G is the intersection of R^G with the Jacobson radical of R. Note that the assumption of $|G|$ being a bijection can be replaced by the hypothesis of $|G|^{-1}$ being an element of R, as explained in [Mon80, Theorem 1.14].

As mentioned before, the above results relied on R having no $|G|$-torsion. Montgomery's next goal was to establish a connection between the ring structures of R, R^G, and $R * G$ with no assumptions about $|G|$ acting on R. It turns out that if R is simple with 1 or a direct sum of simple rings and if G is outer, the assumption of no $|G|$-torsion can often be dropped. But if R is semiprime, G being outer is not enough, so further restrictions are necessary. In 1975, Kharchenko used the Martindale ring of quotients $Q_0(R)$ to generalize the definition of an inner automorphism of a ring, in a way restricting the definition of an outer automorphism (see [Mon80, page 42]):

Definition 3. 1. *An automorphism g is X-inner if there exists a nonzero x in $Q_0(R)$ such that $x(g \cdot r) = rx$ for all r in R.*

2. *A subgroup G of* $\mathrm{Aut}(R)$ *is called X-inner, if every element of G is X-inner, and it is called X-outer if its only X-inner element is* 1.

Note that if G is X-outer then it is always outer, but the converse is not always true.

Kharchenko used this definition to prove that R^G is prime if R is prime and G is X-outer. Montgomery then further developed the method of X-inner automorphisms and investigated the connection between R, R^G, and $R * G$ (see [Mon80, Theorems 3.17, 5.3, 6.9, Corollary 6.10]):

Theorem 4. *Let R be a ring with a finite group G of automorphisms.*

1. *Assuming $R * G$ is semiprime, R^G is also semiprime. Moreover, R is Goldie if and only if R^G is Goldie.*
2. *Assuming R is semiprime and G is X-outer, both $R * G$ and R^G and are semiprime. If, in addition, R^G is Goldie, then R and R^G have the same PI degree.*
3. *Assuming R is prime and G is X-outer, $R * G$ is also prime and R and R^G have the same PI degree.*

The above results were then extended by Montgomery in her joint work with Fisher (see [Mon80, Corollary 3.18]):

Theorem 5. *Let R be a semiprime ring and let G be X-outer.*

1. *If R is simple then $R * G$ is simple.*
2. *If R is primitive and G is finite then $R * G$ is primitive.*
3. *If R is semiprimitive then $R * G$ is semiprimitive.*

Furthermore, Fisher and Montgomery used the method of X-inner automorphisms to prove a "Maschke-type" theorem, effectively answering the question of when the skew group ring is semiprime. Recall that, a classical theorem due to Maschke states that, for a field G, the group algebra kG is semisimple if and only if the characteristic of k does not divide $|G|$. Generalizations of this fundamental result, often referred to as "Maschke-type" theorems, are extremely important; the Fisher–Montgomery theorem provides such generalization from group algebras to skew group rings (see [Mon80, Theorem 7.4]):

Theorem 6 (Fisher–Montgomery). *If G is finite and R is semiprime with no $|G|$-torsion then $R * G$ is semiprime.*

The question about the semiprimeness of a skew group ring naturally led to the one about the semiprimeness of a crossed product. In [MP78], Montgomery and Passman obtained necessary and sufficient conditions for the crossed product to be prime or semiprime, assuming that the ring itself is prime. As a consequence, they proved that in characteristic 0, if R is prime, then $R *_\sigma G$ is semiprime. Because X-inner automorphisms played a very important role in describing the above-mentioned conditions, Montgomery and Passman followed this paper with a series of joint works in which they studied X-inner automorphisms of various rings including group rings and crossed products.

In related work [Mon81], Montgomery studied the connection between the prime ideals of R and R^G, by passing through the skew group ring $R * G$. In order to formalize this correspondence, she introduced certain equivalence relations on $Spec(R)$ and $Spec(R^G)$, the sets of prime ideals of R and R^G, and proved that the sets of equivalence classes are homeomorphic with respect to the quotient Zariski topology.

In a different direction, joint with Small, Montgomery extended Noether's classical theorem on affine rings of invariants from the commutative to non-commutative case:

Theorem 7. [MS81, Theorems 1 and 2] *Let R be a Noetherian ring which is affine (that is, finitely generated) over a commutative Noetherian ring C and let G be a finite group of C-automorphisms of R. Then R^G is affine over C provided that one of the following conditions holds:*

1. $|G|^{-1} \in R$.
2. *R is a domain satisfying a polynomial identity and R^G is Noetherian.*

The authors also provided examples when these results fail if either R is not a domain and $|G|R = 0$ or R is not Noetherian.

3. Hopf Algebras

3.1. Group actions, group gradings, and module algebras.

Since group algebras provide the first example of Hopf algebras, in the beginning of the 1980s, Susan Montgomery got interested in these kinds of algebras and started to work on the topic with Miriam Cohen. As for an action of a group on a ring, one can define an action of a group G on an algebra A over a field k as a homomorphism from G to $\mathrm{Aut}_k(A)$, the group of k-linear automorphisms of A. Having such an action of G on A is equivalent to A being a module algebra over the group algebra kG, and this notion can be generalized to the notion of an H-module algebra over any Hopf algebra H. In their first paper on the subject [CM84], Cohen and Montgomery pointed out that, for a finite group G, a grading of A by G is equivalent to A being a $(kG)^*$-module algebra, where $(kG)^*$ is the Hopf algebra dual to kG; in this case it is possible to define a smash product $A\#(kG)^*$, which we will discuss later. Similar to how the connection between the structures of R, R^G, and $R * G$ were studied in the previous section, Cohen and Montgomery investigated the relationship between A, A_1 (the identity component of the graded algebra A), and $A\#(kG)^*$, in particular, they obtained results about the Jacobson radical, prime ideals, and semiprimeness. Furthermore, the authors proved a Maschke-type theorem, analogous to the original Fisher–Montgomery theorem, but for group gradings instead of group actions (see

[CM84, Theorem 2.9]). The most important results of the paper are the following duality theorems relating group actions and group gradings:

Theorem 8. [CM84, Theorems 3.2 and 3.5] *Let G be a group of order n and A be an algebra.*

1. *If G acts on A, then $A * G$ is naturally G-graded and $(A * G)\#(kG)^* \cong M_n(A)$.*
2. *If A is G-graded, then $A\#(kG)^*$ has a natural G-action and $(A\#(kG)^*) * G \cong M_n(A)$.*

The ideas of duality were inspired partly by results in von Neumann algebras and C^*-algebras. As an application, the authors showed that the graded Jacobson radical $J_G(A)$ is always contained in the usual Jacobson radical $J(A)$, proving a conjecture of Bergman on radicals of graded rings.

Having proved the duality theorems, Cohen and Montgomery asked the natural question whether the analogs of the duality theorems hold not only for group algebras and their duals, but also for other finite-dimensional Hopf algebras. In a series of papers, joint with Blattner and Cohen, Montgomery extended these ideas in several directions. Since then, Susan has worked almost exclusively on topics related to Hopf algebras. In 1992, Susan Montgomery was the Principle Lecturer at the Conference Board of the Mathematical Sciences conference on Hopf Algebras and, in 1993, she published the CBMS monograph [Mon93] *Hopf algebras and their actions on rings*.

Before describing Montgomery's results further, we will discuss the motivation behind these generalizations. A group algebra kG, in addition to being an algebra, has a coalgebra structure; in particular, there is a k-linear map $\Delta : kG \to kG \otimes kG$ called comultiplication and defined via $\Delta(g) = g \otimes g$ for $g \in G$ and extended linearly. This additional structure, used implicitly, allows us to define a kG-action on a tensor product of two kG-modules via

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w) = \Delta(g) \cdot (v \otimes w),$$

that is, we see that a tensor product of two representations is again a representation. This property is not true anymore if the group algebra is replaced by an arbitrary algebra, but it can be extended from group algebras to Hopf algebras H, using the comultiplication map $\Delta : H \to H \otimes H$. In this case the H-module structure is defined on the tensor product of two H-modules via

$$h \cdot (v \otimes w) = \Delta(h) \cdot (v \otimes w) = \sum (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w)$$

using the so-called Sweedler notation for the coproduct to write $\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \in H \otimes H$ for $h \in H$.

For an algebra A over a field k, we can treat its multiplication and unit as k-linear maps $m : A \otimes A \to A$ and $u : k \to A$. When A is finite-dimensional, the notion of an algebra can be dualized in the following way: Consider the dual vector space $C = A^*$ of k-linear maps from A to k. This dual vector space has two k-linear maps $\Delta = m^* : C \to C \otimes C$ and $\varepsilon = u^* : C \to k$; these maps, called comultiplication and counit give C the structure of a coalgebra. We call H a bialgebra if it is both an algebra and a coalgebra and these two structures satisfy a compatibility condition, namely that comultiplication and counit are algebra maps. A bialgebra H becomes a Hopf algebra if in addition it has a map $S : H \to H$, which is called an antipode and satisfies requirements that generalize the ones for the inverse map in groups.

Historically, the first Hopf algebras studied were cocommutative, that is, the ones where for every element h, the coproduct $\Delta(h) = \sum h_{(2)} \otimes h_{(1)}$. Over an algebraically closed base field of characteristic 0, the only finite-dimensional cocommutative Hopf algebras are group algebras. Other examples of cocommutative Hopf algebras include universal enveloping algebras of Lie algebras and restricted Lie algebras. For any Hopf algebra H one can consider all elements g such that $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, called group-like elements of H. They are linearly independent, form a group, denoted by $G(H)$, and generate a cocommutative Hopf subalgebra $kG(H)$ of H.

When A is both an algebra and an H-module over a Hopf algebra H, we say that A is an H-module algebra, or, equivalently, that A is an algebra in the category of left H-modules ${}_H\mathcal{M}$, if its multiplication and unit are H-module maps, that is, $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and $h \cdot 1_A = \varepsilon(h) 1_A$. Similarly to skew group rings arising when groups act on rings as automorphisms, for an H-module algebra A the smash product algebra $A\#H$ is defined to be $A \otimes H$ as a vector space, but with multiplication

$$(a\#h)(a'\#h') = \sum a(h_{(1)} \cdot a')\#h_{(2)}h'$$

where the elements of $A\#H$ are denoted by $a\#h$.

For example, for an algebra $A = A_1 \oplus A_g$, graded by the group $G = \langle g \rangle \cong \mathbb{Z}_2$, the $(kG)^*$-action on A is defined via $p_1 \cdot a = a_1$ and $p_g \cdot a = a_g$, where $\{p_1, p_g\}$ is the dual basis of $(kG)^*$ and $a = a_1 + a_g$ for $a_1 \in A_1$ and $a_2 \in A_2$. Since $\Delta(p_x) = \sum_{yz=x} p_y \otimes p_z$, the multiplication in $A\#(kG)^*$ is determined by $(1\#p_1)(a\#1) = a_1\#p_1 + a_g\#p_g$ and $(1\#p_g)(a\#1) = a_1\#p_g + a_g\#p_1$.

If H is a finite-dimensional Hopf algebra, then the dual vector space H^* is also a Hopf algebra. In particular, the multiplication in H^* is the map dual to the comultiplication in H, and vice versa, as mentioned before. When H is not finite-dimensional, H^* is not a Hopf algebra anymore, but one can use the so-called finite dual H^0 instead. In 1985, Susan Montgomery, in collaboration with her husband Bob Blattner, extended the results of [CM84] from group algebras to infinite-dimensional Hopf algebras H with bijective antipode, where H^* is replaced by a Hopf subalgebra U of the Hopf algebra H^0, and proved that,

Figure 2. Susan Montgomery and Bob Blattner visiting Munich in 1994.

under certain conditions,

$$(A\#H)\#U \cong A \otimes (H\#U)$$

(see [BM85, Theorem 2.1]). In order to obtain these results, Bob and Susan combined their perspectives from functional analysis and noncommutative algebra. The authors then discussed several applications of this theorem, in particular, in the case when H is a universal enveloping algebra of a finite-dimensional Lie algebra or a group algebra of a residually k-linear FC-group. Furthermore, when H is a finite-dimensional Hopf algebra of dimension n, this result yields the generalization of the duality results in Theorem 8 (see [BM85, Corollary 2.7]):

$$(A\#H)\#H^* \cong A \otimes (H\#H^*) \cong A \otimes M_n(k) \cong M_n(A).$$

3.2. **Crossed products.** In the same way as smash products generalize skew group rings, the notion of a crossed product can be extended to the case when groups are replaced by Hopf algebras; such a crossed product of an algebra A with a Hopf algebra H is denoted by $A\#_\sigma H$, where $\sigma : H \otimes H \to A$ is an invertible cocycle. These general crossed products were introduced independently in 1986 by Blattner, Cohen, and Montgomery, and by Doi and Takeuchi (see [Mon93, Chapter 7]). Next, in a 1989 joint paper with Blattner, Montgomery continued her studies of crossed products and further extended the results of the previous papers. In particular, they generalized the duality theorems from smash to crossed products (see [Mon93, Theorem 9.4.17]):

Theorem 9. *Let H be a Hopf algebra of dimension n and $A\#_\sigma H$ be a crossed product. Then*

$$(A\#_\sigma H)\#H^* \cong M_n(A).$$

Another direction was to extend the Maschke-type results to crossed products $A\#_\sigma H$ for semisimple Hopf algebras H. All previous results were obtained either under the assumption that H is a group algebra or its dual or by imposing additional conditions on H. Blattner and Montgomery proved a Maschke-type theorem by restricting the action instead (see [Mon93, Theorem 7.4.7]):

Theorem 10. *Let H be a semisimple Hopf algebra, A be a semiprime algebra, and $A\#_\sigma H$ be a crossed product. Then $A\#_\sigma H$ is semiprime if the action of H is inner.*

Many ideas and results discussed in this section, as well as in Section 2, motivated Montgomery, together with Linchenko and Small, to investigate two related questions about H-module algebras and smash products. In the first question they asked whether, for a semisimple Hopf algebra H and an H-module algebra R, the Jacobson radical is stable under the action of H. As it was mentioned in the beginning of Subsection 3.1, for $H = (kG)^*$, it is equivalent to the question of Bergman about the graded Jacobson radical and it was answered positively in [CM84]. In 2001, Linchenko showed that it is true when R is finite-dimensional and the base field has either characteristic 0 or characteristic $p > \dim R$ with additional condition of H being cosemisimple (that is, H^* being semisimple). In [LMS05, Theorem 3.8], Linchenko, Montgomery, and Small proved that the answer is positive for any infinite-dimensional PI-algebra R which is either affine or algebraic over the base field of characteristic 0. In the case of positive characteristic they showed that the Jacobson radical is H-stable under the additional assumptions of H being cosemisimiple and the characteristic being large enough compared to the dimension of H and the degree of the polynomial identity satisfied by R. Note that, by the Larson-Radford theorem, in characteristic 0 semisimplicity and cosemisimplicity are equivalent.

The second question addressed in [LMS05] was whether $A = R\#H$ is semiprime provided that H is semisimple and R is H-semiprime. This open question was asked by Cohen and Fischman in 1984, under the stronger hypothesis of R being semiprime, and the positive answer would generalize the results of Fisher and Montgomery for $H = kG$ and Cohen and Montgomery for $H = (kG)^*$. In [LMS05, Theorems 2.8 and 2.11], Linchenko, Montgomery, and Small proved that these two questions are connected: First, given a finite-dimensional Hopf algebra H, they established two conditions equivalent to the one that the Jacobson radical of every H-module algebra is H-stable. Then the authors proved that if the first question is answered positively for all H-module algebras R, then the prime radical of every H-module algebra is H-stable and $R'\#H^*$ is semiprime for all H^*-semiprime H^*-module algebras R'. In conclusion, they showed that $A = R\#H$ is semiprimitive provided that H is semisimple, R is an H-semiprime H-module algebra satisfying a polynomial identity, and the base field has characteristic 0 (in the case of positive characteristic

Figure 3. Susan Montgomery with her mathematical siblings Gail Letzter, Daniel Farkas, Lance Small, and Lynne Small in Torrey Pines in 2014.

some extra hypotheses on k and H were needed); in particular, under the above assumption, the second question was answered positively, since semiprimitiveness implies semiprimeness.

3.3. **Coalgebras and comodules.** In the same fashion how dualizing the multiplication and unit of an algebra A, as k-linear maps led to the concept of a coalgebra C, the notion of a C-comodule is dual to the one of an A-module. That is, treating the A-action on a left A-module M as a k-linear map $\cdot : A \otimes M \to M$, one can define a left C-comodule M via the coaction $\rho : M \to C \otimes M$. The category of left C-comodules is denoted by ${}^{C}\mathcal{M}$, while the category of right C-comodules is denoted by $\mathcal{M}^{C}$. If C is finite-dimensional, then left C-comodules are exactly right C^*-modules; furthermore, for a group G, a vector space is a kG-comodule if and only if it is G-graded. Additional background on coalgebras and comodules can be found in [Mon93, Chapter 5].

Note that, in the infinite-dimensional case, the dual vector space A^* of an algebra A is not always a coalgebra, since $(A \otimes A)^*$ is larger than $A^* \otimes A^*$, and therefore there is no one-to-one correspondence between the theories of algebras and coalgebras. Nevertheless, by the fundamental theorem of coalgebras, any finite subset of elements of a coalgebra is contained in a finite-dimensional subcoalgebra, and, thus, every simple coalgebra is finite-dimensional. This fact led Montgomery, as well as the other researchers in the area, to working on the extension of the results from the theory of finite-dimensional algebras to general coalgebras.

In [Mon95] Montgomery used the classical Brauer theorem about the indecomposable finite-dimensional algebras to prove the decomposition theorem for coalgebras. She considered the quiver Γ_C whose vertices are simple subcoalgebras of coalgebra C, showed that it is isomorphic to the so-called Ext quiver whose vertices are the isomorphism classes of simple (right) C-comodules, and proved that any coalgebra can be decomposed as a direct sum of indecomposable components, each of which corresponds to a connected component of Γ_C. Montgomery then applied these results about coalgebras to prove that every pointed Hopf algebra, that is, the one for which every simple subcoalgebra is one-dimensional, can be decomposed as a crossed product. It was shown independently by Cartier and Gabriel and by Kostant in the early 1960's that a pointed cocommutative Hopf algebra is a skew group ring of its group of group-like elements over the irreducible component of the identity element. For an arbitrary pointed Hopf algebra H, Montgomery showed that $H_{(1)}$, the indecomposable component containing 1, is a Hopf subalgebra of H, the group of group-like elements $G = G(H)$ acts on $H_{(1)}$ via conjugation, and the group of group-like elements of $H_{(1)}$, $N = G(H_{(1)})$, is normal in G. She then proved that H is isomorphic to the Hopf algebra $H_{(1)}\#_\sigma k(G/N)$, which has a structure of a crossed product with a certain cocycle σ as an algebra and a structure of a tensor product as a coalgebra.

Then, in [CM97], Chin and Montgomery constructed, for a given coalgebra C, an associated basic coalgebra B for which every simple subcoalgebra is the dual of some finite-dimensional division algebra and proved that categories of C-comodules and B-comodules are equivalent, that is, that C and B are Morita–Takeuchi equivalent. In particular, when the base field k is algebraically closed, every finite-dimensional division algebra and, therefore, every simple subcoalgebra of B is one-dimensional, implying that B is pointed. The authors then applied their results to path coalgebras, and showed that, over an algebraically closed base field, any coalgebra is equivalent to a large subcoalgebra of a path algebra of the Ext quiver.

3.4. **Extensions.** As crossed products, considered in Subsection 3.2, play a fundamental role in the theory of extensions, the next direction of Montgomery's research was to study certain types of these extensions. In the joint paper with Blattner from 1989, she started working on Hopf Galois extensions, which were first introduced in 1969 by Chase and Sweedler for commutative algebras; the general definition was given by Kreimer and Takeuchi in 1980 (see [Mon93, Chapter 8]).

Recall that for an H-module algebra A, the algebra of H-invariant elements A^H is defined via

$$A^H = \{a \in B \mid h \cdot a = \varepsilon(h)a \text{ for all } h \in H\},$$

extending the notion of a fixed subring R^G from Section 2. Dualizing, if A is a (right) H-comodule algebra with coaction $\rho : A \to A \otimes H$ (that is, an algebra in the category $\mathcal{M}^H$), then one can define A^{coH}, its algebra of

H-coinvariant elements, via

$$A^{coH} = \{a \in A \mid \rho(a) = a \otimes 1\}.$$

Using this terminology, $R \subset A$ is called a (right) H-extension if A is a right H-comodule algebra with $R = A^{coH}$ and the H-extension $R \subset A$ is called (right) H-Galois provided that the canonical map $\beta : A \otimes_R A \to A \otimes H$ defined by $\beta(a \otimes b) = (a \otimes 1)\rho(b)$ is bijective. This definition extends the notion of the classical Galois extensions as follows: Let $k \subset E$ be fields, G be a finite group acting as automorphisms of E fixing k, and $F = E^G$ be the set of elements fixed by G. Since E is a kG-module algebra, it becomes a $(kG)^*$-comodule algebra with $E^{co(kG)^*} = F$. Then E/F is a classical Galois field extension with Galois group G if and only if $F \subset E$ is $(kG)^*$-Galois.

Important examples of H-Galois extensions include $R \subseteq R\#_\sigma H$, where $R\#_\sigma H$ is a right H-comodule via coaction $\rho = \mathrm{id}\otimes\Delta$, but not every H-Galois extension can be written as a crossed product. In fact, combining the results of Doi and Takeuchi and of Blattner and Montgomery, one can show that for an H-extension $R \subset A$, the algebra A is isomorphic to $R\#_\sigma H$ if and only if $R \subset A$ is H-Galois with the so-called normal basis property (see [Mon93, Corollary 8.2.5]).

While group crossed products are transitive in the sense that if $R *_\sigma G$ is a crossed product, N is a normal subgroup of G, and $L = G/N$, then there exists a cocycle $\tau : L \times L \to R *_\sigma N$ such that $R *_\sigma G \cong (R *_\sigma N) *_\tau L$, Hopf crossed products are not transitive in general. In order to state the transitivity problem, we first consider an exact sequence of Hopf algebras $K \hookrightarrow H \twoheadrightarrow \overline{H}$, where K is a normal Hopf subalgebra of H and $\overline{H} = H/I$ is the quotient Hopf algebra for the Hopf ideal $I = HK^+ = K^+H$ of H, with $K^+ = \mathrm{Ker}(\varepsilon) \cap K$. This sequence is called an extension of $\overline{H}$ by K; note, however, that not every quotient Hopf algebra arises from a normal Hopf subalgebra. Then, by an example of Schneider, it is not true in general that a crossed product $R\#_\sigma H$, with H being an extension of $\overline{H}$ by K, can always be written as $(R\#_\sigma K)\,\#_\tau \overline{H}$ for some cocycle τ.

One of the advantages of studying Hopf Galois extensions rather than crossed products is that, unlike Hopf crossed products, faithfully flat Hopf Galois extensions are transitive, which enables the use of inductive arguments. This was proved by Montgomery and Schneider in their joint paper [MS99]:

Theorem 11 (Transitivity). *Let $R \subset A$ be a faithfully flat H-Galois extension, H be an extension of $\overline{H}$ by K, and define $B = \rho^{-1}(A \otimes K)$. Then*

1. *$R \subset B$ is faithfully flat K-Galois.*
2. *$B \subset A$ is faithfully flat $\overline{H}$-Galois.*

The main focus of [MS99] was, however, to study prime ideal structure in faithfully flat H-Galois extensions $R \subset A$ for a finite-dimensional Hopf algebra H. First, since there is no H-action on R in this situation, the authors introduced the notion of an H-stable ideal I of R as the one satisfying $AI = IA$; when $A = R\#H$, it coincides with the usual notion of H-stable ideals. Since H is finite-dimensional, A becomes an H^*-module algebra, and the authors applied the Morita equivalence to obtain a bijective correspondence first between the set of H-prime ideals of R, H-Spec(R), and the set of H^*-prime ideals of A, H^*-Spec(A), and then between the sets of H-equivalence classes of Spec(R) and of H^*-equivalence classes of Spec(A). In addition, they proved that H-Spec(R) can be identified with H_0-Spec(R), where H_0 is the coradical of H, that is, the sum of its simple subcoalgebras.

Next they defined the version of the Krull relations, extending the basic relations that hold between the prime ideals of a ring R and the group crossed product $R *_\sigma G$, such as lying over, incomparability, and going up. A Hopf algebra H satisfies one of these six Krull relations (three basic and three dual) if for all faithfully flat H-Galois extensions $R \subset A$ a certain relation between the prime ideals of R and A holds. For example, H has incomparability if for all faithfully flat H-Galois extensions $R \subset A$ and any $P_2 \subset P_1$ in Spec(A) the condition $P_2 \neq P_1$ implies $P_2 \cap R \neq P_1 \cap R$. Furthermore, Montgomery and Schneider analyzed the conditions under which a Hopf algebra satisfies each of these Krull relations; they proved that in order to see whether H has the given Krull relation, it suffices to check the special case of smash product extensions $R \subset R\#H$ for all H-prime H-module algebras R. The strongest results, when H satisfies all six Krull relations, were obtained with the help of the transitivity theorem either when H is solvable and cosolvable (where H being (co)solvable means that it has a normal series in which all of the quotients are (co)commutative) or when H is semisolvable (that is, H has a normal series in which the quotients are either commutative or cocommutative) and (co)semisimple.

In addition, the authors extended the notion of equivalent prime ideals in R^G from [Mon81] to the case of an H-module algebra A with $R = A^H$, satisfying certain conditions, and showed that the sets of equivalence classes of Spec(A) and Spec(R) are in one-to-one correspondence.

Another type of extensions studied by Montgomery, jointly with Fischman and Schneider, were β-Frobenius extensions of subalgebras of a Hopf algebra. One of the equivalent definitions for a finite-dimensional algebra A to be a Frobenius algebra is that A should be isomorphic to A^* as right A-modules. In the beginning of 1960s, this concept was generalized twice: First Kasch introduced the notion of Frobenius extensions of rings, and then Nakayama and Tsuzuku defined β-Frobenius extensions, also called Frobenius extensions of the second type.

One of the main theorems proved by Fischman, Montgomery, and Schneider in [FMS97], states that, under suitable conditions, the property of being a β-Frobenius extension is inherited by the subalgebras of coinvariants. One of these conditions was that a certain Hopf algebra extension was of the so-called (right) integral type; such extensions include Hopf algebra extensions $U \subset W$ when either U is a normal Hopf subalgebra of finite index or W is finite-dimensional. The authors also showed that, if U and W are pointed and U is of finite index, then such an extension is always of integral type.

The general theory, developed in this paper, was then applied to the situation when $B \subset A$ and H are Hopf algebras with bijective antipodes and there exists a Hopf surjection $\pi : A \to H$ which is still surjective when restricted to B. Then A and B become right H-comodules with H-coaction $(\mathrm{id} \otimes \pi) \circ \Delta$ and one can consider the subalgebras of coinvariants, $R = A^{coH}$ and $S = B^{coH}$. Then the general results of the paper imply, under the assumptions that $B \subset A$ is a faithfully flat extension of right integral type and $R \subset A$ and $S \subset B$ are faithfully flat H-Galois extensions, that $S \subset R$ is β-Frobenius. In particular, these results are applicable in the following two important special cases:

1. When $A = R\#_\sigma H$ and $B = S\#_\sigma H$ are crossed products for an invertible cocyle $\sigma : H \otimes H \to S$.
2. When $S \subset R$ are Yetter–Drinfeld Hopf algebras, that is, Hopf algebras in the category ${}^H_H YD$ of all Yetter–Drinfeld modules over H, and $B = S \star H$ and $A = R \star H$ are their Radford biproducts, as described in [Mon93, Section 10.6].

As a corollary, the authors showed that the finite-dimensional Yetter–Drinfeld Hopf algebras themselves are Frobenius algebras and proved a Maschke-type result in this case.

3.5. Representations and Kaplansky's Conjectures. As it was mentioned before, for a Hopf algebra H, the tensor product of two left H-modules is again an H-module. Moreover, for any left H-module V, its dual V^* is also a left H-module via $(h \cdot f)(v) = f(Sh \cdot v)$, where $h \in H$, $f \in V^*$, and $v \in V$. Therefore the category of finite-dimensional H-modules is a rigid monoidal category, and this fact establishes a strong connection between Hopf algebra theory and category theory. Many of the results about Hopf algebras and their representations were later extended to tensor categories, which makes the study of the representation theory of Hopf algebras very important.

In 1975, Kaplansky stated ten conjectures about Hopf algebras, most of which were motivated by corresponding properties of groups and played a fundamental role in the development of Hopf algebra theory. One of them, the sixth conjecture, which will be referred to as simply the Kaplansky Conjecture, appears to be particularly important and is still open. This conjecture suggests that the classical Frobenius theorem for groups extends to semisimple Hopf algebras: Roughly, it states that, for a semisimple Hopf algebra H over an algebraically closed field, the dimension of any irreducible representation of H (that is, a simple H-module) divides the dimension of H. It is true for all known examples of such algebras, including group algebras and their duals, and often the results about semisimple Hopf algebras are proved under the assumption that the Kaplansky Conjecture holds for them. In recent years this conjecture was shown to be true in many special situations, and one of the first results obtained in this direction was by Montgomery and Witherspoon in [MW98]. The authors first established a one-to-one correspondence between irreducible representations of a crossed product $A\#_\sigma kG$ and irreducible representations of certain twisted group algebras of subgroups of G. This so-called Clifford correspondence allowed them to show that for a finite-dimensional algebra A over a field k and a group G, with characteristic of k not dividing the order of G, if the dimension of any irreducible A-module divides the dimension of A then the same is true for $B = A\#_\sigma kG$ or $A\#_\sigma(kG)^*$. The proof of the latter result involves the application of the duality results in Theorem 9. Then the authors showed that, over an algebraically closed field, the above divisibility property still holds if the group algebra or its dual is replaced by a lower or upper semisolvable semisimple Hopf algebra; in particular, every lower or upper semisolvable semisimple Hopf algebra, over an algebraically closed field of characteristic not dividing its dimension, satisfies the Kaplansky Conjecture. Here, the notion of lower semisolvablity coincides with the notion of semisolvablity from [MS99] (see Subsection 3.4), and upper semisolvable Hopf algebras are defined similarly, using the series of quotients instead of normal Hopf subalgebras.

Finally, Montgomery and Witherspoon proved that every semisimple Hopf algebra of prime power dimension over an algebraically closed field of characteristic 0 is both lower and upper solvable and cosolvable and, therefore, satisfies the Kaplansky Conjecture.

4. Frobenius–Schur Indicators

In 2000, Susan Montgomery and her student Vitaly Linchenko showed that the standard trichotomy for group representations given by Frobenius–Schur indicators (real, non real but real valued characters, and totally non real) can be extended to Hopf algebras. This seminal paper played a role of fundamental importance as it laid the groundwork for the theory of Frobenius–Schur indicators and made a huge impact on the field, laying out the directions for further development: First, Kashina, Sommerhäuser, and Zhu developed the theory of higher

Frobenius–Schur indicators, which was then extended to the case of semisimple quasi-Hopf algebras by Mason and Ng and to the case of spherical fusion categories by Ng and Schauenburg (see, for example, [KSZ06] and [NS07]). Frobenius–Schur indicators became a very important categorical invariant and had a lot of different applications ranging from the proof of the analogue of Cauchy's theorem for Hopf algebras to finding the dimensions of simple modules to classification results for Hopf algebras and fusion categories. Montgomery continued studying these indicators in a series of papers with Guralnick, Iovanov, Jedwab, Kashina, Mason, Ng, Vega, and Witherspoon (see [KMM02], [GM09], [JM09], and [IMM14]).

Recall that in group theory the n-th Frobenius–Schur indicator of the character χ of a finite group G is defined as

$$\nu_n(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^n).$$

When $n = 2$, this definition yields the classical Frobenius–Schur indicator, which was used by Frobenius and Schur in the beginning of the 20th century to determine whether an irreducible complex representation of a finite group can be realized by matrices with real entries. Note that, when the characteristic of the base field k does not divide $|G|$, the element $\Lambda = \frac{1}{|G|}\sum_{g\in G} g$ becomes the so-called normalized integral for the Hopf algebra $H = kG$. Linchenko and Mongomery realized that, when the group algebra is replaced by an arbitrary semisimple Hopf algebra, the n-th Frobenius–Schur indicator of the character χ can be defined as $\nu_n(\chi) = \chi(\Lambda^{[n]})$ for the n-th Hopf power of the normalized integral Λ; when $n = 2$ these indicators are just referred to as Frobenius–Schur indicators, while higher Frobenius–Schur indicators stand for the case of $n > 2$. The authors proved the following generalization of the classical Frobenius–Schur theorem:

Theorem 12. [LM00, Theorem 3.1] *Let H be a semisimple Hopf algebra over an algebraically closed field of characteristic not 2 and let χ be the character of an irreducible representation of H corresponding to the simple H-module V; in the case of positive characteristic assume that H is also cosemisimple.*

1. *The Frobenius–Schur indicator $\nu_2(\chi)$ takes on only the values 0, 1, or -1.*
2. *$\nu_2(\chi) \neq 0$ if and only if V is selfdual. Moreover, $\nu_2(\chi) = 1$ (respectively, -1) if and only if V admits a symmetric (respectively, skew-symmetric) nondegenerate bilinear H-invariant form.*
3. *The trace of the antipode is $\operatorname{Tr} S = \sum \nu_2(\chi)\chi(1)$, where the summation is taken over all irreducible representations of H.*

Since Frobenius–Schur indicators proved to be a very useful invariant of Hopf algebras, Montgomery, together with Kashina and Mason in [KMM02], obtained an explicit formula for the indicator of an important class of semisimple Hopf algebras, which are called cocentral abelian extensions and include Drinfeld doubles of group algebras. The authors used this formula to compute the Frobenius–Schur indicators for various examples; in particular they established conditions under which all irreducible representations of the smash product $(kG)^*\#kL$, where G and L are groups, have positive Frobenius–Schur indicators. As a consequence, they showed that the indicator is always positive for Drinfeld doubles of symmetric groups as well as of generalized dihedral groups and their direct products. In group theory, the groups which admit only irreducible representations with positive indicators are of particular interest and are called totally orthogonal. In a joint paper with Guralnick [GM09], Montgomery introduced the same terminology for Hopf algebras and proved that the Drinfeld double of any finite real reflection group is totally orthogonal. In addition, the authors showed that Frobenius–Schur indicators of irreducible representations, taking on only the values 0, 1, or -1, can be defined even in positive characteristic for involutory non-semismple Hopf algebras. Then, together with Jedwab in [JM09], Montgomery studied Frobenius–Schur indicators for bismash products $(kC_n)^*\#kS_{n-1}$ and $(kS_{n-1})^*\#kC_n$, for the cyclic group C_n and the symmetric group S_{n-1}, where k is an algebraically closed field of characteristic 0. The authors showed that $(kC_n)^*\#kS_{n-1}$ is totally orthogonal, while $(kS_{n-1})^*\#kC_n$ admits irreducible representations with indicator 0 as well as 1. In the next paper with Jedwab, Montgomery studied the representations of general bismash products $(kG)^*\#kL$ over algebraically closed fields of positive characteristic not equal to 2 and defined Brauer characters for such bismash products. They proved the bismash product analog of the theorem of Thompson on lifting Frobenius–Schur indicators of group representations from characteristic p to characteristic 0 and deduced that $(kG)^*\#kL$ is totally orthogonal if $(\mathbb{C}G)^*\#\mathbb{C}L$ is totally orthogonal.

While higher Frobenius–Schur indicators of group representations are always integers, this does not hold for Hopf algebras. Since there is a relation between group representations and representations of Drinfeld doubles of groups, an open question was whether this integrality property still holds for these Drinfeld doubles. This turned out to be an interesting group-theoretic question, which was studied by Montgomery together with Iovanov and Mason in [IMM14]. They showed that, while the n-th indicators are integers for $n = 2, 3, 4, 6$ as well as for many important families of groups, such as symmetric groups, alternating groups, the projective special liner groups $PSL_2(q)$, and the sporadic simple groups M_{11} and M_{12}, it is false in general.

We will now discuss some significant applications of Frobenius–Schur indicators in Hopf algebra theory and beyond, for which Montgomery's results played an instrumental role.

In [KMM02, Section 7], Montgomery with her coathors used the Frobenius–Schur indicators for classification purposes as an invariant that helps distinguishing between nonisomorphic semisimple Hopf algebras, for example, by showing that one Hopf algebra has a two-dimensional irreducible representation with indicator -1, and the other one does not. Moreover, since (higher) Frobenius–Schur indicators are categorical invariants, they can be used to show that Hopf algebras are not twist-equivalent to each other, that is, the corresponding representation categories are not equivalent. For example, this line of argument was later used by Kashina to show that all nonisomorphic Hopf algebras from a certain family of dimension 32 are also not twist-equivalent.

In [KSZ06, Section 3.4], Kashina, Sommerhäuser, and Zhu used the (higher) Frobenius–Schur indicators to prove the Hopf algebra analogue of Cauchy's theorem. The original theorem for finite groups states that for every prime divisor p of the order of the group there exists an element of order p and can be reformulated by saying that every prime divisor of the order of the group also divides its exponent. The authors showed that if a prime p does not divide the exponent of a semisimple Hopf algebra H, then the p-th Hopf power of the normalized integral Λ equals Λ itself. Then for the regular representation χ_R, in which H acts on itself via left multiplication, the p-th Frobenius–Schur indicator equals

$$\nu_p(\chi_R) = \chi_R(\Lambda^{[p]}) = \chi_R(\Lambda) = 1.$$

On the other hand, by another formula for indicators, established by the authors, if p divides $\dim H$ then

$$\nu_p(\chi_R) = (\dim H)^{p-1} \equiv 0 \pmod{p}.$$

This establishes the Hopf algebra analogue of Cauchy's theorem: Every prime divisor of the dimension of H divides its exponent.

As it was mentioned in the beginning of this section, Ng and Schauenburg extended the notion of (higher) Frobenius–Schur indicators to category theory. These generalizations were essential to the proofs of several important results, such as the categorical version of Cauchy's theorem for integral fusion categories by Ng and Schauenburg in [NS07] and for spherical fusion categories by Bruillard, Ng, Rowell, and Wang in [BNRW16]. The latter group of authors applied Cauchy's theorem to prove the rank-finiteness theorem, which was a longstanding conjecture of the fourth author. This theorem states that, up to equivalence, there are finitely many modular categories of given finite rank. As modular categories contribute to the mathematical foundation of topological quantum computation and are closely related to conformal field theories and topological quantum field theories, Frobenius–Schur indicators played an important role for the further development of these areas.

5. Concluding Remarks

Susan's research on Hopf algebras and their representations, as well as her work on group and Hopf algebra actions on rings, established the framework and motivated further studies in these areas. She gave numerous invited talks at conferences, seminars, and colloquia around the world, including the AMS invited addresses at the Joint Mathematics Meetings in 1984, the Joint meeting of the AMS and the Israel Mathematical Union in 1995, and the Spring Southeastern sectional meeting in 2005, as well as the plenary lecture at the Canadian Mathematical Society summer meeting in 2009. In 2011, Susan was selected to deliver the Emmy Noether Lecture at the Joint Mathematics Meetings.

Susan Montgomery was a co-organizer of more than 35 conferences, workshops, and special sessions on ring theory, Lie algebras, Hopf algebras, tensor categories and related topics, as well as the noncommutative algebra year at MSRI in 1999–2000. Two Hopf algebra conferences were held in Susan's honor: one at the University of Southern California in 2009 and another one as a special session at the Joint Mathematics Meetings in San Diego in 2018.

Susan served as an editor for numerous mathematical journals, including the *Proceedings of the AMS*, *Advances in Mathematics*, and *Journal of Algebra*.

Susan's impact on noncommutative algebra is not limited to her own research accomplishments and editorial work. She had, and continues to have, a broad influence on the field through mentorship and collaboration with young mathematicians. Thirteen graduate students received their Ph.D. degree under Susan's supervision, and she currently has one Ph.D. student. Many experts in the areas of Hopf algebras, quantum groups, and category theory spent their postdoctoral years at USC working with her. Susan is a wonderful mentor and advisor; she cares deeply about her graduate students and provides a lot of guidance and encouragement throughout their academic careers. I was very fortunate to have Susan as my graduate advisor and to be able to collaborate with her. More than twenty years after graduation, I still feel her strong support.

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Yevgenia Kashina

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