WHAT IS...

a Metaplectic Group?

Martin H. Weissman

Metaplectic groups "cover" symplectic groups, fitting into a short exact sequence like

$$1 \to \{\pm 1\} \to \mathrm{Mp}_{2n}(\mathbb{R}) \to \mathrm{Sp}_{2n}(\mathbb{R}) \to 1.$$

They arise when one realizes that a group of operators *almost* satisfies relations among symplectic matrices. In this formulation, they are creatures of geometry and harmonic analysis. More profoundly, modular forms of half-integer weight like those studied by Shimura [Shi73] come from automorphic representations of metaplectic groups [Gel76]. In this setting, and via the theta correspondence, metaplectic groups belong to number theory and the Langlands program. Here we introduce metaplectic groups from multiple perspectives.

1. Topological

The two-by-two special linear group is the Lie group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1,\ a,b,c,d \in \mathbb{R} \right\}.$$

The group structure here is by matrix multiplication, and geometric structure arises from its embedding as a smooth submanifold of $\mathbb{R}^4$. Its subgroup of rotations is diffeomorphic to the circle,

$$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta < 2\pi \right\}.$$

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Write $\mathfrak{p}$ for the real vector space of symmetric trace-zero matrices,

$$\mathfrak{p} = \left\{ \begin{pmatrix} u & v \\ v & -u \end{pmatrix} : u, v \in \mathbb{R} \right\}.$$

Then multiplication and the matrix exponential give a diffeomorphism

$$\phi : \mathrm{SO}_2(\mathbb{R}) \times \mathfrak{p} \to \mathrm{SL}_2(\mathbb{R}), \quad \phi(r, X) = r \cdot \exp(X).$$

E. Cartan proved such a statement for connected semisimple Lie groups in 1927, though we refer to Mostow [Mos49] for a tidier proof. In such a general context, one finds a diffeomorphism,

$$\phi : K \times \mathfrak{p} \xrightarrow{\text{diffeo}} G, \tag{1.1}$$

where K is a maximal compact subgroup of G, and $\mathfrak{p}$ is a real vector space. From this, it follows that the fundamental groups of K and G (with base point 1) can be identified. In particular, we have

$$\pi_1(\mathrm{SL}_2(\mathbb{R})) \equiv \pi_1(\mathrm{SO}_2(\mathbb{R})) = \pi_1(\bigcirc) = \mathbb{Z}.$$

Associated to the quotient $\mathbb{Z}/2\mathbb{Z}$ of $\pi_1(\mathrm{SL}_2(\mathbb{R}))$, one finds a two-fold covering space of $\mathrm{SL}_2(\mathbb{R})$. This covering space is naturally a Lie group, and the covering map a Lie group homomorphism, with kernel of order two. This "covering group" $\mathrm{Mp}_2(\mathbb{R})$ is the simplest example of a *metaplectic group*, and it fits into a short exact sequence,

$$1 \to \mu_2 \to \mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}) \to 1.$$

Here $\mu_2 := \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ is a group of order two, and it lies in the center of $\mathrm{Mp}_2(\mathbb{R})$.

The term "metaplectic" ("groupe métaplectique") is due to André Weil [Wei64, §34], in a more general context, where metaplectic groups cover symplectic groups. Indeed, Cartan's diffeomorphism (1.1) holds for the symplectic groups $\mathrm{Sp}_{2n}(\mathbb{R})$, where they provide an identification,

$$\pi_1(\mathrm{Sp}_{2n}(\mathbb{R})) = \pi_1(\mathrm{U}(n)) = \mathbb{Z}, \text{ when } n \ge 1.$$

In this way, the previous construction gives a central extension of Lie groups,

$$1 \to \mu_2 \to \mathrm{Mp}_{2n}(\mathbb{R}) \to \mathrm{Sp}_{2n}(\mathbb{R}) \to 1.$$

Note $n = 1$ gives $\mathrm{Mp}_2(\mathbb{R})$, since $\mathrm{SL}_2(\mathbb{R}) = \mathrm{Sp}_2(\mathbb{R})$. These 2-fold coverings $\mathrm{Mp}_{2n}(\mathbb{R})$ of symplectic groups $\mathrm{Sp}_{2n}(\mathbb{R})$ are called metaplectic groups, since Weil's landmark paper.

On the other hand, this topological construction also provides a d-fold covering of $\mathrm{Sp}_{2n}(\mathbb{R})$ for every positive integer d, corresponding to the quotient $\mathbb{Z}/d\mathbb{Z}$ of the fundamental group $\mathbb{Z}$. These are sometimes called "higher metaplectic groups" though naming is not so consistent.

If one attempts to work with $\mathrm{SL}_n(\mathbb{R})$, things change more dramatically. While $\mathrm{SO}_2(\mathbb{R})$ is the circle, with fundamental group $\mathbb{Z}$, the space-rotation group $\mathrm{SO}_3(\mathbb{R})$ has fundamental group of order 2. This corresponds to the famous double cover,

$$1 \to \mu_2 \to \mathrm{SU}(2) \to \mathrm{SO}_3(\mathbb{R}) \to 1.$$

In fact, the fundamental groups $\pi_1(\mathrm{SO}_n(\mathbb{R}))$ are all just $\mathbb{Z}/2\mathbb{Z}$, for $n \geq 3$. Since $\mathrm{SO}_n(\mathbb{R})$ is a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$, one finds only 2-fold covers of $\mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$. Some describe these 2-fold covers of $SL_n(\mathbb{R})$ as "metaplectic covers of $SL_n(\mathbb{R})$" even though the symplectic group is no longer present.

2. Weil's Groups of Operators

We have given a topological *characterization* of $\mathrm{Mp}_2(\mathbb{R})$, but not really a *construction*. One might compare to the case of spin groups, where the fundamental group of $\mathrm{SO}_n(\mathbb{R})$ has order two (assuming $n > 2$). As a result, there is a central extension,

$$1 \to \mu_2 \to \mathrm{Spin}_n(\mathbb{R}) \to \mathrm{SO}_n(\mathbb{R}) \to 1.$$

This "covering group" of $\mathrm{SO}_n(\mathbb{R})$ is the compact group known as the spin group. But a more direct construction of $\mathrm{Spin}_n(\mathbb{R})$ is possible using the Clifford algebra of dimension 2^n. With it, one may embed $\mathrm{Spin}_n(\mathbb{R})$ in the group of invertible $2^m \times 2^m$ matrices with $m = \lfloor (n-1)/2 \rfloor$ – a great cost, to be sure, but at least the spin groups *are* groups of matrices. (Should we call $\mathrm{Spin}_n(\mathbb{R})$ a metaplectic group? I think not, but others may disagree.)

The traditional metaplectic groups $\mathrm{Mp}_{2n}(\mathbb{R})$ admit no such matrix representation. In fact, this is a theorem.

Theorem 2.1. *Let N be a positive integer and let $\rho\colon \mathrm{Mp}_{2n}(\mathbb{R}) \to \mathrm{GL}_N(\mathbb{C})$ be a continuous homomorphism. Then $\mathrm{Ker}(\rho)$ contains μ_2, i.e., ρ factors through $\mathrm{Sp}_{2n}(\mathbb{R})$. In particular, ρ is not injective.*

A proof is difficult to locate in the literature; Bourbaki suggests it as an exercise in [Bou98, Ch.III, Exercises for §6]. It suffices to assume $n = 1$, since $\mathrm{Mp}_2(\mathbb{R}) \subset \mathrm{Mp}_{2n}(\mathbb{R})$ for all $n \geq 1$. Finite-dimensional representations of $\mathrm{Mp}_2(\mathbb{R})$ are determined by associated representations of its Lie algebra, $\mathfrak{mp}_2(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{R})$. Those can all be written down, and the corresponding representations of $\mathrm{Mp}_2(\mathbb{R})$ factor through $\mathrm{Mp}_2(\mathbb{R})/\mu_2 = \mathrm{SL}_2(\mathbb{R})$.

Since $\mathrm{Mp}_{2n}(\mathbb{R})$ does not sit inside a group of finite matrices, one must go further and work with groups of unitary operators on a Hilbert space. Such representations of $\mathrm{Mp}_2(\mathbb{R})$—in fact, of the universal covering of $\mathrm{SL}_2(\mathbb{R})$—were classified by Pukánzsky [Puk64] in 1964. In the same year, Weil [Wei64] published "Sur certains groupes d'opérateurs unitaires," where he constructed metaplectic groups via such operators.

Here we describe the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ in terms of unitary operators $\mathrm{U}(\mathcal{H})$ on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ of square-integrable functions. One learns about various unitary operators in a first course in harmonic analysis. One example is the Fourier transform,

$$[\mathbf{F}f](x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi i x y} dy.$$

It will be convenient to scale the Fourier transform by an eighth root of unity, to define

$$\mathbf{W}f = \zeta \cdot \mathbf{F}f, \text{ and } \zeta = e^{i\pi/4}.$$

Fourier inversion gives the formulae,

$$[\mathbf{F}\mathbf{F}f](x) = f(-x), \text{ so } [\mathbf{W}\mathbf{W}f](x) = if(-x). \tag{2.2}$$

Note that $\mathbf{F}^4 = \mathrm{Id}$, but $\mathbf{W}^4 = -1$, in the group $\mathrm{U}(\mathcal{H})$.

Another operator is obtained by multiplying by a function of absolute value one. For $b \in \mathbb{R}$, define $\sigma_b(x) = e^{\pi i b x^2}$ (a "quadratic sinusoid"), and define unitary operators $\mathbf{E}_+(b)$, $\mathbf{E}_-(b)$ by

$$[\mathbf{E}_+(b)f](x) = \sigma_b(x) \cdot f(x), \text{ and}$$
$$\mathbf{E}_-(b) = \mathbf{W}\mathbf{E}_+(b)\mathbf{W}^{-1}.$$

A computation in Fourier analysis (admittedly not in $\mathcal{L}^2$) shows that $\mathbf{F}\sigma_b = |b|^{-1/2}\zeta^{\mathrm{sgn}(b)}\sigma_{-1/b}$. This allows us to compute

$$[\mathbf{E}_-(b)f](x) = |b|^{-1/2}\zeta^{\mathrm{sgn}(b)} \cdot [\sigma_{-1/b} * f](x).$$

Here the star $*$ denotes convolution.

For $0 \neq u \in \mathbb{R}$, define the operator $\mathbf{H}(u)$ by

$$[\mathbf{H}(u)f](x) = |u|^{1/2} f(ux) \cdot \begin{cases} 1 & \text{if } u > 0; \\ -i & \text{if } u < 0. \end{cases}$$

This is a familiar "time-scaling" operator in Fourier analysis, scaled by $|u|^{1/2}$ to make it unitary, and scaled by 1 or $-i$ to make the following equation true.

$$\mathbf{E}_-(u^{-1})\mathbf{E}_+(u) = \mathbf{E}_+(-u)\mathbf{H}(u)\mathbf{W} \text{ for all } u \in \mathbb{R}^\times.$$

This implies

$$\mathbf{H}(u) = \mathbf{E}_+(u)\mathbf{E}_-(u^{-1})\mathbf{E}_+(u)\mathbf{W}^{-1}.$$

The proofs of these formulae are left as a computational exercise. And if the reader has made it through the zoo of unitary operators, $\mathbf{W}$ and $\mathbf{E}_\pm(b)$, and $\mathbf{H}(u)$, they can find the following relations.

Proposition 2.3. *The operators above satisfy the following relations (for all $a, b \in \mathbb{R}$, $u, v \in \mathbb{R}^\times$).*

i. $\mathbf{W}^2 = \mathbf{H}(-1)$;
ii. $\mathbf{E}_+(a+b) = \mathbf{E}_+(a)\mathbf{E}_+(b)$;
iii. $\mathbf{H}(u)\mathbf{H}(v) = (u,v)_2\mathbf{H}(uv)$;
iv. $\mathbf{H}(u)\mathbf{E}_+(a)\mathbf{H}(u^{-1}) = \mathbf{E}_+(au^2)$.

In (iii), the expression $(u,v)_2 \in \{1,-1\}$ is the Hilbert symbol, defined by

$$(u,v)_2 = \begin{cases} 1 & \text{if } u > 0 \text{ or } v > 0; \\ -1 & \text{if } u < 0 \text{ and } v < 0. \end{cases}$$

We have focused on the operators and relations above, because they are analogous to matrices and relations in SL_2. For any field F, define matrices in $\mathrm{SL}_2(F)$,

$$w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; e_+(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ for all } b \in F.$$

Define also, for all $b \in F$, $u \in F^\times$,

$$e_-(b) := we_+(b)w^{-1} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

$$h(u) := e_+(u)e_-(u^{-1})e_+(u)w^{-1} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

In Shalika's thesis [Sha04, Thm. 1.2.1] he proves:

Proposition 2.4. *Let F be a field. Then $\mathrm{SL}_2(F)$ can be presented by generators $\{w, e_+(b) : b \in F\}$, subject to the following relations.*

i. $w^2 = h(-1)$.
ii. $e_+(a+b) = e_+(a)e_+(b)$ *for all* $a, b \in F$.
iii. $h(u)h(v) = h(uv)$ *for all* $u, v \in F^\times$.
iv. $h(u)e_+(a)h(u^{-1}) = e(au^2)$ *for* $a \in F, u \in F^\times$.

The only difference in the relations, between Propositions 2.3 and 2.4, is the sign in relation (iii). Taken together, we find the following.

Theorem 2.5. *Let M be the subgroup of $\mathrm{U}(\mathcal{H})$ generated by $\mathbf{W}$ and $\{\mathbf{E}_+(b) : b \in \mathbb{R}\}$. Then there is a unique isomorphism $\iota : SL_2(\mathbb{R}) \to M/\{\pm 1\}$ satisfying*

$$\iota\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{W} \text{ and } \iota\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \mathbf{E}_+(b).$$

Indeed, the relations (i-iv) of $SL_2(\mathbb{R})$ are satisfied by $\mathbf{W}$ and $\mathbf{E}_+(b)$, once we quotient out by ± 1. This provides the unique homomorphism from $SL_2(\mathbb{R})$ to $M/\{\pm 1\}$. Surjectivity is clear since M is by definition generated by $\{\mathbf{W}, \mathbf{E}_+(b) : b \in \mathbb{R}\}$. For injectivity, we can use the fact that the only proper nontrivial normal subgroup of $SL_2(\mathbb{R})$ is $\{\pm 1\}$, and

$$\iota\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \iota(h(-1)) = [\mathbf{H}(-1)] \neq [\pm 1].$$

In this way, one can constructively *define* the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ to be the subgroup of $\mathrm{U}(\mathcal{H})$ generated by $\mathbf{W}$ and $\{\mathbf{E}_+(b) : b \in \mathbb{R}\}$. The topological characterization of the first section is replaced by an analytic construction. This construction extends from $\mathrm{Mp}_2(\mathbb{R})$ to $\mathrm{Mp}_{2n}(\mathbb{R})$, when the Hilbert space $\mathcal{L}^2(\mathbb{R})$ is replaced by $\mathcal{L}^2(\mathbb{R}^n)$.

More dramatically, this construction extends from $\mathbb{R}$ to any field where one may perform harmonic analysis, e.g., the fields $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, and their finite extensions (avoiding characteristic two). The Hilbert space $\mathcal{L}^2(\mathbb{R})$ is replaced by $\mathcal{L}^2(F)$ for such a *local field* F. The function $x \mapsto e^{2\pi i x}$ is replaced by a suitable additive character $x \mapsto \mathbf{e}(x) \in \mathrm{U}(1)$, to define a Fourier transform $\mathbf{F}$ and operator $\mathbf{E}_+(b)$ in this context. This defines metaplectic groups $\mathrm{Mp}_{2n}(F)$ for every local field (avoiding characteristic two), following Weil [Wei64] and Shalika's 1966 PhD thesis (published in [Sha04]).

Beyond just constructing the metaplectic group for its own sake, Weil's operator theoretic approach set the stage for decades of results in number theory. Constructing the metaplectic group over all local fields, and putting the local fields together, Weil's construction defines an *adelic* metaplectic group $\mathrm{Mp}_{2n}(\mathbb{A})$. Here $\mathbb{A}$ is the ring of adeles, and $\mathrm{Mp}_{2n}(\mathbb{A})$ is a double-cover of the adelic symplectic group $\mathrm{Sp}_{2n}(\mathbb{A})$.

As is customary in number theory (since Tate's Thesis) and automorphic forms, one considers the subgroup $\mathrm{Sp}_{2n}(\mathbb{Q}) \subset \mathrm{Sp}_{2n}(\mathbb{A})$. The metaplectic cover $\mathrm{Mp}_{2n}(\mathbb{A})$ naturally "splits" over $\mathrm{Sp}_{2n}(\mathbb{Q})$, meaning that the inclusion map $\mathrm{Sp}_{2n}(\mathbb{Q}) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{A})$ lifts to realize $\mathrm{Sp}_{2n}(\mathbb{Q})$ as a subgroup of $\mathrm{Mp}_{2n}(\mathbb{A})$.

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{Mp}_{2n}(\mathbb{A}) \arrow[r] & \mathrm{Sp}_{2n}(\mathbb{A}) \arrow[r] & 1 \\
 & & & \mathrm{Sp}_{2n}(\mathbb{Q}) \arrow[u, "\text{inclusion}"'] \arrow[ul, "\text{splits!}"] &
\end{tikzcd}$$

This natural splitting carries precisely the information of quadratic reciprocity, Thus one finds a basic connection between harmonic analysis (on local fields and the adeles) and reciprocity laws in number theory. The splitting above also set the stage for Gelbart's analysis [Gel76] of automorphic forms and representations of metaplectic groups, e.g., the spectral decomposition of $\mathcal{L}^2(\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}_2(\mathbb{A}))$. This in turn led to the full development of the theta correspondence by Waldspurger. It is too big a subject for treatment here, so we refer to the beautiful survey of Dipendra Prasad [Pra98] and more recent state of the art by Wee Teck Gan and Wen-Wei Li [GL18].

3. Steinberg and Matsumoto

Proposition 2.4 describes $\mathrm{SL}_2(F)$ by generators and relations, and Steinberg generalizes this to Chevalley groups in [Ste16, §6]. While Shalika used relations like (i-iv) to prove that the group of operators $\mathrm{Mp}_2(F)$ is a double-cover of $\mathrm{SL}_2(F)$, one may use generators and relations to directly construct the metaplectic group – without any reference to operators at all! Here we describe this last approach to metaplectic groups as it leads to important generalizations.

Forget about topology and analysis, and consider a simple simply-connected Chevalley group over any field F. These are classified according to Cartan type: the type A_ℓ groups are $\mathrm{SL}_{\ell+1}(F)$, and the type C_ℓ are the symplectic groups $\mathrm{Sp}_{2\ell}(F)$, and the type B_ℓ and D_ℓ are the "split" spin groups $\mathrm{Spin}_{2\ell+1}(F)$ and $\mathrm{Spin}_{2\ell}(F)$ respectively. One has exceptional Chevalley groups too, with types E_6, E_7, E_8, F_4, G_2. Each Chevalley group comes with a root system Φ, which Steinberg uses to present the group: the generators are elements $e_\alpha(x)$ for each root $\alpha \in \Phi$ and all $x \in F$. The relations (see [Ste16, §6]) are essentially those of Proposition 2.4, but for all roots in Φ:

A. $e_\alpha(x+y) = e_\alpha(x)e_\alpha(y)$ for all $x, y \in F$, $\alpha \in \Phi$.

B. A description of the commutator of $e_\alpha(x)$ and $e_\beta(y)$, for all $\alpha, \beta \in \Phi$ with $0 \neq \alpha + \beta$.

$\tilde{\mathrm{B}}$. $w_\alpha(t)e_\alpha(x)w_\alpha(-t) = e_{-\alpha}(-t^{-2}x)$ for all $t \in F^\times$, $x \in F$. Here $w_\alpha(t) \coloneqq e_\alpha(t)e_{-\alpha}(t^{-1})e_\alpha(t)$.

C. $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$ for all $u, v \in F^\times$. Here $h_\alpha(t) \coloneqq w_\alpha(t)w_\alpha(-1)$ for all $t \in F^\times$.

Let $\mathrm{G}(F)$ be the simply-connected Chevalley group with entries in a field F, presented as above with generators $\{e_\alpha(x) : \alpha \in \Phi, x \in F\}$. If one keeps relations A, B, $\tilde{\mathrm{B}}$, and drops relation C, one gets a new sort of covering group $\mathrm{G}'(F)$ with a surjective homomorphism π to $\mathrm{G}(F)$. In this way, one finds a short exact sequence

$$1 \to \varkappa(\mathrm{G}, F) \to \mathrm{G}'(F) \xrightarrow{\pi} \mathrm{G}(F) \to 1. \tag{3.1}$$

Steinberg and Matsumoto prove the following theorem, after a first step by Moore [Moo68] for SL_2.

Theorem 3.2. *Assume $\#F \geq 5$ in general, and $\#F \geq 11$ when $\mathrm{G} = \mathrm{SL}_2$. Then the group $\mathrm{G}'(F)$ is the <u>universal</u> central extension of $\mathrm{G}(F)$. Namely, $\varkappa(\mathrm{G}, F) \coloneqq \mathrm{Ker}(\pi)$ is in the center of $\mathrm{G}'(F)$, and the short exact sequence* (3.1) *is universal among central extensions of $\mathrm{G}(F)$ by abelian groups, as described by the diagram below.*

$$\begin{tikzcd}
1 \arrow[r] & \varkappa(\mathrm{G},F) \arrow[r] \arrow[d, "\exists!"] & \mathrm{G}'(F) \arrow[r] \arrow[d, "\exists!"] & \mathrm{G}(F) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & A \arrow[r] & M \arrow[r] & \mathrm{G}(F) \arrow[r] & 1
\end{tikzcd}$$

Remarkably, the kernel $\varkappa(\mathrm{G}, F)$ in this universal central extension does *not* depend as much on the group $\mathrm{G}(F)$ as one would expect! Steinberg studied the kernel $\varkappa(\mathrm{G}, F)$ extensively, and Matsumoto completed the work to prove the following in [Mat69, Corollaire 5.11].

Theorem 3.3. *Suppose that* G *is* not *of type C, i.e., exclude $\mathrm{G} = \mathrm{Sp}_{2n}$. Then the kernel $\varkappa(\mathrm{G}, F)$ is isomorphic to the Milnor K-theory group $\mathrm{K}_2(F)$,*

$$\varkappa(\mathrm{G}, F) \cong \mathrm{K}_2(F) \coloneqq \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle s \otimes (1-s) : s \in F - \{0,1\}\rangle}.$$

In type C, the Milnor K-theory group $\mathrm{K}_2(F)$ is naturally a quotient of $\varkappa(\mathrm{G}, F)$.

The theorem is a bit anachronistic in the presentation above; Milnor's group $\mathrm{K}_2(F)$ was defined in 1970, a few years after Steinberg and Matsumoto; in fact, Milnor explains in [Mil69] that his definition of $\mathrm{K}_2(F)$ was *motivated* by Steinberg, Moore, and Matsumoto.

With this hindsight, we have now constructed central extensions of Chevalley groups,

$$1 \to \mathrm{K}_2(F) \to \mathrm{G}'(F) \to \mathrm{G}(F) \to 1. \tag{3.4}$$

This construction is purely by generators and relations, using (A,B,$\tilde{\mathrm{B}}$), and makes sense for *every* field F. So how does it relate to our metaplectic groups, like our original group $\mathrm{Mp}_2(\mathbb{R})$?

The connection is through the Hilbert symbol, and we refer to number theory texts, like Cassels and Fröhlich [CF67, Exercises, p.351] for a fuller treatment. As it is usually presented, one begins with a local field F, and assumes that F contains a primitive d^{th} root of unity. In particular, one assumes that $\mathrm{char}(F) \nmid d$, and writes $\mu_d = \{\zeta \in F : \zeta^d = 1\}$ for the resulting cyclic group of order d. The Hilbert symbol is then a function,

$$(\bullet, \bullet) : F^\times \times F^\times \to \mu_d,$$

satisfying the properties

$$(uv, w) = (u, w)(v, w) \text{ and } (u, vw) = (u, v)(u, w),$$

$$(u, v)(v, u) = 1, \text{ and } (s, 1-s) = 1,$$

for all $u, v, w \in F^\times$, and all $s \in F - \{0, 1\}$.

From these properties, the Hilbert symbol $(\bullet, \bullet)$ factors through the Milnor K-theory group $\mathrm{K}_2(F)$. In fact when F is a local field containing a primitive d^{th} root of unity, the Hilbert symbol gives a surjective homomorphism

$$\mathrm{Hilb}_d : \mathrm{K}_2(F) \to \mu_d.$$

Recalling the central extension $\mathrm{G}'(F)$ from (3.4), and defining $\tilde{\mathrm{G}}(F) = \mathrm{G}'(F)/\mathrm{Ker}(\mathrm{Hilb}_d)$, we find a new sort of covering group

$$1 \to \mu_d \to \tilde{\mathrm{G}}(F) \to \mathrm{G}(F) \to 1.$$

When $d = 2$ and $\mathrm{G} = \mathrm{Sp}_{2n}$, this construction gives yet another definition of metaplectic groups,

$$\tilde{\mathrm{G}}(F) = \mathrm{Mp}_{2n}(F)$$

by generators and relations (algebra) along with the Hilbert symbol (number theory).

For other Chevalley groups, the covering groups $\tilde{\mathrm{G}}(F)$ (generalized metaplectic groups) play an important role in number theory and automorphic forms, e.g., the 3-fold cover of SL_3 in [BFG01]. The 2-fold cover of the exceptional group of type F_4 arises in the theta correspondence, e.g., if one wishes to understand half-integral weight modular forms on $SU(1,2)$. The 3-fold cover of the exceptional group G_2 arises if one wishes to understand "minimal" representations—analogues of Weil's operator construction for exceptional groups.

4. Further Generalizations

Work of Steinberg and Matsumoto demonstrates that $K_2(F)$ appears in the study of central extensions of $G(F)$, when G is a Chevalley group. But if one wants to go beyond Chevalley groups—to understand reductive groups like GL_n or "non-split" groups like SU_3—one finds more difficulties. Deodhar had success adapting the Steinberg-Matsumoto techniques to simple quasisplit groups in [Deo75]. In 1984, Prasad and Raghunathan used the geometry of the Bruhat-Tits building to describe the universal topological central extension for isotropic simply-connected groups over local fields in [PR84a, PR84b]. But groups like GL_n and algebraic tori do not have universal central extensions at all.

Brylinski and Deligne created a new framework in [BD01], which incorporates K_2 in the *definition* of the central extension rather than discovering it later. They study central extensions of a reductive algebraic group G by K_2, viewing all "groups" as sheaves of groups on the big Zariski site over a field. At the end of the day, Brylinski and Deligne's framework includes all of the central extensions of Steinberg and Matsumoto, including Weil's metaplectic groups of course. For a number theorist interested in metaplectic groups, the extensions of Brylinski and Deligne seem most convenient and general for extending the Langlands program—this was the goal of the author, with Wee Teck Gan and Fan Gao, in our Asterisque volume [GGW18].

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