In Memory of Earl Jay Taft (1931–2021)

Uma N. Iyer, Susan Montgomery, Siu-Hung Ng, and David Radford

Uma N. Iyer

My first correspondence with Professor Earl J. Taft was when I was a post-doctoral visitor at the Harish-Chandra Research Institute, India, in 1999. I do not remember what exactly we corresponded about, but I do remember the thrill of receiving an answer from him for a question I had within a day. That generosity of a much renowned mathematician towards a post-doc he had never met is what I remember him the most for.

Years later, in 2006, I met him at the City University of New York (CUNY). Professor Taft regularly participated in several seminars at CUNY. In collaborative works with A. Lauve and S. Rodriguez-Romo, he constructed left Hopf algebras that are not Hopf algebras modeled after $SL_q(n)$. Our joint work [2] was a search for a left-Hopf algebra which is not a Hopf algebra containing $U_q(sl_2)$. While we were unsuccessful in identifying such an algebra, we were able to look at one non-example closely. Together with Professor Jonathan D.H. Smith, we were able to study the connections between one-sided Hopf algebras and one-sided quantum quasigroups.

Working with Professor Taft never seemed like work. He talked about various things: about his childhood in New York City, about his travels all around the world, about his amazing wife, Hessy Levinsons Taft, and her family. He could read and speak several languages fluently. He could discuss opera and literature as comfortably as he could discuss mathematics.

At the risk of repeating myself, I am grateful for his kindness and generosity. What a privilege it has been to have worked with him, to have had interacted with him and his wife. My deepest condolences to his family. *Om Shanti.*

Uma N. Iyer

Susan Montgomery

I first met Earl in the fall of 1971, when I visited Rutgers. I also met his wife Hessy and family, who were very welcoming. Of course, Earl suggested that I should look at Hopf algebras. At the time I was not interested; if an algebra had a multiplication, why did it need a comultiplication? Over the years, I would run into Earl at meetings, and again he would suggest I should look at Hopf algebras.

In 1981, I started working with Miriam Cohen on the duality between group actions on rings and rings graded

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by groups; suddenly the fact that we had two Hopf algebras and were looking at their duals made everything clear. Then I was hooked.

This began my 40 year interest in Hopf algebras. Later, I realized that one needed comultiplication in order to take the tensor product of modules, and having a suitable comultiplcation on an algebra A means that $Mod(A)$, the category of A-modules, is a tensor category. Beautiful!

So Earl was right: I should have looked at Hopf algebras earlier.

In the 70s general Hopf algebras, which are neither commutative nor cocommutative, were believed to be pathological as they did not directly arise from algebraic geometry or Lie theory. As a result, some Hopf algebra papers were published as algebraic geometry in disguise.

Drinfeld's fundamental ICM talk in 1986 on quantum groups changed this narrative; he gave solutions to the quantum Yang Baxter equation by using the representations of certain noncommutative, noncocommutative Hopf algebras. Suddenly general Hopf algebras were of wide interest (and some of the naysayers said that in retrospect they had been too harsh). But Drinfeld made one small error; he said his quantum groups were the first examples of non-commutative non-cocommutative Hopf algebras.

But in 1971 Earl constructed infinite families of such objects. He was interested in constructing examples in which the order of the antipode S could be arbitrarily large (In the commutative and cocommutative cases, it is always true that $S^2 = id$). Earl's examples have turned out to be fundamental in the classification of Hopf algebras. They are now known as Taft algebras. In a joint paper with Yevgenia Kashina and Siu-Hung Ng [3], we proved, by computing the 2nd indicators, that the Taft algebras are completely distinguished by their representation tensor categories.

As more people worked in the area, Earl came to all of our meetings and was very encouraging.

In 1988 there was an algebra meeting at the Banach Institute in Warsaw. Earl, Hessy, and I were put up at a hotel with a nice restaurant, but the menus were only in Polish. No Polish-English dictionaries were available in Warsaw. Hessy found a Polish-Spansh dictionary, which solved our problem, as she spoke Spanish fluently. So at meals, she would translate the Polish into Spanish, and then tell us what the items were. We had a lot of fun there.

I owe him a great debt, both mathematically and personally.

Susan Montgomery

Siu-Hung Ng

Some conversations with Earl from when I became his PhD student have stuck in my mind to this day. On several occasions, he expressed to me that he would continue to work as long as he could do the job mentally and physically. Earl sent me a message in November 2016 announcing his retirement in 2017. He passed away on August 9, 2021.

Figure 1. Taft at St. John's, Newfoundland, 2001.

Earl joined Rutgers University in 1959 after his three-year instructorship at Columbia University. I was admitted to Rutgers in 1992 and became Earl's PhD student in 1994. Before asking Earl to be my advisor, I took a graduate abstract algebra course and a course on quantum groups that he taught. He was a serious lecturer, and I liked his style perhaps because of my cultural background. Meanwhile, Earl was a pioneer of Hopf algebras. He discovered a groundbreaking family of Hopf algebras, now called the Taft algebras. I decided to ask Earl to be my PhD supervisor after understanding some of the interplays among Hopf algebras and many other areas of mathematics and physics.

During the time when I was his graduate student, he was working on Lie bialgebras and the Hopf algebra structure of linearly recursive sequences. Since I had some background in mathematics from my master's degree from Hong Kong, Earl showed me what he had been thinking

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in some of his research articles, and I began to work on his proposed problems. We met weekly and I reported what I discovered, even when he was on sabbatical at the Institute for Advanced Study.

In [12], Earl proved that for any Lie algebra L, if $r \in L \wedge L$ satisfies the classical Yang Baxter Equation (CYBE), then the map $\delta_r : L \to L \wedge L$, $\delta_r(x) = [r, x]$ defines a Lie bialgebra structure on L. The pair (L, δ_r) is called a *coboundary* Lie bialgebra. If L is the Witt or Virasoro algebra, then the solutions of the CYBE of the form $a \wedge b \in L \wedge L$ were classified in [12]. With Earl's guidance, I proved in our joint paper [7] that if L is the one-sided Witt algebra (the Lie algebra of derivations of $\mathbb{C}[t]$), then every Lie bialgebra structure on L is a coboundary Lie bialgebra given by a solution of the CYBE r of the form $a \wedge b$. I was thrilled by the result and grateful for Earl's encouragement to complete the paper.

In another paper [8] of Earl's, the set of linearly recursive sequences over a field $\Bbbk$ was proven to be a Hopf algebra isomorphic to $(\Bbbk[x])^\circ$, the dual Hopf algebra of $\Bbbk[x]$. In particular, they are closed under the convolution product; if $(f_i)_{i \geq 0}$ and $(g_i)_{i \geq 0}$ are $\Bbbk$-linearly recursive sequences, then so is $\left(\sum_i \binom{n}{i} f_i g_{n-i}\right)_{n \geq 0}$. He asked whether they are also closed under the q-convolution product, which is similar to the convolution product but using the q-binomial coefficients $\binom{n}{i}_q$ where $q \in \Bbbk$ is a given root of unity. We ended up answering the question affirmatively with two different approaches in [4]: the first one is from the view point of dual Hopf algebras in the category of representations of a finite abelian group and the second one is a direct combinatorial method. This joint work with Earl also brought me to the representation categories of Hopf algebras, which are fundamental examples of finite tensor categories.

In the course of working on these joint articles, Earl demonstrated how I should carry on my career as a mathematician. We tested and discussed some ideas, wrote up notes on our discussions, and organized manuscripts for publication. He carefully edited all the drafts of these articles and meticulously verified the mathematics written on the drafts. In my PhD thesis, I extended the result of [6] on Lie bialgebras to those base fields of finite characteristic. I am deeply grateful for his guidance on my graduate research projects.

After finishing these collaborations, I returned to my focus on Hopf algebras. Earl recommended that I communicate with Susan Montgomery, who first coined the term "Taft algebra." I had a few email communications with Montgomery soon after I graduated from Rutgers, and I attended some seminars at MSRI in 1999, where I was excited by an open question presented by Montgomery (cf. [4]): If p is a prime number, must a nonsemisimple Hopf algebra over $\mathbb{C}$ of dimension p^2 be a Taft algebra? An affirmative answer to the question is of fundamental importance to the theory of Hopf algebras.

Figure 2. Ng, Hoffman, Witherspoon, Richmond, Heckenberger, and Taft at Bowling Green, Kentucky, 2005.

In Earl's 1971 paper [11], he constructed the Hopf algebra $T(q)$, later called the Taft algebra, for any primitive nth root of unity $q \in \Bbbk$. The underlying $\Bbbk$-algebra of $T(q)$ is generated by a, x subjected to the relations $xa = qax$, $x^n = 0$ and $a^n = 1$, and its coalgebra structure (Δ, ϵ) is given by $\Delta(a) = a \otimes a$, $\Delta(x) = a \otimes x + x \otimes 1$, $\epsilon(a) = 1$ and $\epsilon(x) = 0$. In 1998, Andruskiewitsch and Schneider [1] proved Susan's open question for *pointed* Hopf algebras of dimension p^2. Under Earl's encouragement in 2002, long after my graduation, I finally settled this open question in [5] over $\mathbb{C}$ by proving that p^2-dimensional Hopf algebras over $\mathbb{C}$ are pointed. I could not have done this without the continuous support of Earl. The result is not only a mathematical achievement, but also a reminder of Earl's unique place in my heart.

Beside mathematics, Earl had many other interests, particularly in languages. He was fluent in Spanish and French. I still remember that he studied Chinese before his trip to Hong Kong and mainland China in 1997. It was apparent to me that he was entertained by learning new languages. He also immensely enjoyed the city life in New York because of the proximity to good food and Broadway theatre. He took great pleasure in traveling. Before 2010, I met Earl quite often in conferences. He appreciated wine with meals. I usually joined him for a bottle because he was always more relaxed and in good humor after a couple of glasses of wine. Accompanying Earl to a conference was always a delight for me.

In addition to doing mathematics research, Earl promoted mathematics in different communities. He was the founding editor of *Communications in Algebra* in 1974. In his condolence message, Professor Kar-ping Shum, former president of the Southeast Asian Mathematical Society, described Earl's contributions to the Southeast Asian mathematics community. Earl initiated the first *International*

Congress in Algebra and Combinatorics (ICAC) in Hong Kong in 1997 [10], and proposed organizing the ICAC every ten years. He was a keynote speaker at the ICAC in 1997 and 2007, and helped edit the proceedings of the ICAC in 1997, 2007, and 2017, which were all published by World Scientific Publishing. Earl also helped develop the *Southeast Asian Bulletin of Mathematics*, which now publishes six issues per year. Professor Shum emphasized that ICAC and its proceedings have had a great impact on the mathematics community.

After I graduated from Rutgers, Earl provided his down-to-earth advice on every major decision of my career. In retrospect, Earl was my thesis supervisor, my life-long career advisor, and a great friend. I am wholeheartedly thankful to him for his unfettered support all these years and I am proud to have been his student.

Siu-Hung Ng

David Radford

I am forever grateful for the role Earl played in shaping my career. I have very fond memories of visits to Rutgers University and of my association with Earl there, as well as at numerous mathematical meetings. Earl was a gracious host, funny, thoughtful, and always considerate.

He invited me to give a talk in the Algebra Seminar at Rutgers in 1974 and invited me to spend the year there as a visitor in 1975–1976, which turned out to be mathematically very stimulating and productive. Among other things, Earl led the seminar discussion of Warren Nichols's PhD thesis, the origin of the Nichols algebras. Together with Robert Wilson, we wrote a paper on the antipode of a Hopf algebra [9].

That year also provided a great opportunity for me to look for another job with very little stress. It was a fruitful transition. From Rutgers, I went to the University of Illinois Chicago in 1976 where I remained on the faculty for the rest of my academic career. I was a visitor again at Rutgers in 1979–1980.

Earl was the first to construct finite-dimensional Hopf algebras with antipode of a given even order [11], later called the Taft algebras. In this 1971 paper, he describes the important variation of the commutative law, namely $yx = qxy$, where q is a root of unity. This variation is fundamental in his construction and the theory of quantum groups, which are important mathematical structures discovered in the 1980s.

Taft algebras are basic examples of Hopf algebras that have been studied over the years in many contexts. Quantum groups are replete with them.

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