ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

December 2023 Volume 70, Number 11

AMERICAN
MATHEMATICAL
SOCIETY

AMS

Advancing research. Creating connections.

The cover design is based on imagery from "Presbyopia Correction, Differential Geometry, and Free Boundary PDEs," page 1763.

APPLY FOR SUPPORT FOR YOUR SUMMER MATHEMATICS PROGRAMS

AMS Young Scholars Program

supported by the Epsilon Fund

Grants of up to $15,000 support existing summer programs that are successful in nourishing mathematically engaged youth.

Apply October 15–January 15 on MathPrograms.org

American Mathematical Society

Advancing research. Creating connections.

www.ams.org/emp-epsilon

Questions? Contact us at mathcamps@ams.org.

Notices of the American Mathematical Society

December 2023

Cover Credit: The image used in the cover design appears in "Presbyopia Correction, Differential Geometry, and Free Boundary PDEs," p. 1763, and is courtesy of Brocken Inaglory, CC-BY-SA 3.0.

FEATURES

FROM THE AMS SECRETARY

INVITATIONS FROM THE AMS

Notices
of the American Mathematical Society

SUBSCRIPTION INFORMATION

Individual subscription prices for Volume 70 (2023) are as follows: nonmember, US$742, member, US$445.20. (The subscription price for members is included in the annual dues.) For information on institutional pricing, please visit https://www.ams.org/publications/journals/subscriberinfo. Subscription renewals are subject to late fees. Add US$6.50 for delivery within the United States; US$24 for surface delivery outside the United States. See www.ams.org/journal-faq for more journal subscription information.

ADVERTISING

Notices publishes situations wanted and classified advertising, and display advertising for publishers and academic or scientific organizations. Advertising requests, materials, and/or questions should be sent to:
classads@ams.org (classified ads)
notices-ads@ams.org (display ads)

PERMISSIONS

All requests to reprint *Notices* articles should be sent to:
reprint-permission@ams.org.

SUBMISSIONS

The editor-in-chief should be contacted about articles for consideration after potential authors have reviewed the "For Authors" page at **www.ams.org/noticesauthors**.

The managing editor should be contacted for additions to our news sections and for any questions or corrections. Contact the managing editor at: **notices@ams.org**.

Letters to the editor should be sent to: **notices-letters@ams.org**.

To make suggestions for additions to other sections, and for full contact information, see **www.ams.org/noticescontact**.

Supported by the AMS membership, most of this publication, including the opportunity to post comments, is freely available electronically through the AMS website, the Society's resource for delivering electronic products and services. Use the URL **www.ams.org/notices** to access the *Notices* on the website. The online version of the *Notices* is the version of record, so it may occasionally differ slightly from the print version.

The print version is a privilege of Membership. Graduate students at member institutions can opt to receive the print magazine by updating their individual member profiles at **www.ams.org/member-directory**. For questions regarding updating your profile, please call 800-321-4267.

For back issues see **www.ams.org/backvols**. Note: Single issues of the *Notices* are not available after one calendar year.

The American Mathematical Society is committed to promoting and facilitating equity, diversity and inclusion throughout the mathematical sciences. For its own long-term prosperity as well as that of the public at large, our discipline must connect with and appropriately incorporate all sectors of society. We reaffirm the pledge in the AMS Mission Statement to "advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals," and urge all members to conduct their professional activities with this goal in mind. (*as adopted by the April 2019 Council*)

[Notices of the American Mathematical Society (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA, GST No. 12189 2046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to Notices of the American Mathematical Society, PO Box 6248, Providence, RI 02904-6248 USA.] Publication here of the Society's street address and the other bracketed information is a technical requirement of the US Postal Service.

Printed in the United States of America. The paper used in this journal is acid-free and falls within the guidelines established to ensure permanence and durability.

Opinions expressed in signed *Notices* articles are those of the authors and do not necessarily reflect opinions of the editors or policies of the American Mathematical Society.

A WORD FROM...

Karen Saxe, Director of Government Relations for the AMS

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Photo courtesy of Macalester College/David Turner.

By the mid-twentieth century, researchers—including mathematicians—were exploring artificial intelligence (AI) and machine learning. But it wasn't until OpenAI's late 2022 public launch of ChatGPT that widespread public engagement really ramped up. Over the past year, teens have gone wild creating fun images using DALL·E, and professionals across many fields have been engaging AI in their work.

Mathematicians are thinking about many facets of AI. We talk about mathematics' role in pushing innovation in AI, how AI can be used to advance mathematical research, and appropriate uses in our classrooms to enhance learning, and we consider the ethics involved with publishing articles and books that have used AI in some capacity. We also consider our responsibility to others, as publicly accessible AI develops and is regulated.

Congress

It is honestly hard to keep up with all the hearings in Congress about AI. Congressional committees usually post announcements of these hearings only days in advance. And on this particular topic, many different committees are holding hearings including the Senate Judiciary Committee; the Senate Committee on Homeland Security and Government Affairs; the House Science, Space, and Technology Committee; and the House Armed Services Committee. During these hearings members of Congress voice their concerns on a very broad range of topics—from how our own as well as our adversaries' defense and intelligence communities use AI; to providing funding and identifying priorities for AI researchers; to applications in agriculture, climate change, manufacturing, and supply chains; to how our creativity enterprise (music and film industries especially) is threatened.

Senate Leader Schumer has held educational "senator only" sessions about artificial intelligence, which have offered a general overview of AI and its current capabilities, an examination of the research frontiers, and thoughts about how we can maintain American leadership as AI develops. These were held privately so that the senators could ask questions freely, genuinely learn, and avoid political grandstanding.[1]

It is also hard to keep up with all of the bills being introduced. These include the *Creating Resources for Every American To Experiment with Artificial Intelligence Act of 2023* (*CREATE AI Act*), which would establish "the National Artificial Intelligence Research Resource (NAIRR) as a shared national research infrastructure that provides AI researchers and students with greater access to the complex resources, data, and tools needed to develop safe and trustworthy artificial intelligence."[2] Other bills introduced focus on everything from building an AI workforce, to ensuring that human control must always be used to launch nuclear weapons, to regulating how AI is used in political campaigning.

The White House

The White House is also devoting attention to AI and here, as with Congress, it is a developing and moving target. In July, the White House announced a voluntary commitment by major AI companies to help ensure responsible

Karen Saxe is an associate executive director of the AMS and head of the Government Relations Division. Her email address is kxs@ams.org.

DOI: https://doi.org/10.1090/noti2817

[1] https://www.journals.uchicago.edu/doi/abs/10.1086/709147?journalCode=jop

[2] https://www.booker.senate.gov/news/press/booker-heinrich-young-rounds-introduce-bipartisan-bicameral-bill-to-expand-access-to-artificial-intelligence-research

innovation of secure systems that address societal challenges.[3]

The Biden–Harris administration has also issued a *Blueprint for an AI Bill of Rights*,[4] and a *National AI Research and Development Strategic Plan*.[5] Additionally, the President's Council of Advisors on Science and Technology (PCAST) has launched a working group on generative AI, co-led by mathematician Terence Tao.

The NSF

Because the National Science Foundation (NSF) funds mathematics research at a higher level than any other federal agency, it is worth noting that the NSF is investing heavily in AI.[6] Many *Notices* readers will know a lot about this already, through their own research awards and through activities at the NSF-funded Mathematical Sciences Institutes.[7] The NSF also has launched National Artificial Intelligence Research Institutes across the country which cover a very broad range of topics. One in Pennsylvania focuses on AI-enabled materials. An Oklahoma-based institute brings together a cross-disciplinary team to address pressing environmental concerns. A team in Texas is developing foundational tools. Others focus on rural health, law and decision-making, and construction of and operation of buildings.

The National Academies

While the National Academies of Sciences, Engineering, and Medicine are not government entities, they do provide expertise to federal agencies. In July, NASEM's Board on Mathematical Sciences and Analytics held a well-attended workshop on *AI to Assist Mathematical Reasoning*.[8] This workshop was virtual and open to the public; through their website[9] you can listen to that workshop, watch for future activities, and sign up for their email newsletter.

The AMS

Scientific societies such as the AMS are contemplating what AI means for their members as researchers, students, professors, and creators of copyrighted and patented materials.

In July, AMS President Bryna Kra formed the Advisory Group on Artificial Intelligence and the Mathematical Community.[10] According to its charge, "(t)he advisory group will focus on issues that are at the forefront of these developments, including: the role of mathematics in the development and deployment of artificial intelligence, the impact of artificial intelligence on research in mathematics, the use of AI in publications, education, and research, and the impact of AI on our community."

You may know that the AMS sponsors one Congressional Fellow each year.[11] Fellows work for one year in a Senate or House office, and lend their expertise in various ways—by preparing for hearings, crafting legislation, and meeting with constituents. The AMS Congressional Fellowship is in partnership with the American Association for the Advancement of Science (AAAS), and the current cohort of AAAS fellows (and most likely those for the next few years) includes several chosen for their AI expertise. These fellows have a unique opportunity to help shape our nation's AI policies.

JMM 2024

The three AMS Colloquium Lectures on *Machine Assisted Proof* will be given by Terence Tao (UCLA and PCAST).

The AMS Committee on Science Policy[12] is organizing a panel discussion on *Artificial Intelligence in Mathematics, Science, and Society*. The organizers are Gunnar Carlsson, Stanford University (chair); Duane Cooper, Morehouse College; Carla Cotwright-Williams, US Dept. of Defense (and 2012–2013 AMS Congressional Fellow); Fern Hunt, National Institute of Standards and Technology (NIST); and Jerry McNerney, US House of Representatives, retired.

And, of course, our partner societies are also engaged on the topic. The Society for Industrial and Applied Mathematics (SIAM) is running two AI-relevant JMM sessions—one is the SIAM ED Session on *Artificial Intelligence and its Uses in Mathematical Education, Research, and Automation in the Industry*; the other is the SIAM Minisymposium on *Scientific Machine Learning to Advance Modeling and Decision Support*.

---

[3] https://www.whitehouse.gov/briefing-room/statements-releases/2023/07/21/fact-sheet-biden-harris-administration-secures-voluntary-commitments-from-leading-artificial-intelligence-companies-to-manage-the-risks-posed-by-ai/

[4] https://www.whitehouse.gov/ostp/ai-bill-of-rights/

[5] https://www.whitehouse.gov/briefing-room/statements-releases/2023/05/23/fact-sheet-biden-harris-administration-takes-new-steps-to-advance-responsible-artificial-intelligence-research-development-and-deployment/

[6] *Other federal agencies are also investing heavily in AI including, for example, NIST:* https://www.nist.gov/artificial-intelligence

[7] https://mathinstitutes.org/

[8] https://www.nationalacademies.org/our-work/ai-to-assist-mathematical-reasoning-a-workshop

[9] https://www.nationalacademies.org/bmsa/board-on-mathematical-sciences-and-analytics

[10] https://www.ams.org/about-us/governance/committees/comm-all.html

[11] https://www.ams.org/government/government/ams-congressional-fellowship

[12] https://www.ams.org/about-us/governance/committees/csp-home

JMM 2024 Registrants:
Join us for member activities!

JMM 2024 Grand Opening Reception

Wednesday, January 3 | 6:00 p.m.– 8:30 p.m.
Moscone Center South, Hall A

Enjoy music, food stations, and a chance to socialize and connect with your colleagues on this first night in San Francisco.

Find the AMS booth and these special freebies:

6:00 p.m.–8:30 p.m.
Photo booth—take a memory home!

6:00 p.m.–8:00 p.m.
Wine tasting

6:00 p.m.–8:30 p.m.
Virtual Reality Tour—AMS members may enjoy a unique view of "The Golden City" provided to you by the AMS Membership Department!

Special JMM Offers on AMS Membership

JMM 2024 registrant members of one of the JMM partner organizations* who have not been an AMS member since 2021, **are eligible for 50% off a 2024 American Mathematical Society Membership!**

Note: to receive this discount please **use the same email account you used to register for JMM.** This special offer is available until February 1, 2024.**

*Eligible JMM partner organizations: American Statistical Association (ASA), Association for Women in Mathematics (AWM), Mathematical Association of America (MAA), National Association of Mathematicians (NAM), and Society for Industrial and Applied Mathematics (SIAM).

**This offer excludes Affiliate, Life, and Unemployed memberships.

Join or renew your membership on Thursday–Friday at the exhibit booth and receive a complimentary gift!

Professional Portraits for AMS members—back by popular demand!

Thursday, January 4–Friday, January 5

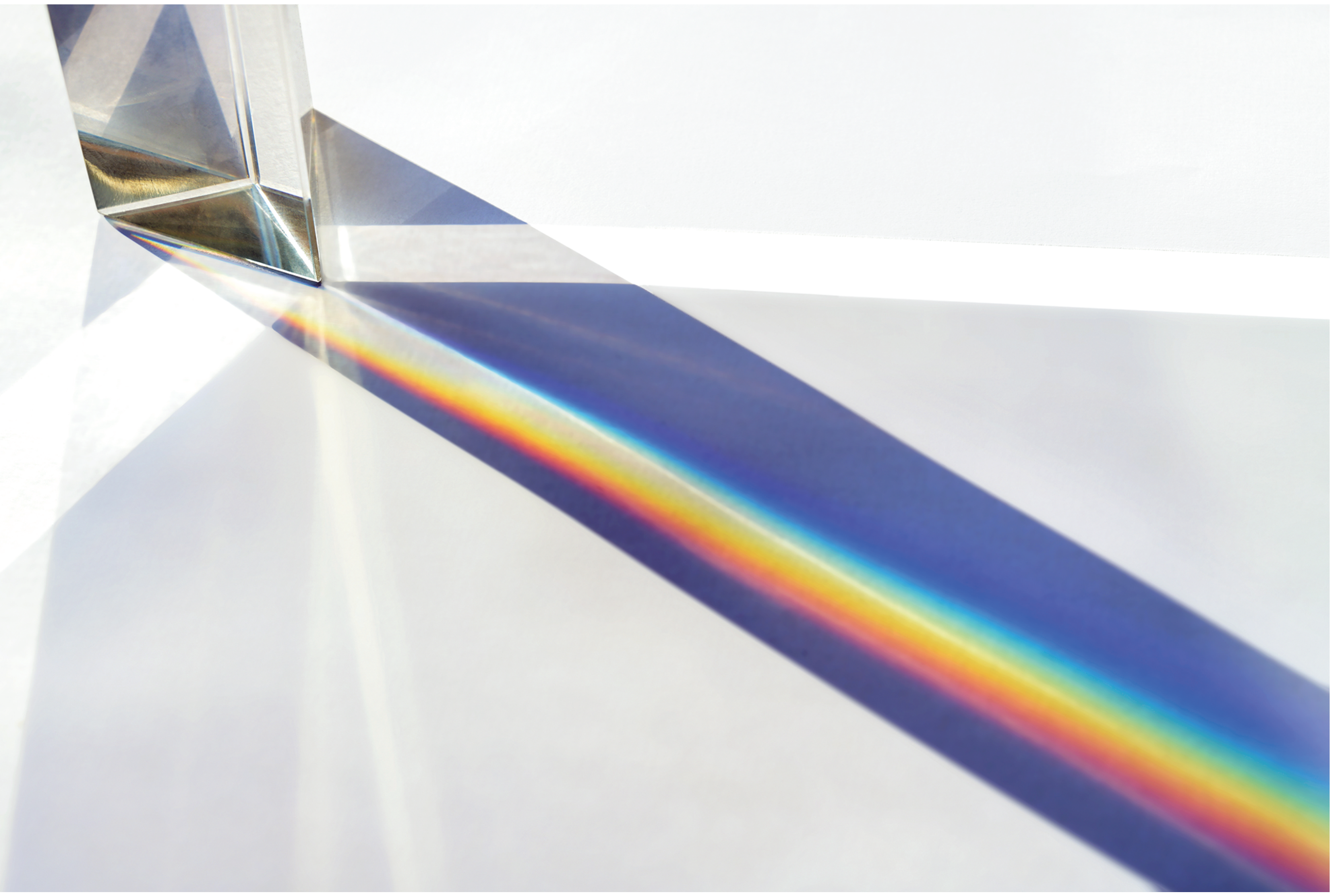
AMS members: Make your appointment to have your professional portrait taken for free. Make an appointment between 9:30 a.m.– 4:25 p.m. PST at the AMS membership booth. Your image will be emailed to you in just a few minutes!

Schedule appointments at: amermathsoc.simplybook.me/v2/

AMS American Mathematical Society
Advancing research. Creating connections.

www.ams.org/membership

Presbyopia Correction, Differential Geometry, and Free Boundary PDEs



Sergio Barbero and María del Mar González

We survey some mathematical tools involved in the design of multifocal surfaces which are used in presbyopia correction. After giving a brief background in geometrical optics, we consider the Willmore functional in differential geometry, summarizing some classical results, and then explaining its significance in this particular optical application.

Sergio Barbero is an optics researcher at Instituto de Óptica from CSIC (Spain). His email address is `sergio.barbero@csic.es`.

María del Mar González is a professor of mathematics at the Universidad Autónoma de Madrid (Spain). Her email address is `mariamar.gonzalezn@uam.es`.

Communicated by Notices *Associate Editor Daniela De Silva.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2828

1. A Basic Introduction to Geometrical Optics

Modeling precisely how light propagates in a physical medium has been, and still is, one of the great challenges of mathematical physics. Fortunately, in many cases of practical interest, light propagation can just be modeled by means of energy transported along *rays*; this is the crux of the so-called *geometrical optics theory*. Beyond geometrical optics, one has the wave optics model, a more sophisticated and accurate theory of light propagation which we will not consider here. For the interested reader, a classical text in optics is [BW65].

In general, rays are space curves in the Euclidean space, which obey a stationary action principle; namely, the trajectories are stationary solutions of path length integrals weighted by the refractive index of the propagating medium (this refractive index is related to the microscopic electromagnetic properties of the material of which the medium is made). Note that rays are straight lines when the refractive index is constant.

Image formation. Within the geometrical optics framework, given a bundle of light rays generated by a punctual source, there exists a normal surface to all the rays containing a fixed point, even if the bundle of rays has suffered multiple refractions or reflections (Malus–Dupin theorem). This surface is called the *wavefront*. Therefore, light propagation can be modeled by propagating either rays or wavefronts. Rays emanating from a point source, within a homogeneous refractive index medium, form a spherical wavefront which can be transformed after refraction or reflection by an optical element.

However, in practice, real objects are not point objects. We owe Ibn al-Haytham (c. 965–c. 1041), latinized as Alhacen, the idea that an extended object can be represented by the union of multiple points and thus, from every point of the surface of an object, light is emitted in all directions in the form of rays. These rays can, for instance after crossing a lens, converge onto another point, which is called *focus*. Following Alhacen, we say that a set of focal points is the image of an object if rays emerging from each point of the latter converge onto points of the former. Notice that the focus is the only geometrical light concept we actually *see* or measure; rays and wavefronts are merely convenient mathematical tools.

Then, an *imaging system* is a set of lenses, mirrors, or other types of optical elements that transform rays, and hence wavefronts, onto converging rays/wavefronts that focus on a light detector—such as a camera sensor or the retina of the human eye—forming a set of images of the object's points. Ideally, in order to have a "perfect image" of an extended object, each spherical wavefront generated by a point of the extended object should be transformed, by the imaging system, into an outgoing spherical wavefront. Such a system is called an absolute instrument. However, as already noted by James Clerk Maxwell in 1858 [Max58], there are some limitations to the existence of absolute instruments and, indeed, forming an image of an extended object with depth into a planar light detector is a specially difficult task.

Optical power. In any case, under the so-called *paraxial approximation*, i.e., when the sine of the angle is approximated by the angle itself in the refraction law, the imaging system's capacity of transforming wavefronts depends on the radii of curvature of both the incoming (r_i) and the outgoing (r_o) wavefronts by means of the *vergence equation*:

$$\frac{n_1}{r_0} = P + \frac{n_2}{r_i},$$

P being the *optical power* of the imaging system, and n_1 and n_2 the refractive indexes at the incoming and outgoing mediums. In consequence, if we have object points located at different distances from the imaging system, its optical power must change accordingly.

The optical power of the imaging system depends on the geometrical properties of the optical element surfaces and the refractive indexes of the mediums at each side (denoted by n_1, n_2). For instance, in the simple case of a single reflective/refractive spherical optical surface S, its optical power is $P = \frac{(n_2 - n_1)}{r_S}$, where r_S is the radius of curvature. Another simple example is given by traditional zoom camera systems, which change the optical power through axial distance shifts between two lenses. Our eyes, however, change the optical power, when looking at near distances–for instance, when reading a book—by means of bending the crystalline lens with the help of the ciliary muscle, thus modifying r_S (see the scheme of a human eye in Figure 1.)

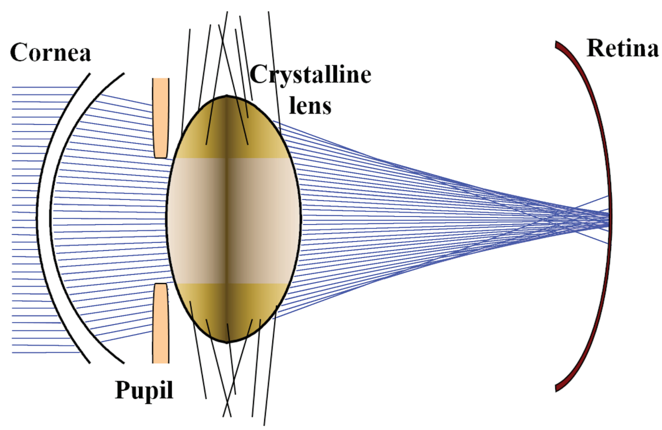


Figure 1. Scheme showing the optical elements of the human eye. Two lenses: the cornea and crystalline lens form images on the retina. The human eye sees nearby objects by means of increasing the optical power of the crystalline lens.

Presbyopia. When we get old we lose the capacity to bend our crystalline lens because of the increase in the stiffness of the crystalline lens. This phenomenon is called *presbyopia*. Then, as shown in Figure 1, our eyes (formed by two lenses: the cornea and the crystalline lens) cannot achieve the required radius of curvature of the outgoing wavefront associated with near vision. The result is that, at the retina, instead of having an image point we perceive a blurred disk. Of course, it is possible to correct that using reading eyeglasses. However, these are monofocal, meaning that they only provide a single optical power and entail shifting between spectacles for far (in case of being myopic or hypermetropic) and near vision tasks. An alternative to that nuisance is to use *multifocal* instruments.

2. Multifocality

Multifocality is the property of a surface—which could be a wavefront or an optical element—of providing light intensity distributions, not concentrated around a single image point (monofocality) but, instead, distributed along different foci or an extended region. While a spherical surface produces a single focus, multifocality requires a surface with nonconstant curvature.

Geometrically, at any point of a smooth surface S in Euclidean space there are two principal curvatures $\varkappa_1$ and $\varkappa_2$, which are the maximal and minimal normal curvatures, respectively. Hence it is reasonable to take the *mean curvature* $H = \varkappa_1 + \varkappa_2$ as an average measurement of the inverse of r_S (radius of curvature) overall normal planes. Another important quantity is the absolute difference between principal curvatures $C = |\varkappa_1 - \varkappa_2|$, known as *cylinder* or *astigmatism*, which quantifies, at second order approximation, the blurring of the image point due to that difference in ray trajectories for each normal plane.

A caustic digression. *Caustics*, the geometrical location of light energy concentration, is an ancient concept in geometrical optics because of the caustic (hence the name) properties of some burning lenses and mirrors known since antiquity. Caustics are ubiquitous and provide us with beautiful patterns (see Figure 2).

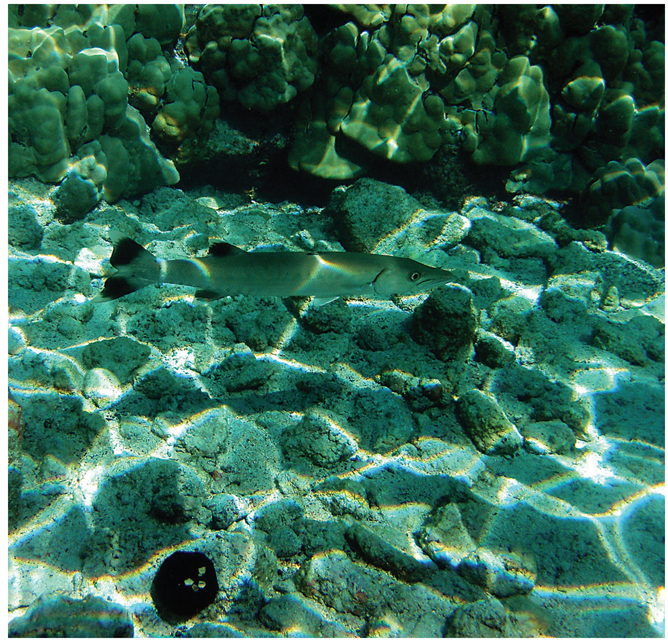


Figure 2. The effect of sunlight on clear sea water.

We underline here the work of Leonardo Da Vinci who, fascinated with light as he was, drew the caustics generated by a circular mirror (these are known as *catacaustics*): see Figure 3 for the original drawings and Figure 4 for a real life example.

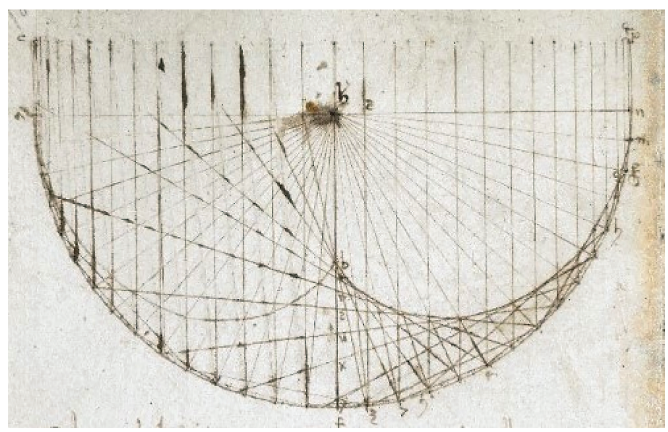


Figure 3. Da Vinci's drawing of a reflected caustic by a spherical mirror.

For a planar curve, Tschirnhausen defined the caustic curve as its evolute or the envelope of the normal family of rays to the original curve. In the case of having a wavefront surface, the analog of the evolute of a plane curve is the so-called *focal set*; then the caustic can be defined as the geometric locus of centers of principal curvatures of the wavefront [KO93]. Since there are two principal curvatures, each wavefront generates two caustic sheets. From a Riemannian geometry perspective, caustics are the set where rays (geodesics in the optical Riemannian space) concentrate, i.e., the *cut locus*, the location where more than one minimizing geodesic arrives.

Ideally, in a perfect multifocal wavefront, the two caustic sheets should coincide and degenerate into a set of points, or a line, because this provides maximum intensity concentration in the regions of interest. However, the two caustic surfaces only coincide when the associated principal curvatures are equal ($\varkappa_1 = \varkappa_2$), i.e., at *umbilical* points where the cylinder vanishes.

Minkwitz theorem. It is known that, within a smooth Euclidean surface, umbilicity is only possible either at isolated points or lines. Günter von Minkwitz, a young mathematician working on progressive addition lenses, deduced in 1963 what came to be known as *Minkwitz's theorem* [Min63], which establishes that if a smooth surface contains an umbilical line, then the cylinder along the orthogonal direction increases twice as quickly as the growth rate of the mean curvature along the umbilical line. The effect of Minkwitz's theorem is illustrated in Figure 5, which

Figure 4. Caustics in a cup of milk.

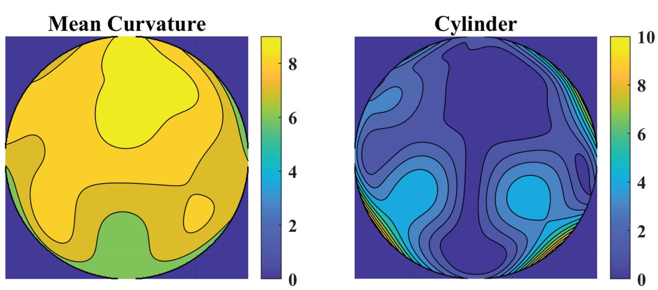


Figure 5. Mean power and cylinder of a real PAL surface.

shows the mean curvature and cylinder distribution across a circular domain of a real PAL surface [RY11].

Recently, a generalization of Minkwitz's theorem has been derived that provides the exact magnitude of the cylinder at any arbitrary point as a function of the geodesic curvature and the ratio of change of a principal curvature along a principal line [BdMG20].

Presbyopia correction: A multifocal approach. There are two ways of applying multifocality for presbyopia correction:

A. Multifocal intraocular or contact lenses.
B. Progressive addition spectacles (PALs, for short).

The former solution (A) is based on generating an outgoing wavefront with a varying mean curvature which, for different viewing distances, provides superimposed images on the retina. Then, the neural system is capable of selecting the correct focus at each moment; this is called the *simultaneous vision principle*.

Figure 6 illustrates this principle: of all rays (green arrow) emerging from a far object (butterfly) only those passing through the central part of the contact lens, converge after refraction onto the retina. On the other hand, of the rays coming from a near object (book) only those (red arrow) passing through the peripheral part of the contact lens converge onto the retina, after passing through the contact lens and the eye. This is possible because the central and peripheral parts of the contact lens are spheres of different radii of curvature: R_1 and R_2 respectively. Thus the main question is how to design the transition zone between both spheres while keeping astigmatism under control.

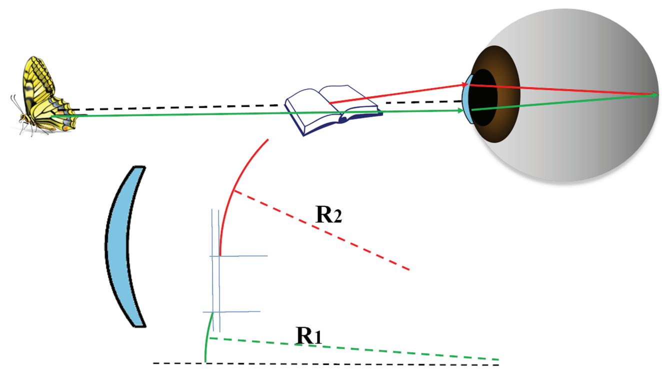


Figure 6. Scheme showing how a multifocal intraocular or contact lens works (in this case a contact lens). Rays from a far object (green arrow emerging from the butterfly) that pass through the central part of the contact lens, where the radius of curvature is R_1, converge onto the retina. On the other hand, rays from a near object (red arrow emerging from the book) only converge onto the retina if they pass through the peripheral part of the contact lens because, there, the radius of curvature (R_2) is different from that of the central part (R_1).

In PALs, the latter solution (B), a surface is designed containing a spatially varying mean curvature. Contrary to contact and intraocular lenses, in PALs, the eye can move independently of the lens, so through gaze movements it chooses, sequentially on time, each viewing area of the PAL providing the required optical power for: far distance vision (center of the lens) and near-view (low nasal portion). Between these two zones, the power varies progressively. Figure 7 shows this.

An ideal progressive surface is one with the prescribed smooth progressive power (mean curvature) and with zero astigmatism (cylinder) everywhere. But, as we have seen above, some amount of astigmatism is unavoidable unless the surface is a plane or a sphere, which does not provide progressive power. In any case, an archetypical model of a PAL is that of an elephant's trunk (see Figure 8), where there is an umbilical line (red solid line) and the circles obtained by cutting the trunk with parallel planes (dashed

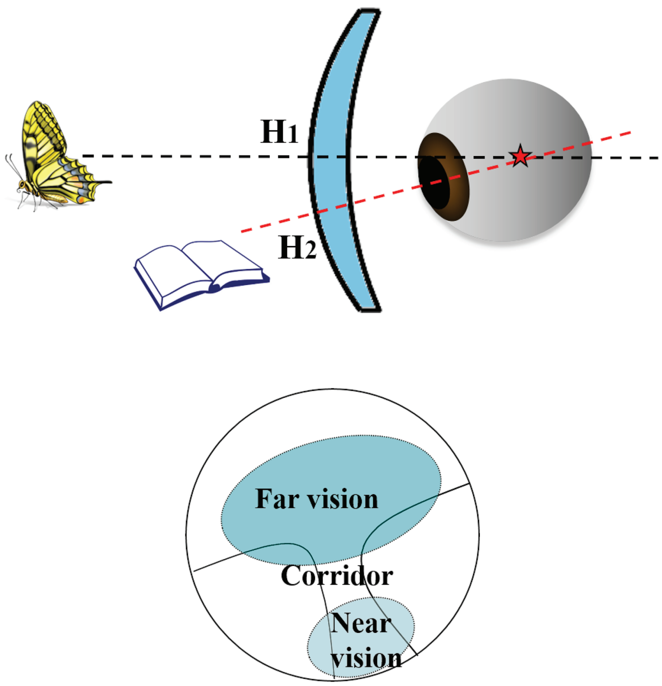


Figure 7. Scheme showing how the eye changes its gazes (top figure) when using different parts of the PAL surface (bottom figure). H_1 and H_2 denote mean curvatures for far and near vision, respectively.

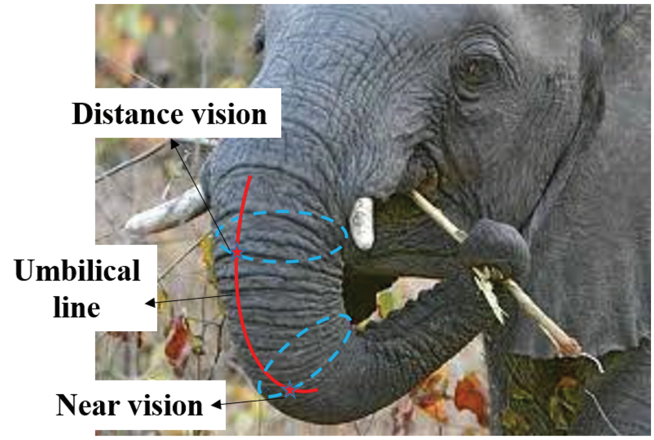


Figure 8. An elephant trunk is a primitive type of PAL surface model with an umbilical line.

blue lines) exhibit reducing radii of curvature when moving from the upper to the lower part of the trunk.

In general, to obtain designs for both solutions, (A) and (B), one should consider the combined action of the eye and the artificial lenses [Rub17]. However, since multifocality is usually restricted to a single surface, both in intraocular, contact, and PALs, the mathematical study can be, as a reasonable approximation, restricted to a single surface. Then, one may use the calculus of variations and PDEs to give optimal designs in both settings (A) and (B).

Yet, there is a crucial difference between solutions (A) and (B). In the former case (intraocular and contact lenses), axial rotational symmetry is, in most cases, applied, which can be mathematically exploited. Then, the problem is finding a surface that interpolates between two mean curvature values (for adjusting far and near vision, respectively) and minimizes astigmatism. On the other hand, for PALs one instead introduces some weights in the variational functional to include this spatial dependence. Both approaches will be explained in Section 4.

3. The Willmore Functional

The research of Thomas J. Willmore (1919–2005) was fundamental in the understanding of the following functional, and this is the reason it bears his name [Wil93]. Let S be a smooth, closed, embedded surface S in $\mathbb{R}^3$. We define *Willmore energy* as

$$\mathcal{W} = \frac{1}{4}\int_S H^2\, da, \tag{3.1}$$

where $H = \kappa_1 + \kappa_2$ is the mean curvature of S and da the surface area differential. The factor 1/4 in front is just a normalization constant.

One can also consider the related functional

$$\widetilde{\mathcal{W}} = \int_S \left(\tfrac{1}{4}H^2 - K\right) da,$$

for $K = \kappa_1\kappa_2$ the Gauss curvature of the surface. Remark that an equivalent expression is given by

$$\widetilde{\mathcal{W}} = \frac{1}{4}\int_S (\kappa_1 - \kappa_2)^2\, da, \tag{3.2}$$

which is the relevant formulation for our application to the design of multifocal surfaces as we are interested in minimizing astigmatism (cylinder). Indeed, we can think of the functional $\widetilde{\mathcal{W}}$ as a measure of the deviation of S from being a sphere (note that, if S is exactly a round sphere, $\widetilde{\mathcal{W}}$ vanishes.)

Taking into account the Gauss–Bonnet formula

$$\int_S K\, da = 2\pi\chi(S),$$

which relates the Euler characteristic $\chi(S)$ of the surface to its total Gauss curvature, one sees that

$$\widetilde{\mathcal{W}} = \frac{1}{4}\int_S H^2\, da - 2\pi\chi(S).$$

Since the last term in the formula above is a topological invariant, in minimization problems for closed surfaces both $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are equivalent.

An important property of the Willmore functional is its *conformal* invariance (recall that conformal map in $\mathbb{R}^3$ is a transformation of space that locally preserves angles.) To prove this statement, one observes that any such conformal transformation can be decomposed into a composition of translations, dilations, and inversions. The precise calculation is due to White [Whi73] although the result was known to Blaschke in 1929.

The Willmore functional in elasticity. Describing elasticity in terms of geometry is an old subject. In this regard, we underline the works of Sophie Germain and Siméon D. Poisson in the 19th century, in which they modeled the *bending* energy of a thin plate as the integral with respect to the surface area of an even, symmetric function of the principal curvatures (note here that the Willmore energy is the simplest possible example).

A classical generalization of the Willmore functional comes from the theory of membranes. Indeed, most of the cell membranes of living organisms are made of a lipid bilayer, modeled as a surface minimizing the Canham–Helfrich bending energy, which is essentially

$$\int_S c(H-c_0)^2\,da,$$

where c and c_0 are given constants. This approach explains, for instance, the biconcave discoid shape of red blood cells. (See [ZCOY99] for an exhaustive discussion on this topic.)

A variational point of view. Going back to the geometric problem, the first question one may ask is how to find the infimum of the Willmore energy $\mathcal{W}$ over the space of closed surfaces S in $\mathbb{R}^3$. Willmore himself showed that the sphere is a minimizer; more precisely,

$$\mathcal{W}(S) \geq 4\pi,$$

among all such S, with equality if and only if S is the round sphere. The proof of this result is quite simple in geometric terms: if we lay the surface on the floor, the point where S touches the ground (so the normal vector is vertical) must be elliptic, that is $K=\kappa_1\kappa_2 \geq 0$. But now we can repeat this process in any direction. So, for any direction of the normal, we can find a point on the surface S such that $K \geq 0$. This means that the image of the Gauss map $N : S \to \mathbb{S}^2$ covers the whole sphere $\mathbb{S}^2$ and, moreover, since the Jacobian of the Gauss map is precisely the Gauss curvature, we have that

$$\int_{S\cap\{K\geq 0\}} K\,da \geq \int_{\mathbb{S}^2} 1 = 4\pi.$$

To conclude, just note that $\frac{1}{4}H^2 \geq K$, with equality iff $\kappa_1 = \kappa_2$.

Willmore's next question was to find a minimizer in a class of fixed topology, for instance, in the class of tori. He proved, in particular, that any torus generated by a small circle moving (perpendicularly) along a closed plane curve C fulfills

$$\mathcal{W}(S) \geq 2\pi^2, \tag{3.3}$$

with equality if the generating curve is a circle and the ratio of the radii is $1/\sqrt{2}$, which corresponds to the parametrization

$$S_0 \;:\; ((\sqrt{2}+\cos u)\cos v, (\sqrt{2}+\cos u)\sin v, \sin u),$$

for $u, v \in \mathbb{R}$. Note that S_0 can be conformally mapped to the Clifford torus.

Willmore conjectured in 1965 that this result could be generalized to all surfaces in $\mathbb{R}^3$. However, this conjecture remained open until the seminal paper of Marques and Neves in 2014:

Theorem 3.1 ([MN14a]). *Every embedded compact surface S in $\mathbb{R}^3$ with positive genus satisfies* (3.3). *Up to rigid motions, the equality holds only for stereographic projections of the Clifford torus.*

Finally, a classical result of Li–Yau [LY82] states that when an immersion $\Psi : S \hookrightarrow \mathbb{R}^3$ has multiplicity k (i.e., k points in S get mapped to the same point by Ψ), then

$$\mathcal{W}(\Psi(S)) \geq k \cdot 4\pi.$$

This settles the Willmore conjecture for all surfaces. For further insight in this topic and additional references, we refer to the beautiful survey [MN14b].

Willmore surfaces. The Euler–Lagrange equation for $\mathcal{W}$ is the fourth order quasi-linear elliptic PDE

$$\Delta_S H + 2H(\tfrac{1}{4}H^2 - K) = 0, \tag{3.4}$$

where Δ_S is the Laplacian on S. Solutions to this equation are known as *Willmore surfaces*, even if not necessarily minimizers.

This is a highly nonlinear fourth-order PDE (see [GGS10] for an introduction to fourth-order boundary value problems.) An important remark here is that these equations do not usually enjoy many of the nice properties of second-order elliptic problems:

- The first issue is that we do not have a priori bounds for the regularity of the solution.
- The second point is the lack of a comparison principle that says that any subsolution of the equation stays below any supersolution. For second-order equations with a variational formulation, the classical trick to prove a maximum principle is to look at u^+, u^- (positive and negative parts of u, respectively). However, these test functions do not have enough regularity in a fourth-order equation, even in a weak sense.

4. Practical Applications of Willmore-Type Functionals

In the previous section, we considered closed surfaces, that is, without boundary. However, lenses have boundaries! So we should let S be a surface with boundary. The natural generalization is, thus, to replace the functional (3.1) by:

$$\mathcal{W}_\partial = \frac{1}{4}\int_S H^2\, da + \int_{\partial S} \kappa_g\, ds, \tag{4.1}$$

where κ_g is the geodesic curvature of the boundary curve and ds the arclength. Some existence results, in the spirit of those of the closed case, are considered by [Sch10].

In the following, we will consider two direct applications of the Willmore functional: designing a PAL surface and a multifocal intraocular (or contact) lens.

Modified Willmore functionals in PALs design. The Willmore functional has a relevant industrial application in PAL design. On a PAL surface, one can first prescribe a desired mean curvature distribution within the domain of the surface to be designed, then measure the difference with the true mean curvature, and finally minimize a functional where both the mean curvature difference and the astigmatism are weighted [KR99, WGS03].

More precisely, let us parameterize the surface as a graph of a function $u : \Omega \to \mathbb{R}$ over a domain Ω in $\mathbb{R}^2$. We denote the desired mean curvature distribution by $H_0 : \Omega \to \mathbb{R}$. One tries to minimize

$$\begin{aligned} I(u) = \int_\Omega &\{\alpha(x,y)(\kappa_1-\kappa_2)^2 \\ &+\beta(x,y)(H-H_0)^2\}\sqrt{1+|\nabla u|^2}dxdy, \end{aligned} \tag{4.2}$$

where α and β are prescribed weights. In fact, in the design of ophthalmic lenses (recall Figure 7), certain regions on the surface are required to have very small astigmatism (where we impose α to be large), while in other regions its more desirable that the lens has the correct power (where we set β large). Nevertheless, this technique involves prescribing a varying mean curvature on the whole surface, which may not be the optimal solution.

A relevant theoretical problem associated with both the functional (4.2) and Willmore energy, is whether it is possible to find its minimum value on open surfaces, as we did for closed surfaces in Section 3. This would offer a kind of Minkwitz theorem *in the large*. Knowing a lower bound for the amount of total cylinder achievable by a PAL surface would offer useful insights to an optical designer about how close his/her PAL design is to an ideal one.

A free boundary problem for Willmore surfaces. As mentioned in Section 2, in intraocular and contact lens multifocal applications, we are interested in the following geometric problem: find a Willmore surface S that interpolates between two spheres S_1 and S_2 of different radius R_1 and R_2, respectively. Of course, one would like to have a minimizer of Willmore energy, but for now, let us be satisfied with a solution of the Euler–Lagrange equation (3.4).

For practical reasons (that is, having sharp jumps on a lens may introduce undesired visual effects for the wearer), we need to impose that this interpolation solution is smooth up to second derivatives at each of the two boundaries (the two contact lines, let us denote them by ℓ_1 and ℓ_2, respectively). This imposes three boundary conditions at each ℓ_1 and ℓ_2, which make equation (3.4) overdetermined and, as a consequence, one cannot prescribe ℓ_1 and ℓ_2 in advance. Remark that we should not expect higher regularity than second derivatives since this is a fourth-order *obstacle*-type problem. For the interested reader, we refer to a 2020 *Notices* paper by D. Danielli [Dan20] on this topic.

In general, such a S is called a *free boundary* Willmore surface. A free boundary problem is PDE (or ODE) defined on a domain whose boundary is a priori unknown and it has to be determined from the solution of the problem a posteriori. The archetypal example of a free boundary model is that of understanding the interface between ice and water when studying the heat equation (*Stefan problem*).

Unfortunately, the free boundary Willmore problem has not been solved yet in full generality. A partial answer can be given if one assumes additional symmetries.

The one-dimensional problem. Assuming translation invariance, one can reduce to finding a curve Γ that minimizes the curvature functional for

$$\mathcal{W}(\Gamma) = \int_\Gamma \kappa^2\, ds,$$

where κ is the curvature of Γ and ds the arclength. Such Γ is called an *elastic curve*, and satisfies the Euler–Lagrange equation

$$\kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0 \quad \text{on } \Gamma. \tag{4.3}$$

If the curve is given by the graph of a function $u : [0,1] \to \mathbb{R}$, then equation (4.3) reduces to the nonlinear ODE

$$\frac{1}{\sqrt{1+u'^2}}\frac{d}{dx}\left(\frac{\kappa'}{\sqrt{1+u'^2}}\right) + \frac{1}{2}\kappa^3 = 0 \text{ in } (0,1), \tag{4.4}$$

where

$$\kappa(x) = \frac{u''(x)}{(1+u'(x)^2)^{\frac{3}{2}}}.$$

There are various sets of boundary conditions that one may impose. Since we are interested in controlling the curvature at the endpoints, let us consider Navier conditions:

$$\begin{cases} u(0) = u(1) = 0, \\ \kappa(0) = \kappa(1) = \alpha. \end{cases} \tag{4.5}$$

In the symmetric case $u(x) = u(1-x)$ one knows the following bifurcation picture (see Figure 9)

Theorem 4.1 ([DG07]). *There exists $\alpha_{max} = 1.34\ldots$ such that for $0 < |\alpha| < \alpha_{max}$, the problem (4.4) with boundary conditions (4.5) has precisely two smooth solutions in the class of smooth symmetric functions. If $|\alpha| = \alpha_{max}$ there exists only* *one such solution, for* $\alpha = 0$ *one only has the trivial solution and no such solutions exist for* $|\alpha| > \alpha_{max}$.

The proof of this result goes back to an observation by Euler in 1952. He found that such a solution must satisfy the conservation law

$$\kappa(x)(1 + u'(x)^2)^{\frac{1}{4}} = cst. \tag{4.6}$$

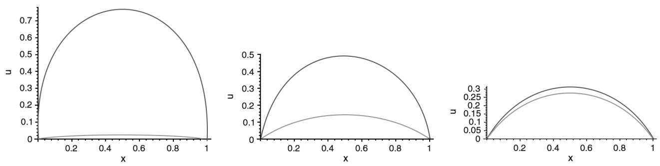


Figure 9. Solutions of the Navier boundary value problem from Theorem 4.1 for $\alpha = 0.2$, $\alpha = 1$, and $\alpha = 1.34$ (left to right). Taken from [DG07].

There are several other pairs of conditions one can impose, for this we refer to the same paper [DG07]. In particular, the nonsymmetric case may be obtained by scaling, translating, and rotating the coordinate system.

Now, going back at our initial question of interpolating between two circles of different radii by a Willmore curve, it is clear that we need to impose boundary conditions on u, u', and u'' at both endpoints of the interval, ℓ_1 and ℓ_2. A possible strategy is, leaving the contact points ℓ_1 and ℓ_2 free, to use the solution from Theorem 4.1 in order to also match the first derivatives of u at the endpoints.

Willmore surfaces of revolution. In this particular case, the surface S is obtained by rotating a curve in the xy-plane around the x axis. We assume that this curve is given by the graph of a function

$$u : [-1, 1] \to (0, \infty)$$

which, in some cases, can be taken to be symmetric $u(x) = u(-x)$. The surface of revolution S is then parametrized by

$$S_u : (x, u(x)\cos\theta, u(x)\sin\theta), \quad x \in [-1, 1], \theta \in [0, 2\pi].$$

After some calculation, we arrive at

$$H = -\frac{u''(x)}{(1 + u'(x)^2)^{\frac{3}{2}}} + \frac{1}{u(x)\sqrt{1 + u'(x)^2}},$$
$$K = -\frac{u''(x)}{u(x)(1 + u'(x)^2)^2},$$

which reveals the highly nonlinear character of (3.4). Unfortunately, conservation laws such as (4.6) are no longer available in this case.

There are many works that explore rotationally symmetric Willmore surfaces with boundaries, imposing different restrictions at the boundary. We recall some existence results when *natural* boundary conditions (from the variational point of view) are considered. These are given by

$$\begin{cases} u(-1) = u(1) = \alpha, \\ H(\pm 1) = \dfrac{\gamma}{\alpha\sqrt{1 + u'(\pm 1)^2}}, \quad \text{for } \gamma \in (0, 1). \end{cases} \tag{4.7}$$

Note that the simplest solution is the circular arc $u(x) = \sqrt{\alpha^2 + 1 - x^2}$ for $\gamma = 1$. In general we have:

Theorem 4.2 ([BDF10])**.** *For each* $\alpha > 0$ *and* $\gamma \in [0, 1]$, *there exists a positive, smooth, and symmetric function* u *such that the surface generated satisfies* (3.4) *with conditions* (4.7).

Obviously, we are interested in the nonsymmetric case. Unfortunately, there are still no results on the associated free boundary problem, despite its potential applications in the optics field.

ACKNOWLEDGMENTS. We would like to thank Jacob Rubinstein and Hans-Christoph Grunau for their valuable comments. S. B. and M. G. are supported by the Spanish Government grant PID2020113596GB-I00.

References

[BDF10] Matthias Bergner, Anna Dall'Acqua, and Steffen Fröhlich, *Symmetric Willmore surfaces of revolution satisfying natural boundary conditions*, Calc. Var. Partial Differential Equations **39** (2010), no. 3-4, 361–378, DOI 10.1007/s00526-010-0313-7. MR2729304

[BdMG20] Sergio Barbero and María del Mar González, *Admissible surfaces in progressive addition lenses*, Opt. Lett. **45** (2020), 5656–5659.

[BW65] Max Born and Emil Wolf, *Principles of optics: Electromagnetic theory of propagation, interference and diffraction of light*, Third revised edition, Pergamon Press, Oxford-New York-Paris, 1965. With contributions by A. B. Bhatia, P. C. Clemmow, D. Gabor, A. R. Stokes, A. M. Taylor, P. A. Wayman and W. L. Wilcock. MR198807

[Dan20] Donatella Danielli, *An overview of the obstacle problem*, Notices Amer. Math. Soc. **67** (2020), no. 10, 1487–1497, DOI 10.1090/noti. MR4201882

[DG07] Klaus Deckelnick and Hans-Christoph Grunau, *Boundary value problems for the one-dimensional Willmore equation*, Calc. Var. Partial Differential Equations **30** (2007), no. 3, 293–314, DOI 10.1007/s00526-007-0089-6. MR2332416

[GGS10] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers, *Polyharmonic boundary value problems: Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics, vol. 1991, Springer-Verlag, Berlin, 2010, DOI 10.1007/978-3-642-12245-3. MR2667016

[KO93] Yu. A. Kravtsov and Yu. I. Orlov, *Caustics, catastrophes and wave fields*, 2nd ed., Springer Series on Wave Phenomena, vol. 15, Springer-Verlag, Berlin, 1999. Translated from the Russian by M. G. Edelev, DOI 10.1007/978-3-642-59887-6. MR1723808

[KR99] Dan Katzman and Jacob Rubinstein, *Method for the design of multifocal optical elements*, 1999, US Patent 6302540B1.
[LY82] Peter Li and Shing Tung Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), no. 2, 269–291, DOI 10.1007/BF01399507. MR674407
[Max58] J. C. Maxwell, *On the general laws of optical instruments*, Q. J. Pure and Appl. Maths (1858), no. 2, 233–247.
[Min63] G. Minkwitz, *Über den ber den flchenastigmatismus bei gewissen symmetrischen asphren*, Optica Acta: International Journal of Optics **10** (1963), no. 3, 223–227.
[MN14a] Fernando C. Marques and André Neves, *Min-max theory and the Willmore conjecture*, Ann. of Math. (2) **179** (2014), no. 2, 683–782, DOI 10.4007/annals.2014.179.2.6. MR3152944
[MN14b] Fernando C. Marques and André Neves, *The Willmore conjecture*, Jahresber. Dtsch. Math.-Ver. **116** (2014), no. 4, 201–222, DOI 10.1365/s13291-014-0104-8. MR3280571
[Rub17] Jacob Rubinstein, *The mathematical theory of multifocal lenses*, Chinese Ann. Math. Ser. B **38** (2017), no. 2, 647–660, DOI 10.1007/s11401-017-1088-3. MR3615509
[RY11] T. W. Raasch, L. Su, and A. Yi, *Whole-surface characterization of progressive addition lenses*, Optometry and Vision Science **88** (2011), no. 2, E217–E226.
[Sch10] Reiner Schätzle, *The Willmore boundary problem*, Calc. Var. Partial Differential Equations **37** (2010), no. 3-4, 275–302, DOI 10.1007/s00526-009-0244-3. MR2592972
[WGS03] Jing Wang, Robert Gulliver, and Fadil Santosa, *Analysis of a variational approach to progressive lens design*, SIAM J. Appl. Math. **64** (2003), no. 1, 277–296, DOI 10.1137/S0036139902408941. MR2029135
[Whi73] James H. White, *A global invariant of conformal mappings in space*, Proc. Amer. Math. Soc. **38** (1973), 162–164, DOI 10.2307/2038790. MR324603
[Wil93] T. J. Willmore, *Riemannian geometry*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. MR1261641
[ZCOY99] Zhong-Can Ou-Yang, Ji-Xing Liu, and Yu-Zhang Xie, *Geometric methods in the elastic theory of membranes in liquid crystal phases*, World Scientific, 1999.

Sergio Barbero

María del Mar González

Credits

The opening image is courtesy of Fermate via Getty.

Figures 1, 4, 5, 6, 7, and 8 are courtesy of Sergio Barbero.

Figure 2 is courtesy of Brocken Inaglory, CC-BY-SA 3.0.

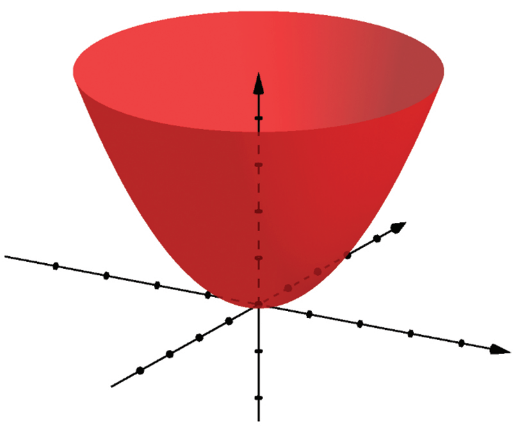
Figure 3 is courtesy of The British Library, folio 87r Codex Arundel.

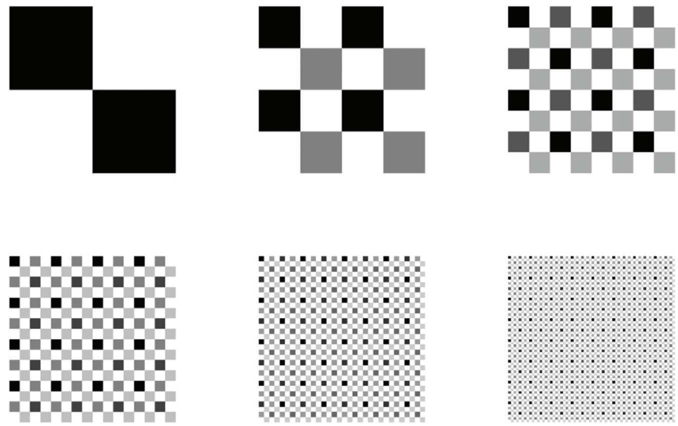
Figure 9 is courtesy of Klaus Deckelnick and Hans-Christoph Grunau, *Boundary value problems for the one-dimensional Willmore equation*, Calc. Var. Partial Differential Equations **30** (2007), no. 3, 293–314, `https://link.springer.com/article/10.1007/s00526-007-0089-6`. Reprinted with permission.

Photo of Sergio Barbero is courtesy of Natalia Barcenilla.

Photo of María del Mar González is courtesy of Jose Pedro Moreno.

Differentiating by Prime Numbers





Jack Jeffries

It is likely a fair assumption that you, the reader, are not only familiar with but even quite adept at differentiating by x. What about differentiating by 13? That certainly didn't come up in my calculus class! From a calculus perspective, this is ridiculous: are we supposed to take a limit as 13 changes?

One notion of differentiating by 13, or any other prime number, is the notion of *p-derivation* discovered independently by Joyal [Joy85] and Buium [Bui96]. p-derivations have been put to use in a range of applications in algebra, number theory, and arithmetic geometry. Despite the wide range of sophisticated applications, and the fundamentally counterintuitive nature of the idea of differentiating by a number, p-derivations are elementary to define and inviting for exploration.

In this article, we will introduce p-derivations and give a few basic ways in which they really do act like derivatives by numbers; our hope is that you will be inspired and consider adding p-derivations to your own toolkit!

Jack Jeffries is an assistant professor of mathematics at the University of Nebraska. His email address is `jack.jeffries@unl.edu`.

Communicated by Notices *Associate Editor Steven Sam.*

DOI: https://doi.org/10.1090/noti2833

p-Derivations on $\mathbb{Z}$

First we want to discuss differentiating one number n, by another, p; i.e., what we will call p-derivations on $\mathbb{Z}$. Before we succeed, we need to abandon the notion of derivative as a limit as the input varies by a small amount: the thing p that we are differentiating by does not vary, and the thing n that we are differentiating does not even have an input! Instead, we take a little inspiration from elementary number theory.

Let p be a prime number. By Fermat's little theorem, for any integer n, we have

$$n \equiv n^p \mod p,$$

so we can divide the difference $n - n^p$ by p. The starting point of our journey is that not only *can* we divide by p here, but we *should*. The p-derivation on $\mathbb{Z}$ is the result of this process. Namely:

Definition 1. For a prime number p, the *p-derivation* on $\mathbb{Z}$ is defined as the function $\delta_p : \mathbb{Z} \to \mathbb{Z}$ given by the formula

$$\delta_p(n) = \frac{n - n^p}{p}.$$

So, in particular, there is the 2-derivation on $\mathbb{Z}$ and the 13-derivation on $\mathbb{Z}$ given respectively by

$$\delta_2(n) = \frac{n - n^2}{2} \quad \text{and} \quad \delta_{13}(n) = \frac{n - n^{13}}{13}.$$

Let's plug in a few values:

n	$\delta_2(n)$	$\delta_3(n)$	$\delta_5(n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
-4	-10	20	204
-3	-6	8	48
-2	-3	2	6
-1	-1	0	0
0	0	0	0
1	0	0	0
2	-1	-2	-6
3	-3	-8	-48
4	-6	-20	-204
5	-10	-40	-624
6	-15	-70	-1554
$\vdots$	$\vdots$	$\vdots$	$\vdots$

A quick look at this table suggests a few observations, easily verified from the definition:

- Numbers are no longer "constants" in the sense of having derivative zero, but at least 0 and 1 are.
- These functions are neither additive nor multiplicative, e.g.:
$$\delta_p(1) + \delta_p(1) \neq \delta_p(2),$$
$$\delta_p(1)\delta_p(2) \neq \delta_p(2).$$
- δ_p is an odd function, at least for $p \neq 2$.
- The outputs of δ_2 are just the negatives of the triangular numbers.

We might also note that the outputs are very large in absolute value, and think that this operation is simply making a mess of our numbers. However, something more informative occurs if we think about largeness of the outputs from the point of view of p, namely, the *p-adic order* of n—the number of copies of p in its prime factorization. Writing $n = p^a m$ with $\gcd(m, p) = 1$, if $a > 0$, we get

$$\delta_p(p^a m) = \frac{p^a m - (p^a m)^p}{p} = p^{a-1} m(1 - p^{ap-a} m^{p-1}).$$

Since $p \geq 2$ and $a \geq 1$, we must have $ap - a \geq 1$, so p does not divide $1 - p^{ap-a} m^{p-1}$. In particular, the p-derivation decreases the p-adic order of a multiple of p by exactly one. This leads to our first comparison with old-fashioned $\frac{d}{dx}$:

Comparison 1 (Order-decreasing property).

- *If $f \in \mathbb{R}[x]$ is a polynomial and $x = r$ is a root of f of multiplicity $a > 0$, then $x = r$ is a root of the polynomial $\frac{d}{dx}(f(x))$ of multiplicity $a - 1$.*
- *If n is an integer and p is a prime factor of n of multiplicity $a > 0$, then p is a prime factor of the integer $\delta_p(n)$ of multiplicity $a - 1$.*

In particular, if r is a simple root or p is a simple factor, then it is no longer a root or factor of $\frac{d}{dx}(f(x))$ or $\delta_p(n)$ respectively.

Let's check this against our table: the numbers $-2, 2$, and 6 that were divisible by 2 but not 4 result in odd numbers when we apply δ_2, whereas ± 4 returned even numbers no longer divisible by 4. Note that this order-decreasing property says nothing about what happens when you apply δ_2 to an odd number, and indeed, based on the table we observe that even and odd numbers can result. You can convince yourself that

$$\delta_2(n) \text{ is } \begin{cases} \text{even if } n \equiv 0, 1 \mod 4 \\ \text{odd if } n \equiv 2, 3 \mod 4. \end{cases}$$

We've observed already that these p-derivations on $\mathbb{Z}$ are not additive. This can be a bit unsettling for those of us (like myself) who are usually accustomed to the luxury of additive operators. However, any function satisfying the order-decreasing property of δ_p above must not be additive, since an additive function has to take multiples of p to multiples of p. However, the error term can be made concrete:

$$\delta_p(m+n) - (\delta_p(m) + \delta_p(n)) = \frac{m^p + n^p - (m+n)^p}{p}$$
$$= -\sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} m^i n^{p-i}.$$

All of the binomial coefficients $\binom{p}{i}$ appearing above are multiples of p, so this expression is, given a particular value of p, a particular polynomial in m and n with integer coefficients; let's call it $C_p(m, n)$ for convenience. This gives us the following "sum rule" for δ_p:

$$\delta_p(m+n) = \delta_p(m) + \delta_p(n) + C_p(m, n). \qquad (+)$$

Products satisfy a rule with a similar flavor:

$$\delta_p(mn) = m^p \delta_p(n) + n^p \delta_p(m) + p\delta_p(m)\delta_p(n). \qquad (\times)$$

The fact that we have rules to break things down into sums and products gives the basis for another comparison with old-fashioned $\frac{d}{dx}$:

Comparison 2 (Sum and product rules).

- *For polynomials $f(x), g(x)$, one can compute each of $\frac{d}{dx}(f+g)$ and $\frac{d}{dx}(fg)$ as a fixed polynomial expression in the inputs $f, g, \frac{d}{dx}(f), \frac{d}{dx}(g)$, namely $\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$ and $\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$.*
- *For integers m, n, one can compute each of $\delta_p(m+n)$ and $\delta_p(mn)$ as a fixed polynomial expression in the inputs $m, n, \delta_p(m), \delta_p(n)$, namely $(+)$ and $(\times)$.*

We might pause to ask whether we could have hoped for a simpler way to differentiate by 13. If we want Comparsion 2 to hold, then the following theorem of Buium provides a definitive answer.

Theorem 1 (Buium [Bui97]). *Any function $\delta : \mathbb{Z} \to \mathbb{Z}$ that satisfies*

- *a sum rule $\delta(m+n) = S(m,n,\delta(m),\delta(n))$ for some polynomial S with integer coefficients*
- *a product rule $\delta(mn) = P(m,n,\delta(m),\delta(n))$ for some polynomial P with integer coefficients*

is of the form

$$\delta(n) = \pm\frac{n - n^{p^e}}{p} + f(n)$$

for some prime integer p, positive integer e, and polynomial f with integer coefficients.

That is, any function satisfying a sum rule and a product rule is a mild variation on a p-derivation.

With the properties of p-derivations we have so far, we can recreate analogues of some familiar aspects of calculus. For example, from the product rule $(\times)$ and a straightforward induction, we obtain a power rule:

$$\delta_p(n^a) = \sum_{i=1}^{a} \binom{a}{i} p^{i-1}\delta_p(n)^i n^{(a-i)p}.$$

Note that the $i = 1$ term in the sum above, $an^{(a-1)p}\delta_p(n)$, looks a bit like the power rule for usual derivatives. If we allow ourselves to extend δ_p to a map on $\mathbb{Q}$, then we get an analogue of the quotient rule:

$$\delta_p\left(\frac{m}{n}\right) = \frac{n^p\delta_p(m) - m^p\delta_p(n)}{n^{2p} + pn^p\delta_p(n)}.$$

Of the main cast of characters in a first class on derivatives, perhaps the most conspicuous one missing at this point is the chain rule. Since there is no way to compose a number with a number, we will need a notion of p-derivations for functions to state a sensible analogue of the chain rule.

p-Derivations for General Commutative Rings

One can define p-derivations for commutative rings with 1.

Definition 2. Let R be a commutative ring with 1 and p a prime integer. A *p-derivation* on R is a function $\delta : R \to R$ such that $\delta(0) = \delta(1) = 0$ and δ satisfies the sum rule $(+)$ and the product rule $(\times)$ above; i.e., for all $r, s \in R$,

$$\delta_p(r+s) = \delta_p(r) + \delta_p(s) - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} r^i s^{p-i} \qquad (+)$$

and

$$\delta_p(rs) = r^p\delta_p(s) + r^p\delta_p(s) + p\delta_p(r)\delta_p(s). \qquad (\times)$$

Evidently, the functions δ_p we defined on $\mathbb{Z}$ above are p-derivations. In fact, for a fixed p, a simple induction and the sum rule show that for any p-derivation δ on a ring R and any n in the prime subring (image of $\mathbb{Z}$) of R, $\delta(n) = \delta_p(n)$.

The other basic example is as follows. Take the ring of polynomials in n variables with integer coefficients, $R = \mathbb{Z}[x_1, \dots, x_n]$. For any polynomial $f(x_1, \dots, x_n)$, we can consider its pth power $f(x_1, \dots, x_n)^p$, or we can plug in pth powers of the variables as inputs to get $f(x_1^p, \dots, x_n^p)$. These are different, but they agree modulo p as a consequence of the "freshman's dream." Namely, in the quotient ring $R/pR \cong \mathbb{Z}/p\mathbb{Z}[x_1, \dots, x_n]$,

$$(f+g)^p = f^p + \sum_{i=1}^{p-1}\binom{p}{i} f^i g^{p-i} + g^p = f^p + g^p,$$

since each $\binom{p}{i}$ is a multiple of p, and

$$(fg)^p = f^p g^p$$

as a consequence of commutativity, so the map $f \mapsto f^p$ is a ring homomorphism in R/pR, called the *Frobenius map*. Thus, in R/pR, taking pth powers before doing polynomial operations is just as good as after. So, back in R we *can* divide the difference by p, and we *will*! Namely, we can define the function

$$\delta(f(x_1, \dots, x_n)) = \frac{f(x_1^p, \dots, x_n^p) - f(x_1, \dots, x_n)^p}{p},$$

and this function is a p-derivation. Just so we can refer to this function later, let's call this the *standard* p-derivation on $\mathbb{Z}[x_1, \dots, x_n]$ and denote it by $\delta_{\mathrm{st},p}$ (though this notation is not at all standard).

For example,

$$\begin{aligned}\delta_{\mathrm{st},2}(x^3 + 5x) &= \frac{(x^2)^3 + 5(x^2) - (x^3+5x)^2}{2} \\ &= -5x^4 - 10x^2.\end{aligned}$$

As this operator $\delta_{\mathrm{st},p}$ measures the failure of the freshman's dream, one might think of this as a freshman's nightmare. In fact, in large generality, p-derivations all arise from some freshman's nightmare. Let's make this precise. Given a ring R, we say that a map $\Phi : R \to R$ is a *lift of Frobenius* if it is a ring homomorphism and the induced map from $R/pR \to R/pR$ is just the Frobenius map, i.e., $\Phi(r) \equiv r^p \mod pR$ for all R. Given a p-derivation $\delta : R \to R$, the map $\Phi : R \to R$ given by

$$\Phi(r) = r^p + p\delta(r)$$

is a lift of Frobenius. Indeed, the congruence condition is automatic, and the sum rule and product rule on δ translate exactly to the conditions that Φ respects addition and multiplication. Conversely, if p is a nonzero divisor on R, and Φ is a lift of Frobenius, then the map $\delta(r) = \frac{\Phi(r) - r^p}{p}$ is a p-derivation: the freshman's nightmare associated to the lift of Frobenius Φ.

It is worth noting that not every ring admits a p-derivation. For a quick example, no ring R of characteristic

p admits a p-derivation, since we would have

$$0 = \delta(0) = \delta(p) = \delta_p(p) = 1 - p^{p-1} = 1$$

in R. Much more subtle obstructions exist, and it is an interesting question to determine which rings admit p-derivations; see [AWZ21] for some recent work on related questions.

Note that the power rule from before follows for any p-derivation on any ring, since we just used the product rule to see it. The order-decreasing property holds in general, too, at least if p is a nonzero divisor on R—this follows from writing $s = p^a r$ with $p \nmid r$ and applying the product rule:

$$\begin{aligned}\delta(p^a r) &= p^{ap}\delta(r) + r^p\delta(p^a) + p\delta(p^a)\delta(r)\\ &= p^{ap}\delta(r) + (p^{a-1} - p^{ap-1})r^p + p(p^{a-1} - p^{ap-1})\delta(r)\\ &= p^{a-1}r^p + p^a(\delta(r) - p^{a(p-1)-1}r^p - p^{a(p-1)}\delta(r)).\end{aligned}$$

Let's wrap up our cliffhanger from the previous section. Now that we have p-derivations of polynomials, we have the ingredients needed for a chain rule: given a polynomial $f(x)$ and a number n, we will think of the number $f(n)$ as the composition of the function f and the number n, and we can try to compare $\delta_p(f(n))$ with $\delta_{\mathrm{st}}(f)$ and $\delta_p(n)$. Here's the chain rule:

$$\delta_p(f(n)) = \delta_{\mathrm{st},p}(f)(n) + \sum_{j=1}^{\deg(f)} p^{j-1}\frac{d^j f}{j!\, dx^j}(n^p)\delta_p(n)^j.$$

This is a bit more complicated than the original, but let's notice in passing that the $j = 1$ term in the sum, $\frac{df}{dx}(n^p)\delta_p(n)$, looks pretty close to the classic chain rule, besides the pth power on n. The curious reader is encouraged[1] to prove the formula above.

We have collected a decent set of analogues for the basics of differential calculus for p-derivations. One can ask how far this story goes, and the short answer is very far. Buium has developed extensive theories of arithmetic differential equations and arithmetic differential geometry, building analogues of the classical (nonarithmetic) versions of these respective theories with p-derivations playing the role of usual derivatives. The reader is encouraged to check out [Bui05, Bui17] to learn more about these beautiful theories, though our story now diverges from these. Instead, we will turn our attention towards using p-derivations to give some algebraic results with geometric flavors.

A Jacobian Criterion

One natural geometric consideration is whether, and where, a shape has *singularities*: points that locally fail to look flat, due to some sort of crossing or crinkle (or some harder-to-imagine higher-dimensional analogue of a crossing or crinkle). For example, the double cone cut out by the equation $z^2 - x^2 - y^2 = 0$ has a singularity at the origin where the two cones meet, but any other point on the cone is not a singularity, see Figure 1.

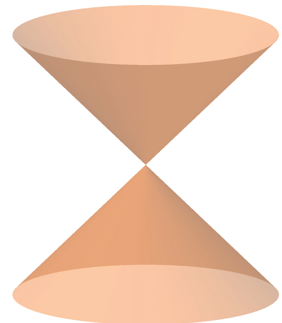


Figure 1. The cone of solutions of $z^2 - x^2 - y^2 = 0$. The origin is the unique singular point.

We are going to consider shapes like this that are cut out by polynomial equations, though to state the classical Jacobian criterion, we will consider their solution sets over the complex numbers.

Since it is difficult to envision higher-dimensional shapes (and impossible to envision what we're doing next!), it will be useful to give a somewhat more algebraic heuristic definition of singularity. We will say that a point x is a *nonsingular* point in X if within X one can locally cut out x by exactly $d = \dim(X)$-many equations without taking roots, and *singular* otherwise. For example, the point $(1, 0, 1)$ in the cone is nonsingular, and I claim that the two equations $y = 0$, $z - 1 = 0$ "work" for our definition: with these two equations and the equation for X, we get

$$y = z - 1 = z^2 - x^2 - y^2 = 0.$$

Substituting in, we get $0 = x^2 - 1 = (x - 1)(x + 1)$, and "near $(1, 0, 1)$," $x + 1$ is nonzero, so we can divide out and get $x - 1 = 0$, so $x = 1, y = 0, z = 1$. On the other hand, $(0, 0, 0)$ is singular, and the two equations $y = 0$, $z = 0$ don't "work" for our definition: we have

$$y = z = z^2 - x^2 - y^2 = 0,$$

so $x^2 = 0$, but we need to take a root to get $x = 0$.

The classical Jacobian criterion gives a recipe for the locus of all singularities of a shape cut out by complex polynomials.

Theorem 2 (Jacobian criterion). *Let $X \subseteq \mathbb{C}^n$ be the solution set of the system of polynomial equations*

$$f_1 = \cdots = f_m = 0.$$

If the dimension of X is $d = n - h$, and $f_1, \ldots, f_m$ generate a

[1] *For a hint, consider the lift of Frobenius on $\mathbb{Z}[x]$ that sends $x \mapsto x^p + p\delta_p(n)$, and use Taylor expansion to rewrite the associated p-derivation in terms of the standard p-derivation and derivatives of f.*

prime ideal[2] *in the polynomial ring* $\mathbb{C}[x_1, \dots, x_n]$, *then the set of singular points is the solution set within* X *of the system of polynomial equations*

$$\text{all } h \times h \text{ minors of } \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = 0.$$

In particular, if f *is irreducible, the set of singular points of the solution set* $f = 0$ *is the solution set of*

$$f = \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0.$$

For example, for the Whitney umbrella cut out by the polynomial $f = x^2 - y^2 z$, the singular locus is cut out by the system

$$x^2 - y^2 z = 2x = 2yz = y^2 = 0,$$

which simplifies to $x = y = 0$; the z-axis is where the shape crosses itself.

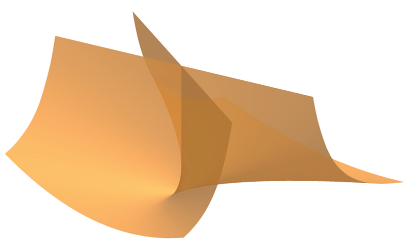


Figure 2. The Whitney umbrella of solutions of $x^2 - y^2 z = 0$. The singular locus consists of the line where it crosses itself.

The notion of (non)singularity in geometry is generalized in algebra by the notion of regular ring. For starters, prime ideals in algebra play the role of points in geometry: this is motivated by Hilbert's nullstellensatz, which says that for a quotient ring of the form[3]

$$\frac{\mathbb{C}[x_1, \dots, x_n]}{(f_1, \dots, f_m)},$$

every maximal ideal is of the form

$$\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$$

for some $a = (a_1, \dots, a_n)$ solution to

$$f_1(a) = \cdots = f_m(a) = 0;$$

including all prime ideals leads to a better theory for general rings. Then we say that a prime ideal in a ring R is *nonsingular* or *regular* at a prime ideal $\mathfrak{q}$ if $\mathfrak{q}$ can be generated "locally" by h equations, where h is the codimension of $\mathfrak{q}$ (how much $\mathfrak{q}$ cuts down the dimension of R), and "locally" means[4] that one can divide by elements outside of $\mathfrak{q}$. In the motivating geometric situation where X is the solution set of $f_1 = \cdots = f_m = 0$ over $\mathbb{C}$, the point $a \in X$ is nonsingular if and only if $R = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$ is nonsingular at the maximal ideal $\mathfrak{m}_a$. A prime ideal is *singular* if it is not nonsingular.

Intuitively, when working over $\mathbb{Z}$ rather than over $\mathbb{C}$ or a field, in addition to the geometric dimensions, there is an arithmetic dimension that corresponds to the prime integers p in $\mathbb{Z}$. To detect singularity, it suffices to include p-derivations as a substitute for derivatives in the p-direction! The following is a special case of a result independently obtained by Saito [Sai22] and Hochster and the author [HJ21].

Theorem 3 (Saito, Hochster–Jeffries). *Let* $R = \frac{\mathbb{Z}[x_1, \dots, x_n]}{(f_1, \dots, f_m)}$ *for some prime ideal* $(f_1, \dots, f_m)$ *of* $\mathbb{Z}[x_1, \dots, x_n]$. *Then the set of singular prime ideals* $\mathfrak{q}$ *of* R *containing a fixed prime integer* p *is exactly the set of prime ideals containing* p *and*

$$\text{all } h \times h \text{ minors of } \begin{bmatrix} \delta_{\mathrm{st},p}(f_1) & (\frac{\partial f_1}{\partial x_1})^p & \cdots & (\frac{\partial f_1}{\partial x_n})^p \\ \vdots & & \ddots & \vdots \\ \delta_{\mathrm{st},p}(f_m) & (\frac{\partial f_m}{\partial x_1})^p & \cdots & (\frac{\partial f_m}{\partial x_n})^p \end{bmatrix},$$

where $h = n + 1 - \dim(R)$. *In particular, if* f *is irreducible, the set of singular prime ideals of* $R = \frac{\mathbb{Z}[x_1, \dots, x_n]}{(f)}$ *containing* p *is exactly the set of primes containing* p *and*

$$\delta_{\mathrm{st},p}(f), \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}.$$

Example 1. Let n be a squarefree integer (excluding 0 and 1), q a prime number, and consider the ring $R = \mathbb{Z}[\sqrt[q]{n}]$. We claim that this admits a singular prime ideal if and only if $\delta_q(-n)$ is a multiple of q. Think[5] of R as $\mathbb{Z}[x]/(x^q - n)$. For a prime number p, the singular prime ideals $\mathfrak{a}$ containing p are those that contain

$$p, \delta_{\mathrm{st},p}(x^q - n), \frac{d}{dx}(x^q - n).$$

Using the sum rule for $\delta_{\mathrm{st},p}$ and the defining equation for R, we have

$$\delta_{\mathrm{st},p}(x^q - n) = \begin{cases} \delta_p(-n) & \text{if } p \neq 2 \\ \delta_p(-n) - nx^q & \text{if } p = 2 \end{cases}$$

in R.

[2] *We recall that an ideal* I *is prime if it is proper and* $gh \in I$ *implies* $g \in I$ *or* $h \in I$. *Experts will recognize this condition as overkill, but something needs to be done to avoid examples like* $(x - y)^2 = 0$ *whose solution set* X *is the (complex) line* $\{(a, a)\}$, *which is nonsingular, but* $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ *for every point in* X, *or* $x^2 - x = xyz = 0$ *whose solution set* Y *is two-dimensional but has a crossing singularity at* $(0, 0, 1)$ *that is not a solution of the* 1×1 *minors of the Jacobian matrix.*

[3] *For a set of elements* $a_1, \dots, a_t$ *in a ring* R, *we use the notation*

$$(a_1, \dots, a_t) := \{r_1 a_1 + \cdots + r_t a_t \mid r_i \in R\}$$

for the ideal generated by $\{a_1, \dots, a_t\}$.

[4] *Precisely, "locally" means that we work in the localization* $R_{\mathfrak{q}}$, *and codimension refers to the height of the ideal* $\mathfrak{q}$.

[5] *This is where excluding* 0 *and* 1 *is necessary.*

For $p \neq q$, from $\frac{d}{dx}(x^q - n) = qx^{q-1}$, we get that $\mathfrak{a}$ must contain x, and from the defining equation, n as well. But if the integers $p, n, \delta_p(n)$ are in a proper ideal $\mathfrak{a}$, since p is a prime number, we must have $p \mid n$ and $p \mid \delta_p(n)$, since 1 would be a linear combination of these numbers otherwise. By the order-decreasing property, $p^2 \mid n$, contradicting that n is squarefree. So, there are no singular prime ideals containing $p \neq q$.

For $p = q \neq 2$ since $\frac{d}{dx}(x^q - n)$ is a multiple of q, using the simplification above, a prime ideal containing q is singular if and only if it contains

$$q, \delta_q(-n).$$

But if q and $\delta_q(-n)$ are in $\mathfrak{a}$, then $q \mid \delta_q(-n)$; a singular prime ideal then occurs if and only if this happens. The analysis for $p = q = 2$ is similar (cf. [HJ21]).

This has a consequence for a familiar object in elementary number theory. By some standard results[6] in commutative algebra, the ring $\mathbb{Z}[\sqrt[q]{n}]$ is a ring of algebraic integers (for its fraction field $\mathbb{Q}(\sqrt[q]{n})$) if and only if it has no singular prime ideals. Thus, we conclude that $\mathbb{Z}[\sqrt[q]{n}]$ is a ring of integers if and only if $q \nmid \delta_q(-n)$. In particular, for $q = 2$, using our earlier observation on when $\delta_2(n)$ is odd or even, we recover the fact that for n squarefree, $\mathbb{Z}[\sqrt{n}]$ is a ring of integers if and only if $n \not\equiv 3 \mod 4$.

A Zariski–Nagata Theorem for Symbolic Powers

Let's recall another classical theorem relating algebra and geometry. To state it, we need the notion of symbolic power of a prime ideal. Over a polynomial ring over a field, or more generally, over a commutative Noetherian ring, any ideal can be written as an intersection

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t \tag{3}$$

where the $\mathfrak{q}_i$ are *primary* ideals: ideals with the property that $xy \in \mathfrak{q} \Rightarrow x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some n. Such an expression is called a *primary decomposition*. The existence of primary decomposition is a famous result of Lasker in the polynomial case and Noether in the Noetherian case. This can be thought of as a generalization of the fundamental theorem of arithmetic: in $\mathbb{Z}$, the primary ideals are just the ideals generated by powers of primes, and a prime factorization

$$n = p_1^{e_1} \cdots p_n^{e_n}$$

corresponds to writing (n) as an intersection of primary ideals:

$$(n) = (p_1^{e_1}) \cap \cdots \cap (p_n^{e_n}).$$

[6] *Namely, $\mathbb{Z}[\sqrt[q]{n}]$ is a ring of integers if and only if it is integrally closed in its fraction field. Since this ring is an integral extension of $\mathbb{Z}$ generated by one element as an algebra, it is a one dimensional Noetherian domain. Such a ring is integrally closed in its fraction field if and only if it is nonsingular.*

There are two important differences with the fundamental theorem of arithmetic, though:

- primary ideals are not powers of prime ideals in general, nor are powers of prime ideals always primary, and
- the collection of primary ideals $\mathfrak{q}_i$ appearing in the decomposition (3) is not unique, but if the decomposition satisfies a simple irredundancy hypothesis, then the components whose radical does not contain any other component's radical are uniquely determined.

In particular, if $\mathfrak{q}$ is a prime ideal and $n > 1$, then according to the first point above, $\mathfrak{q}^n$ may admit an interesting primary decomposition, and as a consequence of the second point, the component with radical $\mathfrak{q}$ is the same in any (irredundant) primary decomposition. This is called the nth *symbolic power* of $\mathfrak{q}$, denoted $\mathfrak{q}^{(n)}$.

Symbolic powers arise in various contexts in algebra and geometry. For example, they arise naturally in interpolation questions, they play a key role in the proofs of various classical theorems such as Krull's principal ideal theorem, and they have enjoyed a resurgence of interest in combinatorics in connection with the packing problem of Conforti and Cornuéjols. The interested reader is recommended to read the survey [DDSG$^+$18].

A classical pair of theorems of Zariski and Nagata gives a geometric description of the symbolic power of an ideal in a polynomial ring over $\mathbb{C}$. The result has various statements; we will give a differential statement.

Theorem 4 (Zariski–Nagata Theorem). *Let $X \subseteq \mathbb{C}^n$ be the solution set of the system of polynomial equations*

$$f_1 = \cdots = f_m = 0.$$

Suppose that $f_1, \dots, f_m$ generate a prime ideal $\mathfrak{q}$. Then $\mathfrak{q}^{(r)}$ is exactly the set of polynomials $f \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$\left.\frac{\partial^{a_1+\cdots+a_n} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}\right|_X \equiv 0 \text{ for all } a_1 + \cdots + a_n < r.$$

The same characterization is doomed to fail in $\mathbb{Z}[x]$: for example, $\mathfrak{q} = (2)$ is a prime ideal with $\mathfrak{q}^{(2)} = (4)$; in particular, $2 \notin (4)$. But 2 satisfies the derivative condition corresponding to the right hand side above: taking $a_1 = 0$, we have $2 \in (2)$, and taking $a_1 = 1$, we have $\frac{\partial 2}{\partial x} = 0 \in (2)$.

If you've been paying attention so far, you should be able to name the missing ingredient. Indeed, $\delta_{\mathrm{st},2}(2) = -1 \notin (2)$, so allowing partial derivatives and a 2-derivation is enough to take this element 2 that isn't in $\mathfrak{q}^{(2)}$ out of $\mathfrak{q}$, suggesting a way to characterize $f \in \mathfrak{q}^{(2)}$ in terms of derivatives (including our "derivative by 2").

In fact, this works in general. The following analogue of the Zariski–Nagata theorem is a special case of a result of De Stefani, Grifo, and the author [DSGJ20].

Theorem 5 (De Stefani–Grifo–Jeffries). *Consider the ring $\mathbb{Z}[x_1, \dots, x_n]$ and let $\mathfrak{q} = (f_1, \dots, f_m)$ be a prime ideal. Suppose that $\mathfrak{q}$ contains the prime integer p. Then $\mathfrak{q}^{(r)}$ is exactly the set of polynomials $f \in \mathbb{Z}[x_1, \dots, x_n]$ such that*

$$\delta_{\mathrm{st},p}^{a_0}\left(\frac{\partial^{a_1+\cdots+a_n} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}\right) \in \mathfrak{q} \text{ for all } a_0 + a_1 + \cdots + a_n < r.$$

Other Applications

p-derivations have appeared in a range of sophisticated applications to number theory and arithmetic geometry. We briefly list a few of these, and encourage the reader to explore further.

Effective bounds on rational points and p-jet spaces. The motivation for Buium's original work on p-derivations was to give bound the number of points on rational points on curves. For a complex curve X defined over $\overline{\mathbb{Q}}$, there is a natural map from X to an algebraic group A, the Jacobian of X; the main result of Buium [Bui96] gives an effective bound, depending only on the genus of X and the smallest prime of a good reduction for X, on the number of points in X that map to torsion elements of A.

To establish these results, Buium constructs *p-jet spaces*. In differential geometry, the *jet* of a function f of order k at a point is the data of all of the values of the derivatives of f up to order k at that point; the *jet space* of order k of a manifold X is a manifold whose points correspond to jets on X. Buium's p-jet spaces are analogues of jet spaces obtained by replacing usual derivatives with p-derivations. The result mentioned above is then obtained by intersection theory on p-jet spaces of curves.

As mentioned earlier, Buium has developed an extensive theory of arithmetic differential geometry in analogy with classical differential geometry, for which p-jet spaces form the starting point. We refer the reader to [Bui05] to learn more.

Relationship with Witt vectors. The Witt vectors are a construction in number theory that generalizes the relationship between the prime field $\mathbb{F}_p$ and the corresponding ring of p-adic integers $\mathbb{Z}_p$. To be precise, there is a functor W from the category of rings to itself, called the functor of (p-typical) Witt vectors, that maps $\mathbb{F}_p$ to $\mathbb{Z}_p$. More generally, for any perfect field F of characteristic p, $W(F)$ is a local ring with maximal ideal (p), and $W(F)/pW(F) \cong F$; in this way, one can think of W as a generalization of a construction of $\mathbb{Z}_p$ from $\mathbb{F}_p$.

p-derivations have many interesting connections with Witt vectors; indeed, they first arise in Joyal's work to study Witt vectors. Namely, Joyal shows that the forgetful functor from the category of rings with a p-derivation to the category of rings is left adjoint to the Witt vector functor.

Philosophy of the field with one element. Various formulas for enumerating basic objects over a finite field $\mathbb{F}_q$ limit to combinatorially meaningful quantities as $q \to 1$; for example,

$$\lim_{q\to 1} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|(\mathbb{F}_q^\times)^n|} = n!\,,$$

where $\mathrm{GL}_n(\mathbb{F}_q)$ is the collection of linear automorphisms of $\mathbb{F}_q^n$, the vector space of n-tuples over $\mathbb{F}_q$, and $n!$ counts the number of permutations of n elements. Partially motivated by this phenomenon, and partially motivated by transferring results over finite fields to other settings, there is a program of inventing a notion of algebraic geometry over a "field with one element:" the mythical field with one element is not literally a field with one element, which would contradict the definition of field, but something with a different structure than a field that admits some sort of geometry analogous to what one would expect over a field with one element for quantitative or various other reasons. On the algebraic side, the field with one element can be thought of as a "deeper base ring" than $\mathbb{Z}$ (though not necessarily itself a ring!).

There are many approaches in the literature to implementing the philosophy of the field with one element. Most relevant for this article is the theory established by Borger [Bor09] as well as the closely related approach of Buium. Roughly speaking, Borger proposes that a model for the field with one element should be $\mathbb{Z}$ equipped with the collection of all its p-derivations δ_p. In particular, in Borger's model, GL_n "over the field with one element" is the symmetric group on n letters, aligning with the numerical coincidence noted in the previous paragraph.

Unifying cohomology theories. Recent work of Bhatt and Scholze [BS22] has employed p-derivations to relate various p-adic cohomology theories (étale, de Rham, crystalline). The key ingredient is the notion of a *prism*, which is a ring equipped with a p-derivation subject to some conditions. We refer the reader to [BS22] to learn more.

ACKNOWLEDGMENTS. The author is grateful to Alessandro De Stefani, Eloísa Grifo, Claudia Miller, Steven Sam, and the referee for many helpful comments on this article.

Jack Jeffries's work is supported by NSF CAREER Award DMS-2044833.

References

[AWZ21] Piotr Achinger, Jakub Witaszek, and Maciej Zdanowicz, *Global Frobenius liftability I*, J. Eur. Math. Soc. (JEMS) **23** (2021), no. 8, 2601–2648. MR4269423

[Bor09] James Borger, *Lambda-rings and the field with one element*, arXiv preprint arXiv:0906.3146 (2009).

[BS22] Bhargav Bhatt and Peter Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) **196** (2022), no. 3, 1135–1275. MR4502597

[Bui05] Alexandru Buium, *Arithmetic differential equations*, Mathematical Surveys and Monographs, vol. 118, American Mathematical Society, Providence, RI, 2005, DOI 10.1090/surv/118. MR2166202

[Bui17] Alexandru Buium, *Foundations of arithmetic differential geometry*, Mathematical Surveys and Monographs, vol. 222, American Mathematical Society, Providence, RI, 2017, DOI 10.1090/surv/222. MR3643159

[Bui96] Alexandru Buium, *Geometry of p-jets*, Duke Math. J. **82** (1996), no. 2, 349–367, DOI 10.1215/S0012-7094-96-08216-2. MR1387233

[Bui97] Alexandru Buium, *Arithmetic analogues of derivations*, J. Algebra **198** (1997), no. 1, 290–299. MR1482984

[DDSG+18] Hailong Dao, Alessandro De Stefani, Eloísa Grifo, Craig Huneke, and Luis Núñez Betancourt, *Symbolic powers of ideals*, Singularities and foliations. geometry, topology and applications, 2018, pp. 387–432. MR3779569

[DSGJ20] Alessandro De Stefani, Eloísa Grifo, and Jack Jeffries, *A Zariski-Nagata theorem for smooth $\mathbb{Z}$-algebras*, J. Reine Angew. Math. **761** (2020), 123–140. MR4080246

[HJ21] Melvin Hochster and Jack Jeffries, *A jacobian criterion for nonsingularity in mixed characteristic*, arXiv preprint arXiv:2106.01996 (2021).

[Joy85] André Joyal, *δ-anneaux et vecteurs de Witt* (French), C. R. Math. Rep. Acad. Sci. Canada **7** (1985), no. 3, 177–182. MR789309

[Sai22] Takeshi Saito, *Frobenius-Witt differentials and regularity*, Algebra Number Theory **16** (2022), no. 2, 369–391, DOI 10.2140/ant.2022.16.369. MR4412577

Jack Jeffries

Credits

The opening image, Figure 1, and Figure 2 are courtesy of Jack Jeffries.

Photo of Jack Jeffries is courtesy of Eloísa Grifo.

THE

NEXT GENERATION

Early-career AMS members take a moment

Evans Boadi
PhD Candidate, SUNY Buffalo

Hobbies: Videography and photography. Making short films

Field of research: Nonlinear Water Waves

How would you describe math to a non-math person?: Math is beyond equations and symbols. It is something you can visualize. The things around us, objects, matter, water, spaces are all visualizations of mathematics. They can all be represented by math equations. So, when you see anything around you, it is mathematics! And that is the beauty of mathematics, it is found everywhere you go.

What does the AMS mean to you?: An AMS Membership is a connection to me. It connects me to other mathematicians and other people like me.

Alexanderia Lacy
Graduate Research Assistant, University of Tennessee

Hobby: Singing

Field of research: Mathematical Biology

Describe a situation when you first fell for math: I fell in love with math in high school, but I fell in "**love love**" during my first summer internship in the 9th grade. I was able to take my love of puzzles and problem solving and apply it to real world applications. At that time, I felt my purpose.

Leah LeJeune
PhD Candidate, University of Louisiana at Lafayette

Hobby: Being outside (running, biking, hiking, swimming)

Field of research: Mathematical Modeling in Epidemiology

What does the AMS mean to you?: The AMS has given me the opportunity to give back to my graduate school community. As president of our graduate student chapter, I have been able to help develop programs geared toward supporting students throughout their graduate careers.

OF MATHEMATICS

to share a little about themselves:

#AMSMember

Hobby: Running

Field of research: Mathematical Biology

What do you love about math?: I love how collaborative math can be and the tangible impacts that can be felt in the real world. Using math to solve medical problems is beautiful and useful and I love that my work may improve quality of life and public health.

What does the AMS mean to you?: Opportunity to meet fellow mathematicians, travel to JMM Conferences, and receive an important math newsletter.

How would you describe math to a non-math person?: I would describe to the person that math is nothing but a game. Learn the rules of the game and practice playing the game to become perfect in it.

Field of research: Statistical Analysis

What do you think is the most important service the AMS offers?: JMM! It is an excellent source to show work, have great discussions, and make fantastic connections.

Favorite color: Blue

Favorite food: Birria

Field of research: Computational Biology

Describe a situation when you first fell for math: Seeing how you use math to predict physical sciences absolutely helped me learn that math is everywhere.

www.ams.org/membership

On Bernoulli-Type Elliptic Free Boundary Problems

Daniela De Silva

1. Free Boundary Problems

The modern theory of free boundary problems embodies the interplay between sophisticated mathematical analysis and the applied sciences. The discipline has been flourishing since the works of several illustrious mathematicians, such as Lewy, Stampacchia, Lax, Lions, in the early 1950s until the early 1970s. Examples of free boundaries arise in flame propagation, image reconstructions, jet flows, optimal stopping problems in financial mathematics, tumor growth, and in many other different contexts.

Daniela De Silva is a professor of mathematics at Barnard College, Columbia University. Her email address is `desilva@math.columbia.edu`.

Communicated by Notices *Associate Editor Chikako Mese.*

DOI: https://doi.org/10.1090/noti2829

In a free boundary problem one or more unknown functions must be determined in different domains with a common boundary portion, and each function is governed in its domain by a set of state laws expressed as partial differential equations (PDEs). The domains are a priori unknown, and their boundaries are therefore called *free boundaries* and must be determined thanks to a number of *free boundary conditions* dictated by physical laws or other constraints governing the phase transition.

The origin of the modern mathematical discipline owes much to the *Stefan problem*, named after the physicist Stefan who toward the end of the 19th century studied ice formations in the polar seas. The problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, for example ice melting into water: this is accomplished by coupling the heat equation in the water with a transmission condition, the free boundary condition, on the evolving boundary between its two phases. Interestingly, the classical Stefan problem can be reduced to the so-called parabolic (time-dependent) *obstacle problem*. The time-independent obstacle problem is a classical problem that belongs to a wide class of free boundary problems with variational structure. It consists of finding the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle φ under the action of gravity.

To better understand what a variational problem is, we pose the classical question of the celebrated theory of minimal surfaces: we seek for the function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with given boundary value $u = g$ on $\partial\Omega$, whose graph has the smallest area

$$\int_\Omega \sqrt{1 + |\nabla u|^2}$$

among all possible such graphs. A simpler question, arising from the linearization of the area functional, consists in minimizing the so-called Dirichlet integral

$$\int_\Omega |\nabla u|^2 dx.$$

Via the classical method of small perturbations, a PDE, known as the Euler–Lagrange equation, can be associated to such minimization problems. In the case of the Dirichlet integral, the corresponding equation is the Laplace equation $\Delta u = 0$.

One fundamental problem in the *Calculus of Variations* consists in studying critical points for an energy functional of the type

$$J(u, \Omega) = \int_\Omega \left(|\nabla u|^2 + W(u)\right) dx,$$

that is, solutions to its associated *semilinear* equation

$$\triangle u = W'(u),$$

where $W : \mathbb{R} \to [0, \infty)$ represents a given potential with minimum 0. Certain classes of potentials have been extensively studied in the literature. One such example is the double-well potential $W(t) = (1-t^2)^2$ and the corresponding Allen–Cahn equation which appears in the theory of phase-transitions and minimal surfaces.

When the potential W is not of class C^2 near one of its minimum points, then a minimizer can develop constant patches where it can take that value, and this leads to a free boundary problem. Two such potentials were investigated in great detail. The first one is the Lipschitz potential $W(t) = t^+$ which corresponds to the classical obstacle problem with obstacle $\varphi = 0$, and we refer the reader to the book of Petrosyan, Shahgholian, and Uraltseva for an introduction to this subject. The second one is the discontinuous potential $W(t) = \chi_{\{t>0\}}$ with its associated *Alt–Caffarelli* energy, which is known as the *Bernoulli* free boundary problem [AC81].

In this note we present an overview of the theory for minimizers of the Bernoulli problem, which has inspired much of the recent research in the area of free boundaries. This is by far not an exhaustive account, but it should offer an insight on some essential techniques and results in elliptic free boundary problems, and also on some of the interesting open questions in the theory. In the last section, we will describe some related problems and current research directions, and the content will be slightly more technical.

2. The Bernoulli Problem

2.1. Introduction. The classical Bernoulli (or Alt–Caffarelli) one-phase free boundary problem arises from the minimization of the energy functional

$$J(u, \Omega) := \int_\Omega (|\nabla u|^2 + \chi_{\{u>0\}}) dx, \tag{2.1}$$

among all competitors $u \geq 0$, $u \in H^1(\Omega)$, with a given boundary data ϕ on $\partial\Omega$. Here Ω is a bounded domain in $\mathbb{R}^n$ (say with Lipschitz boundary), and $H^1(\Omega)$ denotes the usual Sobolev space of L^2 functions with weak derivatives in L^2 as well (see for example [GT83] for an account on basic properties of such spaces).

This minimization problem was first investigated systematically by Alt and Caffarelli in the pioneer work [AC81], and originates in two-dimensional fluid dynamics. Indeed, in 2D the Euler equations for the motion of an incompressible, irrotational, inviscid fluid of constant density, and in absence of gravity, can be reinterpreted in the steady case in terms of a *stream* function u. Under the assumption that the pressure on the surface of the liquid is constant, the system is reduced to a Bernoulli free boundary problem for u.

Existence of minimizers with a given boundary data is easily obtained via the classical *Direct Method* of the Calculus of Variations, relying on the *lower semicontinuity* of the energy J and the compactness properties of Sobolev Spaces.

In order to understand the free boundary problem that such a minimizer solves, let us perform a simple one-dimensional computation of the Euler–Lagrange equation associated to J. Let $u \geq 0$ (a "nice" function) minimize

$$\int_{-1}^{1} (u'^2 + \chi_{\{u>0\}}) dx$$

among competitors with the same boundary data, and let us consider first a small perturbation of u, $u + \epsilon\varphi$, with φ a smooth (i.e., C^∞) function compactly supported in $\{u > 0\}$, and $\epsilon > 0$ small. Then,

$$\chi_{\{u>0\}} = \chi_{\{u+\epsilon\varphi\}}$$

and the minimization property gives (we drop the dependence of J on the domain),

$$0 = \lim_{\epsilon\to 0} \frac{J(u+\epsilon\varphi) - J(u)}{\epsilon} = 2\int_{-1}^{1} u'\varphi'\, dx$$

By integration by parts, and the fact that φ is arbitrary, we conclude that

$$u'' = 0, \quad \text{in } \{u > 0\} \cap (-1, 1).$$

Thus, u is linear in its positive phase and we need to determine its slope. Say $0 \in \partial\{u > 0\} \cap (-1, 1)$ and $u = \alpha x^+$ for some $\alpha > 0$. In order to determine α we compare the energy of u with the energy of the competitor $v(x) := \frac{\alpha}{1-\epsilon}(x-\epsilon)^+$, and obtain

$$J(u) \le J(v) \Rightarrow \alpha^2 + \epsilon \le (\frac{\alpha}{1-\epsilon})^2(1-\epsilon) \sim \alpha^2(1+\epsilon),$$

hence

$$\alpha \ge 1.$$

Similarly, by comparing the energy of u with the energy of $w(x) := \frac{\alpha}{1+\epsilon}(x+\epsilon)^+$, we get that $\alpha \le 1$, and conclude that u has slope 1 at a free boundary point. Heuristically, denoted by

$$\Omega^+(u) := \Omega \cap \{u > 0\},$$

the positivity set of u, and by

$$F(u) := \Omega \cap \partial\Omega^+(u),$$

the free boundary of u, we expect that in the n-dimensional case minimizers of $J(u, \Omega)$ will solve the following free boundary problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega^+(u), \\ |\nabla u| = 1, & \text{on } F(u), \end{cases} \tag{2.2}$$

which is indeed the case. Clearly, this has to be understood in an appropriate *weak* sense, as a minimizer $u \in H^1(\Omega)$, and we have no information about the regularity of the set $F(u)$.

The notion of weak solutions to elliptic equations is well established, and it is known that a weak solution to the interior equation

$$\Delta u = 0, \quad \text{in } \Omega^+(u),$$

is a classical solution to the equation (and in fact it is analytic in $\Omega^+(u)$), see for example [GT83]. The novelty here consists in the interpretation of the free boundary condition in a suitable weak sense. The simplest interpretation, which connects to our one-dimensional computation above, is provided by the notion of *viscosity* solution introduced by Caffarelli, see for example [CS05]. Precisely, let $u \ge 0$ be a continuous function in Ω (minimizers will turn out to be continuous), and let $x_0 \in F(u)$. We say that x_0 is a regular point for $F(u)$ if there exists a tangent ball B to $F(u)$ at x_0, fully contained either in the positive or zero set of u. It turns out that if u is harmonic in its positive set, then if x_0 is regular

$$u(x) = \alpha((x - x_0)\cdot\nu)^+ + o(|x - x_0|), \quad \text{as } x \to x_0,$$

in any nontangential region, for some $0 \le \alpha \le +\infty$. Here ν is the normal to B at x_0 pointing toward the positivity set, and a nontangential region is contained in a cone with vertex at x_0 strictly included in the half-space $(x - x_0)\cdot\nu > 0$. The free boundary condition in (2.2) can then be interpreted as the Neumann-type boundary condition

$$\alpha = 1.$$

The main question then becomes: *is it possible to prove that u and F(u) are sufficiently smooth, so that the free boundary condition can be understood in the classical (pointwise) sense?* We answer this question in the following section; we will focus on minimizers and we will not provide details of proofs but rather we will exhibit a successful roadmap to attack such questions, displaying tools which are useful in a variety of free boundary problems.

2.2. Regularity. The first question that we want to address is the regularity of a minimizer u to $J(\cdot, \Omega)$. For simplicity, we focus on interior regularity and work with *local minimizers*. Precisely, u is a (local) minimizer for J in Ω, if $u \in H^1_{loc}(\Omega)$ and for any domain $D \subset\subset \Omega$ and every function $v \in H^1_{loc}(\Omega)$ which coincides with u in a neighborhood of $\Omega \setminus D$ we have

$$J(u, D) \le J(v, D).$$

From the Euler–Lagrange equation (2.2), given the jump discontinuity in the gradient across the free boundary, we expect that the optimal regularity for u is Lipschitz continuity. Indeed, the following theorem is proved in [AC81]. Here and in what follows, $B_r(x_0)$ denotes a ball of radius $r > 0$ centered at x_0. When $x_0 = 0$, we drop the dependence on x_0. Also, constants depending only on dimension are called *universal*.

Proposition 2.1. *Let $u \ge 0$ minimize J in Ω. Then $u \in C^{0,1}(\Omega)$. If $x_0 \in F(u)$ and $B_r(x_0) \subset \Omega$, then*

$$\|\nabla u\|_{L^\infty(B_{r/2}(x_0))} \le C,$$

for a universal constant $C > 0$.

The proof of Lipschitz continuity in this one-phase context is reasonably straightforward and relies on comparison with an explicit subsolution.

Moreover, minimizers satisfy the following so-called *nondegeneracy* property, which is an essential tool in analyzing the regularity of the free boundary [AC81].

Proposition 2.2. *Let $u \geq 0$ minimize J in Ω. Then, for $x \in \Omega^+(u)$,*

$$u(x) \geq c\, d(x), \quad d(x) := dist(x, F(u)),$$

if $B_{d(x)}(x) \subset \Omega$, for some $c > 0$ universal.

Lipschitz regularity and nondegeneracy yield positive density property for $\{u > 0\}$ and $\{u = 0\}$ at free boundary points, i.e, for $x_0 \in F(u), B_r(x_0) \subset \Omega$,

$$\delta \leq \frac{|B_r(x_0) \cap \Omega^+(u)|}{|B_r(x_0)|} \leq 1 - \delta,$$

for $\delta > 0$ small universal. In particular, the free boundary cannot exhibit a cusp-like behavior. Also, these properties are crucial in obtaining the following important compactness result for minimizers [AC81].

Proposition 2.3. *Assume u_k are minimizers to J in Ω and $u_k \to u$ locally uniformly. Then u is a minimizer to J, $\chi_{\{u_k>0\}} \to \chi_{\{u>0\}}$ locally in L^1, and $F(u_k) \to F(u)$ locally in the Hausdorff distance.*

Analyzing the regularity of the free boundary is a much harder task (which exploits the propositions above) and in fact, it is the heart of the matter in most free boundary problems.

One first regularity result in the Geometric Measure Theory (GMT) sense follows from the observation that $F(u)$ has *finite perimeter*. This can be seen formally from integration by parts as if u solves (2.2)

$$0 = \int_{\Omega^+(u) \cap B_r} -\Delta u \, dx = \int_{\Omega^+(u) \cap \partial B_r} u_\nu d\sigma + \int_{F(u) \cap B_r} u_\nu d\sigma,$$

with ν the unit inner normal to $\partial\Omega^+(u)$. Notice that, the first term on the right-hand side is bounded in view of the Lipschitz continuity, while, by the free boundary condition, the second term represents $\mathcal{H}^{n-1}(F(u) \cap B_r)$.

In particular, one can define the reduced part of the free boundary, $F^*(u)$, and in view of the density estimates $\mathcal{H}^{n-1}(F(u) \setminus F^*(u)) = 0$. For the notion of perimeter of a set and of reduced boundary we refer to Giusti [Giu84].

The main purpose of this section will be to explain the strategy behind the following "state of the art" strong regularity result for the free boundary $F(u)$ of a minimizer u. It contains contributions of several authors [AC81, CJK04, JS15, KN77] and we will make that precise in what follows.

Theorem 2.4. *Let $u \geq 0$ be a minimizer of $J(\cdot, \Omega)$. Then, $F(u)$ is locally an analytic surface, except on a closed singular set Σ_u of Hausdorff dimension $n - 5$, i.e.,*

$$\mathcal{H}^s(\Sigma_u) = 0, \quad \textit{for } s > n - 5.$$

The strategy to prove Theorem 2.4 is based on a *blow-up analysis*. Let us assume for simplicity that $0 \in F(u)$ and let us analyze the behavior of u near 0. A crucial feature of this problem is its invariance under Lipschitz rescaling

$$u_r(x) = r^{-1}u(rx), \tag{2.3}$$

that is u is a minimizer if and only if u_r is a minimizer. We thus consider a sequence of rescalings $u_k := u_{r_k}$, with $r_k \to 0$ as $k \to \infty$, and in view of the Lipschitz continuity (Proposition 2.1) we obtain by Ascoli–Arzela (up to extracting a subsequence):

$$u_k \to u_0, \quad \text{locally uniformly in } \mathbb{R}^n.$$

By nondegeneracy $u_0 \not\equiv 0$ and is called a *blow-up limit*. Moreover, by the compactness, Proposition 2.3,

$$u_0 \text{ is a global minimizer,}$$

i.e., it minimizes J over any compact set in $\mathbb{R}^n$. Finally, again by Proposition 2.3, $F(u_k) \to F(u_0)$ in the appropriate sense. Therefore, *properties of $F(u_0)$ can be transferred to $F(u)$ near* 0. Our objective is now to better understand blow-up global minimizers. A fundamental property follows from the next result, a monotonicity formula due to Weiss [Wei98]:

Theorem 2.5. *If u is a minimizer to J in B_R, then*

$$\Phi_u(r) := r^{-n}J(u, B_r) - r^{-n-1}\int_{\partial B_r} u^2, \quad 0 < r \leq R,$$

is increasing in r. Moreover Φ_u is constant if and only if u is homogeneous of degree 1.

Indeed the rescaling $u_\lambda(x)$ defined in (2.3) satisfies

$$\Phi_{u_\lambda}(r) = \Phi_u(\lambda r). \tag{2.4}$$

Thus, if $u_k := u_{\lambda_k} \to u_0$ we obtain that:

$$\Phi_{u_k}(r) = \Phi_u(\lambda_k r)$$

and as $k \to \infty$

$$\Phi_{u_0}(r) = \Phi_u(0^+), \quad \forall r,$$

hence u_0 is homogeneous of degree 1. Thus, *we need to classify global minimizers that are homogeneous of degree one*. In view of (2.2), the obvious candidate is (up to rotation) x_n^+. In fact, the following classification theorem holds.

Theorem 2.6 (Classification of Global Homogenous minimizers in low dimensions.). *In dimension $n = 2, 3, 4$ the only global 1-homogeneous minimizer is (up to rotations) x_n^+.*

In dimension $n = 2$ this was already obtained in [AC81]. The case $n = 3$ was settled by Caffarelli, Jerison, and Kenig in [CJK04], while Jerison and Savin established the result in dimension $n = 4$ [JS15]. Furthermore, in collaboration with Jerison [DSJ09], we provided an explicit example of a global homogeneous minimizer with a singularity at the origin, and we will discuss it in more detail in the next section. The question of whether Theorem 2.6 holds in dimension $n = 5, 6$ remains open.

Now, to settle the original question of the regularity of the free boundary, we need to investigate the following perturbative issue:

Assume u is "close" to the optimal configuration x_n^+, what can we deduce about the regularity $F(u)$?

Since the optimal configuration has a free boundary which is a hyperplane, we refer to this question as a

"flatness implies regularity"

question. The answer is contained in the next theorem, first established in [AC81] and then extended by Caffarelli to the context of free boundary viscosity solutions in his breakthrough trilogy in the late 80's (for an account of such works see for example [CS05]).

Theorem 2.7. *Let $u \geq 0$ be a minimizer to J in B_1. There exists a universal constant $\bar{\epsilon} > 0$, such that if*

$$\|u - x_n^+\|_{L^\infty(B_1)} \leq \bar{\epsilon}$$

and

$$\{x_n \leq -\bar{\epsilon}\} \subset B_1 \cap \{u = 0\} \subset \{x_n \leq \bar{\epsilon}\},$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

With this theorem combined with the blow up analysis, we can conclude that minimizers in dimension $n = 2, 3, 4$ have $C^{1,\alpha}$ free boundaries. This is the desired regularity as by classical elliptic theory we conclude also that u is $C^{1,\alpha}$ up to $F(u)$, hence the free boundary condition, $|\nabla u| = 1$ on $F(u)$, can be understood in a pointwise sense. Moreover, once $C^{1,\alpha}$ regularity of the free boundary is proven, then the celebrated work of Kindherleher and Nirenberg [KN77] yields analyticity of the free boundary. This is achieved via a hodograph transform that changes the problem into a fixed boundary nonlinear elliptic problem. Once full regularity in dimension $n = 2, 3, 4$ is established, a standard dimension reduction argument due to Federer (see for example [Giu84]) leads to the conclusive claim in Theorem 2.4. More precisely, the singular set has Hausdorff dimension $n - k$, where k is the first dimension in which a singular homogeneous minimizer occurs. In view of the result in [DSJ09], we have that k is either 5, 6, or 7.

In conclusion we have identified the key steps in proving (partial or full) regularity of the free boundary, and this strategy is applicable in a variety of free boundary problems:

- Optimal regularity and compactness of minimizers;
- Weiss-type monotonicity formula;
- Classification of global homogeneous solutions;
- Flatness implies regularity;
- Dimension reduction & GMT techniques;
- Higher regularity.

2.3. Singular minimizers.

In this section we address the question of better understanding global nontrivial solutions to (2.2), which are homogeneous of degree 1. We focus on those which are generated by the so-called *Lawson cones*. For example, consider the (unique up to scalar multiple) positive harmonic function U in the axis-symmetric cone

$$\Gamma := \{|x_n| < t_n |x|\}$$

which is 0 on $\partial\Gamma$. Γ is connected and its complement consists of two circular cones with central axis in the direction $\pm e_n$. For $n \geq 3$, there exists a unique $t_n > 0$ such that U is homogeneous of degree 1. The inner normal derivative of U, U_ν, is homogeneous of degree zero and by rotational symmetry it is constant. Thus we can choose U so that $U_\nu = 1$ on $\partial\Gamma \setminus \{0\}$. Then, after extending it to 0 outside Γ, U is clearly a global critical point to our Bernoulli problem (2.2). In dimension $n = 3$, it was ruled out in [AC81] that U is a minimizer. Later, Caffarelli, Jerison, and Kenig showed that the same is true in dimension $n \leq 6$ [CJK04]. In collaboration with Jerison, we proved in [DSJ09] that U is a minimizer in dimension $n = 7$, exhibiting an explicit minimizing solution with a singular free boundary. The proof is based on comparison with explicit families of sub- and supersolutions, expressed in terms of certain hypergeometric series.

In [Hon15], Hong studied other Lawson-type cones, showing that they are all *unstable* in dimension $n \leq 6$. Roughly speaking, stability in a region D where $u, F(u)$ are smooth, consists in requiring minimality with respect to the appropriate class of small perturbations supported in D. It amounts to establish whether the following inequality holds true for all test functions $\phi \in C_0^\infty(D)$:

$$\int_{D\cap\{u>0\}} |\nabla\phi|^2 dx + \int_{D\cap F(u)} u_{\nu\nu}\phi^2 \, d\sigma \geq 0.$$

Here ν denotes the normal to $F(u)$ pointing toward the positive phase. Notice that, at a free boundary point,

$$\Delta u = u_{\nu\nu} - u_\nu H,$$

where H is the mean curvature of $F(u)$ oriented toward the positive phase, thus the free boundary condition yields $u_{\nu\nu} = H$. While this stability analysis is supporting evidence that $n = 7$ is the first dimension in which singularities occur, the question remains open.

In a recent work with Jerison and Shahgholian [DSJS22], we observed that in fact the singularity of U can be perturbed away. Precisely, we proved the following result. Here we use the notation: for $t > 0$

$$u_t(x) = \frac{1}{t}u(tx), \; \Gamma(u_t) := \{(x, u_t(x)) : x \in \overline{\{u_t > 0\}}\}.$$

Theorem 2.8. *Let U_0 be a global minimizer to J homogeneous of degree 1, with singular set $\Sigma_{U_0} = \{0\}$. There exist $\overline{U}, \underline{U}$ global minimizers to* (2.1), *such that*

i. $\underline{U} \leq U_0 \leq \overline{U}$, *and* $dist(F(\overline{U}),\{0\}) = dist(F(\underline{U}),\{0\}) = 1$;

ii. $F(\overline{U}), F(\underline{U})$ *are analytic hypersurfaces;*

iii. *for any* $x \in \{U_0 = 0\}^o$, *the ray* $\{tx,\ t > 0\}$, *intersects* $F(\overline{U})$ *in a single point and the intersection is transverse; similarly, for any* $x \in \{U_0 > 0\}$, *the ray* $\{tx,\ t > 0\}$, *intersects* $F(\underline{U})$ *in a single point and the intersection is transverse;*

iv. *the graphs* $\overline{\Gamma}_t := \Gamma(\overline{U}_t), \underline{\Gamma}_t := \Gamma(\underline{U}_t)$ *foliate the half-space* $\mathbb{R}^n \times [0,+\infty)$, *i.e.,*

$$\mathbb{R}^n \times [0,+\infty) = \bigcup_{t\in(0,\infty)} (\overline{\Gamma}_t \cup \underline{\Gamma}_t) \cup \Gamma(U_0);$$

v. *if* V *is a global minimizer to* J *and* $V \geq U_0$ *(resp.* $V \leq U_0$*), then*

$$V \equiv \overline{U}_t, \quad (resp.\ V \equiv \underline{U}_t),$$

for some $t > 0$, *unless* $V \equiv U_0$.

In fact, we can provide precise asymptotic developments for $\underline{U}, \overline{U}$ in terms of the homogeneous solutions of the so-called *linearized equation* associated to our problem. However, this is rather technical and beyond the scope of this note.

Finally, the question of whether or not blowups at singular points are unique remains open. A priori, it is possible that the free boundary around a singular point asymptotically approaches one singular cone at a certain set of small scales, but approaches a different singular cone at another set of scales. In [ESV20], Engelstein, Spolaor, and Velichkov proved uniqueness of blowup and $C^{1,log}$-regularity at points where one blowup has an isolated singularity.

2.4. **Connection with minimal surfaces.** The regularity theory described above parallels very well the theory for area minimizing sets (in the sense of De Giorgi), both in results and techniques. For a comprehensive treatment of such theory we refer the reader to Giusti's monograph [Giu84]. In the context of the perimeter functional $Per_\Omega(E)$, Theorem 2.4 can be stated as follows.

Theorem 2.9. *Let* $E \subset \mathbb{R}^n$ *be a Caccioppoli set that minimizes* $Per_\Omega(\cdot)$. *Then,* ∂E *is locally an analytic surface, except on a singular set* Σ *of Hausdorff dimension* $n-8$, *i.e.,*

$$\mathcal{H}^s(\Sigma) = 0, \quad \text{for } s > n-8.$$

Moreover, this result is sharp as, in dimension $n = 8$, the Simons cone $\{x_1^2 + ... x_4^2 = x_5^2 + ... + x_8^2\}$ is a global minimizing cone with a singularity at the origin. Also in this context the singularity can be perturbed away as proved by Hardt and Simon.

In dimension $n = 2$ an actual connection between the two problems was unveiled by Traizet in [Tra14]. Precisely, he established a one-to-one correspondence between the following two classes of objects:

- critical points of J with a smooth free boundary and such that $|\nabla u| < 1$ in $\Omega^+(u)$,
- complete, embedded minimal surfaces M in $\mathbb{R}^3$ which are symmetric with respect to the horizontal plane $x_3 = 0$ and such that $M_+ = M \cap \{x_3 > 0\}$ is a graph over the unbounded domain in the plane bounded by $M \cap \{x_3 = 0\}$ (minimal bigraph).

Further results for critical points which resemble those available in the theory of minimal surfaces are also known (due to Hauswirth, Helein and Pacard, Khavinson, Lundberg and Teodorescu, Jerison and Kamburov). However, for the sake of brevity, in this note we only focus on minimizers.

2.5. **Two-phase problems.** In several applications it is important to consider minimizers u of J, or of slightly more general classes of energy functionals, which change sign, for instance when modeling the flow of two liquids in jet and cavities. In the case of the basic functional J, the corresponding Euler–Lagrange equation reads,

$$\begin{cases} \Delta u = 0, & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 & \text{on } F(u), \end{cases} \tag{2.5}$$

where

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\},$$
$$\Omega^-(u) := \{x \in \Omega : u(x) < 0\},$$
$$F(u) := \partial\Omega^+(u) \cap \Omega = \partial\Omega^-(u) \cap \Omega,$$

while u_ν^+ and u_ν^- denote the normal derivatives of u in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively. This so-called *two-phase* problem was investigated by Alt, Caffarelli, and Friedman in [ACF84]. Again, the free boundary condition can be interpreted in the following viscosity sense (say for a continuous solution). Let $x_0 \in F(u)$; we say that x_0 is a regular point if there exists a tangent ball B to $F(u)$ at x_0 such that either $B \subset \Omega^+(u)$ or $B \subset \Omega^-(u)$. At such a point, if say $B \subset \Omega^+(u)$, u satisfies (in every nontangential region):

$$u^+(x) = \alpha((x-x_0)\cdot\nu)^+ + o(|x-x_0|) \quad \text{in } B,$$

and

$$u^-(x) = \beta((x-x_0)\cdot\nu)^- + o(|x-x_0|) \quad \text{in } B^c,$$

where ν is the unit normal to ∂B at x_0 inward to $\Omega^+(u)$, and $\alpha \in (0,+\infty], \beta \in [0,\infty)$.

At regular points, the well-defined constants α and β can be used to give a weak notion for the normal derivatives, u_ν^+ and u_ν^-. The prescribed free boundary condition then becomes

$$\alpha^2 - \beta^2 = 1.$$

According to our roadmap, the first step in analyzing this two-phase problem consists in proving the optimal regularity of a minimizer, which again is expected to be Lipschitz continuity. However, in this two-phase context the proof is far from trivial, and the authors introduced a powerful new monotonicity formula, which we state below.

Theorem 2.10 (ACF monotonicity formula.). *Let u_1, u_2 be nonnegative subharmonic functions in $C(B_1)$. Assume that $u_1 \cdot u_2 = 0$ and that $u_1(0) = u_2(0) = 0$. Set,*

$$\Phi(r) := \left(\frac{1}{r^2}\int_{B_r}\frac{|\nabla u_1|^2}{|x|^{n-2}}dx\right)\left(\frac{1}{r^2}\int_{B_r}\frac{|\nabla u_2|^2}{|x|^{n-2}}dx\right),$$

for $0 < r < 1$. Then $\Phi(r)$ is bounded and it is an increasing function of r.

Observe that if the supports of u_1 and u_2 were separated by a smooth surface with normal ν at $x = 0$ then, by taking the limit as $r \to 0$, we could deduce that

$$\left(u_{1_\nu}(0)\right)^2 \cdot \left(u_{2_\nu}(0)\right)^2 \le C\,\Phi(1).$$

Hence, when applied to $u_1 = u^+, u_2 = u^-$, $\Phi(r)$ "empirically" gives a control in average of the product of the normal derivatives of a solution u to (2.5) at the origin. This combined with the free boundary condition in (2.5) leads to the Lipschitz continuity of u.

Moreover, this formula allows us to classify a blow-up limit u_0 as either a so-called *two-plane* solution (up to rotations):

$$u_0 = U_\beta := \alpha x_n^+ - \beta x_n^-, \quad \alpha^2 - \beta^2 = 1, \quad \beta > 0$$

or a solution to the one-phase problem, i.e.,

$$u_0^- \equiv 0.$$

Thus, in order to apply our strategy for investigating the regularity of the free boundary, we need to understand *flatness implies regularity* results in which u is either a perturbation of U_β with $\beta > 0$ or of a one-plane solution $U_0 := x_n^+$. This was first established by Alt, Caffarelli, and Friedman in [ACF84] in the variational context and then extended by Caffarelli to the viscosity set-up [CS05]. The proof is more delicate in this two-phase case, as when α and β are comparable, closeness to U_β gives a nice control on the location of $F(u)$, but when $\beta \ll \alpha$ only a one side control of $F(u)$ is possible.

Following these steps, one deduces that the thesis of Theorem 2.4 still holds true even when u is allowed to change sign.

A related two-phase energy can be considered, allowing for a "double discontinuity" at zero. Precisely, let

$$J(u,\Omega) = \int_\Omega (|\nabla u|^2 + \lambda_+^2\chi_{\{u>0\}} + \lambda_-^2\chi_{\{u<0\}})dx,$$

with $\lambda_\pm > 0$. In this case, $\partial\Omega^+(u)$ may not coincide with $\partial\Omega^-(u)$ and the zero level set $\{u = 0\}$ might have nonempty interior. This introduces a new element in the analysis of the free boundary, which can now switch from one-phase to two-phase at the so-called branching points, at which the zero level set looks like a cusp. The regularity theory in this context was studied by De Philippis, Spolaor, and Velichkov in [DPSV21].

3. Some Related Problems

In this final section, we will present some free boundary problems which are related to the Bernoulli problem and are the object of current investigation. We will focus mostly on the question of regularity of the free boundary, and certain discussions may be slightly more technical in nature.

3.1. The thin Bernoulli problem. Consider minimizers u of the energy functional J

$$J(u,\Omega) := \int_\Omega |\nabla u|^2 dX + \mathcal{H}^n(\{u(x,0) > 0\} \cap \Omega), \tag{3.1}$$

where $\Omega \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and points in $\mathbb{R}^{n+1}$ are denoted by $X = (x, x_{n+1})$. Assume also that

$$\Omega \text{ is symmetric with respect to } \{x_{n+1} = 0\},$$
$$u \ge 0 \quad \text{is even with respect to } x_{n+1}.$$

This problem is closely related to the classical Bernoulli problem where the second term of the energy J is replaced by $\mathcal{H}^{n+1}(\{u > 0\})$. In this setting, the zero set $\{u = 0\}$ occurs on the lower-dimensional subspace $\mathbb{R}^n \times \{0\}$ and the free boundary, defined as

$$F(u) := \partial_{\mathbb{R}^n}\{u(x,0) > 0\} \cap \Omega \subset \mathbb{R}^n,$$

is expected to be $n-1$-dimensional. For this reason, we refer to it as the *thin* Bernoulli problem. The associated Euler–Lagrange equation (up to a multiplicative constant) reads as

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ \dfrac{\partial u}{\partial\sqrt{t}} = 1 & \text{on } F(u). \end{cases} \tag{3.2}$$

where we used the notation

$$\frac{\partial u}{\partial\sqrt{t}}(Z) := \lim_{t\to 0^+}\frac{u(z + t\nu, 0)}{t^{1/2}}, \quad Z = (z,0) \in F(u),$$

and ν denotes the outward normal to the free boundary $F(u)$ in $\mathbb{R}^n$. The role of the one-dimensional solution x_n^+ in this context is played by the explicit solution

$$Re\, z^{\frac{1}{2}}, \qquad z = x_n + ix_{n+1}.$$

The thin one-phase problem was first introduced by Caffarelli, Roquejoffre and Sire in [CRS10] as a variational problem involving fractional H^s norms. For example, if

u is a local minimizer of J defined in $\mathbb{R}^{n+1}$ then its restriction to the n-dimensional space $\mathbb{R}^n \times \{0\}$ minimizes locally an energy of the type

$$c_n \|u\|^2_{H^{1/2}} + \mathcal{H}^n\{u > 0\}.$$

Such problems are relevant in classical physical models in mediums where long range (nonlocal) interactions are present, see [CRS10] for further motivation.

The main difficulty in the thin-one phase problem occurs near the free boundary where all derivatives of u blow up and the problem becomes degenerate. The state of the art result on the regularity of $F(u)$ is contained in the following theorem (see [DSS15] and references therein).

Theorem 3.1. *Let u be a minimizer for J. The free boundary $F(u)$ is locally a C^∞ surface, except on a singular set $\Sigma_u \subset F(u)$ of Hausdorff dimension $n-3$, i.e.,*

$$\mathcal{H}^s(\Sigma_u) = 0 \quad \text{for } s > n-3.$$

Moreover, $F(u)$ has locally finite $\mathcal{H}^{n-1}$ measure.

As a corollary, in dimension $n = 2$, free boundaries of minimizers are always C^∞.

While the path leading to the proof of Theorem 3.1 is the one delineated in the previous section, the strategy to accomplish each of the steps varies greatly. The key tool is a geometric Harnack inequality that localizes the free boundary well, and allows to linearize the problem. This strategy was introduce in [DS11] to deal with inhomogeneous Bernoulli problems.

Concerning *higher regularity*, as explained in the previous section, the analyticity of $C^{1,\alpha}$ free boundaries of the classical Bernoulli problem was obtained via a hodograph transform to reduce the problem to a nonlinear Neumann problem with fixed boundary. In the thin case a new method has been developed that avoids the hodograph transformation and it makes use of Schauder estimates in *slit domains* for both a Dirichlet and a Neumann problem [DSS15]. The method is flexible and robust and applies also to other problems, for example to the *thin obstacle problem*. On the other hand, the details are rather technical and beyond the scope of this note. The classification of cones in dimension greater than 3 remains an open question. In the case of axially symmetric cones, Ros-Oton and Fernandez-Real [FRRO] showed that stables cones are trivial in dimensions $n \leq 5$.

The thin two-phase problem, that is when u is allowed to change sign, was considered by Allen and Petrosyan in [AP12]. They showed that the positive and negative phases are always separated thus the problem reduces locally back to a one-phase problem. They also obtained a Weiss-type monotonicity formula for minimizers and proved that, in dimension $n = 2$, the free boundary is C^1 in a neighborhood of a regular point.

3.2. The Alt–Phillips problem.

The obstacle problem (mentioned in the introduction) and the Bernoulli problem can be seen as part of the more general family of Alt–Phillips energies

$$J(u,\Omega) := \int_\Omega \left(|\nabla u|^2 + u^\gamma \chi_{\{u>0\}}\right) dx, \qquad \gamma \in [0,2).$$

Nonnegative minimizers $u \geq 0$ of J were studied by Alt and Phillips in [AP86]. They showed that $F(u)$ has finite $n-1$ Hausdorff measure and established the regularity of the reduced part of the free boundary, following the roadmap we described above.

In the recent series of papers (see [DSS23] and the references therein) we studied the regularity properties of nonnegative minimizers of J and their free boundaries for potentials of negative powers

$$W(t) = t^\gamma \chi_{\{t>0\}}, \quad \gamma \in (-2,0).$$

The bound $\gamma > -2$ is necessary for the existence of functions with bounded energy.

The negative power potentials are natural in modeling sharper transitions of densities u between their zero set and positivity set. With respect to the classical Bernoulli model, $\gamma = 0$, in which the energy penalizes the measure of the set $\{u > 0\}$ uniformly, in the negative power setting this penalization is stronger when u is positive and small. This means that minimizers transition faster from the regions where $u \sim 1$ to their zero set and are not expected to be Lipschitz continuous near the free boundary. Despite this, we show that the free boundaries possess sufficiently nice regularity properties.

Theorem 3.2. *Let u be a nonnegative minimizer for J in B_1. Then*

$$\mathcal{H}^{n-1}(F(u) \cap B_{1/2}) \leq C(n,\gamma)$$

and $F(u)$ is locally $C^{1,\beta}$ except on a closed singular set $\Sigma_u \subset F(u)$ of Hausdorff dimension $n-k(\gamma)$, where $k(\gamma) \geq 3$ is the first dimension in which a nontrivial α-homogenous minimizer exists.

The change of sign in the exponent makes the problem more degenerate and the free boundary condition takes a different form. Some of the difficulties can be seen from a direct analysis of the one-dimensional case that we sketch below. Consider the ODE

$$u'' = \frac{\gamma}{2} u^{\gamma-1} \qquad \text{in} \quad (0,\delta) \subset \{u > 0\}, \qquad u(0) = 0.$$

The explicit homogenous solution

$$u_0 = c_\gamma t^\alpha, \quad \alpha := \frac{2}{2-\gamma},$$

for an appropriate constant c_γ, plays an important role in the analysis. In general, it follows that u has an expansion of the form

$$u(t) = a\,t + o(t^{1+\delta}), \qquad \text{if } \gamma \geq 0,$$
$$u(t) = c_\gamma t^\alpha + a\,t^{2-\alpha} + o(t^{2-\alpha+\delta}), \quad \text{if } \gamma < 0,$$

with $a \in \mathbb{R}$ a free parameter. If in addition u minimizes J in $[-\delta, \delta]$, then $a = 0$ in the case $\gamma > 0$ and $a = 1$ when $\gamma = 0$. These can be viewed as Neumann conditions at the free boundary point $t = 0$ and imply that the first nonzero term in the expansion of u near 0 is given precisely by the homogenous explicit solution $c_\gamma t^\alpha$, $\alpha \geq 1$.

On the other hand, when $\gamma < 0$, all solutions have $c_\gamma t^\alpha$ as the first nonzero term in the expansion. In this case the minimality condition imposes that the coefficient of the second order term in the expansion must vanish, i.e., $a = 0$. Since $\alpha < 1$, u' becomes infinite at 0, and this free boundary condition cannot be easily detected by integrations by parts or domain variations as in the case of nonnegative exponents.

In our work we analyze minimizers by introducing an appropriate notion of viscosity solutions, and use the method of calibrations in order to handle the singularity of the PDE and of the free boundary condition. Moreover, we also establish the "Gamma-convergence" of a suitable rescaled energy functional J_γ to the perimeter of the positivity set $Per_\Omega(\{u > 0\})$ as $\gamma \to 2$. This unveils yet another correspondence of the Bernoulli problem with minimal surfaces as the Bernoulli functional and the perimeter functional can be achieved as the endpoints of the same class of energies. With a more careful analysis (whose technicalities are beyond the purpose of this note) exploiting this connection we can conclude that if γ is close to 2 then we inherit the properties of minimal surfaces and find $k(\gamma) \geq 8$, and when γ is close to 0 then we inherit the properties of the Bernoulli minimizers and obtain $k(\gamma) \geq 5$. As a consequence we have the following full regularity result for $F(u)$.

Theorem 3.3. *Let $u \geq 0$ be a minimizer of J. Then $F(u)$ is of class $C^{1,\delta}$ if*

$$n \leq 7 \text{ and } \gamma \in (2-\eta, 2) \text{ or } n \leq 4 \text{ and } \gamma \in (0, \eta),$$

where $\delta, \eta > 0$ are small constants.

3.3. Concluding remarks. Many other problems are modeled by one-phase energies comparable to the Bernoulli one. Related problems appear for example in the study of cooperative systems of species, in optimization problems for spectral functions, in optimal partition problems, or in the study of harmonic functions with junctions. Several evolution problems are also connected to the Bernoulli problem, like the Stefan problem already mentioned in the introduction, or the Hele-Shaw problem used to describe an incompressible flow lying between two nearby horizontal plates. Moreover, the study of these type of problems has contributed to a better understanding of the question of regularity in a variety of other contexts, for example in fully nonlinear elliptic problems. This note serves as an introduction to the vast literature on such free boundary problems.

References

[AP12] Mark Allen and Arshak Petrosyan, *A two-phase problem with a lower-dimensional free boundary*, Interfaces Free Bound. **14** (2012), no. 3, 307–342, DOI 10.4171/IFB/283. MR2995409

[AC81] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144. MR618549

[ACF84] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461, DOI 10.2307/1999245. MR732100

[AP86] H. W. Alt and D. Phillips, *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math. **368** (1986), 63–107. MR850615

[CS05] Luis Caffarelli and Sandro Salsa, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2005, DOI 10.1090/gsm/068. MR2145284

[CJK04] Luis A. Caffarelli, David Jerison, and Carlos E. Kenig, *Global energy minimizers for free boundary problems and full regularity in three dimensions*, Noncompact problems at the intersection of geometry, analysis, and topology, Contemp. Math., vol. 350, Amer. Math. Soc., Providence, RI, 2004, pp. 83–97, DOI 10.1090/conm/350/06339. MR2082392

[CRS10] Luis A. Caffarelli, Jean-Michel Roquejoffre, and Yannick Sire, *Variational problems for free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1151–1179, DOI 10.4171/JEMS/226. MR2677613

[DPSV21] Guido De Philippis, Luca Spolaor, and Bozhidar Velichkov, *Regularity of the free boundary for the two-phase Bernoulli problem*, Invent. Math. **225** (2021), no. 2, 347–394, DOI 10.1007/s00222-021-01031-7. MR4285137

[DS11] D. De Silva, *Free boundary regularity for a problem with right hand side*, Interfaces Free Bound. **13** (2011), no. 2, 223–238, DOI 10.4171/IFB/255. MR2813524

[DSJ09] Daniela De Silva and David Jerison, *A singular energy minimizing free boundary*, J. Reine Angew. Math. **635** (2009), 1–21, DOI 10.1515/CRELLE.2009.074. MR2572253

[DSJS22] Daniela De Silva, David Jerison, and Henrik Shahgholian, *Inhomogeneous global minimizers to the one-phase free boundary problem*, Comm. Partial Differential Equations **47** (2022), no. 6, 1193–1216, DOI 10.1080/03605302.2022.2051187. MR4432955

[DSS15] D. De Silva and O. Savin, *C^∞ regularity of certain thin free boundaries*, Indiana Univ. Math. J. **64** (2015), no. 5, 1575–1608, DOI 10.1512/iumj.2015.64.5632. MR3418452

[DSS23] Daniela De Silva and Ovidiu Savin, *Compactness estimates for minimizers of the Alt-Phillips functional of negative exponents*, Adv. Nonlinear Stud. **23** (2023), no. 1, Paper No. 20220055, 19, DOI 10.1515/ans-2022-0055. MR4567389

[ESV20] Max Engelstein, Luca Spolaor, and Bozhidar Velichkov, *Uniqueness of the blowup at isolated singularities for the Alt-Caffarelli functional*, Duke Math. J. **169** (2020), no. 8, 1541–1601, DOI 10.1215/00127094-2019-0077. MR4101738

[FRRO] X. Fernández-Real and X. Ros-Oton, *Stable cones in the thin one-phase problem*, `arXiv:2010.01064`.

[GT83] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983, DOI 10.1007/978-3-642-61798-0. MR0737190

[Giu84] Enrico Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984, DOI 10.1007/978-1-4684-9486-0. MR775682

[Hon15] Guanghao Hong, *The singular homogeneous solutions to one phase free boundary problem*, Proc. Amer. Math. Soc. **143** (2015), no. 9, 4009–4015, DOI 10.1090/S0002-9939-2015-12553-1. MR3359589

[JS15] David Jerison and Ovidiu Savin, *Some remarks on stability of cones for the one-phase free boundary problem*, Geom. Funct. Anal. **25** (2015), no. 4, 1240–1257, DOI 10.1007/s00039-015-0335-6. MR3385632

[KN77] D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 2, 373–391. MR440187

[Tra14] Martin Traizet, *Classification of the solutions to an overdetermined elliptic problem in the plane*, Geom. Funct. Anal. **24** (2014), no. 2, 690–720, DOI 10.1007/s00039-014-0268-5. MR3192039

[Wei98] Georg S. Weiss, *Partial regularity for weak solutions of an elliptic free boundary problem*, Comm. Partial Differential Equations **23** (1998), no. 3-4, 439–455, DOI 10.1080/03605309808821352. MR1620644

Daniela De Silva

Credits

The opening image is courtesy of Svproduction via Getty.
Photo of Daniela De Silva is courtesy of Ioan Savin.

Some Arithmetic Properties of Complex Local Systems

Hélène Esnault

Le niveau uniforme du varech sur toutes les roches marquait la ligne de flottaison de la marée pleine et de la mer étale.

(The uniform level of kelp on all the rocks marked the waterline of full tide and slack [étale] sea.)

—Victor Hugo, *Les travailleurs de la mer*, 1866, p. 257

Hélène Esnault is Einstein Professor of Mathematics emerita at Freie Universität Berlin, Germany, a part-time professor at the Københavns Universitet, Denmark, and an associate professor at Harvard University, USA. Her email address is `esnault@math.fu-berlin.de`.

Communicated by Notices *Associate Editor Han-Bom Moon.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2830

1. Introduction

A group π is said to be *finitely generated* if it is spanned by finitely many letters, that is, if it is the quotient $F \to \pi$ of a free group F on finitely many letters. It is said to be *finitely presented* if the kernel of such a quotient is itself finitely generated. This does not depend on the choice of generation chosen. For example the trivial group $\pi = \{1\}$ is surely finitely presented as the quotient of the free group in 1 generator by itself (!). The following finitely presented group shall play a role in the note:

Example 1. The group Γ_0 is generated by two elements (a, b) with one relation $b^2 = a^2ba^{-2}$.

There are groups which are finitely generated but not finitely presented, see the interesting MathOverflow elementary discussion on the topic (`https://tinyurl.com/3cavr69a`).

The finitely presented groups appear naturally in many branches of mathematics. The fundamental group $\pi_1(M, m)$ of a topological space M based at a point m is defined to be the group of homotopy classes of loops centered at m. A group is finitely presented *if and only if* it is the fundamental group $\pi_1(M, m)$ *of a connected finite CW-complex* M based at a point m. This is essentially by definition of a CW (C =closure-finite, W =weak) complex which is a topological space defined by an increasing sequence of topological subspaces, each one obtained by gluing cells of growing dimension to the previous one. So the 1-cells glued to the 0-cell m yield the loops on which we take the free group F, and the relations come from the finitely many 2-cells glued to the loops.

If X is a smooth connected quasi-projective complex variety, its complex points $X(\mathbb{C})$ form a topological manifold which has the homotopy type of a connected finite CW-complex M.

The difference between X and its complex points $X(\mathbb{C})$ is subtle, and crucial for the note. If X is projective for example, when we say X we mean the set of defining homogeneous polynomials in finitely many variables with coefficients in $\mathbb{C}$. This collection of polynomials is called a *scheme*. On the other hand, only finitely many of those polynomials are necessary to describe them all (this is the Noetherian property of the ring of polynomials over a field), so in fact there is a ring R of finite type over $\mathbb{Z}$ which contains all the coefficients of those finitely many polynomials. We write $X_{\mathbb{C}}$ to remember $\mathbb{C}$, X_R to remember R. We can then take any maximal ideal $\mathfrak{m}$ in R. The residue field $R/\mathfrak{m}$ is finite, say $\mathbb{F}_q$, and has characteristic $p > 0$. Then we write $X_{\mathbb{F}_q}$ for the scheme defined by this collection of polynomials where the coefficients are taken modulo $\mathfrak{m}$. Fixing an algebraic closure $\mathbb{F}_q \subset \bar{\mathbb{F}}_p$, and thinking of the polynomials as having coefficients in $\bar{\mathbb{F}}_q$ we write $X_{\bar{\mathbb{F}}_p}$ etc.

When we say $X(\mathbb{C})$, we mean the complex solutions of the defining polynomials. (Of course there is the similar notion $X_R(R), X_{\mathbb{F}_q}(\mathbb{F}_q), X_{\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p) = X_{\mathbb{F}_q}(\bar{\mathbb{F}}_p)$, etc.)

The notion of a quasi-projective complex variety X is easily understood on its complex points $X(\mathbb{C})$. They have to be of the shape $\bar{X}(\mathbb{C}) \setminus Y(\mathbb{C})$ where both $\bar{X}$ and $Y (\subset \bar{X})$ are projective varieties.

We do not know how to characterize the fundamental groups $\pi_1(X(\mathbb{C}), x)$, where $x \in X(\mathbb{C})$, among all possible $\pi_1(M, m)$. In this small text, we use the following terminology:

Definition 1. A finitely presented group π is said to come from geometry if it is isomorphic to $\pi_1(X(\mathbb{C}), x)$ where X is a smooth connected quasi-projective complex variety and $x \in X(\mathbb{C})$.

The aim of this note is to describe a few obstructions for a finitely presented group to come from geometry.

2. Classical Obstructions: Topology and Hodge Theory

A classical example comes from the uniformization theory of complex curves: any *free* group on n letters, where n is a natural number, is the fundamental group of the complement of $(n+1)$-points on the Riemann sphere $\mathbb{P}^1$. This is because we understand exactly $\pi_1(X(\mathbb{C}), x)$ if X has dimension 1, that is if $X(\mathbb{C})$ is a Riemann surface. The simplest possible example is the Riemann sphere $X = \mathbb{P}^1$. Then $\pi_1(X(\mathbb{C}), x) = \{1\}$ as any loop centered at x can be retracted to a point, see Figure 1.

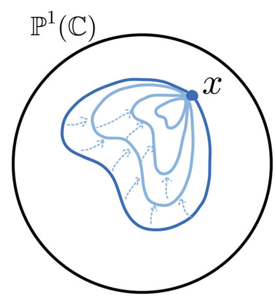


Figure 1. Any loop is retracted on $\mathbb{P}^1(\mathbb{C})$.

The same holds true on $X = \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. The first interesting example is $X = \mathbb{P}^1 \setminus \{0, \infty\}$. Then $X(\mathbb{C}) = \mathbb{C} \setminus \{0\}$, and $\pi_1(X(\mathbb{C}), 1) = \mathbb{Z} \cdot \gamma$ where $\gamma : [0\ 1] \to \mathbb{C} \setminus \{0\}$, $t \mapsto \exp(2\pi\sqrt{-1}t)$ is the circle turning around the origin $\{0\}$, see Figure 2.

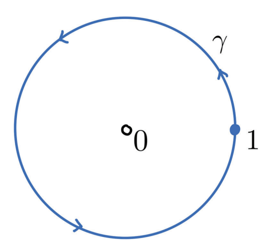


Figure 2. A nontrivial loop γ on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$.

More generally, if a smooth compactification $\bar{X}$ of X has genus g, topologically $\bar{X}(\mathbb{C})$ is a donut with g holes. Then $\pi_1(\bar{X}(\mathbb{C}), x)$ is spanned by $2g$ elements $(a_i, b_i), i = 1, \dots, g$ with one relation $\prod_{i=1}^{g} [a_i, b_i] = 1$. If $(\bar{X} \setminus X)(\mathbb{C})$ consists of $(n+1)$ points, $\pi_1(X(\mathbb{C}), x)$ is spanned by $2g+n+1$ elements $(a_i, b_i), i = 1, \dots, g,\ c_j, j = 1, \dots, n+1$ with one relation $\prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n+1} c_j = 1$. The literature is full of beautiful colored pictures visualizing this classical computation.

Beyond Riemann surfaces, that is, for X of dimension ≥ 2, our understanding is very limited.

The 2 in the $2g$ in the previous example is more general: by the fundamental structure theorem on finitely generated $\mathbb{Z}$-modules, the maximal abelian quotient $\pi_1(X(\mathbb{C}), x)^{\mathrm{ab}}$, that is, the abelianization of $\pi_1(X(\mathbb{C}), x)$, is isomorphic to a direct sum of $\mathbb{Z}^{\oplus b}$ for some natural number b and of a finite abelian group T.

Any abelian finitely presented group $\mathbb{Z}^b \oplus T$ comes from geometry: Serre's classical construction [Ser58] realizes any finite group as the fundamental group of the quotient Z of a complete intersection of large degree in the projective space of large dimension, while the fundamental group of $(\mathbb{P}^1 \setminus \{0, \infty\})(\mathbb{C})$ is equal to $\mathbb{Z}$, see Figure 2. As the fundamental group of a product variety is the product of the fundamental groups of the factors (Künneth formula), we can take $X = (\mathbb{P}^1 \setminus \{0, \infty\})^b \times Z$ and there is *no* obstruction for $\mathbb{Z}^b \times T$ to be the abelianization of the fundamental group of a smooth connected quasi-projective complex variety. If we require X to be projective, then *Hodge theory*, more precisely, *Hodge duality* implies that b is *even*. This is the only obstruction as we can then take $X = E^{\frac{b}{2}} \times Z$ instead, where E is any elliptic curve, so $E(\mathbb{C})$ is a donut with one hole, so $\pi_1(X(\mathbb{C}), x) = \mathbb{Z}^2$, see Figure 3.

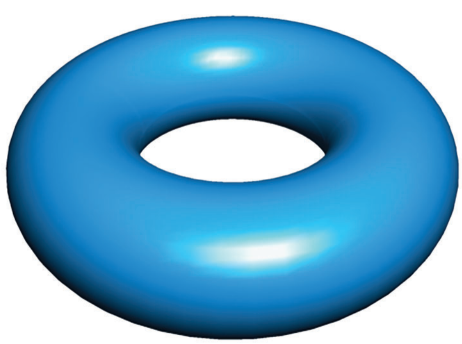


Figure 3. Riemann surface of genus $g = 1$.

In the same vein, but *much deeper* is the fact that the pronilpotent completion of $\pi_1(X(\mathbb{C}), x)$ (also called Malčev completion) is endowed with a *mixed Hodge structure*. While so far we commented the *topological* structure of $X(\mathbb{C})$, Hodge theory studies in addition the *analysis* stemming from the *complex structure*, and the more refined properties, packaged in the notion of *Kähler geometry and harmonic theory*, which come from the property that X is defined algebraically by complex polynomials. A modern way (due to Beilinson) to think of it is to identify the Malčev completion with the cohomology of an (infinite) simplicial complex scheme and to apply the classical Hodge theory on its truncations. We do not elaborate further.

3. Profinite Completion: The Étale Fundamental Group

Thus the difficulty lies in the kernel of the group to its abelianization. To study it, we first introduce the classical notion:

Definition 2. A *complex local system* $\mathbb{L}_\rho$ is a complex linear representation

$$\rho : \pi_1(X(\mathbb{C}), x) \to \mathrm{GL}_r(\mathbb{C}),$$

considered *modulo conjugacy* by $\mathrm{GL}_r(\mathbb{C})$. The local system $\mathbb{L}_\rho$ is said to be irreducible if its underlying representation ρ (thus defined modulo conjugacy) is irreducible.

Why modulo conjugacy? A path γ_{xy} from x to y defines an isomorphism $\gamma_{yx}^{-1}\pi_1(X(\mathbb{C}), y)\gamma_{yx} = \pi_1(X(\mathbb{C}), x)$. This isomorphism is not unique, any other path from x to y differs from this one by left multiplication by a loop $\gamma_x \in \pi_1(X(\mathbb{C}), x)$ centered at x, which thus conjugates the isomorphism by γ_x. Thus *not fixing the base point* forces us to consider representations modulo conjugacy.

As $\pi_1(X(\mathbb{C}), x)$ is finitely presented, thus in particular finitely generated, ρ factors through $\pi_1(X(\mathbb{C}), x) \xrightarrow{\rho_A} \mathrm{GL}_r(A)$ where $A \subset \mathbb{C}$ is a ring of finite type. Any such A can be embedded into the ring of ℓ-adic integers $\mathbb{Z}_\ell$ for some prime number ℓ, say $\iota : A \subset \mathbb{Z}_\ell$. (For example if $A = \mathbb{Z}$, ι has to be the natural pro-ℓ-completion for any choice of ℓ. If $A = \mathbb{Z}[T]$ we take in $\mathbb{Z}_\ell$ a transcendental element over $\mathbb{Q}$ and send T to it, etc. The main point is that the field of fractions $\mathbb{Q}_\ell$ of $\mathbb{Z}_\ell$, that is the field of ℓ-adic numbers, has infinite transcendence degree over $\mathbb{Q}$). Thus the datum of ρ is equivalent to the one of $\iota \circ \rho_A$ whose range $\mathrm{GL}_r(\mathbb{Z}_\ell)$ is profinite. In particular, $\iota \circ \rho_A$ factors through the profinite completion

$$\mathfrak{prof} : \pi_1(X(\mathbb{C}), x) \to \pi_1(X(\mathbb{C}), x)\hat{}$$

and induces

$$\hat{\rho} : \pi_1(X(\mathbb{C}), x)\hat{} \to \mathrm{GL}_r(\mathbb{Z}_\ell),$$

a representation which is *continuous* for the profinite topology on both sides. Recall that the profinite completion $\mathfrak{prof} : \pi \to \hat{\pi}$ of an abstract group π is the projective limit over all finite quotients $\pi \to H$. It inherits the profinite topology compatible with the group structure for which a basis of open neighborhoods of 1 is defined to be the inverse image of $1 \in H$ by one of those projections.

However, Toledo in [Tol93] constructed a smooth connected complex projective variety X with the property that $\mathfrak{prof}$ is *not* injective. It answered a problem posed by Serre. It is an important fact which in particular implies that *the study of complex local systems ignores* $\mathrm{Ker}(\mathfrak{prof})$. This leads us in two different directions.

The invariants b and T of the abelianization are seen on the complex abelian algebraic group

$$\mathrm{Hom}(\pi_1(X(\mathbb{C}),x),\mathrm{GL}_1(\mathbb{C}))\cong(\mathbb{C}^\times)^b\times\mathrm{Hom}(T,\mathbb{C}^\times).$$

Here the notation $\mathbb{C}^\times$ means the set $\mathbb{C}\setminus\{0\}$ endowed with the (abelian) multiplicative group structure.

More generally the finite generation of $\pi_1(X(\mathbb{C}),x)$ enables one to define a "moduli" (parameter) space $M_B^{\mathrm{irr}}(X,r)$ of all its irreducible local systems $\mathbb{L}_\rho$ in a given rank r. It is called the *Betti moduli space* of X of irreducible local systems in rank r or the *character variety* of $\pi_1(X(\mathbb{C}),x)$ of irreducible local systems in rank r. It is a complex quasi-projective scheme of finite type. Its study is the content of Simpson's *non-abelian Hodge theory* developed in [Sim92]. It is an *analytical theory* relying on harmonic theory, as is classical Hodge theory.

The second direction relies on the profinite completion homomorphism $\mathfrak{prof}$. By the Riemann existence theorem, a finite topological covering is the complexification of a finite étale cover. Thus $\pi_1(X(\mathbb{C}),x)\hat{}$ is identified with the *étale fundamental group* $\pi_1(X_\mathbb{C},x)$ of the *scheme* $X_\mathbb{C}$ defined over $\mathbb{C}$, based at the complex point x, as defined by Grothendieck in [Gro71]:

This profinite group is defined by its representations in finite sets. A representation of $\pi_1(X_\mathbb{C},x)$ in finite sets is "the same" (in the categorial sense) as a pointed (above x) finite étale cover of X.

We denote by

$$\rho_{\mathbb{C},\ell}\,:\,\pi_1(X_\mathbb{C},x)\xrightarrow{\hat{\rho}}\mathrm{GL}_r(\mathbb{Z}_\ell)\hookrightarrow\mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$$

the composite morphism. Here $\bar{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$.

The notion of a complex local system (Definition 2) generalizes naturally:

Definition 3. An *ℓ-adic local system* $\mathbb{L}_{\rho_\ell}$ on the variety $X_\mathbb{C}$ is a continuous linear representation

$$\rho_\ell\,:\,\pi_1(X_\mathbb{C},x)\to\mathrm{GL}_r(\bar{\mathbb{Q}}_\ell),$$

considered *modulo conjugacy* by $\mathrm{GL}_r(\bar{\mathbb{Q}}_\ell)$. The local system $\mathbb{L}_{\rho_\ell}$ is said to be irreducible if its underlying representation ρ_ℓ (thus defined modulo conjugacy) is irreducible.

As the kernel of the projection $\mathrm{GL}_r(\mathbb{Z}_\ell)\to\mathrm{GL}_r(\mathbb{F}_\ell)$ is a pro-ℓ-group (that is all its finite quotients H have order of power of ℓ), Grothendieck's specialization theory in *loc. cit.* implies that the specialization homomorphism

$$sp_{\mathbb{C},\bar{\mathbb{F}}_p}\,:\,\pi_1(X_\mathbb{C},x)\to\pi_1^t(X_{\bar{\mathbb{F}}_p},x)$$

induces an isomorphism on the image of $\rho_{\mathbb{C},\ell}$ for p larger than the order of $\mathrm{GL}_r(\mathbb{F}_\ell)$. Here $X_{\bar{\mathbb{F}}_p}$ is a reduction of $X_\mathbb{C}$ as explained in the introduction, and is *good*, that is smooth, as well as the stratification of the boundary divisor if X is not projective. The upper script t refers to the tame quotient of $\pi_1(X_{\bar{\mathbb{F}}_p},x)$ in case X was not projective. We do *not* detail with precision the tameness concept, for which we refer to [KS10]. This roughly works as follows. Representations in finite sets of the étale fundamental group $\pi_1(X_{\bar{\mathbb{F}}_p},x)$ which factor through the tame quotient $\pi_1^t(X_{\bar{\mathbb{F}}_p},x)$ have base change properties "as if" $X_{\bar{\mathbb{F}}_p}$ were proper. We can contract the fundamental group of X over a p-adic ring R with residue field $\bar{\mathbb{F}}_p$ to the one over $\bar{\mathbb{F}}_p$ in the way we do topologically in order to identify the topological fundamental group of a tubular neighborhood of a compact manifold to the one of the compact manifold. The natural identification of $\pi_1(X_\mathbb{C},x)$ with $\pi_1(X_K,x)$ where K is an algebraic closure of the field of fractions of R (this is called base change property) enables us to define $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$. Grothendieck computes that $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ induces an isomorphism on all finite quotients of $\pi_1(X_\mathbb{C},x)$ and $\pi_1^t(X_{\bar{\mathbb{F}}_p},x)$ of order prime to p.

The factorization defines the irreducible ℓ-adic local system $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$ on $X_{\bar{\mathbb{F}}_p}$ from which $\mathbb{L}_{\mathbb{C},\ell}$ comes. This leads us to study $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$ in order to derive *arithmetic properties* of the initial $\mathbb{L}_\rho$. We can remark that again we know extremely little on the kernel of $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ and that the study of complex local systems ignores them as well, for a chosen $\iota\,:\,A\to\mathbb{Z}_\ell$ and p large as before.

On the other hand, $X_\mathbb{C}$ is defined over a field of finite type over $\mathbb{Q}$, thus with a huge Galois group, and $X_{\bar{\mathbb{F}}_p}$ is defined over a finite field $\mathbb{F}_q$ of characteristic $p>0$, with a very small Galois group isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, topologically spanned by the Frobenius φ of $\mathbb{F}_q$. Nonetheless, we shall see that this small Galois group yields nontrivial information.

Our goal now is twofold. First we shall illustrate how to go back and forth between the Hodge theory side and the arithmetic side on a *particular example*. This by far does not cover the whole deepness of the theory, but we hope that it gives some taste on how it functions. Then we shall mention on the way and at the end more general theorems to the effect that deep arithmetic properties stemming from the Langlands program, notably the "integrality" illustrated on this particular example, enable one to find a new obstruction for the finitely presented group to come from geometry.

4. An Example to Study

Let X be a smooth connected quasi-projective complex variety. If X is not projective, we fix a smooth projective compactification $X\hookrightarrow\bar{X}$ so that the divisor at infinity $D=\bar{X}\setminus X=\cup_{i=1}^M D_i$ is a strict normal crossings divisor (so its irreducible components D_i are smooth and meet transversally). For each i we fix r roots of unity $\mu_{ij},\,j=1,\dots,r$, possibly with multiplicity. They uniquely determine a conjugacy class T_i of a semi-simple matrix of finite order.

The normal subgroup spanned by the conjugacy classes of small loops γ_i around the components D_i is identified with the kernel of the surjection $\pi_1(X(\mathbb{C}), x) \to \pi_1(\bar{X}(\mathbb{C}), x)$. We fix an extra natural number $\delta > 0$.

We make the following assumption

Assumption $(\star)_r$: For a given rank $r \geq 2$, there are at most finitely many irreducible rank r complex local systems $\mathbb{L}_\rho$ on X such that the determinant of $\mathbb{L}_\rho$ has order dividing δ, and, if X is not projective, such that the semisimplification of $\rho(\gamma_i)$ falls in T_i.

It is simple to describe $(\star)_r$: in the Betti moduli space $M_B^{irr}(X, r)$ we have the subscheme $M_B^{irr}(X, r, \delta, T_i)$ defined by the conditions $\{\delta, T_i\}$. The condition $(\star)_r$ means precisely that $M_B^{irr}(X, r, \delta, T_i)$ is 0-dimensional. Equivalently $M_B^{irr}(X, r, \delta, T_i)(\mathbb{C})$ consists of finitely many points, or is empty.

Note the condition on δ depends only on $\pi_1(X(\mathbb{C}), x)$ so could be expressed on the character variety, not however the condition on T_i. For this we have to know which γ_i in $\pi_1(X(\mathbb{C}), x)$ come from the boundary divisor, so we need the geometry.

If $r = 1$, we drop the condition on the determinant, and assume for simplicity that X is projective. So the assumption becomes that there are finitely many irreducible rank 1 complex local systems $\mathbb{L}_\rho$ on X. This then forces b to be 0, so $\pi_1(X(\mathbb{C}), x)^{\mathrm{ab}}$ to be finite.

Consequently, those finitely many $\mathbb{L}_\rho$ of rank 1 have finite monodromy (i.e., $\rho(\pi_1(X(\mathbb{C}), x)^{\mathrm{ab}})$ is finite). This implies that the $\mathbb{L}_\rho$ *come from geometry*, that is there is a smooth projective morphism $g : Y \to U \subset X$ where U is a Zariski dense open in X (in our case $U = X$), such that $\mathbb{L}_\rho$ restricted to U is a subquotient of the local system $R^i g_* \mathbb{C}$ coming from the representation of $\pi_1(U(\mathbb{C}), x)$ in $GL(H^i(g^{-1}(x), \mathbb{C}))$ for some i (in our case g is finite étale and $i = 0$).

A different way of thinking of finiteness is using Kronecker's analytic criterion [Esn23]: the set of the rank 1 local systems is invariant under the action of the automorphisms of $\mathbb{C}$ acting on $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. Finiteness of the monodromy is then equivalent to the monodromy being *unitary* (i.e., lying in $S^1 \subset \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$) and being *integral* (i.e., lying in $\mathrm{GL}_1(\bar{\mathbb{Z}}) \subset \mathrm{GL}_1(\mathbb{C})$). We now discuss the generalization of these two properties: *unitarity and integrality*.

We first observe that $(\star)_r$ implies that the irreducible rank r complex local systems are *rigid* if we preserve the $\{\delta, T_i\}$ conditions. As the terminology says, it means that we can not "deform" nontrivially the local system $\mathbb{L}_\rho$. Precisely it says that a formal deformation

$$\rho_t : \pi_1(X(\mathbb{C}), x) \to \mathrm{GL}_r(\mathbb{C}[[t]])$$

of $\rho = \rho_{t=0}$ with the same $\{\delta, \mu_{ij}\}$ conditions does not move $\mathbb{L}_\rho$, that is there is a $g \in \mathrm{GL}_r(\mathbb{C}((t)))$ such that in $\mathrm{GL}_r(\mathbb{C}((t)))$ the relation

$$\rho_t = g \rho_{t=0} g^{-1}$$

holds.

A classical example where $(\star)_r$ is fulfilled is provided by Shimura varieties of real rank ≥ 2. Margulis super-rigidity [Mar91] implies that all complex local systems are semisimple and all irreducible ones are rigid. *While by super-rigidity they all are integral* (i.e., the image of the representations lie in $\mathrm{GL}_r(\bar{\mathbb{Z}})$ up to conjugacy), *we do not know whether they come from geometry.*

Another example is provided by connected smooth projective complex varieties X with the property that all symmetric differential forms, except the functions, are trivial. In this case, non-abelian Hodge theory implies $(\star)_r$ is fulfilled. Indeed, the Betti moduli space of semisimple rank r complex local systems is affine, while the moduli space of semistable Higgs bundles with vanishing Chern classes (which we discuss below) admits a projective morphism to the so-called Hitchin base. The latter consists of one point under our assumption. As by a deep theorem of Simpson [Sim92], both spaces are real analytically isomorphic, they are both affine and compact, thus are 0-dimensional. It is proven in [BKT13], using Hodge theory, the period domain and birational geometry, that all the $\mathbb{L}_\rho$ have then finite monodromy. This yields a positive answer to a conjecture I had formulated. As the proof uses Hodge theory, it is analytic. As of today, there is no arithmetic proof of the theorem.

5. Non-abelian Hodge Theory

We first assume that X is projective. We discuss a little more the notion of Higgs bundles mentioned above. Simpson in [Sim92] constructs the moduli space $M_{Dol}^{\mathrm{s}}(X, r, \delta)$ of stable Higgs bundles (V, θ) with vanishing Chern classes, where V is a vector bundle of rank r, $\theta : V \to \Omega_X^1 \otimes V$ is a $\mathcal{O}_X$-linear operator fulfilling the integrality condition $\theta \wedge \theta = 0$, such that $\det(V, \theta)$ has finite order dividing δ. (The integrality notion here is for the Higgs field θ, and is not related to the integrality of a linear representation mentioned in Section 4). The stability condition is defined on the pairs (V, θ), that is one tests it on Higgs subbundles. The finite order of $\det(V, \theta)$ implies that the underlying Higgs field of $\det(V, \theta)$ is equal to 0, so $\det(V, \theta) = (\det(V), 0)$. The moduli space $M_{Dol}^{\mathrm{s}}(X, r, \delta)$ is a complex scheme of finite type. It has several features.

There is a real analytic isomorphism $M_B^{\mathrm{irr}}(X, r, \delta) \xrightarrow{\cong} M_{Dol}^{\mathrm{s}}(X, r, \delta)$. So $(\star)_r$ implies that $M_{Dol}^{\mathrm{s}}(X, r, \delta)$ consists of finitely many points.

Simpson defines on Higgs bundles the algebraic $\mathbb{C}^\times$-action which assigns $(V, t\theta)$ to (V, θ) for $t \in \mathbb{C}^\times$. It preserves stability near $1 \in \mathbb{C}^\times$ and semistability in general. Thus under the assumption $(\star)_r$, the $\mathbb{C}^\times$ -action stabilizes

$M^{s}_{Dol}(X, r, \delta)$ pointwise. Simpson proves in *loc. cit.* that $\mathbb{C}^{\times}$-fixed points correspond to polarized complex variations of Hodge structure (PCVHS). Mochizuki in [Moc06] generalized this part of Simpson's theory to the smooth quasi-projective case so the conclusion remains valid in general.

We summarize this section: *The assumption $(\star)_r$ implies that the irreducible $\mathbb{L}_\rho$ of rank r, with determinant of order diving δ and semisimplification of $\rho(\gamma_i)$ falling in T_i, underlie a PCVHS. This property is the analog of the unitary property in rank $r = 1$.*

We cannot expect more as already on Shimura varieties of real rank ≥ 2, not all local systems are unitary. If they all were, as they are integral, they would have finite monodromy. This is not the case.

6. Arithmeticity

Again we fix r. Once we obtain the finitely many local systems $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$ on $X_{\bar{\mathbb{F}}_p}$ by specialization as in Section 3, also taking p large enough so it is prime to the orders of δ (and the μ_{ij} in case X is not projective), we consider Grothendieck's homotopy exact sequence

$$1 \to \pi_1(X_{\bar{\mathbb{F}}_p}, x) \to \pi_1(X_{\mathbb{F}_q}, x) \to \hat{\mathbb{Z}} \cdot \varphi \to 1$$

[Gro71]. Here the finite field $\mathbb{F}_q \subset \bar{\mathbb{F}}_p$ is chosen so $X_{\bar{\mathbb{F}}_p}$ is defined over $\mathbb{F}_q$ and φ is the Frobenius endomorphism of $\bar{\mathbb{F}}_p$ sending λ to λ^q.

Let us first discuss the meaning of the sequence in terms of finite étale covers. The surjectivity on the right says that if $\mathbb{F}_q \subset \mathbb{F}_{q'}$ is a finite field extension, then the induced finite étale cover $X_{\mathbb{F}_{q'}} \to X_{\mathbb{F}_q}$ has no section. The injectivity on the left says that any finite étale cover of $X_{\bar{\mathbb{F}}_p}$ can be dominating by one induced by a finite étale cover of $X_{\mathbb{F}_q}$. The exactness in the middle says that if a finite étale cover of $X_{\mathbb{F}_q}$ acquires a section on $X_{\bar{\mathbb{F}}_p}$, then the induced cover of $X_{\bar{\mathbb{F}}_p}$ is completely split.

The kernel of $\pi_1(X_{\bar{\mathbb{F}}_p}, x)$ to its tame quotient $\pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$ is normal in $\pi_1(X_{\mathbb{F}_q}, x)$. Thus a lift of φ to $\pi_1(X_{\mathbb{F}_q}, x)$, which is well defined up to conjugation by $\pi_1(X_{\bar{\mathbb{F}}_p}, x)$, acts by conjugation on $\pi_1^t(X_{\bar{\mathbb{F}}_p}, x)$, therefore on ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$ and respects tameness.

This action preserves r, irreducibility, δ and the T_i. Thus φ acts as a bijection on the finite set $\{\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}\}$. We conclude that replacing q by some nontrivial finite power q^t, all $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$ descend to ℓ-adic local systems $\mathbb{L}_{\mathbb{F}_{q^t},\ell}$ on $X_{\mathbb{F}_{q^t}}$. We say that the $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$ are *arithmetic*. (This argument is adapted from [EG18]).

We summarize this section: *The assumption $(\star)_r$ implies that the local systems $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$ are arithmetic.*

More generally, without the assumption $(\star)_r$ being fulfilled, Simpson proves in [Sim92] in all generality that the $\mathbb{L}_{\mathbb{C},\ell}$ *coming from irreducible rigid local systems are arithmetic*, that is they descend to ℓ-adic local systems on X_F where F is a field of finite type over $\mathbb{Q}$.

7. ℓ-adic Companions and Integrality

Quoted from [Esn23], with adapted notation:

"Given a field automorphism τ of $\mathbb{C}$, we can postcompose the underlying monodromy representation of a complex local system $\mathbb{L}_\rho$ by τ to define a *conjugate* complex local system $\mathbb{L}_\rho^\tau$. Given a field automorphism σ of $\bar{\mathbb{Q}}_\ell$, which then can only be continuous if it is the identity on $\mathbb{Q}_\ell$, or more generally given a field isomorphism σ between $\bar{\mathbb{Q}}_\ell$ and $\bar{\mathbb{Q}}_{\ell'}$ for some prime number ℓ', the postcomposition of a *continuous* nonfinite monodromy representation is no longer continuous (unless $\ell = \ell'$ and σ is the identity on $\mathbb{Q}_\ell$), so *we cannot define a conjugate* $\mathbb{L}^\sigma_{\mathbb{C},\ell}$ of an ℓ-adic local system by this simple postcomposition procedure."

However, when $X_{\mathbb{C}}$ is replaced by $X_{\bar{\mathbb{F}}_p}$, Deligne conjectured in Weil II [Del80] that we can. Let us first state the conjecture.

By the Čebotarev density theorem, an *irreducible* ℓ-adic sheaf $\mathbb{L}_\ell$ defined by an irreducible continuous representation $\rho_\ell : \pi_1(X_{\mathbb{F}_q}, x) \to GL_r(\bar{\mathbb{Q}}_\ell)$ considered modulo conjugacy by $GL_r(\bar{\mathbb{Q}}_\ell)$, is determined *uniquely* by the characteristic polynomials

$$P(\mathbb{L}_\rho, y, T) = \det(T - \rho_\ell(Frob_y))$$

for all closed point y of $X_{\mathbb{F}_q}$, where $Frob_y$ is the arithmetic Frobenius at y. This expression just means that the closed point y has a residue field $\kappa(y) \subset \bar{\mathbb{F}}_p$ which is a finite extension of $\mathbb{F}_q$, say of degree m_y. Then y is a rational point of $X_{\kappa(y)}$, thus the conjugacy class of φ^{m_y} is well defined as a subgroup in $\pi_1(X_{\mathbb{F}_q}, x)$.

The first part of the conjecture predicts that if the determinant of ρ has finite order, then

$$P(\mathbb{L}_\ell, y, T) \in \bar{\mathbb{Q}}[T] \subset \bar{\mathbb{Q}}_\ell[T].$$

In particular, its σ-conjugate

$$P(y, T)^\sigma \in \bar{\mathbb{Q}}[T] \subset \bar{\mathbb{Q}}_{\ell'}[T]$$

is defined.

The second part of the conjecture predicts the existence of an irreducible ℓ'-adic local system $\mathbb{L}_\ell^\sigma$ with the property that

$$P(\mathbb{L}_\ell^\sigma, y, T) = P(\mathbb{L}, y, T)^\sigma.$$

Again Čebotarev density theorem implies unicity up to isomorphism once we know the existence.

The two parts have been proven on smooth curves $X_{\mathbb{F}_q}$ by Drinfeld in rank $r = 2$ [Dri80] and L. Lafforgue in any rank [Laf02] as a corollary of the Langlands conjecture over functions fields. It thus uses automorphic forms. Drinfeld's Shtukas also imply that all $\mathbb{L}_\ell$ on $X_{\mathbb{F}_q}$ come from geometry. The existence of companions has been extended to smooth quasi-projective $X_{\mathbb{F}_q}$ of any dimension by Drinfeld by arithmetic-geometric methods [Dri12], reducing to curves. The reduction method goes back initially to Götz Wiesend.

Deligne's initial conjecture is for normal varieties. To go from smooth to normal varieties is still an open problem.

So coming back to $(\star)_r$, we see that under this assumption, and abusing notation replacing q^t by q, it holds that for any $\ell' \neq \ell, \ell' \neq p$, we have as many $\mathbb{L}^{\sigma}_{\mathbb{F}_q,\ell|X_{\bar{\mathbb{F}}_p}}$ as $\mathbb{L}_{\bar{\mathbb{F}}_p,\ell}$. (The companion formation preserves the irreducibility on $X_{\bar{\mathbb{F}}_p}$, the T_i, and δ as we had taken p prime to all those orders, so in particular it also preserves the tameness).

We now use this fact to prove that *the assumption $(\star)_r$ implies that all irreducible rank r local systems with the conditions (δ, μ_{ij}) are integral.*

The initial irreducible complex $\mathbb{L}_\rho$ being rigid, they are defined over $\mathcal{O}_K[N^{-1}]$ where K is a number field and N is a positive natural number. Precomposing the $\mathbb{L}^{\sigma}_{\mathbb{F}_q,\ell|X_{\bar{\mathbb{F}}_p}}$ with $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ and then by $\mathfrak{prof}$ yields as many irreducible local systems as the $\mathbb{L}_\rho$ with the conditions given by δ and μ_{ij}. They all are integral at the place above ℓ' determined by

$$K \subset \bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_\ell \xrightarrow{\sigma} \bar{\mathbb{Q}}_{\ell'}.$$

This construction is performed for all $\ell' \neq \ell$, $\ell' \neq p$, all isomorphisms σ between $\bar{\mathbb{Q}}_\ell$ and $\bar{\mathbb{Q}}_{\ell'}$, with p large enough (larger than N, δ, the order of the μ_{ij}, and such that $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ is well defined and surjective). This finishes the proof of the integrality under the assumption $(\star)_r$.

This proof is taken from [EG18]. It is shown in *loc. cit.* that the argument applies for cohomologically rigid local systems (a notion we do not detail here) without the assumption $(\star)_r$. The assumption $(\star)_r$ does not imply cohomological rigidity, and cohomological rigidity does not imply $(\star)_r$.

8. The $(\star)_r$ Condition for an Abstract Finitely Presented Group

In [dJE23] we report on the example 1 constructed by Becker–Breuillard–Varljú. We quote from *loc. cit.* adapting the notation:

For $r = 2$ and $\delta = 1$, that is for SL_2 representations, the authors compute that Γ_0 has exactly two irreducible complex representations modulo conjugacy. The first one $\mathbb{L}_1$ is defined by

$$\rho_1(a) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \ \rho_1(b) = \begin{pmatrix} j & 0 \\ 0 & j^2 \end{pmatrix},$$

where j is a primitive 3-rd root of unity. It is defined over $\mathbb{Q}(j)$. The local system $\mathbb{L}_2$ is Galois conjugate to $\mathbb{L}_1$. The authors compute

$$\rho_1(ab) = \frac{j}{\sqrt{2}}\begin{pmatrix} 1 & j \\ -1 & j \end{pmatrix}.$$

As $\mathrm{Trace}(\rho_1(ab)) = -\frac{1}{\sqrt{2}}$, $\mathbb{L}_1$ is not integral at $\ell = 2$, so $\mathbb{L}_2$ is not integral at $\ell = 2$ either. Furthermore, $\rho_1(a)$ does not preserve the eigenvectors of $\rho_1(b)$, so $\mathbb{L}_1$ and thus $\mathbb{L}_2$ are irreducible with dense monodromy in $\mathrm{SL}_2(\mathbb{C})$. They also compute that those representations are cohomologically rigid."

So we see that Γ_0 can not be isomorphic to $\pi_1(X(\mathbb{C}), x)$ for a connected smooth *projective* complex variety X. We conclude that *the integrality property in Section 7 is an obstruction for a finitely presented group to come from projective geometry.*

Jakob Stix remarks that Γ_0^{ab} is isomorphic to $\mathbb{Z}$, which has rank 1, so Γ_0 obeys the Hodge theoretic obstruction mentioned in Section 2 as well.

The rest of the note is devoted to indicating how to extend the obstruction based on integrality to *all* connected quasi-projective varieties. This is the content of [dJE23].

9. de Jong's Conjecture

If $X_{\bar{\mathbb{F}}_p}$ is a connected normal quasi-projective variety, and $\ell \neq p$ is a prime number, de Jong conjectured in [dJ01] that an irreducible representation

$$\pi_1(X_{\bar{\mathbb{F}}_p}, x) \to \mathrm{GL}_r(\overline{\mathbb{F}_\ell((t))})$$

which is arithmetic is in fact constant in t, thus in particular has finite monodromy. Here $\mathbb{F}_\ell((t))$ is the Laurent power series field over the finite field $\mathbb{F}_\ell$ and $\overline{\mathbb{F}_\ell((t))}$ is an algebraic closure.

He shows that *assuming the conjecture, irreducible representations $\pi_1(X_{\bar{\mathbb{F}}_p}, x) \to \mathrm{GL}_r(\bar{\mathbb{F}}_\ell)$ always lift to arithmetic ℓ-adic local systems if $X_{\bar{\mathbb{F}}_p}$ is a smooth connected curve.*

Drinfeld in [Dri01] applied this argument to produce over a connected normal complex quasi-projective variety $X_{\mathbb{C}}$ ℓ-adic local systems with the property that via $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ for p large they are arithmetic over $X_{\bar{\mathbb{F}}_p}$.

de Jong's conjecture has been proved by Böckle–Khare in specific cases and Gaitsgory [Gai07] in general for $\ell \geq 3$. The latter proof uses the geometric Langlands program.

10. Weak Integrality for Groups

Let Γ be a finitely presented group, together with natural numbers $r \geq 1, \delta \geq 1$. We define in [dJE23] the following notion: Γ has the *weak integrality property with respect to* (r, δ) if, assuming there is an irreducible representation $\rho : \Gamma \to \mathrm{GL}_r(\mathbb{C})$ with determinant of order δ, then for any prime number ℓ, there is a representation $\rho_\ell : \Gamma \to \mathrm{GL}_r(\bar{\mathbb{Z}}_\ell)$ which is irreducible over $\bar{\mathbb{Q}}_\ell$ and of determinant of order δ.

The main theorem of *loc. cit.* is that *if X is a connected smooth quasi-projective complex variety, then $\Gamma = \pi_1(X(\mathbb{C}), x)$ does have the weak integrality property with respect to any (r, δ).*

Using now the example by Becker–Breuillard–Varjú presented in Section 8, we see that their Γ_0 *does not come from geometry* at all, whether the desired X is assumed to be projective or quasi-projective.

So we conclude that *the weak integrality property for Γ with respect to all (r, δ) is an obstruction for Γ to come from geometry.* This new kind of obstruction does not rest on analytic methods, but on arithmetic properties, more specifically *the arithmetic Langlands program for the existence of companions* and *the geometric Langlands program for de Jong's conjecture*, as we briefly discuss in the next and last section.

11. Weak Arithmeticity and Density

The main theorem of *loc. cit.* is proven by combining

(1) the method discussed in Section 7 to show integrality once we have ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$ which are arithmetic;

(2) and the use de Jong's conjecture discussed in Section 9, roughly as Drinfeld did in [Dri01], to produce many such arithmetic ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$.

By Grothendieck's classical "quasi-unipotent monodromy at infinity" theorem [ST68], arithmetic tame ℓ-adic local systems on $X_{\bar{\mathbb{F}}_p}$ have quasi-unipotent monodromies at infinity. So their pull-back to $X(\mathbb{C})$ via $sp_{\mathbb{C},\bar{\mathbb{F}}_p}$ and $\mathfrak{prof}$ do as well.

In order to apply the method described in Section 7 involving the existence of ℓ-companions ultimately yielding integrality, we need quasi-unipotent monodromies at infinity. The method developed in [dJ01] shows that those in $M_B^{\mathrm{irr}}(X, r, \text{torsion})$ are Zariski dense, where "torsion" refers to the determinant of $\mathbb{L}_\rho$ being torsion. This is precisely this fact which enables one to "forget" the quasi-unipotent conditions at infinity and to develop the argument.

ACKNOWLEDGMENT. I thank Johan de Jong, Michael Groechenig, and Moritz Kerz. The material in this expository note relies on joint work or discussions with them. I thank Alexander Beilinson, François Charles, and Jakob Stix for friendly and helpful remarks on a first writing. I thank the AMS for soliciting this contribution. I thank Han-Bom Moon for a careful reading and for suggestions aiming at making the note accessible to a wider audience. The figures were drawn by him.

References

[BKT13] Y. Brunebarbe, B. Klingler, and B. Totaro, *Symmetric differentials and the fundamental group*, Duke Math. J. **162** (2013), no. 14, 2797–2813, DOI 10.1215/00127094-2381442. MR3127814

[Del80] P. Deligne, *La conjecture de Weil II*, Inst. Hautes Études Sci. Publ. Math. **52** (1980), 137–252. MR0601520

[dJ01] J. de Jong, *A conjecture on arithmetic fundamental groups*, Israel J. Math. **12** (2001), 61–84. MR1818381

[dJE23] J. de Jong and H. Esnault, *Integrality of the Betti moduli space*, Trans. of the Am. Math. Soc. **to appear** (2023), 18 pp.

[Dri01] V. Drinfeld, *On a conjecture of Kashiwara*, Math. Res. L. **8** (2001), 713–728. MR1879815

[Dri12] V. Drinfeld, *On a conjecture of Deligne* (English, with English and Russian summaries), Mosc. Math. J. **12** (2012), no. 3, 515–542, 668, DOI 10.17323/1609-4514-2012-12-3-515-542. MR3024821

[Dri80] V. G. Drinfel'd, *Langlands' conjecture for* GL(2) *over functional fields*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 565–574. MR562656

[EG18] H. Esnault and M. Groechenig, *Cohomologically rigid local systems and integrality*, Selecta Math. **24** (2018), 4279–4292.

[Esn23] H. Esnault, *Lectures on Local Systems in Algebraic-Arithmetic Geometry*, Lectures Notes in Mathematics, Springer Verlag **2337** (2023), 70 pp.

[Gai07] D. Gaitsgory, *On de Jong's conjecture*, Israel J. Math. **157** (2007), 155–191. MR2342444

[Gro71] *Revêtements étales et groupe fondamental* (French), Lecture Notes in Mathematics, vol. 224, Springer-Verlag, Berlin-New York, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1); Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. MR0354651

[KS10] M. Kerz and A. Schmidt, *On different notions of tameness in arithmetic geometry*, Math. Ann. **346** (2010), 641–668. MR2578565

[Laf02] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands* (French, with English and French summaries), Invent. Math. **147** (2002), no. 1, 1–241, DOI 10.1007/s002220100174. MR1875184

[Mar91] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991, DOI 10.1007/978-3-642-51445-6. MR1090825

[Moc06] T. Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application* (English, with English and French summaries), Astérisque **309** (2006), viii+117. MR2310103

[Ser58] J.-P. Serre, *Sur la topologie des variétés algébriques en caractéristique p* (French), Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, México, 1958, pp. 24–53. MR98097

[Sim92] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95. MR1179076

[ST68] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517. MR0236190

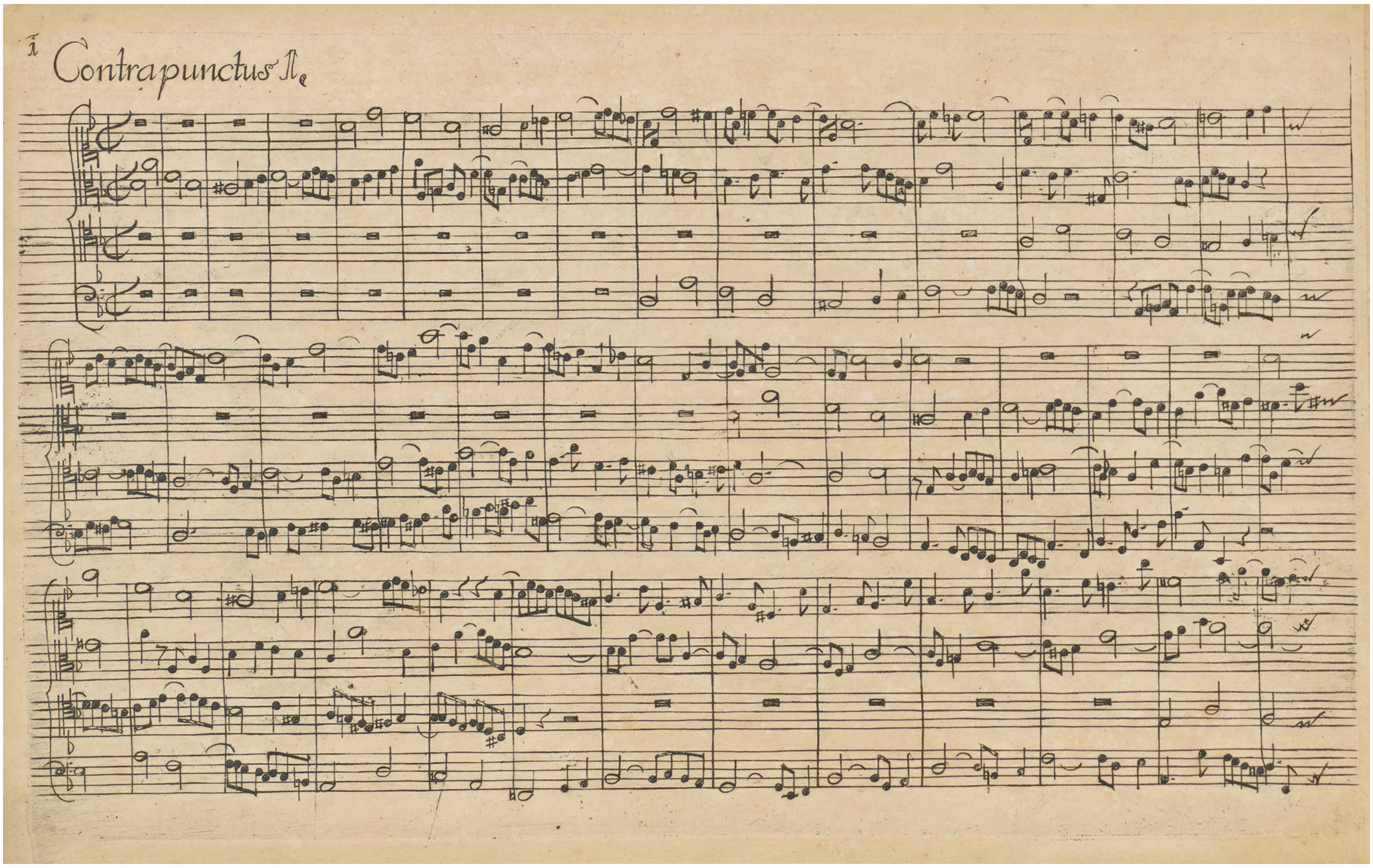
[Tol93] D. Toledo, *Projective varieties with non-residually finite fundamental group*, Inst. Hautes Études Sci. Publ. Math. **77** (1993), 103–119. MR1249171

Hélène Esnault

Credits

All images are courtesy of Hélène Esnault.

Crossed Modules



Johannes Huebschmann

1. Introduction

Modern chatbot software poses a threat to the health of our field. A scholarly article had better pass through the heads of at least two parties, cf. [Huf10, p. 47].[1] Here we undertake the endeavor of writing such an article about crossed modules with an eye toward the past, this past being ignored by the recent activity in this area.

In ancient cultures, *symmetry* arose as repetition of patterns. The human being perceives such symmetry as harmonious and beautiful proportion and balance (music, art, architecture, etc.). The variety of patterns is untold. Symmetry enables us to structure this variety by recognizing repetitions, as Eurynome's dancing structured chaos. The symmetries of an object are encoded in transformations that leave the object invariant. In modern mathematics, abstracting from the formal properties of such transformations led to the idea of a group. A typical example is the group of symmetries of the solutions of an equation, in modern mathematics termed *Galois group*, or each of the 17 plane symmetry groups. The operations that form these groups were already known to the Greeks. Besides in mathematics, symmetry and groups play a major role in physics, chemistry, engineering, materials science, crystallography, meteorology, etc. A group itself admits symmetries: the automorphisms of a group constitute a group. The *crossed*

Johannes Huebschmann is a professeur emérite in the Département de mathématiques at the Université de Lille, Sciences et Technologies, France. His email address is `Johannes.Huebschmann@univ-lille.fr`.

Dedicated to the memory of Donald J. Collins.

Communicated by Notices *Associate Editor Steven Sam.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2831

[1] *"It appears that the reporter has passed along some words without inquiring what they mean, and you are expected to read them just as uncritically for the happy illusion they give you of having learned something. It is all too reminiscent of an old definition of the lecture method of classroom instruction: a process by which the contents of the textbook of the instructor are transferred to the notebook of the student without passing through the head of either party."*

module concept arises by abstracting from the formal properties which this pair enjoys. For two groups N and Q, the technology of crossed modules allows for a complete classification of the groups E having N as a normal subgroup and quotient group E/N isomorphic to Q. This settles the *extension problem* for groups, raised by Hoelder at the end of the nineteenth century. Schreier explored this problem in terms of factor sets, and Turing implicitly noticed that it admits a solution in terms of crossed modules. Given N and Q as symmetry groups, interpreting an extension E as a symmetry group is an interesting task, as is, given the extension group E as a symmetry group, interpreting N and Q as symmetry groups. Crossed modules occur in mathematics under various circumstances as a means to structure a collection of mathematical patterns. A special example of a crossed module arises from an ordinary module over a group. While the concept of an ordinary module is lingua franca in mathematics, this is not the case of crossed modules.

Beyond some basic algebra and algebraic topology, we assume the reader familiar with some elementary category theory ("categorically thinking" suffices). This article is addressed to the nonexpert. We keep sophisticated technology like group cohomology, homotopy theory, and algebraic number theory at a minimum. We write an injection as $\rightarrowtail$ and a surjection as $\twoheadrightarrow$.

The opener image of this article displays the beginning of the first counterpoint (contrapunctus) of the original printed version of J. S. Bach's "Kunst der Fuge". The order of the counterpoints (of the second part thereof) had been lost and, 100 years ago, Wolfgang Graeser, before enrolling as a mathematics student at Berlin university, restored an order (perhaps the original sequence) by means of symmetry considerations. This order is the nowadays generally accepted performance practice.

2. Definition and Basic Examples

A crossed module arises by abstraction from a structure we are all familiar with when we run into a normal subgroup of a group or into the group of automorphisms of a group: Denote the identity element of a group by e and, for a group G and a G-(operator) group H (a group H together with an action of G on H from the left by automorphisms of H) we write the action as $(x, y) \mapsto {}^{x}y$, for $x \in G$ and $y \in H$. Consider two groups C and G, an action of G on C from the left, view the group G as a G-group with respect to conjugation, and let $\partial : C \to G$ be a homomorphism of G-groups. The triple (C, G, ∂) constitutes a *crossed module* if, furthermore, for every pair (x, y) of members of C,

$$xyx^{-1}({}^{\partial x}y)^{-1} = e, \tag{1}$$

that is, the members xyx^{-1} and ${}^{\partial x}y$ of C coincide. For a crossed module (C, G, ∂), the image $\partial(C)$ of C in G is a normal subgroup, the kernel $\ker(\partial)$ of ∂ is a central subgroup of C, and the G-action on C induces an action of the quotient group $Q = G/(\partial(C))$ on $Z = \ker(\partial)$ turning Z into a module over this group, and it is common to refer to the resulting exact sequence

$$Z \rightarrowtail C \longrightarrow G \twoheadrightarrow Q \tag{2}$$

as a *crossed* 2*-fold extension of* Q *by* Z. A morphism of crossed modules is defined in the obvious way. Thus crossed modules constitute a category. The terminology "crossed module" goes back to [Whi49]. The identities (1) appear in [Pei49] and have come to be known in the literature as *Peiffer identities* [Lyn50] (beware: not "Peiffer identity" as some of the present day literature suggests). There is a prehistory, however, [Bae34, Tur38]; in particular, the Peiffer identities occur already in [Tur38]. The reader may consult [Hue21, Section 3] for details.

For a group Q and a Q-module M, the trivial homomorphism from M to Q is a crossed module structure and, more generally, so is any Q-equivariant homomorphism ϑ from M to Q such that the image $\vartheta(M)$ of M in Q acts trivially on M. Relative to conjugation, the injection of a normal subgroup into the ambient group is manifestly a crossed module structure. Also, it is immediate that the homomorphism $\partial : G \to \mathrm{Aut}(G)$ from a group G to its group $\mathrm{Aut}(G)$ of automorphisms which sends a member of G to the inner automorphism it defines turns $(G, \mathrm{Aut}(G), \partial)$ into a crossed module. It is common to refer to the quotient group $\mathrm{Out}(G) = \mathrm{Aut}(G)/(\partial(G))$ as the group of *outer automorphisms* of G.

3. Identities Among Relations

An "identity among relations" is for a presentation of a group what a "syzygy among relations", as considered by Hilbert, is for a presentation of a module: Consider a presentation $\langle X; R\rangle$ of a group Q. That is to say, X is a set of generators of Q and R a family of (reduced) words in X and its inverses such that the canonical epimorphism from the free group F on X to Q has the normal closure N_R of (the image of) R in F as its kernel. Then the F-conjugates of the images in F of the members r of R generate N_R, that is, (with a slight abuse of the notation R,) the family R generates N_R as an F-operator group: With the notation ${}^{y}r = yry^{-1}$ $(r \in R,\ y \in F)$, we can write any member w of N_R in the form

$$w = \prod_{j=1}^{m} {}^{y_j}r_j^{\varepsilon_j},\ r_j \in R,\ y_j \in F,\ \varepsilon_j = \pm 1,$$

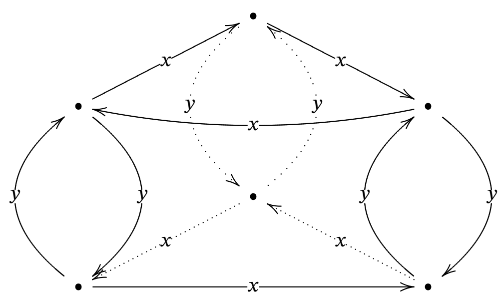
but w does not determine such an expression uniquely. Thus the issue of understanding the structure of N_R as an F-operator group arises. Heuristically, an *identity among relations* is such a specified product that recovers the identity element e of F. For example, consider the presentation

$$\langle x, y; r, s, t\rangle,\ r = x^3, s = y^2, t = xyxy, \tag{3}$$

of the symmetric group S_3 on three letters. Straightforward verification shows that

$$ts^{-1}({}^{x^{-1}}t)({}^{x^{-1}}s^{-1})({}^{x^{-1}y}r^{-1})({}^{x^{-2}}t)({}^{x^{-2}}s^{-1})r^{-1} \tag{4}$$

is an identity among the relations in (3). A standard procedure enables us to read off this identity and others from the following prism-shaped tesselated 2-sphere:



(5)

The reader will notice that reading along the boundaries of the faces recovers the relators in (3). The projection, to a plane, of this prism-shaped 2-sphere with one of the triangles removed is [BH82, Fig. 2 p. 155]. In Section 10 of [BH82], the reader can find precise methods to obtain such identities from "pictures", see in particular [BH82, Fig. 12 p. 194] for the case at hand, and in [CH82] from diagrams etc. The corresponding term in [Pei49] is "Randwegaggregat".

To develop a formal understanding of the situation, let $\widehat{C}_R$ be the free F-operator group on R. The kernel of the canonical epimorphism from the free group on the (disjoint) union $X \cup R$ to F realizes $\widehat{C}_R$. Let $\hat\partial_R \colon \widehat{C}_R \to F$ denote the canonical homomorphism. The members of the kernel of $\hat\partial_R \colon \widehat{C}_R \to F$ are the *identities among the relations* (or *among the relators*) for the presentation $\langle X; R\rangle$ [Pei49, Rei49, BH82]; Turing refers to them as "relations between the relations" [Tur38, Section 2].

The *Peiffer elements* $xyx^{-1}({}^{\hat\partial_R x}y)^{-1}$, as x and y range over $\widehat{C}_R$, are identities that are always present, independently of any particular presentation under discussion. Following [Rei49], let C_R be the quotient group $\widehat{C}_R/P$ of $\widehat{C}_R$ modulo the subgroup P in $\widehat{C}_R$, necessarily normal, which the Peiffer elements generate. The Peiffer subgroup P is an F-subgroup, whence the F-action on $\widehat{C}_R$ passes to an F-action on C_R, and the canonical homorphism $\hat\partial_R$ induces a homomorphism $\partial_R \colon C_R \to F$ that turns (C_R, F, ∂_R) into a crossed module. In particular, the kernel $\pi = \ker(\partial_R)$, being central in C_R, is an abelian group, the F-action on π factors through a Q-module structure, and π parametrizes equivalence classes of "essential" or *nontrivial* identities associated to the presentation $\langle X; R\rangle$ of Q [Rei49]. We shall see below that there are interesting cases where π is trivial, that is, the Peiffer identities generate all identities.

Given a group G, a *G-crossed module* is a G-group C together with a homomorphism $\partial \colon C \to G$ of G-groups such that (C, G, ∂) is a crossed module. Given a set R and a set map $\varkappa \colon R \to G$ into a group G, the *free crossed G-module on $\varkappa$* is the crossed module $(C_\varkappa, G, \partial_\varkappa)$ enjoying the following property: Given a G-crossed module (C, G, ∂) and a set map $\beta \colon R \to C$, there is a unique homomorphism $\beta_\varkappa \colon C_\varkappa \to C$ of G-groups such that $(\beta_\varkappa, \mathrm{Id}) \colon (C_\varkappa, G, \partial_\varkappa) \to (C, G, \partial)$ is a morphism of crossed modules. A standard construction shows that this free G-crossed module always exists. By the universal property, such a free G-crossed module is unique up to isomorphism, whence it is appropriate to use the definite article here. The F-crossed module (C_R, ∂_R) just constructed from a presentation $\langle X; R\rangle$ of a group Q is plainly the free F-crossed module on the injection $R \to F$. The resulting extension $\pi \rightarrowtail C_R \twoheadrightarrow N_R$ of F-operator groups then displays the F-crossed module N_R as the quotient of a free F-crossed module and thereby yields structural insight. It is also common to refer to (C_R, F, ∂_R) as the *free crossed module on $\langle X; R\rangle$*.

4. Group Extensions and Abstract Kernels

Given two groups N and Q, the issue is to parametrize the family of groups G that contain N as a normal subgroup and have quotient G/N isomorphic to Q, or equivalently, in categorical terms, the family of groups G that fit into an exact sequence of groups of the kind

$$N \rightarrowtail G \twoheadrightarrow Q. \tag{6}$$

Given an extension of the kind (6), conjugation in G induces a homomorphism φ from Q to $\mathrm{Out}(N)$. It is common to refer to a triple (N, Q, φ) that consists of two groups N and Q together with a homomorphism $\varphi \colon Q \to \mathrm{Out}(N)$ as an *abstract kernel* or to the pair (N, φ) as an *abstract Q-kernel*. In [Bae34] the terminology is "Kollektivcharakter". We have just seen that a group extension determines an abstract kernel. Given two groups N and Q together with an abstract kernel structure $\varphi \colon Q \to \mathrm{Out}(N)$, the extension problem consists in *realizing* the abstract kernel, that is, in parametrizing the extensions of the kind (6) having φ as its abstract kernel provided such an extension exists, and the abstract kernel is then said to be *extendible*.

When N is abelian, the group of outer automorphisms of N amounts to the group $\mathrm{Aut}(N)$, an abstract Q-kernel structure on N is equivalent to a Q-module structure on N, and the semidirect product group $N \rtimes Q$ shows that this abstract kernel is extendible. When N is nonabelian, not every abstract Q-kernel (N, φ) is extendible, however, [Bae34, p. 415]. We shall shortly see that nonextendible abstract kernels abound in mathematical nature.

For two homomorphisms $G_1 \to Q$ and $G_2 \to Q$, let $G_1 \times_Q G_2$ denote the *pullback* group, that is, the subgroup of $G_1 \times G_2$ that consists of the pairs (x_1, x_2) that have the same image in Q.

For a crossed module (C, G, ∂), the action $G \to \mathrm{Aut}(C)$ of G on C plainly defines an abstract kernel structure $\varphi\colon G/(\partial(C)) \to \mathrm{Out}(C)$. On the other hand, given the abstract kernel (N, Q, φ), the pullback group $G^\varphi = \mathrm{Aut}(N) \times_{\mathrm{Out}(N)} Q$ and the canonical homomorphism $\partial^\varphi : N \to G^\varphi$ which the crossed module structure $\partial : N \to \mathrm{Aut}(N)$ induces combine to the crossed module $(N, G^\varphi, \partial^\varphi)$, and $\ker(\partial^\varphi)$ coincides with the center of N. In fact, this correspondence is a bijection between abstract kernels and crossed modules (C, G, ∂) having $\ker(\partial)$ as the center of C. The group G^φ occurs in [Bae34] as the "Aufloesung des Kollektivcharakters" φ.

We will now explain the solution of the extension problem: Consider an abstract kernel (N, Q, φ), and let $(N, G^\varphi, \partial^\varphi)$ be its associated crossed module. Let $\langle X; R\rangle$ be a presentation of the group Q and, exploiting the freeness of F, choose the set maps $\alpha : X \to G^\varphi$ and $\beta : R \to N$ in such a way that (i) the composite $F \to Q$ of the induced homomorphism $\alpha_F : F \to G^\varphi$ with the epimorphism from G^φ to Q coincides with the epimorphism from F to Q and (ii) the composite $R \to G^\varphi$ of α_F with the injection from R to F coincides with the composite of ∂ with β. In view of the universal property of the free F-crossed module (C_R, ∂_R), the set map β induces a morphism $(\beta_R, \alpha_F)\colon (C_R, F, \partial_R) \to (N, G^\varphi, \partial^\varphi)$ of crossed modules defined on the free crossed module (C_R, F, ∂_R) on $\langle X; R\rangle$. Let $\pi = \ker(\partial_R)$ and display the resulting morphism of crossed modules as

$$\begin{tikzcd} \pi \arrow[r, tail] \arrow[d] & C_R \arrow[r, "\partial_R"] \arrow[d, "\beta_R"'] & F \arrow[r, two heads] \arrow[d, "\alpha_F"] & Q \arrow[d, equal] \\ Z \arrow[r, tail] & N \arrow[r, "\partial_\varphi"'] & G^\varphi \arrow[r, two heads] & Q. \end{tikzcd} \tag{7}$$

(This is Diagram 8 in [Hue21, Section 3]). The following is a version of [Tur38, Theorem 4 p. 356] in modern language and terminology; see [Hue21, Section 3] for details. Crossed modules, in particular free ones, exact sequences, commutative diagrams, pullbacks, etc. were not at Turing's disposal, however, and he expressed his ideas in other ways. [2]

Theorem. *The abstract kernel (N, Q, φ) is extendible if and only if, once α has been chosen, the set map β can be chosen in such a way that, in Diagram 7, the restriction of β_R to π is zero.*

[2] *A. Turing and J.H.C. Whitehead worked as WWII codebreakers at Bletchley Park (UK) but we shall never know whether they then discussed the idea of a crossed module.*

Proof. It is immediate that the condition is necessary. To establish the converse, suppose, α having been chosen, there is a choice of β with the asserted property. Let $N_R = \partial_R(C_R) \subseteq F$ denote the normal closure of R in F. From (7), we deduce the morphism

$$\begin{tikzcd} N_R \arrow[r, tail, "\iota"] \arrow[d, "\widetilde{\beta}"'] & F \arrow[r] \arrow[d, "\alpha_F"] & Q \arrow[d, equal] \\ N \arrow[r, "\partial_\varphi"'] & G^\varphi \arrow[r] & Q \end{tikzcd} \tag{8}$$

of crossed modules. Since (8) is a morphism of crossed modules, in the semidirect product group $N \rtimes F$, the members of the kind $(\widetilde{\beta}(y), \iota(y^{-1}))$ as y ranges over N_R constitute a subgroup, necessarily normal. The quotient group E of $N \rtimes F$ by this normal subgroup yields the group extension E of Q by N we seek. $\square$

This modern proof of the theorem is in [Hue80a, Section 10] (phrased in terms of the coequalizer of $\widetilde{\beta}$ and ι). The diligent reader will notice that the proof is complete, i.e., no detail is left to the reader. It is an instance of a common observation to the effect that *mathematics consists in continuously improving notation and terminology*. Even nowadays there are textbooks that struggle to explain the extension problem and its solution in terms of lengthy and unilluminating cocycle calculations, and it is hard to find the above theorem, only recently dug out as a result of Turing's [Hue21].[3] Eilenberg–MacLane (quoting [Tur38] but apparently not understanding Turing's reasoning) developed the obstruction for an abstract kernel to be extendible in terms of the vanishing of the class of a group cohomology 3-cocycle of Q with values in Z [EM47, Theorem 8.1]. The unlabeled arrow $\pi \to Z$ in Diagram 7 recovers this 3-cocycle, and the condition in the theorem says that the class of this 3-cocycle in the group cohomology group $\mathrm{H}^3(Q, Z)$ is zero. Indeed, once a choice of α and β has been made, the vanishing of that 3-cocycle is equivalent to the unlabeled Q-module morphism $\pi \to Z$ in (7) admitting an extension to a homomorphism $\gamma\colon C_R \to Z$ of F-groups, and setting $\widetilde{\beta}_R(y) = \beta_R(y)\gamma(y)^{-1}$ and substituting $\widetilde{\beta}_R$ for β_R, we obtain a diagram of the kind (7) having $\pi \to Z$ zero.

To put flesh on the bones of the last remark, recall the *augmentation* map $\varepsilon\colon \mathbb{Z}\Gamma \to \mathbb{Z}$ of a group Γ defined on the integral group ring $\mathbb{Z}\Gamma$ of Γ by $\varepsilon(x) = 1$ for $x \in \Gamma$ and write the *augmentation ideal* $\ker(\varepsilon)$ as $I\Gamma$. The F-action on C_R induces a Q-action on the abelianized group $C_{R,\mathrm{ab}} = C_R/[C_R, C_R]$ in such a way that the set R constitutes a set of free generators; thus $C_{R,\mathrm{ab}}$ is canically isomorphic to the free Q-module $\mathbb{Z}Q[R]$ freely generated by R (with a slight abuse of the notation R). The induced morphism

[3] *Turing would have slipped through an evaluation system based on bibliographic metrics.*

$\pi \to C_{R,\mathrm{ab}}$ of Q-modules is still injective. For $r \in R$ and $x \in X$, using the fact that $\{x - 1 \in \mathrm{I}F; x \in X\}$ is a family of free F-generators of $\mathrm{I}F$, define $r_x \in \mathbb{Z}F$ by the identity $r - 1 = \sum r_x(x-1) \in \mathrm{I}F$, and let $[r_x] \in \mathbb{Z}Q[X]$ denote the image. Defining d_2 by $d_2(r) = \sum [r_x]x \in \mathbb{Z}Q[X]$ where r ranges over R and x over X, and d_1 by $d_1(x) = [x] - 1 \in \mathbb{Z}Q$ where $[x] \in \mathbb{Z}Q$ denotes the image of x, as x ranges over X, renders the sequence

$$\pi \rightarrowtail \mathbb{Z}Q[R] \xrightarrow{d_2} \mathbb{Z}Q[X] \xrightarrow{d_1} \mathbb{Z}Q \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z} \tag{9}$$

exact, and the three middle terms thereof together with the corresponding arrows constitute the beginning of a free resolution of $\mathbb{Z}$ in the category of Q-modules. When we take $\langle X; R\rangle$ to be the *standard* presentation having X the set of all members of Q distinct from e, we obtain the first three terms of the *standard resolution*, and the composite of the Q-module epimorphism $B_3 \to \pi$ defined on the fourth term B_3 of the standard resolution with the above unlabeled arrow $\pi \to Z$ verbatim recovers the 3-cocycle in [EM47, Theorem 8.1].

A *congruence* between two group extensions of a group Q by a group N is a commutative diagram

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, equal] & E_1 \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ N \arrow[r, tail] & E_2 \arrow[r, two heads] & Q \end{tikzcd} \tag{10}$$

with exact rows in the category of groups. For a group N with center Z, the multiplication map restricts to a homomorphism $\mu\colon N \times Z \to N$.

Complement. *Let (N, Q, φ) be an extendible abstract kernel and realize it by the group extension $N \overset{j_1}{\rightarrowtail} E_1 \twoheadrightarrow Q$. The assignment to an extension $Z \overset{j}{\rightarrowtail} E \twoheadrightarrow Q$ of Q by the center Z of N realizing the induced Q-module structure on Z of the group extension $N \overset{j_2}{\rightarrowtail} E_2 \twoheadrightarrow Q$ which the commutative diagram*

$$\begin{tikzcd} N \times Z \arrow[r, tail, "{(j_1, j)}"] \arrow[d, "\mu"'] & E_1 \times_Q E \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ N \arrow[r, tail] & E_2 \arrow[r, two heads, "j_2"] & Q \end{tikzcd} \tag{11}$$

with exact rows in the category of groups characterizes induces a faithful and transitive action of $\mathrm{H}^2(Q, Z)$ on the congruence classes of extensions of Q by N realizing the abstract kernel (N, Q, φ).

As for the proof we note that, as before, since the left-hand and middle vertical arrows in (11) constitute a morphism of crossed modules, in the group $N \rtimes (E_1 \times_Q E)$, the triples $(y_1y_2, j_1(y_1)^{-1}, j(y_2)^{-1})$, as y_1 ranges over N and y_2 over Z, form a normal subgroup, and the group E_2 arises as the quotient of $N \rtimes (E_1 \times_Q E)$ by this normal subgroup. When N is abelian, it coincides with its center, Diagram 11 displays the operation of "Baer sum" of two extensions with abelian kernel, and the complement recovers the fact that the congruence classes of abelian extensions of Q by N constitute an abelian group, the group structure being induced by the operation of Baer sum [Bae34], indeed, the group cohomology group $\mathrm{H}^2(Q, N)$.

5. Combinatorial Group Theory and Low-dimensional Topology

Consider a space Y and a subspace X (satisfying suitable local properties, Y being a CW complex and X a subcomplex would suffice), and let o be a base point in X. The standard action of the fundamental group $\pi_1(X, o)$ (based homotopy classes of continuous maps from a circle to X with a suitably defined composition law) on the second relative homotopy group $\pi_2(Y, X, o)$ (based homotopy classes of continuous maps from a disk to Y such that the boundary circle maps to X with a suitably defined composition law) and the boundary map $\partial\colon \pi_2(Y, X, o) \to \pi_1(X, o)$ turn $(\pi_2(Y, X, o), \partial)$ into a crossed $\pi_1(X, o)$-module [Whi41]. See [BHS11] for a leisurely introduction to homotopy groups building on an algebra of composition of cubes. Suffice it to mention that the higher homotopy groups of a space acquire an action of the fundamental group. In [Whi41], J.H.C. Whitehead in particular proved that, when the space Y arises from X by attaching 2-cells, the crossed $\pi_1(X, o)$-module $(\pi_2(Y, X, o), \partial)$ is free on the homotopy classes in $\pi_1(X, o)$ of the attaching maps of the 2-cells.

The *Cayley graph* of a group Q with respect to a family X of generators is the directed graph having Q as its set of vertices and, for each pair $(x, y) \in X \times Q$, an oriented edge joining y to xy. The oriented graph that underlies (5) is the Cayley graph of the group S_3 with respect to the generators x and y in (3). The *geometric realization* $K = K\langle X; R\rangle$ of a presentation $\langle X; R\rangle$ of a group Q is a 2-dimensional CW complex with a single zero cell o, having 1-cells in bijection with X and 2-cells in bijection with R in such a way that the fundamental group $\pi_1(K^1, o)$ of the 1-skeleton amounts to the free group on X and that the attaching maps of the 2-cells define, via the boundary map $\partial\colon \pi_2(K, K^1, o) \to \pi_1(K^1, o)$, the members of R. By construction, the fundamental group $\pi_1(K, o)$ of K is isomorphic to Q, the fundamental group $\pi_1(K^1, o)$ is canonically isomorphic to the free group F on the generators X and, by Whitehead's theorem, the $\pi_1(K^1, o)$-crossed module $(\pi_2(K, K^1, o), \partial)$ is free on the attaching maps of the 2-cells and hence canonically isomorphic to the $\pi_1(K^1, o)$-crossed module written above as (C_R, ∂_R). The 1-skeleton $\overline{K}^1$ of the universal covering space $\widetilde{K}$ of K then has fundamental group isomorphic to the normal closure N_R of the relators in F and, together with the appropriate orientation of its edges, then amounts to the Cayley graph of Q

with respect to X. Every 2-dimensional CW complex with a single 0-cell is of this kind. Since the higher homotopy groups of K^1 are zero, the long exact homotopy sequence of the pair (K, K^1) reduces to the crossed 2-fold extension

$$\pi_2(K, o) \rightarrowtail \pi_2(K, K^1, o) \xrightarrow{\partial} \pi_1(K, o) \twoheadrightarrow Q. \tag{12}$$

Thus the group of "essential identities" among the relations R amounts to the second homotopy group $\pi_2(K, o)$ of K (based homotopy classes of continuous maps from a 2-sphere to K, with a suitably defined group structure). For illustration, consider the 1-skeleton of the prism-shaped tesselated 2-sphere (5). For each 2-gon and each 4-gon, attach two copies of the disk bounding it and, likewise, for each triangle, attach three copies of the disk bounding it. Thus we obtain a prism-shaped 2-complex having, beyond the six vertices and twelve edges, six faces bounded by 2-gons, six faces bounded by 4-gons, and six faces bounded by triangles, and this 2-complex realizes the universal covering space $\widetilde{K}$ of the geometric realization K of (3). By its very construction, it has eleven "chambers" (combinatorial 2-spheres). Since $\widetilde{K}$ is simply connected, we conclude its second homotopy group is free abelian of rank eleven. For the inexperienced reader we note that choosing a maximal tree in $\overline{K}^1 = \widetilde{K}^1$ and deforming this tree to a point is a homotopy equivalence, and the result is a bunch of eleven 2-spheres, necessarily having second homology group free abelian of rank eleven. Since $\widetilde{K}$ is simply connected, the Hurewicz map from $\pi_2(\widetilde{K}, o)$ to this homology group is an isomorphism, and so is the homomorphism $\pi_2(\widetilde{K}, o) \to \pi_2(K, o)$ which the covering projection induces. Hence $\pi_2(K, o)$ is free abelian of rank eleven. Under the isomorphism $\pi_2(K, K^1, o) \to C_R$ of crossed F-modules, the based homotopy class of the innermost chamber goes to the class of the identity (4) in C_R, necessarily nontrivial, as are the identities that correspond to the other chambers. Another interesting piece of information we extract from (5) is that $\pi_1(\overline{K}^1, o) \cong N_R \subseteq F$, being the fundamental group of the Cayley graph $\overline{K}^1$ of S_3 is, as a group, freely generated by any seven among the eight constituents of (4).

The paper [Pei49] arose out of a combinatorial study of 3-manifolds: The present discussion applies to the 2-skeleton M^2 of a cell decomposition of a 3-manifold M, and attaching 3-cells to build the 3-manifold under discussion "kills" some of the essential identities associated with M^2: The commutative diagram

$$\begin{tikzcd}
\pi_2(M^2, o) \arrow[r, two heads] \arrow[d, two heads] & \pi_2(M^2, M^1, o) \arrow[r, tail, "\partial"] \arrow[d, two heads] & \pi_1(M^1, o) \arrow[r, two heads] \arrow[d, equal] & \pi_1(M, o) \arrow[d, equal] \\
\pi_2(M, o) \arrow[r, tail] & \pi_2(M, M^1, o) \arrow[r, tail, "\partial_M"'] & \pi_1(M^1, o) \arrow[r, two heads] & \pi_1(M, o)
\end{tikzcd} \tag{13}$$

with exact rows displays how the $\pi_1(M^1, o)$-crossed module $(\pi_2(M, M^1, o), \partial_M)$ arises from the free $\pi_1(M^1, o)$-crossed module $(\pi_2(M^2, M^1, o), \partial)$. For these and related issues, see, e.g., [BH82] and the literature there. For illustration, consider a finite subgroup Q of $\mathrm{SU}(2)$ such as, e.g., $Q = C_n$, the cyclic group of order $n \geq 2$, or Q the quaternion group of order eight. The orbit space $M = \mathrm{SU}(2)/Q$, a *lens space* when $Q = C_n$, is a 3-manifold having $\pi_2(M^2, o)$ nontrivial but $\pi_2(M, o)$ trivial since the 3-sphere has trivial second homotopy group. Via Papakyriakopoulos's *sphere theorem*, the second homotopy group of a general 3-manifold being nontrivial is equivalent to the manifold being geometrically splittable, similarly to what we hint at for tame links below. Thus a 3-manifold is geometrically splittable if and only if attaching the 3-cells of a cell decomposition does not kill all the essential identities arising from its 2-skeleton.

A 2-complex K is said to be *aspherical* when its degree ≥ 2 homotopy groups are trivial. This is equivalent to the second homotopy group being trivial. Lyndon's identity theorem [Lyn50] implies that, when K has a single 2-cell, the 2-complex K is aspherical if and only if the relator r arising as the boundary of the 2-cell is not a proper power in $F = \pi_1(K, o)$. Thus the F-crossed module structure of the normal closure $N_{\{r\}}$ of $\{r\}$ in F is free in this case. For example, a closed surface distinct from the 2-sphere is aspherical, but this is an immediate consequence of the universal covering being the 2-plane. In the same vein, the exterior of a tame link in the 3-sphere is homotopy equivalent to a 2-complex—the operation of "squeezing" the 3-cells of a cell decomposition achieves this—and the link is geometrically unsplittable if and only if the exterior and hence the 2-complex is aspherical. Here "tame" means that the link belongs to a cellular subdivision. Consequently, for the Wirtinger presentation $\langle X; R \rangle$ [BH82, Fig. 3 Section 9 p. 183] of the link group, as a crossed F-module, the normal closure N_R of the relators R in F is free if and only if, by the already quoted result of Papakyriakopoulos, the link is geometrically unsplittable [BH82, Theorem (P) Section 9 p. 183]. In particular, a tame knot is a geometrically unsplittable link. In [Whi41], Whitehead raised the issue, still unsettled, whether any subcomplex of an aspherical 2-complex is itself aspherical. This is equivalent to asking whether, for a presentation $\langle X; R \rangle$ of a group, when the normal closure N_R of R in F is free as an F-crossed module, this is still true of the normal closure $N_{\widetilde{R}}$ of a subset $\widetilde{R}$ of R. See [BH82, Section 9 p. 181 ff.] for more details and literature on partial results. In [Whi41], in the proof of the freeness of the crossed module arising from attaching 2-cells to a space, Whitehead interpreted the Peiffer identities as Wirtinger relations associated to the link arising from a null homotopy; see [BH82, Section 10 p. 187 ff.] for details and more references.

6. Interpretation of the Third Group Cohomology Group

The notion of congruence for group extensions extends to crossed 2-fold extensions, and congruence classes of crossed 2-fold extensions of the kind (2), together with a suitably defined operation of composition arising from a generalized Baer sum, constitute the third group cohomology group $\mathrm{H}^3(Q,Z)$. Under this interpretation, the crossed 2-fold extension associated to the crossed module $(Z,Q,0)$ represents the identity element. See [Mac79] for the history of this interpretation; it parallels the result in [EM47] saying that suitably defined equivalence classes of abstract Q-kernels having Z as its center are in bijection with the members of $\mathrm{H}^3(Q,Z)$. Under this interpretation, the zero class corresponds to the extendible Q-kernels. This is essentially Turing's theorem in a new guise. Thus the crossed 2-fold extension (2) associated to a G-crossed module (C,∂) defines a *characteristic class* in $\mathrm{H}^3(Q,Z)$ with $Q = G/\partial(C)$. In particular, such a G-crossed module having $\ker(\partial)$ equal to the center Z of C defines an abstract Q-kernel, and this Q-kernel is extendible if and only if its characteristic class is zero. The characteristic class in $\mathrm{H}^3(\pi_1(K,o),\pi_2(K,o))$ of the $\pi_1(K^1,o)$-crossed module $(\pi_2(K,K^1,o),\partial)$ associated with a CW complex K recovers the (first) k- (or *Postnikov*) invariant; when K is 2-dimensional, $\pi_1(K,o)$, $\pi_2(K,o)$ and this k-invariant determine the homotopy type of K. A CW complex with nontrivial fundamental group and nontrivial second homotopy group typically has nonzero k-invariant but, to understand the present discussion, there is no need to know anything about k-invariants beyond the fact that the crossed 2-fold extension associated with the corresponding crossed module represents it. The crossed module associated with the geometric realization of (3) provides an explicit example of a nontrivial k-invariant; see below. Here is an even more elementary example, with explicit verification of the nontriviality of its k-invariant: The free crossed module $(C_{\{r\}},C_x,\partial)$ associated with the presentation $\langle x,r\rangle$ with $r = x^n$ of the finite cyclic group C_n of order $n \geq 2$ has C_x the free cyclic group generated by x and $C_{\{r\}}$ the free abelian C_n-group which r generates, equivalently, the free $\mathbb{Z}C_n$-module which r generates when we use additive notation, the action of C_x on $C_{\{r\}}$ is the composite of the projection $C_x \to C_n$ with the action coming from the C_n-group structure on $C_{\{r\}}$, and ∂ sends ${}^{x^k}r$, $0 \leq k \leq n-1$, to $x^n \in C_x$. The n identities

$$i_1 = {}^x r r^{-1}, i_2 = {}^x i_1 = {}^{x^2} r({}^x r^{-1}), \dots, i_n = {}^{x^{n-1}} i_1 = r({}^{x^{n-1}} r^{-1}),$$

generate the kernel $\pi = \ker(\partial)$ as an abelian group, subject to the relation $i_1 i_2 \dots i_n = 1$. Thus i_1 generates π as an abelian C_n-group, equivalently, as a C_n-module when we write π additively. The experienced reader will recognize π as being, as a C_n-module, isomorphic to the augmentation ideal $\mathrm{I}C_n$ of C_n. Sending each i_j, for $1 \leq j \leq n$, to the generator of (a copy of) C_n defines an epimorphism $\pi \to C_n$; indeed, this is the epimorphism that arises by dividing out the C_n-action on π. Let $(\widehat{C}_{\{r\}},\hat{\partial})$ denote the C_x-crossed module which requiring the diagram

$$\begin{tikzcd}
\pi \arrow[r, tail] \arrow[d] & C_{\{r\}} \arrow[r, "\partial"] \arrow[d] & C_x \arrow[r, two heads] \arrow[d, equal] & C_n \arrow[d, equal] \\
C_n \arrow[r, tail] & \widehat{C}_{\{r\}} \arrow[r, "\hat{\partial}"'] & C_x \arrow[r, two heads] & C_n
\end{tikzcd} \tag{14}$$

with exact rows to be commutative characterizes. Let $C_r \subseteq C_x$ denote the free cyclic subgroup which $r = x^n$ generates. As an abelian group, $\widehat{C}_{\{r\}} \cong C_n \times C_r$ and, with the notation u for the generator of the copy of C_n, the rules ${}^x r = ur$ and ${}^x u = u$ characterize the C_x-group structure. Let v denote a generator of the cyclic group C_{n^2} of order n^2. With respect to the trivial actions, the homomorphism $\cdot^n : C_{n^2} \to C_{n^2}$ which sends v to v^n defines a C_{n^2}-crossed module structure on C_{n^2}. Sending u to v^n, r to v and x to v we obtain a congruence morphism

$$\begin{tikzcd}
C_n \arrow[r, tail] \arrow[d, equal] & \widehat{C}_{\{r\}} \arrow[r, "\hat{\partial}"] \arrow[d] & C_x \arrow[r, two heads] \arrow[d] & C_n \arrow[d, equal] \\
C_n \arrow[r, tail] & C_{n^2} \arrow[r, "\cdot^n"'] & C_{n^2} \arrow[r, two heads] & C_n
\end{tikzcd} \tag{15}$$

of crossed 2-fold extensions. The upper row of (14) represents the k-invariant in $\mathrm{H}^3(C_n,\pi)$ of the geometric realization of the presentation $\langle x;r\rangle$ of the group C_n while the lower row of (15) represents a generator of $\mathrm{H}^3(C_n,C_n) \cong C_n$. Diagram (14) says that, under the induced map $\mathrm{H}^3(C_n,\pi) \to \mathrm{H}^3(C_n,C_n)$, that k-invariant goes to a generator of a cyclic group of order $n \geq 2$. Hence that k-invariant is necessarily nontrivial. It is also worthwhile noting that the left-hand copy of C_n in (15) amounts to the third group homology group $\mathrm{H}_3(C_n)$ of C_n and that suitably exploiting the *Yoneda* interpretation of the traditional definition of $\mathrm{H}^3(C_n,C_n)$ as $\mathrm{Ext}^3_{C_n}(\mathbb{Z},C_n)$ identifies the class of the bottom row in (15) with the corresponding member of $\mathrm{Ext}^3_{C_n}(\mathbb{Z},C_n)$. Restricting the crossed 2-fold extension associated with the geometric realization of (3) to any of the cyclic subgroups of S_3 and playing a bit with the data, one can also show that the crossed module associated with the geometric realization of (3) has nonzero k-invariant. Thus crossed modules having nonzero characteristic class and in particular nonextendible abstract kernels abound.

7. Higher Group Cohomology Groups

Suitably extended, the interpretation in terms of crossed 2-fold extensions leads, for $n \geq 1$, to an interpretation of the group cohomology group $\mathrm{H}^{n+1}(Q,Z)$ in terms of "crossed

n-fold extensions". See [Mac79] for the history of this interpretation. For example, the crossed n-fold extension associated with a cell decomposition of an n-dimensional CW-complex X with nontrivial fundamental group and trivial homotopy groups $\pi_j(X)$ for $2 \leq j < n$ represents the first nonzero k-invariant in $\mathrm{H}^{n+1}(X, \pi_n(X))$ of X [Hue80b]. For illustration, as in Section 5, consider the orbit space $M = \mathrm{SU}(2)/Q$ for a (nontrivial) finite subgroup Q of $\mathrm{SU}(2)$. We will now use, without further explanation, some classical material which the reader can find in standard textbooks. The 2-skeleton of a suitable cell decomposition of M yields the geometric realization of the presentation of Q resulting from the cell decomposition, the geometric realization of the presentation $\langle x; r\rangle$ of C_n, with $r = x^n$, when $Q = C_n$. For a general finite subgroup Q of $\mathrm{SU}(2)$, the exact homotopy sequence of the pair (M, M^2), necessarily one of Q-modules, takes the form

$$\begin{aligned} \dots \pi_3(M^2, o) \to \pi_3(M, o) \to \pi_3(M, M^2, o) \\ \to \pi_2(M^2, o) \twoheadrightarrow \pi_2(M, o). \end{aligned} \tag{16}$$

With respect to the epimorphism from $\pi_1(M^1, o)$ to Q, (16) becomes a sequence of $\pi_1(M^1, o)$-modules. The composite

$$\pi_3(M, M^2, o) \to \pi_2(M^2, M^1, o) \to \pi_2(M^2, M^1, o)_{\mathrm{ab}} \tag{17}$$

of $\pi_3(M, M^2, o) \to \pi_2(M^2, o)$ with the injection $\pi_2(M^2, o) \to \pi_2(M^2, M^1, o)$ and, thereafter, with abelianization, amounts to the boundary operator $C_3(S^3) \to C_2(S^3)$ of the Q-equivariant cellular chain complex $C_*(S^3)$ of the (cellularly decomposed) 3-sphere S^3 that underlies $\mathrm{SU}(2)$, and this boundary operator has the homology group $\mathrm{H}_3(S^3)$ as its kernel. Using the Hurewicz isomorphism $\pi_3(S^3, \hat{o}) \to \mathrm{H}_3(S^3)$ and the covering projection isomorphism $\pi_3(S^3, \hat{o}) \to \pi_3(M, o)$ (with the notation $\hat{o}$ for a preimage in S^3 of the base point o of M), we deduce that the Q-morphism $\pi_3(M, o) \to \pi_3(M, M^2, o)$ in (16) is injective. (Beware: We must be circumspect at this point since $\pi_3(M^2, o)$ is nontrivial when Q is not the trival group, and we cannot naively deduce that injectivity from the exactness of (16).) Since the second homotopy group of the 3-sphere S^3 is trivial, so is $\pi_2(M, o)$. Consequently the Q-morphism $\pi_3(M, M^2, o) \to \pi_2(M^2, o)$ in (16) is surjective. Hence splicing (12), with M^2 substituted for K, and (16) yields the crossed 3-fold extension

$$\pi_3(M, o) \rightarrowtail \pi_3(M, M^2, o) \to \pi_2(M^2, M^1, o) \xrightarrow{\partial} \pi_1(M^1, o) \twoheadrightarrow Q \tag{18}$$

of Q by the free cyclic group $\pi_3(M, o) \cong \mathrm{H}_3(S^3)$ (as Q-modules), and the Q-action on $\pi_3(M, o)$ is trivial since this action amounts to the induced Q-action on $\mathrm{H}_3(S^3)$, necessarily trivial. The group $\mathrm{H}^4(Q, \pi_3(M, o))$ is cyclic of order $|Q|$ (the number of elements of Q), and the crossed 3-fold extension (18) represents a generator of $\mathrm{H}^4(Q, \pi_3(M, o))$ and thence the first nonzero k-invariant of M. This k-invariant, the fundamental group $\pi_1(M, o) \cong Q$, and the third homotopy group $\pi_3(M, o)$ determine the homotopy type of M. Furthermore, the group Q has periodic cohomology, of period 2 when Q is cyclic and of period 4 otherwise. Thus, the operation of cup product with the class of (18) induces isomorphisms $\mathrm{H}^s(Q, \cdot) \to \mathrm{H}^{s+4}(Q, \cdot)$, for $s \geq 1$, for any integer s when we interpret the notation H as *Tate* cohomology. When Q is a cyclic group C_n of order $n > 1$, the extension $\pi_3(M, o) \rightarrowtail \pi_3(M, M^2, o) \twoheadrightarrow \pi = \ker(\partial)$ of C_n-modules amounts to the familiar extension $\mathbb{Z} \rightarrowtail \mathbb{Z}C_n \twoheadrightarrow \mathrm{I}C_n$ of C_n-modules arising from the standard small free resolution of $\mathbb{Z}$ in the category of C_n-modules, and it is immediate that the C_n-module structure on $\pi_3(M, o)$ is trivial.

8. Generalization

The description of group cohomology in terms of crossed n-fold extensions ($n \geq 1$) is susceptible to generalizations where cocycles are not necessarily available. For example, for an extension of a topological group G by a continuous G-module whose underlying bundle is nontrivial, (global) continuous cocycles are not available. See [Hue21, Section 3] and the literature there for more situations where this happens.

9. Lie Algebra Crossed Modules

The axiom (1) makes perfect sense for Lie algebras, and the interpretation of the $(n+1)$th Lie algebra cohomology group in terms of crossed n-fold extensions ($n \geq 1$) is available. Also the abstract kernel concept extends to Lie algebras in an obvious manner, as does the equivalence between abstract kernels and crossed modules with the central kernel constraint explained above, and there is an analogue of Turing's theorem. Indeed, much of the above material carries over to Lie algebras. See [Hue21, Section 5] and the literature there for details.

10. Normality of a Noncommutative Algebra Over Its Center

Crossed modules arise in Galois theory. We will now briefly delve into this. See [Hue21, Section 4] for the history and [Hue18b] for a more complete account and references:

Let S be a commutative ring and A an S-algebra having S as its center. Let Q be a group of operators on S. The development of a Galois theory for such algebras leads to the following question: *Does every automorphism in Q extend to an automorphism of A*? The algebra A is said to be *Q-normal* when this happens to be the case. We formalize the situation as follows:

Denote by $\mathrm{Aut}(A)$ the group of ring automorphisms of A and by $\mathrm{U}(A)$ its group of units. The obvious homomorphism $\partial : \mathrm{U}(A) \to \mathrm{Aut}(A)$ assigns to a unit of A the associated inner automorphism of A, the obvious action of $\mathrm{Aut}(A)$ on $\mathrm{U}(A)$ turns the triple $(\mathrm{U}(A), \mathrm{Aut}(A), \partial)$ into a crossed module, and $\ker(\partial) = \mathrm{U}(S)$, the group of units of S. Write $\mathrm{Out}(A) = \mathrm{Aut}(A)/(\partial \mathrm{U}(A))$. Each inner automorphism of A leaves S elementwise fixed whence the restriction map $\mathrm{Aut}(A) \to \mathrm{Aut}(S)$ induces a homomorphism $\mathrm{res}: \mathrm{Out}(A) \to \mathrm{Aut}(S)$. Let Q be a group and $\kappa : Q \to \mathrm{Aut}(S)$ an action of Q on S by ring automorphisms. Define a *Q-normal structure* on the central S-algebra A *relative to the given action* $\kappa : Q \to \mathrm{Aut}(S)$ *of Q on S* to be a homomorphism $\sigma : Q \to \mathrm{Out}(A)$ that lifts the action $\kappa : Q \to \mathrm{Aut}(S)$ of Q on S in the sense that the composite of σ with $\mathrm{res}: \mathrm{Out}(A) \to \mathrm{Aut}(S)$ coincides with κ. A *Q-normal S-algebra* is, then, a central S-algebra A together with a Q-normal structure $\sigma : Q \to \mathrm{Out}(A)$.

Let (A, σ) be a Q-normal S-algebra, let $G^\sigma = \mathrm{Aut}(A) \times_{\mathrm{Out}(A)} Q$, and let G^σ act on $\mathrm{U}(A)$ via the canonical homomorphism from G^σ to $\mathrm{Aut}(A)$. The homomorphism $\partial^\sigma : \mathrm{U}(A) \to G^\sigma$ which $\partial : \mathrm{U}(A) \to \mathrm{Aut}(A)$ induces turns $(\mathrm{U}(A), G^\sigma, \partial^\sigma)$ into a crossed module, and the crossed 2-fold extension

$$\mathrm{e}_{(A,\sigma)} : \mathrm{U}(S) \rightarrowtail \mathrm{U}(A) \xrightarrow{\partial^\sigma} G^\sigma \twoheadrightarrow Q \qquad (19)$$

represents a class $[\mathrm{e}_{(A,\sigma)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$, the *Teichmueller class* of (A, σ). For the special case where S is a field, a cocycle description of this class (independently of any crossed module) is in [Tei40].[4]

Define a *Q-equivariant* S-algebra to be a central S-algebra A together with a homomorphism $\rho : Q \to \mathrm{Aut}(A)$ that induces the Q-action κ on S. For example, let $R = S^Q$, the subring of Q-invariants in S; the S-algebra $A = B \otimes_R S$ for some central R-algebra B plainly admits a canonical Q-equivariant structure. Consider a Q-normal S-algebra (A, σ). We can then ask, as did Teichmueller in the situation he considered, whether the Q-action on S lifts to a Q-equivariant structure. When such a lift exists, it induces a congruence between $\mathrm{e}_{(A,\sigma)}$ and the crossed 2-fold extension arising from the crossed module $(Z, Q, 0)$, and hence the class $[\mathrm{e}_{(A,\sigma)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$ is zero. As for the converse, let $\mathrm{M}_I(A)$ denote the $(I \times I)$ matrix algebra over A for an index family I; when I is not finite, we interpret $\mathrm{M}_I(A)$ as being the endomorphism ring of $\oplus_I A^{\mathrm{op}}$. The algebra $\mathrm{M}_I(A)$ is again a central S-algebra. It is obvious that an automorphism of A yields one of $\mathrm{M}_I(A)$ in a unique way, and the obvious map $A \to \mathrm{M}_I(A)$ is a ring homomorphism. Hence a Q-normal structure $\sigma : Q \to \mathrm{Out}(A)$ on A determines one on $\mathrm{M}_I(A)$, and we denote this structure by $\sigma_I : Q \to \mathrm{Out}(\mathrm{M}_I(A))$. By [Hue18a, Theorem 6.1], the Teichmüller class of a Q-normal S-algebra (A, σ) is zero if and only if, for $I = Q$, the Q-normal structure σ_I on the matrix algebra $\mathrm{M}_I(A)$ comes from an equivariant one. Thus the class $[\mathrm{e}_{(A,\sigma)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$ is the obstruction for the Q-normal algebra (A, σ) to be equivalent to a Q-equivariant one in the sense just explained. In general, we cannot have (A, σ) itself to be equivariant. See (22) below for a special case.

Suppose S is a field. Then we are running into ordinary Galois theory. Here is a family of explicit examples of a Q-normal algebra having nontrivial Teichmueller class: Consider a field K and let $L = K(\zeta)$ be a normal extension having Galois group N cyclic of order $n \geq 2$ (say). Let τ denote a generator of N, let $\eta \in \mathrm{U}(K)$, and consider the cyclic central simple K-algebra $D(\tau, \eta)$ generated by $L = K(\zeta)$ and some (indeterminate) u subject to the relations

$$u\lambda = {}^\tau\lambda u,\ u^n = \eta,\ \lambda \in L = K(\zeta). \qquad (20)$$

In $D(\tau, \eta)$, the member u is a unit having inverse $u^{-1} = u^{n-1}\eta^{-1}$ and, for $\lambda \in L$, we get $u\lambda u^{-1} = {}^\tau\lambda$, that is, the action of the Galois group N on L extends to the inner automorphism of $D(\tau, \eta)$ which u determines. The algebra $D(\tau, \eta)$ is a crossed product of N with L relative to the $\mathrm{U}(L)$-valued 2-cocycle of N determined by η but this fact need not concern us here. The field L is a maximal commutative subalgebra of $D(\tau, \eta)$. The capital D serves as a mnemonic for the fact that Dickson explored such algebras.

Distinct choices of $\eta \in \mathrm{U}(K)$ may lead to the "same" algebra of the kind $D(\tau, \eta)$: The assignment to $\vartheta \in \mathrm{U}(L)$ of $\prod_{j=0}^{n-1} {}^{\tau^j}\vartheta$ defines the classical *norm map* $\nu : \mathrm{U}(L) \to \mathrm{U}(K)$. Let $\vartheta \in \mathrm{U}(L)$. Then $u \mapsto u_\vartheta = \vartheta u$ induces an isomorphism $\alpha_\vartheta : D(\tau, \eta) \to D(\tau, \nu(\vartheta)\eta)$ which restricts to the identity of L, an automorphism α_ϑ of $D(\tau, \eta)$ if and only if $\nu(\vartheta) = 1$. Furthermore, for $\eta = \nu(\vartheta)$ with $\vartheta \in \mathrm{U}(L)$, the algebra $D(\tau, \eta)$ comes down to the algebra of $(n \times n)$-matrices over K. Thus the cokernel $\mathrm{coker}(\nu)$ of the norm map $\nu : \mathrm{U}(L) \to \mathrm{U}(K)$ parametrizes classes of algebras of the kind $D(\tau, \eta)$ such that $D(\tau, \eta_1)$ and $D(\tau, \eta_2)$ belong to the same class if and only if an isomorphism which restricts to the identity of L carries $D(\tau, \eta_1)$ to $D(\tau, \eta_2)$. The expert will recognize that $\mathrm{coker}(\nu)$ amounts to $\mathrm{H}^2(N, \mathrm{U}(L))$ and $\ker(\nu)$ to the group of multiplicatively written $\mathrm{U}(L)$-valued 1-cocycles of N.

Let $\mathrm{Aut}_L(D(\tau, \eta))$ denote the group of automorphisms of $D(\tau, \eta)$ that restrict to an automorphism of $L|\mathfrak{k}$. For $\chi \in \mathrm{U}(L)$, the member $\vartheta_\chi = {}^\tau\chi\chi^{-1}$ of $\mathrm{U}(L)$ lies in the kernel of ν. *The assignment to χ of $\alpha_{\vartheta_\chi} \in \mathrm{Aut}_L(D(\tau, \eta))$ induces an embedding of* $\mathrm{U}(L)/\mathrm{U}(K)$ *into* $\mathrm{Aut}_L(D(\tau, \eta))$ *onto the subgroup of automorphisms that restrict to the identity of L.* Indeed, every automorphism α of $D(\tau, \eta)$ that restricts to the identity of L necessarily satisfies the identity

$$\begin{aligned} \alpha(u)u^{-1}\lambda u\alpha(u)^{-1} &= \alpha(u)({}^{\tau^{-1}}\lambda)\alpha(u)^{-1} \\ &= \alpha(u({}^{\tau^{-1}}\lambda)u^{-1}) = \alpha(\lambda) = \lambda \end{aligned}$$

[4] *Teichmueller worked as WWII codebreaker for the high command of the German army.*

for every $\lambda \in L$, and this implies $\alpha(u)u^{-1} \in \mathrm{U}(L)$ since L is a maximal commutative subalgebra of $D(\tau,\eta)$; then $\vartheta = \alpha(u)u^{-1}$ belongs to the kernel of ν. By Hilbert's "Satz 90", every member ϑ of $\ker(\nu)$ is of the kind ϑ_χ, for some $\chi \in \mathrm{U}(L)$.

Let Q be a finite group of operators on K and let $\mathfrak{k} = K^Q$, so that $K|\mathfrak{k}$ is a Galois extension. Suppose that, furthermore, $L|\mathfrak{k}$ is a Galois extension, let $G = \mathrm{Gal}(L|\mathfrak{k})$, and suppose that the resulting group extension of Q by N is central. The Q-action on $\mathrm{U}(K)$ passes to an action of Q on $\mathrm{coker}(\nu)$. In terms of classes of algebras of the kind $D(\tau,\eta)$, the assignment to $D(\tau,\eta)$ of $D(\tau, {}^x\eta)$, as x ranges over Q, induces this action.

A little thought reveals that the following are equivalent: (i) *The algebra $D(\tau,\eta)$ is Q-normal*; (ii) *the restriction $\mathrm{Aut}_L(D(\tau,\eta)) \to G$ is an epimorphism*; (iii) *for $x \in Q = \mathrm{Gal}(K|\mathfrak{k})$, there is a unit $\vartheta \in \mathrm{U}(L)$ such that $\nu(\vartheta) = {}^x\eta\eta^{-1}$. Furthermore, under the circumstances of* (iii), *the unit ϑ is unique up to multiplication by a unit in K.*

Condition (iii) plainly characterizes the members $[\eta]$ of the subgroup $\mathrm{coker}(\nu)^Q$ of Q-invariants of the cokernel of the norm map $\nu\colon \mathrm{U}(L) \to \mathrm{U}(K)$. Let $\sigma\colon Q \to \mathrm{Aut}(K)$ denote the Galois action. The assignment to the class $[\eta] \in \mathrm{coker}(\nu)^Q$ of the crossed 2-fold extension $\mathrm{e}_{(D(\tau,\eta),\sigma)}$ of Q by $\mathrm{U}(K)$ associated to a chosen representative η defines a map $t\colon \mathrm{coker}(\nu)^Q \to \mathrm{H}^3(Q,\mathrm{U}(K))$, the *Teichmueller map* associated with the data. Since for $\eta_1,\eta_2 \in \mathrm{U}(K)$, the tensor product algebra $D(\tau,\eta_1) \otimes_K D(\tau,\eta_2)$ is the algebra of $(n\times n)$-matrices over $D(\tau,\eta_1\eta_2)$, the Teichmueller map t is a homomorphism of abelian groups.

Up to this stage the discussion is elementary except, perhaps, the quote of Hilbert's "Satz 90". Now we borrow some classical algebra: Recall the Brauer group $\mathrm{B}(K)$ of K consists of classes of central simple K-algebras, two such algebras being equivalent when they are matrix algebras over the same division algebra, the inverse being induced by the assignment to an algebra of its opposite algebra. A field $L|K$ *splits* the central simple K-algebra A when $A \otimes_K L$ is L-isomorphic to a matrix algebra over L. It is common to denote by $\mathrm{B}(L|K)$ the subgroup of Brauer classes that are split by L. For rings more general than fields, the appropriate equivalence relation is Morita equivalence. The properties of being Q-normal and Q-equivariant are properties of the Brauer classes, the Q-action on K induces an action of Q on $\mathrm{Br}(K)$, and the Q-invariants $\mathrm{Br}(K)^Q$ constitute the subgroup of Brauer classes of Q-normal central simple K-algebras. In terms of the canonical isomorphisms $\mathrm{H}^2(Q,\mathrm{U}(K)) \to \mathrm{Br}(K|\mathfrak{k})$, $\mathrm{H}^2(G,\mathrm{U}(L)) \to \mathrm{Br}(L|\mathfrak{k})$ and $\mathrm{H}^2(N,\mathrm{U}(L)) \to \mathrm{Br}(L|K)$, with the notation sc for "scalar extension", the classical five-term exact sequence in the cohomology of the (central) group extension of Q by N with coefficients in $\mathrm{U}(L)$ takes the form

$$\mathrm{Br}(K|\mathfrak{k}) \rightarrowtail \mathrm{Br}(L|\mathfrak{k}) \xrightarrow{\mathrm{sc}} \mathrm{Br}(L|K)^Q \xrightarrow{t} \mathrm{H}^3(Q,\mathrm{U}(K)) \xrightarrow{\inf} \mathrm{H}^3(G,\mathrm{U}(L)). \tag{21}$$

To reconcile this sequence with the above remarks about the vanishing of the Teichmueller class of a Q-equivariant central S-algebra we note that, by "Galois descent", every Q-equivariant central simple K-algebra arises by scalar extension from a central simple $\mathfrak{k}$-algebra.

To arrive at explicit examples, let K be an algebraic number field (a finite-dimensional extension of the field $\mathbb{Q}$ of rational numbers) and, as before, let Q be a finite group of operators on K and $\mathfrak{k} = K^Q$. Let m denote the l.c.m. of the local degrees $[K_{\mathcal{P}} : \mathfrak{k}_p]$ as p ranges over the primes of $\mathfrak{k}$ and $\mathcal{P}$ over extensions thereof to K. By [Mac48, Theorem 3], the cokernel of scalar extension $\mathrm{sc}\colon \mathrm{Br}(\mathfrak{k}) \to \mathrm{Br}(K)^Q$ is a finite cyclic group of order $s = \frac{[K:\mathfrak{k}]}{m}$. Let $L = K(\zeta)$ be a cyclotomic extension having Galois group N cyclic of order $n \ge 2$ (say), that is, ζ is a primitive ℓth root of unity for some ℓ prime to n, and the Galois group N of order n acts faithfully and transitively on the primitive ℓth roots of unity. The field $L|\mathfrak{k}$ coincides with the composite field $\mathfrak{k}(\zeta)K$ in L, and the canonical action of the pullback group $\mathrm{Gal}(\mathfrak{k}(\zeta)|\mathfrak{k}) \times_{\mathrm{Gal}(\mathfrak{k}(\zeta)\cap K|\mathfrak{k})} Q$ on $\mathfrak{k}(\zeta)K$ identifies this group with a finite group of operators on $L = \mathfrak{k}(\zeta)K$ having $\mathfrak{k}$ as its fixed field. Hence $L|\mathfrak{k}$ is a Galois extension having Galois group G canonially isomorphic to the pullback group $\mathrm{Gal}(\mathfrak{k}(\zeta)|\mathfrak{k})\times_{\mathrm{Gal}(\mathfrak{k}(\zeta)\cap K|\mathfrak{k})} Q$, and G is a central extension of Q by the cyclic group $N = \mathrm{Gal}(K(\zeta)|K) \cong \mathrm{Gal}(\mathfrak{k}(\zeta)|(\mathfrak{k}(\zeta)\cap K))$ of order n, a split extension if and only if $\mathfrak{k}(\zeta)\cap K = \mathfrak{k}$.

Every class in $\mathrm{Br}(K)$ has a cyclic cyclotomic splitting field but, beware, this only says that, for a central simple K-algebra A, some cyclic cyclotomic field splits a matrix algebra over A. On the other hand, it implies that every class in $\mathrm{Br}(K)$ and in particular in $\mathrm{Br}(K)^Q$ has a representative of the kind $D(\tau,\eta)$. Thus we may choose a member η of K and, for some $n \ge 2$, a cyclic degree n cyclotomic Galois extension $L = K(\zeta)$ of K such that the image $t[D(\tau,\eta] \in \mathrm{H}^3(Q,\mathrm{U}(K))$ of the class $[D(\tau,\eta)] \in \mathrm{Br}(L|K)^Q$ of the Q-normal central K-algebra $D(\tau,\eta)$ generates the group $\mathrm{H}^3(Q,\mathrm{U}(K))$; such a member η is unique up to multiplication by some $\lambda \in \mathfrak{k}$ and by $\nu(\vartheta)$ for some $\vartheta \in L$. From the exactness of (21) we deduce that the Teichmueller map t fits into the exact sequence

$$\mathrm{Br}(\mathfrak{k}) \xrightarrow{\mathrm{sc}} \mathrm{Br}(K)^Q \overset{t}{\twoheadrightarrow} \mathrm{H}^3(Q,\mathrm{U}(K)) \cong \mathbb{Z}/s \tag{22}$$

but beware, exactness only implies that the class of a Q-normal algebra having trivial Teichmueller class arises by scalar extension, not necessarily the algebra itself. An example of such an algebra that does not arise by scalar extension while its class does is in [Tei40]. The images $t[D(\tau,\eta)], t[D(\tau,\eta^2)], \dots, t[D(\tau,\eta^{n-1})] \in \mathrm{H}^3(Q,\mathrm{U}(K))$

exhaust the nontrivial members of the group $\mathrm{H}^3(Q, \mathrm{U}(K))$, and the algebras $D(\tau, \eta^j\lambda)$, for $1 \leq j \leq n-1$ and $\lambda \in \mathfrak{k}$, cover all Brauer classes of Q-normal algebras split by L with nontrivial Teichmueller class and, when we let L vary, we obtain all Brauer classes of Q-normal algebras with nontrivial Teichmueller class. Thus, to get examples, all we need is a Galois extension $K|\mathfrak{k}$ having $s > 1$. While, in view of the Hilbert–Speiser theorem, this is impossible when the Galois group Q is cyclic, for example, the fields $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$ or $K = \mathbb{Q}(\sqrt{2}, \sqrt{17})$ have as Galois group the four group and $s = 2$.

Again we see that crossed modules having nonzero characteristic class and in particular nonextendible abstract kernels abound.

11. Outlook

A topological group is a group in the category of topological spaces. Groups in the category Cat of small categories—equivalently, categories internal to groups—constitute a category, that of 2-*groups*, and there is an equivalence of categories between crossed modules and groups in Cat, observed by the Grothendieck school in the mid 1960s (unpublished), see [BHS11, Section I.1.8, p. 29; Section 2.7, p. 58]. It is an interesting exercise to see how the Peiffer identities fall out from this equivalence.

Crossed modules, variants, and generalizations thereof are nowadays very lively in mathematics; see [BHS11], [Hue21, Section 3] and the references there. The equivalence of crossed modules (in the category of groups) and 2-groups is relevant in string theory, see [Hue21, Section 3] for references. Suffice it to mention that when we pass from particles to strings, we add an extra dimension, and replacing groups by groups in Cat reflects this adding an extra dimension.

For a leisurely introduction and survey of the state of the art at the time consider [BH82]. A particularly important work is [BHS11], with its special emphasis on foundational issues.

ACKNOWLEDGMENTS. I am indebted to AMS *Notices* editor Erica L. Flapan for encouraging me to compose this article and to render it accessible to the non-expert, and to the referee and to Jim Stasheff for a number of comments. This work was supported in part by the Agence Nationale de la Recherche under grant ANR-11-LABX-0007-01 (Labex CEMPI).

References

[Bae34] Reinhold Baer, *Erweiterung von Gruppen und ihren Isomorphismen* (German), Math. Z. **38** (1934), no. 1, 375–416, DOI 10.1007/BF01170643. MR1545456

[BH82] R. Brown and J. Huebschmann, *Identities among relations*, Low-dimensional topology (Bangor, 1979), London Math. Soc. Lecture Note Ser., vol. 48, Cambridge Univ. Press, Cambridge-New York, 1982, pp. 153–202. MR662431

[BHS11] Ronald Brown, Philip J. Higgins, and Rafael Sivera, *Nonabelian algebraic topology*, EMS Tracts in Mathematics, vol. 15, European Mathematical Society (EMS), Zürich, 2011. Filtered spaces, crossed complexes, cubical homotopy groupoids; With contributions by Christopher D. Wensley and Sergei V. Soloviev, DOI 10.4171/083. MR2841564

[CH82] D. J. Collins and J. Huebschmann, *Spherical diagrams and identities among relations*, Math. Ann. **261** (1982), no. 2, 155–183, DOI 10.1007/BF01456216. MR675732

[EM47] Samuel Eilenberg and Saunders MacLane, *Cohomology theory in abstract groups. II. Group extensions with a non-Abelian kernel*, Ann. of Math. (2) **48** (1947), 326–341, DOI 10.2307/1969174. MR20996

[Hue80a] Johannes Huebschmann, *Crossed n-fold extensions of groups and cohomology*, Comment. Math. Helv. **55** (1980), no. 2, 302–313, DOI 10.1007/BF02566688. MR576608

[Hue80b] Johannes Huebschmann, *The first k-invariant, Quillen's space BG^+ and the construction of Kan and Thurston*, Comment. Math. Helv. **55** (1980), no. 2, 314–318, DOI 10.1007/BF02566689. MR576609

[Hue18a] Johannes Huebschmann, *Normality of algebras over commutative rings and the Teichmüller class. I*, J. Homotopy Relat. Struct. **13** (2018), no. 1, 1–70, DOI 10.1007/s40062-017-0173-3. MR3769365

[Hue18b] Johannes Huebschmann, *Normality of algebras over commutative rings and the Teichmüller class. III*, J. Homotopy Relat. Struct. **13** (2018), no. 1, 127–142, DOI 10.1007/s40062-017-0175-1. MR3769367

[Hue21] Johannes Huebschmann, *On the history of Lie brackets, crossed modules, and Lie-Rinehart algebras*, J. Geom. Mech. **13** (2021), no. 3, 385–402, DOI 10.3934/jgm.2021009. MR4322147

[Huf10] Darrel Huff, *How to Lie with Statistics*, W. W. Norton, 1954, 2010. Pictures by Irving Geis.

[Lyn50] Roger C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. of Math. (2) **52** (1950), 650–665, DOI 10.2307/1969440. MR47046

[Mac48] Saunders MacLane, *Symmetry of algebras over a number field*, Bull. Amer. Math. Soc. **54** (1948), 328–333, DOI 10.1090/S0002-9904-1948-08996-0. MR25442

[Mac79] Saunders MacLane, *Historical note*, J. Algebra **60** (1979), no. 2, 319–320.

[Pei49] Renée Peiffer, *Über Identitäten zwischen Relationen* (German), Math. Ann. **121** (1949), 67–99, DOI 10.1007/BF01329617. MR32640

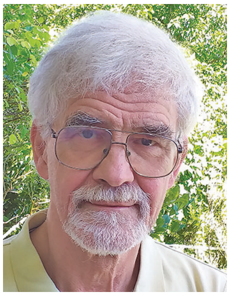
[Rei49] Kurt Reidemeister, *Über Identitäten von Relationen* (German), Abh. Math. Sem. Univ. Hamburg **16** (1949), no. nos. 3-4, 114–118, DOI 10.1007/BF03343521. MR32639

[Tei40] Oswald Teichmüller, *Über die sogenannte nichtkommutative Galoissche Theorie und die Relation $\xi_{\lambda,\mu,\nu}\xi_{\lambda,\mu\nu,\pi}\xi^{\lambda}_{\mu,\nu,\pi} = \xi_{\lambda,\mu,\nu\pi}\xi_{\lambda,\mu,\nu,\pi}$* (German), Deutsche Math. **5** (1940), 138–149. MR2858

[Tur38] A. M. Turing, *The extensions of a group*, Compositio Math. **5** (1938), 357–367. MR1557005

[Whi41] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2) **42** (1941), 409–428, DOI 10.2307/1968907. MR4123

[Whi49] J. H. C. Whitehead, *Combinatorial homotopy. II*, Bull. Amer. Math. Soc. **55** (1949), 453–496, DOI 10.1090/S0002-9904-1949-09213-3. MR30760



Johannes Huebschmann

Credits

The opening image is courtesy of Bach-Archiv Leipzig, Leihgabe der Leipziger Städtischen Bibliotheken / Musikbibliothek Peters (D-LEb Peters PM 5695).

Photo of Johannes Huebschmann is courtesy of Karin Huebschmann.

Applications Open for

AMS CONGRESSIONAL FELLOWSHIP 2024–2025

Apply your mathematics knowledge toward solutions to societal problems.

The American Mathematical Society will sponsor a Congressional Fellow from September 2024 through August 2025.

The Fellow will spend the year working on the staff of either a member of Congress or a congressional committee, working in legislative and policy areas requiring scientific and technical input.

The Fellow brings his/her/their technical background and external perspective to the decision-making process in Congress. Prospective Fellows must be cognizant of and demonstrate sensitivity toward political and social issues and have a strong interest in applying personal knowledge toward solutions to societal problems.

Now in its 19th year, the AMS Congressional Fellowship provides a unique public policy learning experience, and demonstrates the value of science–government interaction. The program includes an orientation on congressional and executive branch operations, and a year-long seminar series on issues involving science, technology, and public policy.

"Every day on the Hill is new and exciting! Whether I am writing press releases, synthesizing complicated ideas into one-pagers, or meeting influential individuals, I learn something new every day."

—Duncan Wright,
AMS Congressional
Fellow 2022–2023

Applicants must have a PhD or an equivalent doctoral-level degree in the mathematical sciences by the application deadline (February 1, 2024). Applicants must be US citizens. Federal employees are not eligible.

The Fellowship stipend is US$100,479 for the Fellowship period, with additional allowances for relocation and professional travel, as well as a contribution toward health insurance.

Applicants must submit a statement expressing interest and qualifications for the AMS Congressional Fellowship as well as a current curriculum vitae. Candidates should have three letters of recommendation sent to the AMS by the February 1, 2024 deadline.

For more information and to apply, please go to **www.ams.org/ams-congressional-fellowship**.

Deadline for receipt of applications: **February 1, 2024**

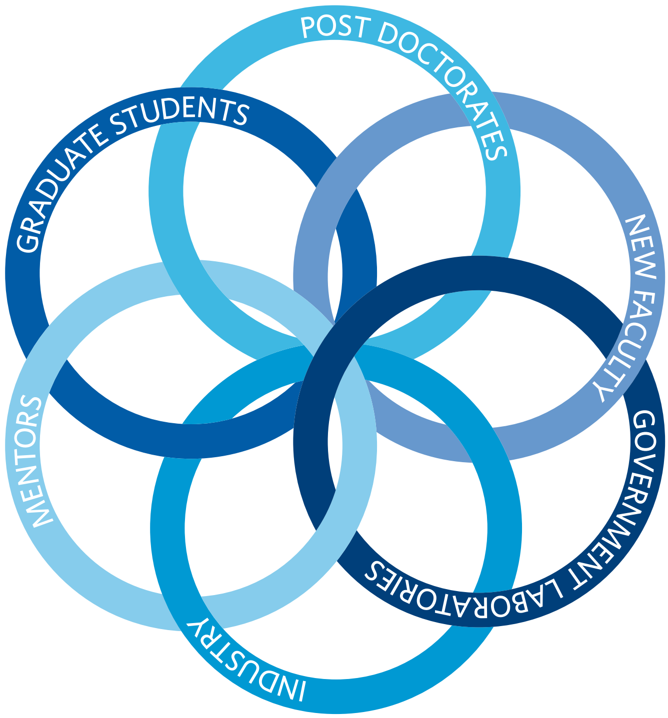
AMS
AMERICAN MATHEMATICAL SOCIETY
Advancing research. Creating connections.

jmm 2024

Learn more at the JMM 2024 session on DC-based Policy and Communications Opportunities to be held Friday, January 5, 2024 at 4:30 p.m.

EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be What Does a Mathematician Do?



Math Institutes

Introduction

Throughout our careers, we (the Early Career editors) have benefited greatly from participating in programs at math institutes both in the US and abroad. In this issue's Early Career, we have compiled a collection of short Q&As with seven major institutes in the hope that Early Career readers might become better acquainted with some of the opportunities available to them. We talked to the following US-based institutes:

- AIM in Pasadena, CA
- ICERM in Providence, RI
- IMSI in Chicago, IL
- IPAM in Los Angeles, CA
- SLMath (formerly MSRI) in Berkeley, CA

as well as the international institutes

- BIRS, with locations in Canada (Banff), Mexico (Oaxaca), China (Hangzhou), Spain (Granada) and India (Chennai)
- IHES in France (Bures-sur-Yvette).

This is by no means a comprehensive collection, and we may run occasional Q&As in future issues should more institutes like to be involved.

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2825

The American Institute of Mathematics (AIM)

David Farmer

The American Institute of Mathematics (AIM) is an NSF Mathematical Science Institute located in the Richard Merkin Center for Pure and Applied Mathematics on the Caltech campus in Pasadena, California.

1. What is the Mission of Your Institute? What Makes it Unique?

The mission of AIM is to advance mathematical knowledge through collaboration, to broaden participation in the mathematical endeavor, and to increase the awareness of the contributions of the mathematical sciences to society. In practical terms, this means that AIM sponsors research programs for mathematicians and researchers in related fields, while supporting outreach programs for K–12 students and teachers, as well as the general public.

The research programs include week-long focused workshops in all areas of the mathematical sciences to make progress on current open problems; the SQuaREs program which brings together a group of 4–6 people to AIM for a week, with the possibility of returning in subsequent years for a total of three visits; and more recently, the AIM

David Farmer is the director of programs at AIM. His email address is `farmer@aimath.org`.

DOI: https://doi.org/10.1090/noti2819

Research Communities program, which are long-term virtual collaborative efforts involving at least 40 people; they are organized around a particular area of mathematics research.

The outreach programs include the Math Circle Network for both students and teachers (`https://mathcircles.org/`) and Math Communities, which is a coalition of organizations fostering joyful math experiences (`https://mathcommunities.org/`).

AIM's commitment to focused collaborative research, combined with its commitment to supporting diversity in every respect, means that our programs are relevant to all researchers. Because Workshops and SQuaREs are week-long programs, we are able to host more than 800 researchers each year, and many mathematicians with more demanding teaching and service responsibilities are still able to participate in AIM programs. Furthermore, Workshops bring together a diverse group of participants, and the workshop model fosters building new collaborations. Workshop activities are deliberately designed to connect mathematicians from different career stages, backgrounds, and locations.

2. What Kinds of Programs are Run Each Year?

Every year AIM runs approximately 20 Workshops, 60 SQuaRE meetings, several AIM Research Communities, and numerous outreach activities.

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply for?

All AIM workshops have open applications to participate, and early career researchers are encouraged to apply. Typically, participants have completed their PhD or are near the end of their graduate program. Approximately 25–30% of workshop participants are chosen from the open application process.

SQuaRE participants have completed their PhD and apply as a group of 4–6 people. The membership of the SQuaRE is fixed at the time the proposal is accepted.

Some AIM Research Communities accept applications. For each of these research activities, early career mathematicians have had proposals accepted. AIM outreach activities welcome contributions from anyone interested in shared mathematical experiences, with specific activities tailored to the participant.

4. What Type of Support is Available (Including Childcare Support)?

Full support (travel, accommodation, and meals) is provided to all funded Workshop and SQuaRE participants. Some Research Community participants receive a stipend to support their participation, which can be used for equipment (e.g., a tablet to access shared virtual whiteboards), childcare, or other needs. For participants with childcare needs, AIM offers additional financial support (`https://aimath.org/visitors/child-care/`).

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

Every **AIM Workshop** has spots reserved for open applications. This is an excellent opportunity for early career researchers to broaden their scholarly focus, identify new problems and research directions, begin new projects during the week, and ideally leave with an active new collaboration. If there is a workshop in your research area, please apply!

SQuaREs programs support ambitious long-term mathematical endeavors in small groups of 4–6 researchers. Like workshops, SQuaRE groups are encouraged to include collaborators who are diverse in terms of gender, race and ethnicity, career stage, and institution type. Many SQuaREs have been formed by collaborations that started from an AIM workshop; however, the reverse is also true and work done in a SQuaRE can lead to the creation of an AIM Workshop. The SQuaRE program was created because of the successes from our Workshops and to provide a place for groups to work where their only responsibility is collaborating on research with minimal distractions. The multi-year focus of a SQuaRE encourages researchers to work on ambitious projects.

AIM's **Research Communities** program offers an opportunity to participate remotely as a member of an active, virtual scholarly community. Whether it is regular seminars, discussions, research group meetings, or social gatherings, it is a valuable way to remain connected and engaged with research colleagues during the academic year when travel is more limited. For early career researchers at smaller institutions that have few (or no) colleagues who share their professional interests, this is an excellent way to maintain research activity.

The **Math Communities** organization provides a valuable gateway into outreach involvement. It matches professional mathematicians and educators with schools, higher education institutions, and community organizations who serve pre-K–12 students, families, and teachers with a focus on joyful, collaborative activities. This is especially important for early career mathematicians who are seeking a meaningful element of outreach consistent with their research program, particularly when applying for funding from an agency which expects an outreach component to a research proposal.

Credits

Logo is courtesy of The American Institute of Mathematics.

Banff International Research Station (BIRS)

Malabika Pramanik

1. What is the Mission of Your Institute? What Makes it Unique?

The Banff International Research Station (BIRS)[1] hosts programs that advance the frontiers of research in all aspects of the mathematical sciences. These programs encompass the most fundamental challenges and breakthroughs in pure and applied mathematics, theoretical and applied computer science, statistics and data science, mathematical physics, financial and industrial mathematics, as well as the mathematics of information technology, environmental and life sciences.

BIRS's signature is its mission of international scientific collaboration, as evidenced by its presence in five countries: Canada (Banff), Mexico (Oaxaca), China (Hangzhou), Spain (Granada) and India (Chennai). Each centre is located in a unique retreat-like atmosphere, providing a distraction-free and intellectually stimulating environment that facilitates complete immersion in scientific discourse. Each year, an international panel of distinguished scientific experts representing a wide range of subject areas select proposals to run in these centres, with a view to serving researchers of all career stages, and bringing regional expertise in contact with global ones. Accommodation, meals, and access to research and conference facilities are provided at no cost to event participants.

2. What Kinds of Programs are Run Each Year?

BIRS hosts many different types of programs, tailored to ensure maximum flexibility for audiences with diverse needs.

- The principal BIRS activities consist of *5-day workshops*, held year-round (48 weeks) at BIRS's primary centre in Banff, Alberta. Other BIRS-affiliate centres host 10–20 workshops per year. These workshops showcase mathematical discovery and innovation at the highest level.
- *2-day weekend workshops* are suitable for small regional conferences, academic or policy-making summits, or academia–industry collaborations.
- *Research in Teams (RIT)* and *Focused Research Groups (FRG)* are popular BIRS programs, where small groups of participants live and conduct research together without a conference format for periods of 1–2 weeks.
- *Summer schools* and *graduate training camps* are held at BIRS once or twice per year.
- BIRS hosts *K–12 math camps* in collaboration with Alberta schools, *teaching and pedagogy workshops* for mathematics educators, and also hosts *training sessions for Team Canada* in International Mathematics Olympiads.
- 5-month-long *hybrid thematic programs* are part of BIRS's post-pandemic offerings. The goal of a hybrid thematic program is to build global communities in exciting new fields in mathematical sciences, through a long-term immersive experience. It consists of a regular flow of purely virtual events, interspersed with up to two in-person events at Banff, five days each.
- *BIRS Now!* events cover topics that are timely, urgent foci of unexpected developments and in need of rapid response from scientific research. Event formats are flexible to accommodate different needs. Possible topics include, but need not be limited to, clean energy and Net Zero Accelerator, infectious disease modelling, vaccine economics and biomanufacturing, responsible development of artificial intelligence (AI), assistive AI in mathematical reasoning, national quantum strategy, cybersecurity, blockchain, digital identity, and other strategic and emerging fields.
- *PIMS/BIRS Team Up!* provides opportunities for in-person collaboration to teams of mathematical scientists with some representation from universities affiliated with the Pacific Institute for the Mathematical Sciences (PIMS). This program targets researchers whose research programs have been disproportionately affected by various obstacles like family obligations, professional isolation, access to funding, and the COVID-19 pandemic. This includes, but need not be limited to, women, people of color, gender-expansive and other minoritized groups, Indigenous scholars, individuals with visible/invisible challenges, and early career researchers with limited resources.

See the general program descriptions webpage[2] and links therein for a full list of BIRS programs, possible venues and eligibility criteria.

Malabika Pramanik is a professor of mathematics at the University of British Columbia and scientific director of BIRS. Her email address is `malabika@math.ubc.ca`.

DOI: https://doi.org/10.1090/noti2820

[1]*BIRS is supported by the National Science Foundation (NSF), Natural Sciences and Engineering Research Council of Canada (NSERC) and Alberta Jobs, Economy and Innovation.*

[2]`https://www.birs.ca/programs/general-program-descriptions`

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply For?

All BIRS events are expected to include early career researchers (ECR-s). An ECR is defined as a graduate student (masters or PhD), or a researcher within ten years of their doctoral degree. Since 2022, at least 40% of all BIRS visitors have been ECR-s.

- 5-day workshops are *required* to have at least one ECR on its leadership team (i.e., the organizing committee). Training and mentoring of ECR-s are critical components of these workshops, as can be seen from the assessment rubrics available on the proposal guidelines webpage.[3] A strong representation of ECR-s in the participant list and a program structure that incorporates professional support and guidance for ECR-s are two key metrics in the evaluation of BIRS proposals. Participation in 5-day BIRS workshops is by invitation only; ECR-s interested in participating should contact the workshop organizers directly to check availability of spots.
- Summer schools and training camps are intended for graduate students and/or advanced undergraduates. Instructional assistance for these schools and camps is often provided by postdoctoral fellows and tenure-track faculty, who serve as role models to the students, and offer invaluable career advice and mentorship along with mathematical training. The summer schools are also beneficial to the professional development of the ECR instructors.
- Postdoctoral fellows and tenure-track faculty can apply as organizers and/or participants for RIT, FRG, and TeamUp programs. Such applications have concrete outcomes that are achievable in the short term, and are therefore particularly helpful to ECR-s looking to finish a project or submit a time-sensitive grant proposal with multiple stakeholders.

4. What Type of Support is Available (Including Childcare Support)?

- All BIRS centres offer support in the form of *free accommodation, meals, and access to research and videoconferencing facilities* for the duration of the program. BIRS *cannot* offer travel support, but organizers sometimes secure external travel funding for participants who do not hold research grants. Participants are also encouraged to apply for travel funding opportunities through partnership agreements if they are eligible. The travel support webpage[4] contains more information on this.
- Lectures and supplementary workshop material are *recorded, live-streamed, and uploaded to a publicly accessible domain* for all 5-day BIRS workshops.
- BIRS provides *technical support for a seamless hybrid experience* for all workshops.
- Some funding is available for workshop participants who are primary caregivers. *Childcare support*[5] at BIRS in Banff needs preapproval and is subject to the availability of funds. Applications should be submitted at least six weeks before the event start date and are assessed on a case-by-case basis. BIRS day-care support covers the duration of the workshop. It extends to daycare facilities within the town of Banff, or to individuals hired by workshop participants to care for their children on BIRS premises. Such hires are made at the sole discretion of the participating caregiver.

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

ECR-s are key architects of scientific progress. Their fresh perspectives are essential for transformative breakthroughs. The success of any research area relies on effective training, mentoring, and support of its ECR-s. BIRS attracts, nurtures, and showcases young talent in a variety of ways.

- ECR-s enrich the program structure of 5-day BIRS workshops through activities like *poster presentations, lightning talks, video contests, hackathons, professional panels, and one-on-one or group mentoring sessions*. If you are an ECR participating in an upcoming BIRS workshop, consider proposing an ECR-friendly activity to the organizers.
- Annual BIRS *summer schools* are dedicated exclusively to the training of ECR-s. The summer schools are great opportunities for networking, peer-to-peer interaction and community-building. The most recent summer schools hosted by BIRS were on geometry, combinatorics, and optimization[6] (2022) and number theory[7] (2023) respectively.
- BIRS often organizes *industry-academia collaborations* like *"Career & Innovation Hub,"*[8] *career fairs, recruitment booths, training sessions for improving effective communication skills*[9] to assist the future mathematical workforce in finding job opportunities that demand advanced quantitative and analytical skills.
- New research collaborations involving a team of young researchers initiated at a BIRS 5-day workshop

[3] https://proposals.birs.ca/guidelines

[4] https://www.birs.ca/participants/travel-support/

[5] https://www.birs.ca/participants/childcare

[6] https://www.birs.ca/events/2022/summer-schools/22ss199

[7] https://www.birs.ca/events/2023/summer-schools/23ss001

[8] https://careerandinnovationhub.ca/

[9] https://caims.ca/news/caims-birs-effective-communication-skills-for-applied-mathematicians-workshop/

or at a BIRS summer school are eligible for subsequent *group follow-ons* hosted by BIRS, in the form of FRG, RIT, or TeamUp programs. These programs enable ECR-s to take on an organizational role in a small group setting, and gain experience in proposing small steps towards bigger goals. These collaborative follow-ons create a sustained support framework for ECR-s stemming from a previously held BIRS event, leading to longer-term professional interactions and greater impact.

Credits

Logo is courtesy of Banff International Research Station.

The Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University

Brendan Hassett

The Institute for Computational and Experimental Research in Mathematics (ICERM) is a research institute at Brown University, with core funding from the Division of Mathematical Sciences of the National Science Foundation[1] and major support from the Simons Foundation. It was started in 2010 by Brown professors Jill Pipher (founding director), Jeffrey Brock (now at Yale), Jan Hesthaven (now at the École polytechnique fédérale de Lausanne), Jeffrey Hoffstein, and Bjorn Sandstede.

1. What is the Mission of Your Institute? What Makes it Unique?

The vision for ICERM is encapsulated in its mission statement:

> The mission of the Institute for Computational and Experimental Research in Mathematics (ICERM) is to support and broaden the relationship between mathematics and computation: specifically, to expand the use of computational and experimental methods in mathematics, support theoretical advances related to computation, and address problems posed by the existence and use of the computer through mathematical tools, research and innovation.

ICERM supports its mission by developing and hosting research programs and activities that:

- Encourage the creation of new computational methods to advance mathematical understanding.
- Foster a deeper understanding of algorithms and computational tools.
- Expose program participants to the use of simulation, visualization, experiments, or computer-assisted proofs.
- Catalyze new directions of mathematical research through synergistic collaborations across disciplinary areas and research communities.
- Advance the training and mentoring of graduate students and early career postdoctoral researchers through exposure to new mathematical areas and computational methods.

ICERM is committed to disseminating the full range of mathematical sciences scholarship, including algorithms, code and software packages, computational methods, computer-assisted proofs, databases of mathematical objects, and examples and counterexamples. Our programs do include lectures on proofs of new theorems, but also demonstrations of software and algorithms. These activities offer students and early career researchers many avenues for participation in research.

Our programs address topics in both pure and applied mathematics, as well as applications of mathematical and statistical techniques to fields like computer science, physics, mathematical biology, climate modeling, neuroscience, etc. Some of our most publicly visible programs have focused on challenges outside STEM, like the *Illustrating Mathematics* semester in 2019 (focusing on design and the arts) and our program *Data Science and Social Justice: Networks, Policy, and Education* which ran in two parts over the summers of 2022 and 2023.

2. What Kinds of Programs are Run Each Year?

Our largest programs are the fall and spring semester programs. These occupy the full resources of the institute and run roughly in parallel with academic semesters. Examples of recent and upcoming events include:

- Spring 2022: Braids[2]
- Fall 2022: Harmonic Analysis and Convexity[3]
- Spring 2023: Discrete Optimization: Mathematics, Algorithms, and Computation[4]

Brendan Hassett is the Jonathan Nelson University Professor of Mathematics at Brown University and the director of the Institute for Computational and Experimental Research in Mathematics. His email address is `brendan_hassett@brown.edu`.

DOI: https://doi.org/10.1090/noti2822

[1]*ICERM is supported by the National Science Foundation under Grant No. DMS-1929284.*

[2]`https://icerm.brown.edu/programs/sp-s22/`

[3]`https://icerm.brown.edu/programs/sp-f22/`

[4]`https://icerm.brown.edu/programs/sp-s23/`

- Fall 2023: Math + Neuroscience: Strengthening the Interplay Between Theory and Mathematics[5]
- Spring 2024: Numerical PDEs: Analysis, Algorithms, and Data Challenges[6]
- Fall 2024: Theory, Methods, and Applications of Quantitative Phylogenomics[7]
- Spring 2025: Geometry of Materials, Packings, and Rigid Frameworks[8]

Semester programs have many long-term visitors, including program organizers, postdocs hired for the duration of the term, research fellows designated by the organizers, and participants chosen based on applications to the program. Research fellows are typically established in the field; along with the organizers, they set scientific directions for the program and guide early career researchers. Graduate students and postdocs are welcome as long-term visitors; all are assigned mentors from the program.

There are also shorter-term visitors, including people in residence for one of the three embedded workshops associated with each semester program. In fall 2023, these included:

- Mathematical Challenges in Neuronal Network Dynamics[9]
- Topology and Geometry in Neuroscience[10]
- Neural Coding and Combinatorics[11]

Recently we have introduced multi-week research programs during the summer. In addition to the Social Justice program mentioned above, we have offered pandemic-era semester programs the opportunity for summer reunions:

- Model and Dimension Reduction in Uncertain and Dynamic Systems (reunion for spring 2020)[12]
- Advances in Computational Relativity (reunion for fall 2020)[13]
- Combinatorial Algebraic Geometry (reunion for spring 2021)[14]

Historically, our longest summer research programs are our undergraduate research programs, called Summer@ICERM. Examples include:

- 2021: Computational Polygonal Billiards[15]
- 2022: Computational Combinatorics[16]
- 2023: Mathematical Modeling of DNA Self-Assembly[17]

ICERM also hosts free-standing workshops, not associated with longer programs. In summer 2023, these included:

- Dynamics, Rigidity, and Arithmetic in Hyperbolic Geometry[18]
- Tangled in Knot Theory[19]
- Mathematical and Scientific Machine Learning[20]
- Mathematical and Computational Biology[21]
- Modern Applied and Computational Analysis[22]
- Murmurations in Arithmetic[23]
- LMFDB, Computation, and Number Theory (LuCaNT)[24]
- Acceleration and Extrapolation Methods[25]

Some of these are five-day Topical Workshops, scheduled well in advance to bring together leading experts or support collaborative networks. Others are short Hot Topics Workshops, running on short notice to highlight recent breakthroughs or emerging fields.

Finally, Collaborate@ICERM is a mechanism for small groups of researchers (3–6 people) to spend a week at the institute to develop or complete a collaborative research project aligned with institute missions.

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply for?

All ICERM programs are open to graduate students and postdocs! Most attend semester programs and workshops by applying. For shorter stays, these are handled through Cube, ICERM's participant management system. However, postdoctoral positions for semester programs are advertised through MathJobs during the fall hiring season. ICERM has a handful of yearlong Institute Postdocs that permit a postdoc affiliated with one of our semester programs the opportunity to spend the full academic year in residence.

4. What Type of Support is Available (Including Childcare Support)?

Hired postdocs receive a stipend for the duration of their affiliation. Other long-term semester program visitors receive lodging and travel reimbursements. For workshops, applicants receive reimbursements for lodging and travel

[5] https://icerm.brown.edu/programs/sp-f23/
[6] https://icerm.brown.edu/programs/sp-s24/
[7] https://icerm.brown.edu/programs/sp-f24/
[8] https://icerm.brown.edu/programs/sp-s25/
[9] https://icerm.brown.edu/programs/sp-f23/w1/
[10] https://icerm.brown.edu/programs/sp-f23/w2/
[11] https://icerm.brown.edu/programs/sp-f23/w3/
[12] https://icerm.brown.edu/events/re-22-sp20/
[13] https://icerm.brown.edu/events/re-22-f20/
[14] https://icerm.brown.edu/events/re-23-s21/
[15] https://icerm.brown.edu/summerug/2021/
[16] https://icerm.brown.edu/summerug/2022/
[17] https://icerm.brown.edu/summerug/2023/
[18] https://icerm.brown.edu/topical_workshops/tw-23-drahg/
[19] https://icerm.brown.edu/topical_workshops/tw-23-tkt/
[20] https://icerm.brown.edu/topical_workshops/tw-23-msml/
[21] https://icerm.brown.edu/topical_workshops/tw-23-mcb/
[22] https://icerm.brown.edu/topical_workshops/tw-23-maca/
[23] https://icerm.brown.edu/events/htw-23-ma/
[24] https://icerm.brown.edu/events/sc-23-lucant/
[25] https://icerm.brown.edu/topical_workshops/tw-23-aem/

in most cases, although we cannot fully cover the expenses of every applicant, especially those traveling from outside the United States.

ICERM offers financial support for childcare and other dependent care expenses for selected programs. We cannot charge these costs to our grants but can use gifts from individuals and corporations for fellowships applicable to these expenses. However, we only have enough funding to offer these for a fraction of our programs.

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

Essentially all ICERM programs originate from proposals from the mathematical sciences community. We are always seeking innovative and creative programs that will support the growth of new collaborations and networks. Early career researchers, whose connections with collaborators and networks are growing most rapidly, are at the heart of most events we host.

Most Collaborate@ICERM groups involve early career researchers, including graduate students. This is a good "entry-level" institute proposal. Workshop organizing committees often include untenured faculty who are aware of recent developments made by their peers. Summer@ICERM organizing committees usually have a mix of tenured and tenure-track faculty, and sometimes postdoctoral researchers; we hire graduate students and postdocs each year as TAs to support the undergraduate research groups.

If you have an idea, contact ICERM's directors!

Credits

Logo is courtesy of The Institute for Computational and Experimental Research in Mathematics.

Institut des Hautes Études Scientifiques (IHES)

Giulia Foffano

The Institut des Hautes Études Scientifiques (IHES) is an advanced research institute in mathematics, theoretical physics, and all related sciences. It is located in Bures-sur-Yvette, France.

1. What is the Mission of Your Institute? What Makes it Unique?

IHES aims to promote and encourage theoretical scientific research in the fields of mathematics, theoretical physics, and all related theoretical disciplines, by providing the Institute's researchers, both permanent and invited, with the resources required to undertake research, as well as by organizing conferences and seminars.

IHES revolves around a small group of six permanent professors, who are at the very top of their respective research fields worldwide. Since its founding in 1958, IHES has been able to count nine Fields Medalists among its professors, three of whom are currently at the Institute. IHES is also a founding member of Université Paris-Saclay, ranked first university worldwide in mathematics and first in physics in Europe, according to the Academic Ranking of World Universities.

In addition to its exceptional scientists and environment, with complete freedom of research, what makes IHES unique is also its small size, which allows for numerous opportunities for researchers of different backgrounds to meet informally—over lunch or tea, or at the Ormaille residence, where most of them are housed. These moments are an incredible source of new ideas and collaborations.

2. What Kinds of Programs are Run Each Year?

General visiting program.[1] At the core of the Institute's activity is its general visiting program. Every year IHES welcomes about 200 researchers from across the world who come to the Institute to benefit from its unique working conditions and the possibility to interact with some of the best minds in their domain. In 2022 IHES welcomed

Giulia Foffano is the communication and development officer of IHES. Her email address is giulia.foffano@ihes.fr.

DOI: https://doi.org/10.1090/noti2823

[1] https://www.ihes.fr/en/applications/#invited-researchers-2

researchers from 31 different countries and representing 36 nationalities who spent about three months on average at the Institute.

Visiting researchers are selected by the Institute's Scientific Council, which gathers twice a year, once in June and once in December, using research excellence as the sole criterion.

In addition to its general visiting program, IHES runs two special programs: these are the Carmin program and the Simons program for researchers from African countries.

The **Carmin program**[2] is a partnership with other French research institutes in mathematics (CIRM,[3] CIMPA,[4] IHP[5]) within a laboratory of excellence financed by the French National Research Agency's "Investments for the Future" program.[6]

Every year, three six-month positions are offered as part of this program. Successful candidates participate in the scientific life of IHES and attend one of the three thematic quarters that IHP organizes every year.

The **Simons program for researchers from African countries**[7] was launched in 2022 thanks to a generous donation by the Simons Foundation that will be carried forward over ten years. The aim of the program is to make the opportunity to visit IHES available to the largest number of researchers, including those from African countries, who are still very underrepresented at the Institute.

Postdoctoral programs.[8] The Institute also runs several programs dedicated to postdoctoral researchers. Successful candidates are selected by the Scientific Council every year in December. Post-docs are invited for two years, starting from the following academic year, with a possible extension of one year.

Seminars and conferences.[9] Every year, IHES organizes several international conferences and lecture series, such as the Cours de l'IHES,[10] designed to present recent and important scientific results in various fields of mathematics and theoretical physics. They are given by experienced researchers, not necessarily affiliated with the IHES.

Every year IHES hosts a **summer school**.[11] The series of lectures and courses that are run over an average of two weeks are great occasions for PhD students, postdocs, and young researchers to get an overview of the latest developments in key areas of mathematics and theoretical physics. Researchers who are interested in organizing a Summer School at IHES can submit their proposals to the Scientific Council. Successful proposals are selected in June.

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply for?

Researchers with a PhD in mathematics, theoretical physics, or any related subject can apply to the IHES **visiting program**[12] to spend a limited period of time at the Institute.

Early career researchers can also apply to a **postdoctoral program**.[13] Only the very best candidates are selected by the Scientific Council and are invited to come to IHES for a period of two years, that can be extended for an extra year. They are offered the possibility to live at the Ormaille residence, which is very conveniently located at a ten-minute walking distance from IHES, with all the logistical and social advantages of an on-campus residence. At the Institute, they join a very lively community of young colleagues who actively participate in the IHES scientific activity and get a chance to interact and collaborate with the IHES permanent members as well as with the numerous visiting professors at the Institute. IHES strongly encourages motivated and enthusiastic early career mathematicians and theoretical physicists to apply to its postdoctoral programs.

PhD students and postdoctoral researchers can apply to take part in the **IHES Summer School**, if the topic of the School is of their interest. All other conferences and events that take place at the Institute are generally open to all, with no need to apply, and early career researchers are very welcome to attend.

Early career researchers who are willing to organize a summer school at IHES can also submit their project[14] to the Scientific Council.

IHES also has a very ambitious hiring policy and is always searching for promising researchers to join its faculty. Both **permanent and junior professors** are actively chosen by the Scientific Council, whose members look for the very best candidates for these positions. Junior professorships in particular are prestigious five-year positions that combine the stability of a higher salary with the flexibility of a fixed-term contract, but on a longer duration compared to the typical postdoctoral fellowships. IHES first opened these positions in 2021 to attract highly talented researchers at the beginning of their career.

[2] https://www.ihes.fr/en/applications/post-docs/carmin-programme/
[3] https://www.cirm-math.com/
[4] https://www.cimpa.info/en/
[5] https://www.ihp.fr/en
[6] https://anr.fr/ProjetIA-10-LABX-0059
[7] https://www.ihes.fr/en/applications/invited-researchers-2/visiting-program-african-countries/
[8] https://www.ihes.fr/en/applications/#post-docs
[9] https://www.ihes.fr/en/events/
[10] https://www.ihes.fr/evenements/#cours
[11] https://www.ihes.fr/en/applications/#summer-school
[12] https://www.ihes.fr/en/applications/#invited-researchers-2
[13] https://www.ihes.fr/en/applications/#post-docs
[14] https://www.ihes.fr/en/applications/#summer-school

4. What Type of Support is Available (Including Childcare Support)?

Specific cases set aside, visiting researchers are granted free accommodation for themselves and their family, as well as the possibility to have lunch at the IHES cafeteria, an office, and technical assistance. They also receive administrative assistance before and during their trip, to ensure that they can devote themselves entirely to their research.

IHES does its best to assist researchers who come to the Institute with their children in their search for childcare services. Scientists on a long visit to France can take advantage of the French school system: French kindergardens, or *écoles maternelles*, accept children from three to six years of age and are free of charge. All children in France aged six or older have access to primary, middle, and high school free of charge. IHES assists families with the registration process in the local school system. Extra childcare services might nevertheless be needed for after-school hours or holidays.

It can be hard to find suitable childcare for children under three years of age, especially for scientists coming to IHES on a relatively short visit. Whenever possible, IHES helps visitors to get in touch with local childcare services. IHES is also currently working on the possibility to organize on-site childcare for children under the age of six. At this stage, parents will need to cover the costs, but IHES hopes to set up a dedicated fund to participate in childcare costs.

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

In addition to all the possibilities listed above, IHES warmly encourages early career researchers to explore the opportunities offered by the "Fondation Mathématique Jacques Hadamard"[15] (FMJH), which coordinates graduate and post-graduate training in mathematics within the Paris-Saclay area. While the research carried out at IHES is fundamental, the FMJH offers programs in various areas of mathematics, including applied mathematics.

In 2023 the FMJH launched the "MathTech meetings,"[16] a series of events hosted by IHES and aimed at doctoral and postdoctoral researchers who want to learn more about the possibilities available to them outside of academia. The first edition[17] of MathTech took place on January 25, 2023, and a second edition is scheduled for January 25, 2024.

In collaboration with the FMJH, the FSMP (Fondation Sciences Mathématiques de Paris), and the IPhT (Institut de Physique Théorique) every year IHES organizes a special day open to all postdoctoral researchers in mathematics and theoretical physics working in the Paris region, which is a great opportunity for them to meet and discuss, as well as to learn more about future career opportunities.

Credits

Logo is courtesy of Institut des Hautes Études Scientifiques.

[15]https://www.fondation-hadamard.fr/en/

[16]https://www.fondation-hadamard.fr/en/articles/2022/12/15/mathtech-meetings/

[17]https://www.ihes.fr/en/maths-en-herbe-and-mathtech-fmjh-ihes/

Institute for Mathematical and Statistical Innovation (IMSI)

Kevin Corlette

The Institute for Mathematical and Statistical Innovation (IMSI) is an NSF Mathematical Science Institute managed by the University of Chicago, Northwestern University, the University of Illinois at Chicago, and the University of Illinois at Urbana-Champaign, and hosted at the University of Chicago.

1. What is the Mission of Your Institute? What Makes it Unique?

The essential mission of IMSI is to apply rigorous mathematics and statistics to urgent, complex scientific and societal problems, and to spur transformational change in the mathematical sciences and the mathematical sciences community. This mission is based on a vision with three fundamental elements: innovation, communication, and diversity.

IMSI is animated by a commitment to furthering innovation in the mathematical sciences. This commitment is rooted in a recognition of the pervasive role of the mathematical sciences in the wider enterprise of research, science, and technology, and the influence of that enterprise on the society we live in. Insights which arise in the abstract precincts of the mathematical sciences can have profound effects on society. IMSI aims to facilitate innovation in the mathematical sciences with awareness of and responsiveness to this interdependence. Scientific activity at IMSI is focused on applications of the mathematical sciences, with an emphasis on questions of importance to society at large. Much of the activity at IMSI aligns with a set of

Kevin Corlette is a professor of mathematics at the University of Chicago and the director of IMSI. His email address is director@imsi.institute.

DOI: https://doi.org/10.1090/noti2824

scientific themes which have been chosen as focal points for research at IMSI. These themes will evolve over time; current IMSI themes are Climate & Sustainability, Data & Information, Health Care and Medicine, Materials Sciences, Quantum Computing and Information, and Uncertainty Quantification.

The second element of the vision for IMSI is a focus on effective communication about the mathematical sciences. Research in the mathematical sciences is generally couched in the precise and technical language developed by experts to communicate among themselves. This form of communication is essential to advancing research in the mathematical sciences, but it is often a barrier to the dissemination of ideas across disciplinary boundaries, as well as to audiences outside of the mathematical sciences. IMSI works to help participants in its programs improve their skill at communicating with new audiences as well as with potential collaborators in other disciplines. Organizing committees for long programs are expected to formulate and implement plans for facilitating communication among members of the typically interdisciplinary groups of researchers they assemble.

The third element of the vision is a focus on diversity. Research in the mathematical sciences is an activity that has the potential to bring meaning and fulfillment into the lives of those who participate in it. It achieves this through the satisfaction of doing curiosity-driven research in theoretical mathematical sciences, through the interplay of the mathematical sciences with other disciplines and domains of inquiry, and through beneficial contributions to society at large. This potential is most likely to be realized through the participation of people with a variety of insights, experiences, and perspectives. One of the fundamental intentions behind IMSI is to offer broad and equitable avenues of access to its programming and to create an environment that promotes the fullest possible flourishing of science and the human beings who engage in it.

2. What Kinds of Programs are Run Each Year?

Our research activity is generally classified into four categories: long programs, workshops, interdisciplinary research clusters, and research collaboration workshops. Long programs focus on a research area related to one or more of IMSI's research themes, and usually last about three months. We typically host two of these each year, one in the fall and one in the spring. Workshops last anywhere from a day to a week, and typically focus on a topic in applied or applicable mathematical science. Workshops need not be connected to IMSI themes, although many are. With rare exceptions, workshops are held in a hybrid format which allows both in-person and virtual participation. Interdisciplinary research clusters (IRCs) involve small groups of researchers coming to IMSI for a period of time (typically a week or two) to work intensively on a specific project. While we are particularly interested in IRCs which connect to one of our themes, this is not required. Finally, research collaboration workshops (RCWs) are activities which involve both a research and a training component. Typically, RCWs involve teams consisting of one or two researchers with experience in a particular area working with a group of early career researchers or researchers with less experience in the area on projects which are expected to be achievable within the timeframe for the activity.

The activity we host is generated by proposals from across the mathematical sciences research community. We generally attempt to accommodate proposals for activities that do not follow standard templates if there are good reasons for nonstandard formats. We are open to proposals from researchers at all career stages. Proposals are considered in two annual cycles, with deadlines of March 15 and September 15 each year.

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply for?

All of our long programs, most of our workshops, and all of our research collaboration workshops are open for applications from researchers at all career stages, including graduate students. Visits of any duration up to the full length of the relevant activity are possible. IMSI does not have a standalone postdoctoral program to which early career researchers can apply.

4. What Type of Support is Available?

In most cases, IMSI can offer financial support for travel, lodging, and meals to participants who need it. In situations where participants have needs which are not addressed by these categories of support, we do our best to work with them to find solutions which make it possible for them to participate. The specific solutions we can offer depend on the situation, and must satisfy any constraints imposed by university and NSF policies.

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

Early Career researchers are welcome in the vast majority of the activities we host, but there are a number of activities that are specifically intended for them. These include a summer internship program for PhD students in mathematics and statistics which offers exposure to research in other disciplines and outside of academia, and summer schools such as the AI+Science Summer School which IMSI has hosted twice in collaboration with the Data Science Institute at the University of Chicago. There are also introductory and tutorial based workshops aimed at Early

Career researchers such as those associated with our 2023–2024 programs *Algebraic Statistics and Our Changing World* and *Data-Driven Materials Informatics*, and workshops such as the *Modern Math Workshop*, which takes place in conjunction with the annual SACNAS NDiSTEM Conference and is organized collectively by the mathematical sciences research institutes under the umbrella of the MSIDI initiative. In addition, many IMSI workshops offer opportunities such as poster sessions in which early career researchers can present their work. We also offer a series of communications boot camps on topics such as Storytelling & Narrative Structure, How to Write for a General Audience, and Job Talks & Stage Presence which might be of interest.

Credits

Logo is courtesy of IMSI.

Institute for Pure and Applied Mathematics (IPAM)

Dimitri Shlyakhtenko

The Institute for Pure and Applied Mathematics (IPAM) is an NSF Mathematical Science Institute[1] located on the UCLA campus in Los Angeles, California.

1. What is the Mission of Your Institute? What Makes it Unique?

IPAM fosters the interaction of mathematics with a broad range of science and technology, builds new inclusive interdisciplinary research communities, promotes mathematical innovation, and engages and transforms the world through mathematics. IPAM's programs connect mathematics and other disciplines, or multiple areas of mathematics. IPAM's programs are designed to initiate and promote collaboration between groups of researchers. Our programs also feature a relatively short lead time, with even the longest programs planned about two years in advance. Many of our programs have the effect of vastly expanding the collaboration network of our participants. For many past participants, an IPAM program enabled them to break out into a new field, or to learn important techniques and results from other fields to be used in their own research. IPAM has played a key role in a number of spectacular advances in mathematics, including topics such as compressed sensing, machine learning and AI, and applications of mathematics to physical and material science.

2. What Kinds of Programs are Run Each Year?

IPAM runs two 14-week-long programs per year, September–December and March–June. Each program has a specific scientific focus. Through an application process, each long program recruits approximately 50 long-term visitors that work collaboratively throughout the program. Accepted participants are provided with funding for housing and travel to and from the program. Each program starts with a week of tutorials whose purpose is to break down linguistic and notational barriers. Throughout the program, participants self-organize working group and seminar series. As part of the long program, we also run 3–4 workshops that bring in additional speakers and visitors for shorter terms. Active participants in our long programs are invited back for two reunion conferences that take place 1.5 and 2.5 years after the end of the long program at the UCLA Lake Arrowhead Conference Center in the San Bernardino Mountains east of Los Angeles. In this way we do our best to sustain and nourish collaborations that get started during our long programs. It is not an overstatement to say that many of our long-term participants gain a lot at IPAM—new ideas, new research directions, new collaborators, and new lifelong friends.

Besides long programs, IPAM runs stand-alone workshops (usually in the winter months, January–February, though sometimes we squeeze one in when we have availability at other times of the year). In most cases, registration is open to anyone interested; there is an application process for participants seeking financial support to attend. Many of the workshops feature poster sessions in addition to talks.

IPAM also runs graduate summer and winter schools (these are usually 1–3 weeks in duration). These offer a combination of talk series by eminent researchers in a particular field as well as opportunities for participants to work on concrete problems, and collaborate with others. Despite their name, these schools could be attractive to early career mathematicians who have already obtained their degree but are interested in learning about a new subject, or branching out in new research directions. There is an application process to participate, and financial support is available.

We also have programs aimed at undergraduate and graduate students who are interested in applied mathematics research in the industry setting. Those interested should check out our RIPS and GRIPS programs, several of which are run in collaboration with international partners.

Dimitri Shlyakhtenko is a professor of mathematics at UCLA and the director of IPAM. His email address is `Shlyakht@ipam.ucla.edu`.

DOI: https://doi.org/10.1090/noti2826

[1]*NSF grant: DMS-1925919.*

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply for?

We absolutely encourage the participation of early career researchers in all of our programs. IPAM doesn't really have special positions; for the most part, our programs just have participants.

Our long programs recruit their participants through an open application, and most of the participants receive funding for their stay (unless they don't need it, e.g., they are local). We start making decisions on a rolling basis approximately six months before the program starts. Participation in our long program can have long-lasting (and sometimes career-changing) positive impact on our visitors, especially the early career visitors. Approximately 2/3 of our long program participants are mid- or early career; many of them are graduate students or postdocs. The application process is not particularly complicated. The main components are a (short) research statement explaining why the applicant is interested and is a good fit for the program, as well as a (very short) letter of reference from their research advisor or mentor.

For most of our workshops, prospective participants have the option of registering (if they have their own funding), or applying for funding, in which case there is an application process similar to that of a long program. The workshops are organized by a committee and scientific directors at IPAM working together. We create opportunities for active engagement of early career attendees through short lightning-round talks, stimulating panel discussions, or encouraging them to contribute to the poster session. Graduate summer schools and (G)RIPS have similar application processes with slightly different deadlines and requirements.

Early career, as well as more senior researchers at primarily undergraduate and minority serving institutions, may also be interested in the recent NSF Partnerships for Research Innovation in the Mathematical Sciences (PRIMES) program, which not only offers funding for a research visit to an NSF institute, but provides additional funding for things like teaching buyouts.

4. What Type of Support is Available (Including Childcare Support)?

IPAM's goal is to ensure that its participants can come to IPAM and focus on science 100% of the time—or as near to it as possible. For example, our normal support package for long-term participants involves an allowance for their living expenses, as well as roundtrip airfare from their home institution. In addition, we work with all participants (short and long-term) based on their individual needs to find a suitable support package. This includes special circumstances such as short- or long-term childcare, which is of course more complex than just financial support—prospective participants should reach out to us about their particular situation.

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

One new initiative that we are starting, aimed specifically at early career mathematicians, is Applied Mathematics skills Improvement for Graduate studies Advancement (AMIGAs). Its aim is to help graduate students interested in applied mathematics through a critical transition period as they move more heavily into research at the start of their second or third year of graduate school. The inaugural program ran in summer 2023; in the future it will be organized jointly with other math institutes within the Mathematical Sciences Institutes Diversity Initiative (MSIDI).

There are several other ways to get involved with IPAM that are perhaps less widely known. One opportunity is for mid-career mathematicians (usually recently tenured associate professors). It is to come to IPAM to serve as an associate director. It's a 2–3 year rotator position that gives insights into scientific program management, but is also an opportunity to help create exciting mathematical programs.

And, last but not least, what about organizing a program at IPAM? A number of our organizer teams at IPAM have in the past included junior researchers, often teaming up with more senior colleagues. There are instructions on how to propose a program on our web site; in most cases a (not very complicated) proposal is due in late September of every year. We are happy to discuss your ideas and help you formulate a proposal—feel free to email us.

Credits

Logo is courtesy of Institute for Pure and Applied Mathematics.

Simons Laufer Mathematical Sciences Institute (SLMath)

Tatiana Toro

SLMath (formerly MSRI) is an NSF Mathematics Institute in Berkeley, California.

1. What is the Mission of Your Institute? What Makes it Unique?

A unique aspect of SLMath is its empowerment of and support for early career mathematicians to ensure they are well poised for successful futures in the profession.

The overarching mission of the Simons Laufer Mathematical Sciences Institute (SLMath) is to:

- foster and communicate mathematical research in a broad range of fundamental topics and applications,
- develop mathematical talent and cultivate a sense of belonging and engagement, and
- inspire an appreciation of the power, beauty, and joy of mathematics.

SLMath encourages curiosity, welcomes new communities, and provides a platform for collaborative mathematical research to thrive. In close collaboration with mathematicians, including those who serve on the Scientific and Broadening Participation advisory committees, we implement programs that enhance the mathematical sciences and respond to the diverse needs and interests of the community at large.

2. What Kinds of Programs are Run Each Year?

SLMath hosts a robust slate of scientific programs throughout the academic year and summer, including semester-long programs, workshops, 10-12 Summer Graduate Schools (SGS), MSRI-UP, MAY-UP, ADJOINT, SRiM, and CIME.

Semester-long programs: SLMath organizes and hosts semester-long programs on leading mathematical topics. Mathematicians worldwide come to SLMath to engage in research, making the Institute the world center of activity in the fields showcased during any given semester. We strive to bring together cross-generational communities to fuel innovation in mathematics. As part of this aim, SLMath invites postdoctoral fellows, who receive one-on-one mentorship from senior researchers, which facilitates fruitful, long-lasting relationships.

Tatiana Toro is a professor of mathematics at the University of Washington and the director of SLMath. Her email address is `ttoro@msri.org`.

DOI: https://doi.org/10.1090/noti2827

Workshops: During the academic year, SLMath hosts 1–2 Hot Topics workshops that highlight vibrant research areas in mathematics. We also host a yearly *Critical Issues in Mathematics Education (CIME) Workshop*. This series addresses key problems in education today. CIME workshops are designed to engage mathematicians, mathematics education researchers, and K–12 teachers as participants. CIME 2023 featured the topic Mentoring for Equity,[1] led by organizers Aris Winger, Pamela E. Harris, and Michael Young.

Summer programs: During the summer SLMath opens its doors (in California and around the world in partnership with other institutions) to programs serving a broad group of mathematicians at varying stages of their careers.

- *SLMath Undergraduate Program (MSRI-UP):* Founded in 2006, the award-winning undergraduate program MSRI-UP,[2] perhaps the nation's premier undergraduate research experience, is primarily for students from underrepresented groups. In its first 14 years, 85% of MSRI-UP's alumni have enrolled in graduate programs in the mathematical sciences. Learn more about the MSRI-UP experience in this video: `https://vimeo.com/779419764` and at `http://www.msri.org/up`.
- *SLMath Mathematically Advancing Young Undergraduates Program (MAY-UP):* In May 2023, the pilot year of Mathematically Advancing Young Undergraduates Program (MAY-UP) took place at Morehouse College, Spelman College, and Clark Atlanta University. The program's goal is to open first-year students' horizons to the many possibilities of a career in mathematics. Learn more about the MAY-UP experience in this video: `https://vimeo.com/846755162`.
- *ADJOINT* is a yearlong program that provides opportunities for US mathematicians—especially those from the African Diaspora—to form collaborations with distinguished research leaders on topics at the forefront of mathematical and statistical research. Learn more at `http://www.msri.org/adjoint` and in these video interviews: `https://vimeo.com/503623927` and `https://vimeo.com/761580270`.
- *Summer Research in Mathematics (SRiM):* SLMath's Summer Research in Mathematics (SRiM) program provides space, funding, and the opportunity for in-person collaboration to small groups of mathematicians, especially women and gender-expansive individuals, whose ongoing research may have been disproportionately affected by various obstacles including family obligations, professional isolation, or access to funding. Through this effort, SLMath aims to

[1] `https://www.msri.org/workshops/1059`

[2] `https://www.msri.org/web/msri/education/for-undergraduates/msri-up`

mitigate the obstacles faced by these groups, improve the odds of research project completion, and deepen their research experience. Learn more at: http://www.msri.org/summer.

3. Within Those Programs, What Types of Positions Can Early Career Mathematicians Apply for?

Programs of particular interest to Early Career mathematicians include:

Summer Graduate Schools (SGS): Mathematics graduate students are the future leaders of the profession and thus providing opportunities for national and international networking and collaboration early in their careers is critical. We aim to provide a motivational, exciting, and accessible experience for graduate students with a wide variety of backgrounds and preparation.

SLMath organizes 10–12 summer graduate schools, held at SLMath and partner institutions across the globe. Attending these schools is often a pivotal experience for students, who have remarked that they feel like "professional" mathematicians for the first time.

Mathematics graduate chairs nominate students to the schools, and acceptance is on a first come, first served basis. Departments nominating students from groups traditionally underrepresented in the mathematical sciences receive additional nominations. Admitted students attend the schools free of charge.

Postdoctoral Fellowships: SLMath awards 32–36 semester-long postdoctoral fellowships annually. These career-defining opportunities provide support for five months, including a monthly stipend, a travel subsidy, funds for research travel, and health insurance. SLMath's postdoctoral program also includes a generous number of endowed named fellowships.

Research Members: Interested individuals at all stages of their careers may apply for research memberships during the semester-long programs for stays of 1–4 months. These awards include per diem support.

4. What Type of Support is Available (Including Childcare Support)?

SLMath is distinguished by its holistic approach to supporting mathematicians at all career stages as well as their families.

In addition to the support outlined in question 3, SLMath endeavors to maintain a diverse community by actively assisting mathematicians with children. To enable full immersion in its scientific activities, SLMath offers researchers with children under the age of 17 the opportunity to apply for childcare grants. These flexible awards may be used for the reimbursement of childcare expenses, including travel, lodging, and meals for children and a caregiver. They may also be used for day-cares, individual childcare providers, and summer camps in Berkeley or the participant's home location.

Additionally, a dedicated family services consultant assists with finding schools and childcare as well as family-friendly networking opportunities in Berkeley.

5. Are There Any Particular Opportunities That You Want Early Career Readers to Know About?

SLMath's 2023–2024 programmatic workshops and summer 2024 programs are excellent opportunities for Early Career readers. The Connections, Introductory, and Topical workshops are open to all (registration required; please see below links):

- **Algorithms, Fairness, and Equity** (fall 2023): https://www.msri.org/programs/353#workshops
- **Mathematics and Computer Science of Market and Mechanism Design** (fall 2023): https://www.msri.org/programs/333#workshops
- **Commutative Algebra** (spring 2024): https://www.msri.org/programs/343#workshops
- **Noncommutative Algebraic Geometry** (spring 2024): https://www.msri.org/programs/356#workshops

Opportunities for summer 2024 include:

- **ADJOINT**: https://www.msri.org/web/msri/scientific/adjoint
- **Summer Research in Mathematics (SRiM)**: https://www.msri.org/web/msri/scientific/summer-research-in-mathematics
- **Summer Graduate Schools**: https://www.msri.org/web/msri/scientific/workshops/summer-graduate-school

Applications for 2024–2025 semester programs open **September 1, 2023**:

- **New Frontiers in Curvature: Flows, General Relativity, Minimal Submanifolds, and Symmetry** (fall 2024): https://www.msri.org/programs/344
- **Special Geometric Structures and Analysis** (fall 2024): https://www.msri.org/programs/361
- **Probability and Statistics of Discrete Structures** (spring 2025): https://www.msri.org/programs/348
- **Extremal Combinatorics** (spring 2025): https://www.msri.org/programs/375

Credits

Logo is courtesy of SLMath.

Dear Early Career

Some of my office mates write really long papers, but some write short papers. I have just started writing my first preprint, and wonder if it is better to write long papers or short papers?

—Curious

Dear Curious,

The first thing to consider is, given your research problem, do you have a choice? If you are fortunate enough to write short, self-contained articles proving interesting results, then this is undoubtedly a wonderful situation to be in, but most working mathematicians make progress by "standing on the shoulders of giants," and it differs from paper to paper how this manifests.

The existing work that you would like to develop or apply may have already been perfectly packaged for your requirements, and you can describe your innovations in a concise manner and then let the references take care of the rest. On the other hand, perhaps you need a lemma which is "sort of like Lemma 3.6" from one paper, and then apply an argument which is "essentially the same as, but not explicitly written" in that other paper. In the latter case, trying to make an argument too short could make the article incomprehensible for anyone who hasn't spent the same amount of time as you have working on the same problem with the same background, which is probably a lot to expect of a reader. Airing on the side of providing more complete arguments, even if they are relegated to an appendix, is likely to be a sound decision, especially for a junior researcher. In our experience this is typically how the length of a paper starts snowballing, and in many cases it is inevitable.

It's best not to make comparisons across mathematics as a whole, as different research areas are different. Instead consider the papers that have served as an inspiration to you, and ask yourself if the level of detail you are giving is appropriate relative to them. Ideally, a greater level of detail should be provided in the most novel components of the proof where your readers are likely to be less comfortable.

In our experience, publishing lengthy papers can be a bit of an ordeal,[a] as referees will typically take a long time to write their reports, and journals often have a higher bar to publish longer papers (you are also limited in terms of the journals that you can submit to). If there is a natural way to split a long paper into shorter papers then it makes sense to explore doing that, but in this situation each of the parts should be (just as) interesting in their own right.

If your intend to apply for a postdoc, then it is possible that shorter papers already accepted to appear in a journal could be more beneficial than a longer paper which has been under review for a long time. However, this can be balanced out by getting strong recommendation letters from interested researchers, especially ones outside of your home institution. Looking longer term, if you end up at a tenure-track position in academia, then you should discuss with your mentors about whether there is a minimum number of papers required for tenure or promotion, as this could result in some decisions regarding how to present your work in order to fulfill these requirements (hopefully without changing the work itself in any significant way). This certainly feels a bit antiquated given the many different ways someone can make an impact in their department, but promotion and tenure rules can be very difficult to update and are often a reflection of where a department was decades ago. Regardless, for both postdoc and tenure applications, it is typically the external evaluations of your work that carry the most weight in evaluating your research, and these are likely to focus on your most significant work.

Many times we have started work on a problem without having a particularly good idea what would come out of it. When we decide to publish, it is (ideally) because we are excited to share an idea. To maximize the opportunity for an idea to be impactful, a paper should be accessible to researchers in the field. The length of the paper then falls into place, and you might be able to make your heuristics precise in one page, or it might take 100 pages.

—Early Career Editors

Have a question that you think would fit into our Dear Early Career column? Submit it to `Taylor.2952@osu.edu` or `bjaye3@gatech.edu` with the subject Early Career.

DOI: https://doi.org/10.1090/noti2821

[a]Publishing shorter papers can also be a bit of an ordeal—a renowned analyst once told us that if they had a paper rejected three times then they knew they were really on to something...

Memorial Article for Yuri Manin

Edited by Fedor Bogomolov and Yuri Tschinkel

Yuri Manin was born in Simferopol, Crimea, on February 16, 1937, and died in Bonn on January 7, 2023. His father perished on the front fighting against Germany. His mother struggled to survive and raise him through the war and the hardships of the postwar years. Even though Manin lived far away from the main centers of academia, he developed a serious interest in number theory rather early. He was admitted to Moscow State University (MGU) in 1953, and received his PhD in 1961, at which time he was already one of the leading experts in what is now known as *arithmetic geometry*. Even before receiving his PhD, he became a member of the Steklov Institute of Mathematics of the Russian Academy of Sciences, and a few years later, a professor at Moscow State University. After the collapse of the Soviet Union, Manin left Russia for faculty positions at MIT and Northwestern, but soon after he settled in Bonn, joining the Max Planck Institute for Mathematics as one of the directors.

Manin supervised over 50 PhD students, in Russia, the USA, and Germany, many of whom are now professors in leading universities all over the world. Manin's scientific interests spanned an enormous range of fields in mathematics and mathematical physics. His early interests in literature and art shaped his truly unique vision of science, culture, and society. With his deep intuitive understanding of interconnections between different fields he had striking insights even in areas not directly related to his research. One of the most spectacular examples was his idea of quantum computing, which he developed around 1981. He closely collaborated with more than 100 mathematicians. His brilliant lectures were special events for students and faculty alike. He published more than 20 carefully crafted books on very different subjects in mathematics, physics, and philosophy of science.

Through many dramatic changes of circumstances he kept his inner balance and independence of thought. He was lucky to have had unconditional support from his wise and loving wife Xenia Glebovna.

Figure 1. Yuri Manin and his wife Xenia Glebovna, September 2011.

Perhaps the best description of his own view of mathematics was in his quote of Georg Cantor's words, spoken at the ICM in Berlin:
The essence of mathematics lies in its Freedom.

We are grateful to the *Notices* for encouraging us to gather personal memories of some of his students and colleagues.

Fedor Bogomolov is a professor of mathematics at New York University. His email address is bogomolo@cims.nyu.edu.

Yuri Tschinkel is a professor of mathematics at New York University. His email address is tschinkel@cims.nyu.edu.

DOI: https://doi.org/10.1090/noti2814

MEMORIAL TRIBUTE

Alexander Beilinson

It rarely happens, but if one is lucky enough to see a mathematical problem from the right perspective, the solution suddenly becomes clear: the known pieces by themselves join into an unexpected whole, taking on form and meaning without any effort on your part. Manin's seminars developed in a similar way. Yuri Ivanovich Manin had a wonderfully light personality—he did not wish to be a leader—and new themes and subjects, as if in gratitude, came to life by themselves, in front of our eyes.

Many shades of blue above him.
Green below him, and the world
Is a giant bird before him,
Warbling, trilling, full of songs.[1]

The time of my youth in Russia was benevolent to those who accepted Pushkin's poem "From Pindemonti"[2] as part of their souls. Manin's seminars, like the books of Yuri Koval and the animated movies of Yuri Norstein, were parts of the happiness of that dandelion-light world, of its deeper Liberty.

If one were to formulate how Yuri Ivanovich viewed things, one might first notice his ability to connect facts to arrive at conclusions that often contradicted everyone else's. Or, perhaps, the absence of desire to belong to any association, and to have sway over any other person. Or the rejection of any kind of malice and greed in human relations. And, certainly, the clear kindness.

The 90s, years of dark poverty in Russia, left almost untouched those of us who flew away, like fluff on the wind, into the opening new world. We met far more rarely than in Moscow, where we would see each other or talk on the phone almost every day. Our attitude toward what was around us was changing, and, during the NATO assault on Serbia, I heard from Yuri Ivanovich a grim foreboding of what has happened since and continues to unfold now.

In Manins' Bad Godesberg apartment a glass wall opened onto the Rhine. It seems to me that when you watch a great river from day to day, you become its relative: it starts flowing through you as time flows, always remaining itself, as memory does. And this is happiness.

I am so grateful to Yuri Ivanovich for his gift of joy, full of sun; it became a part of myself.

When I was very young, my most beloved book was *Winnie the Pooh*; Yuri Ivanovich loved it too: "So they went off together. But wherever they go, and whatever happens to them on the way, in that enchanted place on the top of the Forest, a little boy and his Bear will always be playing."

Alexander Beilinson

Alexander Beilinson is a David and Mary Winton Green University Professor of Mathematics at the University of Chicago. His email address is sasha@math.uchicago.edu.

[1]*Bagritsky's poem "A bird-catcher"*

[2]*Nabokov's translation:* https://ireaddeadpeople.wordpress.com/2014/11/06/alexander-pushkin-to-stroll-in-ones-own-wake

Vladimir Berkovich

One day in the Fall of 1969, my second year at Moscow University, I was talking with Misha Mandel, a fourth-year computational mathematics student about a problem I had. I had until the end of that school year to find an academic advisor, as every student must have one beginning with their third year. I wanted to have an advisor who is an outstanding mathematician with broad and deep knowledge and taste in mathematics. I hoped that everything else would follow. Misha quietly listened to me and then said there was a young mathematician in mech-math, Yuri Ivanovich Manin, about whom he had heard many good things. I had never heard about Manin and soon found that he would give a course on advanced commutative algebra during the second semester. I attended the first lecture with my friend from high school, Anas Nasybullin. The auditorium was packed, we found seats in the last row, and within a couple of minutes, a short man with a strong voice entered in a suit and a white tie; he did not seem young to us (he was already 33!). His lecture with precise definitions, formulations, and arguments was perfect, and I felt a mathematical aura emanating from him. After the lecture, both Anas and I, said to ourselves "we have found our academic advisor." Of course, there was a remaining nuance that he must agree to have us as his students, but his guidance had already started without him being aware of it.

Somebody gave me a rotaprint of Yuri Ivanovich's lectures on algebraic geometry. I was already slightly familiar with basic notions of classical algebraic geometry and could not say I felt comfortable with it. I also heard about Grothendieck's new approach but never had the chance to learn it, and I thought it might be something

Vladimir Berkovich is a professor of mathematics at the Weizmann Institute of Science. His email address is Vladimir.Berkovich@weizmann.ac.il.

complicated. From the first pages of that rotaprint, Yuri Ivanovich wonderfully explained the transition from classical algebraic varieties to Grothendieck's schemes. He presented the latter in such a natural and simple way, only he could do it. At the same time, I found Yuri Ivanovich's paper on etale cohomology, Algebraic topology of algebraic varieties, in Uspehi. Again, his explanations were so natural and straightforward. All this strengthened my confidence in choosing him as my academic advisor, and it remained only to overcome that little nuance.

In the Spring of 1970, near the end of the semester, Anas and I told Yura Zarhin and Pasha Kurchanov, our friends from high school, about the choice of Yuri Ivanovich Manin as an advisor. I do not know if this influenced their decision or if they came to it independently, but one day at the end of May, all four of us came up to him, and Pasha said, "Yuri Ivanovich, take us as your students." Yuri Ivanovich laughed and then said that he could not decide this immediately, but he had just finished a course on commutative algebra, offered to give us an exam on it, and after that, he would see. In this way, we became Manin's students. Slightly later, our friend Kolya Chebotarev approached Yuri Ivanovich on the same subject. As Kolya told us, Yuri Ivanovich asked if he had a relation to the famous Russian mathematician Nikolai Chebotarev. Yes, Kolya was his grandson; this was his entrance ticket for joining our group of Manin's students.

A period of intensive learning and immersion in the beautiful world of mathematics started for all of us: class field theory, Grothendieck's *EGA* and *SGA*, works of Tate, Serre, and other mathematicians. The direction of our study was naturally generated by Yuri Ivanovich's lectures, which were, as always, perfect and introduced us to many new things. Especially beneficial for me were Yuri Ivanovich's requests to give a talk in his seminar about a particular paper or so, which forced me to concentrate on the subject and, as a result, do something new. The close presence of brilliant older and younger students of Yuri Ivanovich was highly motivating.

The last three years of study flew by like a dream, and by the end, Anas, Pasha, and I were accepted for graduate studies. (Yura, the brightest of us, could not hope for this because he had poor marks in Marxist–Leninist wisdom.) Besides other things, a graduate student had to pass an exam on a specific big subject, which was usually chosen to be in the student's research area. But Yuri Ivanovich told us that since we would continue to study our fields in any case, each of us had to choose something distant from our research areas. He suggested complex analysis, functional analysis, and mathematical logic. I grabbed the latter since I knew nothing about it, and besides standard textbooks, I had a chance to read his book on mathematical logic before it was published.

Three more years flew quickly by, and we entered life with all its sorrows, surprises, and joys. I attended Yuri Ivanovich's seminar at the university sporadically because I worked as a computer programmer. For several years, I was seriously thinking about devoting myself to this occupation, but mathematics acquired under Yuri Ivanovich's mentorship was boiling in me and, finally, burst out and took me to freedom in Israel. Eventually, Yuri Ivanovich and most of his former students emigrated from the Soviet Union. We did not communicate much since then, but the precious memories of those years and gratitude to Yuri Ivanovich are always with me.

Vladimir Berkovich

Jean-Louis Colliot-Thélène

At the beginning of the 1970s, I started mathematical research and was quickly attracted to work by the Russian school around Shafarevich and Manin. My first encounter had been with (the French translation of) *Theory of Numbers* by Borevich and Shafarevich.

In the period 1963–1972, Manin published many papers on the geometry and arithmetic of (geometrically) rational surfaces, and more generally "varieties close to the rational ones." In the celebrated 1965 Shafarevich seminar on algebraic surfaces, which I read in the German translation, bought in East Berlin, Manin contributed to the section on linear systems with base points, a topic which went back to Max Noether and Beniamino Segre, and which after the famous Manin–Iskovskikh paper on quartic threefolds developed into the study of rigidity.

In that period, it took time for papers written in Russian to be translated. I learned enough basic Russian to be able to decipher papers by Manin, Iskovskikh, Voskresenskiĭ, Bogomolov, and by the Minsk school on linear algebraic groups, which I closely studied with J.-J. Sansuc.

Jean-Louis Colliot-Thélène is a directeur de recherche émérite at the Université Paris-Saclay. His email address is `jean-louis.colliot-thelene@universite-paris-saclay.fr`.

Ten years later this also helped me to read the papers of the quite distinct algebraic K-theory school in Leningrad (Sankt-Petersburg).

Yuri Manin was among the people who created a bridge between old diophantine problems and the modern algebraic geometry of Serre and Grothendieck. Among the many topics Manin helped develop: birational classification of rational surfaces (which went back to the Italian school, and was then developed into Mori theory and MMP), study of the Cremona group, use of the Brauer group in the study of rationality of varieties and in the study of rational points over number fields.

In 1966, Manin published his big IHÉS paper on Rational Surfaces (in Russian). In 1970, his Nice ICM talk (in French) launched the study of what we now call the Brauer–Manin obstruction. In 1972, Manin published his very attractive book *Cubic forms, algebra, geometry, arithmetic* (in Russian), where among others we can find hints in the direction of descent. One also finds the notion of R-equivalence on rational points, which started its independent life there. Special cubic surfaces, investigated by François Châtelet in 1960, as a possible analogue of descent on elliptic curves, feature in the book. As Manin (and also Peter Swinnerton-Dyer) had predicted, they have turned out to be a testing ground for descent and also for the study of rationality of varieties. In the early 80s, Sansuc and I elaborated an appropriate theory of descent (the full text appeared in the DMJ 1987 volume on the occasion of Manin's 50th birthday) and in works with Daniel Coray and with Swinnerton-Dyer, managed to solve many questions on these surfaces raised in Manin's book. Since then, progress by many people has been achieved—but we still have no systematic algorithm to decide if a given cubic surface over the rationals has a rational point.

Communication with Russia was not simple in the days I am alluding to. In 1982, Jean-Jacques Sansuc and I travelled as "tourists" to Moscow (over two days by train each way), stayed at the old Hotel National where more illustrious people had stayed, a few yards away from Red Square, and met Manin and other Russian mathematicians, for the first time, in the old Steklov Institute. We were smuggled into the main building of MGU, while the guard was not looking. In Manin's seminar, with a packed audience, Sansuc lectured in French on rational points on intersections of two quadrics, with Manin (who spoke perfect French) translating. We were very nicely invited to Manin's home and managed to get hold for him of a copy of the then hardly available book *Master and Margarita*, by Mikhaïl Boulgakov, in a Beriozka shop (opened only to foreigners).

This was for me the first of a series of longer stays in Moscow, where I started a long-lasting collaboration with Alexei Skorobogatov, himself a student of Manin, and also had contacts with Fedor Bogomolov.

At the beginning of the 80s, Manin's manifold interests led him away from "varieties close to rational ones" but by the end of the 80s he had become interested in counting points of bounded height on such varieties, and in works with Batyrev, Franke, and Tschinkel produced conjectures on the behavior of the counting function, with main term depending on the geometry of the underlying variety. Getting the "right" constant in front of the main term was a challenge back in 1990. This was achieved by E. Peyre. The questions Manin then raised, at the interface of analytic number theory and complex algebraic geometry in the MMP style, are still very much open, but they have generated a whole area of research.

Over the years, I met Yuri Ivanovich on various occasions, in Paris and in Bonn. In March 2021, I gave an online talk at his Bonn seminar, where I had the opportunity to recall a 1967 result of his, based on heights, on the lack of finite-to-one parametrizations for rational points of cubic surfaces. My last slide was an image of his handwritten 1982 Moscow dedication of his book on Cubic Forms to Sansuc and me.

Jean-Louis Colliot-Thélène

Vladimir Drinfeld

I was a student of Yuri Ivanovich Manin at Moscow university. Attending his lectures and seminars was a substantial and very happy part of my mathematical life then. This started in 1970/1971, even before I became a student of his. That year he gave a course on the language of schemes, which also included more advanced topics like flat descent. Around that time I also read Manin's brief 1965 survey on étale cohomology, which made me very excited. (By the way, I learned about the existence of Winnie-the-Pooh from the epigraph of the survey.)

Vladimir Drinfeld is a Harry Pratt Judson Distinguished Service Professor at the University of Chicago. His email address is `drinfeld@math.uchicago.edu`.

In 1971/1972, Manin and Piatetskii-Shapiro organized a seminar on modular and automorphic forms. Its goal was essentially to learn the representation-theoretic approach to automorphic forms, which was brand new then. This was another formative experience for me.

Rather than describing the later events year by year, let me say that the mathematical subject on which I am now working is very close to Manin's works of the 1960s on formal groups and the Gauss–Manin connection.

Many years ago, I asked Yuri Ivanovich to become my advisor, and it was great luck for me that he agreed. It was especially great luck because of his human qualities. There was a difficult period in my life when Yuri Ivanovich's practical and moral support was crucial for my survival.

Manin was a great lover of literature. As far as I understand, "The 4 quartets" by T. S. Eliot was a favorite poem of his; in particular, he liked the following line:

> For us, there is only the trying. The rest is not our business.

As I am growing old, I cannot help thinking about this line again and again.

Let me now describe some of Manin's contributions to mathematics.

Manin triples: At the beginning of the 1980s, I told Manin about the notion of Poisson–Lie group, which I managed to extract from the works by L. D. Faddeev's school (especially, by E. K. Sklyanin). I also told him about Lie bialgebras, which are infinitesimal analogs of Poisson–Lie groups. A Lie bialgebra is a vector space $\mathfrak{g}$ equipped with a structure of Lie algebra and that of Lie coalgebra so that the two structures are compatible in the following sense: the Lie cobracket $\mathfrak{g} \to \wedge^2\mathfrak{g}$ is a 1-cocycle.

Manin immediately asked me the following question. Consider triples $(\mathfrak{a}, \mathfrak{g}, \mathfrak{g}')$, where $\mathfrak{a}$ is a Lie algebra equipped with a nondegenerate invariant symmetric bilinear form and $\mathfrak{g}, \mathfrak{g}' \subset \mathfrak{a}$ are transversal Lagrangian Lie subalgebras. Then the vector space $\mathfrak{g}$ is dual to $\mathfrak{g}'$, so the Lie bracket on $\mathfrak{g}'$ induces a Lie coalgebra structure on $\mathfrak{g}$ (in addition to the Lie algebra structure). Can it be that $\mathfrak{g}$ is a Lie bialgebra and that one thus obtains an equivalence between the groupoid of finite-dimensional triples as above and the groupoid of finite-dimensional Lie bialgebras?

A simple computation showed that Manin's guess was correct. It played a very important role in the development of the theory of Poisson–Lie groups and quantum groups (in particular, it led to the notion of "quantum double").

Instantons: In 1978, Manin, Atiyah, Hitchin, and I found all G-instantons on the sphere S^4 for all classical compact groups G; this work is known as ADHM. Since I am one of the authors, I cannot praise the work itself, but I can praise algebraic geometry. It is definitely true that ADHM demonstrated the power of 20th century algebraic geometry to theoretical physicists. Indeed, the problem was formulated by physicists who considered it to be very important and the best theoretical physicists tried to solve it. A complete solution was formulated by ADHM in terms of linear algebra, so they could easily understand it. On the other hand, physicists were unable to guess the answer (although they found some nontrivial solutions) because they did not have the relevant tools at their disposal. The reaction of theoretical physicists to ADHM was adequate: they began to learn algebraic geometry very seriously, and by 1990 they knew it, in some sense, better than algebraic geometers.

How did physicists learn algebraic geometry and how did (some) mathematicians learn (some) physics in the period 1975–1990? At least in Moscow, Manin played a crucial role in this process by writing articles and surveys, giving lectures, organizing seminars, and in other ways. As a scientist, Maxim Kontsevich was brought up in this atmosphere.

Somewhat unexpectedly, the ADHM work led to a revival of interest in homological algebra in Moscow (and probably elsewhere). A crucial step was the description of the derived category of coherent sheaves on the projective space obtained in 1978 by Beilinson and Bernstein–Gelfand–Gelfand. Their motivation was to find a conceptual explanation of the technique of Horrocks and Barth used by ADHM. These works led to a spread of the culture of derived categories (and then differential graded categories) in Moscow. Manin actively participated in this cultural change; in particular, he wrote a textbook on homological algebra jointly with S. Gelfand.

Vladimir Drinfeld

Alexander Goncharov

In 1976, I was fortunate to be admitted to Mechmat MGU[3] bypassing the notorious entrance exams,[4] and on the first Monday of September, shortly after arriving from Ukraine to Moscow, I went to Gelfand's seminar. It was there that I first saw Manin, who gave two talks in February 1977 on soliton equations.

At that time, Moscow was an exciting place where curious students could meet great mathematicians, find all kinds of math and clandestine literature, and discover the math they liked the most.

I came to Manin's seminar in September of 1977. It was the only time when the seminar was designed for beginners. Its main goal was Griffiths's paper "Variations on a theorem of Abel." Among the other topics which Manin suggested at the first meeting was the monodromy of the dilogarithm function. The dilogarithm is the simplest integral of algebraic geometric origin beyond the Abelian integrals. It satisfies Abel's relation—an analog of addition laws for Abelian integrals—which characterizes it uniquely. Manin mentioned that the dilogarithm showed up recently in several unrelated areas, e.g., in Gabrielov–Gelfand–Losik's work on the combinatorial formula for the first Pontryagin class. My fascination with polylogarithms goes back to that time. A year later, Sasha Beilinson gave a series of talks on Manin's seminar on Bloch's 1978 Irvine lectures, where the dilogarithm was one of the key characters. These talks led to Beilinson's conjectures on regulators and crystallization of the very idea of mixed motives.

Manin looked at math through the glasses magnifying the underlying Algebraic Structure. Gelfand presented himself as an analyst. Yet anything he did in math always led, sometimes entirely unexpectedly, to representation theory. For example, his works with Dikii on the KdV equation led to W-algebras—higher analogs of the Virasoro algebra. His works with Ponomarev on classification of linear algebra problems led to quiver varieties, which transformed representation theory, etc. It feels as if they sensed different aspects of the dilogarithm. As we see today, polylogarithms and the relations they satisfy describe the structure of the motivic Galois group, while the quantum deformation of the dilogarithm plays pivotal role in the emerging cluster representation theory of quantum groups.

Yuri Ivanovich influenced the education of generations of mathematicians in a multitude of ways. In Moscow, he conducted two 1.5-hour seminars each week on different subjects, one right after the other. Yuri Ivanovich possessed an amazing intellectual ability to learn any kind of math he found exciting. His attitude toward math was highly social. After mastering a subject, he would give a carefully crafted course, with meticulously prepared notes for each lecture. He never lectured on the same advanced topic twice. Manin's courses attracted a huge audience. And then Manin used his notes to write a book or an expository paper. On top of this, Manin participated in the selection of foreign books to be translated into Russian. He would often either translate a book, or write a beautiful foreword. Since only a small fraction of foreign books were translated, we learned math by reading the books he selected.

After we moved to the West, my relationship with Yuri Ivanovich become much more personal. His kindness, dignity, and decency stand out in my memories.

Alexander Goncharov

Alexander Goncharov is a professor of mathematics at Yale University. His email address is alexander.goncharov@yale.edu.

[3]*Abbreviation for the Faculty of Mechanics and Mathematics of the Moscow State University*

[4]*I do not know any Jewish applicant who survived the entrance exams that year. More than 400 students were admitted.*

Michael Harris

It was common knowledge when I was a Princeton undergraduate that Moscow was not merely an extraordinary mathematical center but that it also possessed a magical aura, where the most unexpected developments were routine. Mathematicians' fascination with Russia was only enhanced by the country's exotic inaccessibility at this stage of the Cold War, and no figure exercised a greater hold on the imaginations of my teachers in number theory and arithmetic geometry than Yuri Ivanovich Manin. For my junior project, Spencer Bloch had me write a report on his 1961 paper on the Hasse–Witt matrix. Nick Katz, my senior advisor, gave me Manin's paper on the Dieudonné modules of formal groups to read. For my senior thesis he suggested I try to answer a question of Manin—in a

Michael Harris is a professor of mathematics at Columbia University. His email address is harris@math.columbia.edu.

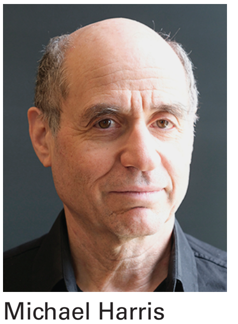
letter he had written to Nick, I think—on an analogue for elliptic curves of the Wieferich criterion for Fermat's last theorem—a question that is still unsolved, as far as I know. My years as a graduate student were marked by a fruitful long-distance exchange between Manin and my advisor Barry Mazur in developing and applying the theory of modular symbols and the construction of p-adic L-functions of modular forms. At the same time, Manin served as a conduit for mathematical news from the Soviet capital.

So it was only natural that, one year past my PhD, I stopped by the Steklov Institute on my way to the Helsinki ICM, to pay a courtesy call on the stranger who had already done so much to shape my mathematical taste. I may have written to warn Manin of my coming but, before she allowed me to wait for the Professor in the library, the woman at the desk by the front door made it clear to me that I had no business being there.[5] Much later I learned that my unplanned visit had subjected Manin to a lengthy bureaucratic nightmare, and that it could have been worse. He must have realized this immediately, but he gave no sign of concern, instead inviting me to join him in an extended conversation about mathematics in Cambridge and Paris, opera in Moscow, and the upcoming ICM. Practically no one from the Moscow school would be allowed to travel to Helsinki, but Manin urged me to spread the word at the Congress that a contingent of colleagues from Moscow would be gathering and hoping to meet visitors in Leningrad, which at the time was accessible for a day trip without a visa. When our conversation ended I drove him home in the Citroën 2 CV, rented in Brussels, that I parked across from the Steklov Institute—after I changed a flat tire, an operation that Manin found incomprehensible.

Eleven years later, Manin hosted my stay for an entire academic year at the Steklov Institute through the National Academy of Sciences exchange program. The optimism of the first years of perestroika had definitely faded and Manin spent most of the year traveling. I mainly experienced his influence second-hand, by attending his 2–3-hour-long seminar at Moscow State University, where Russian colleagues of my generation constantly interrupted each other; someone entering in the middle of a session would have been at a loss to determine who was the speaker and who the audience, a Bakhtinian polyphony that was inspirational but also deeply disorienting. At that point, Manin was mainly thinking about what I thought of as mathematical physics, and I was not. So when we did meet our conversations were mainly about literature and culture in general, as well as politics, about which—unlike most of the Russians I met that year—he had few illusions. He was a big fan of *The Bonfire of the Vanities*, which was still new at the time, and wanted to know more about Tom Wolfe. I cautiously recommended *The Electric Kool-Aid Acid Test* but I don't know whether or not he ever read it.

I have to confess that our conversations that year, and in subsequent years, always left me vaguely uneasy. Manin had a habit of ending his sentences, whatever the subject, with an expectant look. This may just have been Russian body language, but I read it as an invitation to continue the conversation with as much insight and authority as he had just done; the result, I'm afraid, was generally an awkward silence. This only happened infrequently after I moved to Paris in the 1990s, but I did begin a fruitful indirect dialogue with Manin, or rather with his way of thinking, notably through his book *Mathematics as Metaphor*. Faced with predictions of a future dominated by digital technologies and their corporate masters, I take heart from this quotation from an interview[6] published in 2015:

> *Think! Otherwise no Google will help you.*



Michael Harris

Nicholas M. Katz

Manin's 1958 paper "Algebraic curves over fields with differentiation," written while he was still an undergraduate, was transformative, as was his use of those ideas in his 1963 paper on the function field version of the Mordell conjecture. It had been known since Legendre (1811) that the periods of a family of elliptic curves satisfy a differential equation. Once one has a contour integral representation of the periods, one obtains the differential equation by "differentiating under the integral sign." Although this is already present in Fuchs (1869), it is not until Manin that this process of differentiation is explicitly applied, in

[5] *I waited long enough to read an article by Andrianov and Kalinin in the latest issue of* Mat. Sbornik*; this got me started on the projects that would occupy my attention for the next five years.*

[6] *Yuri Manin: "My Life Is Not a Conveyor Belt," in* The Human Face of Computing*, Advances in Computer Science and Engineering, C. S. Calude, ed., Singapore: World Scientific, 2015, 277–286.*

Nicholas M. Katz is a professor of mathematics at Princeton University. His email address is nmk@math.princeton.edu.

the case of curves, to the space of differentials of the second kind mod exact, a space which in characteristic zero gives a purely algebraic way to view the H^1 of a smooth, projective, geometrically connected curve. In a July, 1965, footnote to his paper on de Rham cohomology, Grothendieck says that Manin's idea "strongly suggests" the existence of (what came to be called) the Gauss–Manin connection in complete generality. One could argue that the theory of D-modules has its beginnings here.

Manin's 1963 paper on commutative formal groups in characteristic $p > 0$ was also hugely influential. It made available to a wide audience the theory of Dieudonné modules, their classification up to isogeny, and their relations to Newton polygons for abelian varieties. It clearly suggested their relation to what would become the crystalline H^1 in the case of abelian varieties.

In 1978, I delivered Manin's plenary address at the Helsinki ICM, since he hadn't been allowed to attend. I didn't have the pleasure of meeting Manin in person until 1988, when he gave a lecture at U. Penn., but by that time I had already been deeply influenced by his work for 25 years.

Nicholas M. Katz

Ralph Kaufmann

My first encounter with Yuri Ivanovich Manin was through his writings. In particular, his book on quantum groups was a revelation for me and exemplified just the right balance between motivation and precise abstract structures. In this form of beauty, mathematical knowledge transcends and can become a metaphor. It was clear to me, when Manin came to Bonn to be a director of the Max Planck Institute, that he would be an ideal PhD advisor for me. At that time, there was a seminar that was run by Yuri Ivanovich, Werner Nahm, and Don Zagier on the interactions of mathematics and physics in which I delivered a talk on the Virasoro algebra. After this formal encounter, I mustered the courage to ask him about becoming his student. He informed me that, since he had not yet known me for an extended period of time, I would have to come back in two weeks for a kind of oral entrance exam. Daunting as that was, I passed and began working with him—not, as I had initially thought, on noncommutative geometry, but rather on quantum cohomology.

My first results were on the product structure of quantum cohomology and were included by Yuri Ivanovich in the appendix of a paper he wrote with Maxim Kontsevich. As Yuri Ivanovich put it, pointing to the title page: "you do not appear here, but here." One of the proudest memories of this time was when Yuri Ivanovich wrote me a note saying that my results for higher Weil–Petersson volumes were "beautiful mathematics." This turned into a joint work of Yuri Manin, Don Zagier, and myself, which was the start of a program of study of such volumes and generalizations such as those by Mirzakhani. The years in Beuel, where the Max Planck Institute was located at the time, were full of interactions which shaped me mathematically and philosophically.

I am often asked how it was to have Yuri Ivanovich as an advisor. My response is that for me he was the ideal advisor, the frequency of discussions and mutual understanding being completely in resonance. There was one period, where I did avoid my advisor, and this was when I took some time to write a master thesis in philosophy. I was asked what I was doing, and after three months of dodging the question, when I was finished with the thesis, I came clean. His reaction captures his outlook on the world and his role as an advisor. He first asked what it was about. My answer, "On Frege," evoked an appreciative expression and, as I remember, some words like "aha very interesting" that were followed by a more stern look and the directive to get back to mathematics voiced in the instruction "no more Frege."

Through the years, I was often back in Bonn at MPI which then moved across the Rhine. Entering into Yuri Ivanovich's office was like entering into the hallowed halls of mathematics and culture. His signature shelving of mathematical papers juxtaposed with a collection of books from a wide range of fields reflecting his broad interests. I vividly remember the discussions about mathematics, logic, and the world in his office, with light flooding through the windows overlooking Beethoven and the Münsterplatz.

On a personal level, Yuri Ivanovich and Xenia Glebovna opened their home to my wife Birgit and myself as well as to our sons Julian and Adrian. We still have a book entitled *Leibniz für Kinder* at home which they brought as a gift. These common times and stimulating conversations are many of my happy memories. The first time I met Yuri

Ralph Kaufmann is a professor of mathematics with courtesy appointments in the Department of Physics and Astronomy and the Department of Philosophy at Purdue University. His email address is rkaufman@math.purdue.edu.

Ivanovich, the image of Hesse's Siddhartha becoming wise while contemplating the river directly came to my mind. Seeing him and Xenia Glebovna in their apartment on the Rhine is an almost prophetic fulfillment of this initial association.

In closing, I will give an excerpt of a speech he delivered when he was inducted into the Order Pour le Mérite for Sciences and Arts (Orden Pour le Mérite für Wissenschaften und Künste)—which he had entrusted me to translate into German for him—that characterizes Yuri Ivanovich perfectly.

> "All my intellectual life was molded by a noble tradition which I somewhat carelessly called the Enlightenment Project. The base of this tradition is the belief that human reason has the highest value, and that the dissemination of science and education will help produce better human beings than we are, who will be living in a better society than we live." (Manin 2007)

Ralph Kaufmann

Matilde Marcolli

It was a long voyage and a beautiful one. Our collaboration started almost immediately after I joined the MPI faculty in the summer of 2000 and lasted to the very end, with our last two joint papers posted on the arXiv less than a month before Yuri died. From the very beginning it was Yuri who started to refer to this as "the last voyage of Ulysses," from Dante's *Divine Comedy*, which he liked to read in the original Italian. If you download from the arXiv the source file of our very first joint paper "Continued fractions, modular symbols, and non-commutative geometry" (math/0102006), you will find it right there at the beginning, hidden by a % in the tex file, *"de remi facemmo ali…,"* the last voyage of Ulysses. We read that canto of the *Divine Comedy* together, and I memorized it to be able to recite it for him, which he asked me to do a few times over the years.

We ended up with 25 joint papers, written over a span of 23 years, though several of them were in fact concentrated in these last few years. Curiously, but perhaps not entirely surprisingly, we collaborated a lot more intensely after I left the Max Planck Institute and relocated to California, than we did during the years when we were both working in the same institute and seeing each other daily. The way I like to think of all these papers is as the outer windows to an inner space, to a very personal place, where a long dialog was unfolding through these two decades, a continuing conversation that cut across the boundaries of different fields and disciplines, across our distance in space and time, and the passing of the years. It was a very special and stable place, filled with its own very special affective as well as intellectual intensity.

We had been in the habit of spending New Year's Eve together at his home in Bonn, every year since I first arrived in Bonn. We continued with our regular New Year events after I moved to California. Every year I returned to Bonn in December, right at the end of our fall term, usually in time to give the last talk of the year in Yuri's "Algebra, Geometry, and Physics" seminar, and I would stay until early January, when I made my return for the start of the next term at Caltech. I always tried to make sure to have something new for the talk I would be giving for him upon arriving in Bonn, something that would be different, surprising, and entertaining. Year after year, I brought back cosmology, linguistics, information theory, and various unexpected motivic incarnations. During those winter breaks, Yuri and I would finish up our current ongoing project (many of our papers are posted to the arXiv on the first of January) and we would start discussing the next thing to think about. We met everyday to work together, including on Christmas day, first mostly at the MPI and in the later years mostly at his home. Sometimes a new project would start in relation to whatever little mathematical trophy I was bringing back from my previous year in California. That's how we ended up writing our own linguistics paper in 2016, for example, just after I had taught my first linguistics class at Caltech in 2015 and had written my first linguistics papers out of that experiment. Other times a new idea came up as a way of returning to previous conversations that had remained dormant for some years. To me it always felt like going home, to a unique place that was always reliably there…until that one time when suddenly it was forever gone.

In Homer's *Odyssey*, the voyages of Ulysses draw a chart of encounters with the multiform liminal creatures of Greek mythology, composite figures that cross the designated boundaries of the realms of nature, the human,

---

Matilde Marcolli is a Robert F. Christy Professor of Mathematics and Computing and Mathematical Sciences at California Institute of Technology. Her email address is matilde@caltech.edu.

the animal, and the divine. Our mathematical cartography is usually similarly split into supposedly impenetrable boundaries, and yet there too a pantheon of hybrid chimeras can be envisioned, tantalizing, elusive, luring like the siren's song: noncommutative boundaries of arithmetic varieties, fields with one element, Arakelov holography, phase transitions and noncomputability in spaces of codes, categorified dynamics of neuronal networks, modular and elliptic curves in cosmology, Grassmannian semantics, and other such magical creatures. In Dante's last voyage, Ulysses sails right through the pillars of Hercules, the established and impassable frontier of the system of knowledge of the ancient world, embarked on a heroic, but not solitary, intellectual quest. The voyage ends tragically in a final storm, with the rising mountain of the netherworld looming large on the distant horizon.

There was no grandiose plan guiding our long voyage of exploration, no holy grail hypothesis to chase. It was a peaceful state of meditation, a voyage of curiosity rather than a conquering campaign. It meant a lot to me to be able to finally discover, through our joint work, that mathematics does not need to be a bloody battlefield out there, does not have to be governed by aggression, territoriality, violence, like I have too often experienced it elsewhere. It can also be that peaceful shared inner space and that long shared voyage of discovery beyond *"dov' Ercule segnò li suoi riguardi acciò che l'uom più oltre non si metta."*

Matilde Marcolli

Ivan Penkov

I first met Yuri Ivanovich Manin at the beginning of September 1978, when I started my second year as an undergraduate mathematics major at MGU. My father, a Bulgarian probabilist and statistician who had good connections to some of his Soviet colleagues (despite being firmly anti-Soviet in his political thinking), had attended a lecture of Manin in the spring of 1975 and was deeply impressed by his intellectual power and personal charisma.

Ivan Penkov is an adjunct professor at Constructor University, Bremen. His email address is ipenkov@constructor.university.

In September 1978 I had just returned to Moscow after two very miserable years of military service in Bulgaria and was looking for an intellectual challenge, when I remembered my father's remarks and quickly decided to give Professor Manin's seminars a try. Certainly, I was not free of fear that this might turn into a humiliating experience as fellow students had told me that only the most brilliant students can keep up with the pace of Professor Manin. Nevertheless, my curiosity was strong and I went to the opening meeting of his seminar on number theory. That fall, the seminar's main topic was Mordell's conjecture (which was proved five years later, in 1983 by Gerd Faltings). Manin gracefully planned out several consecutive talks on the subject and distributed them among his advanced students. The very first introductory talk, basically following material from the book *Algebraic Number Theory* (J.W.S Cassels and A. Fröhlich, eds.), remained unassigned, when I raised my hand and said that I could try to understand and prepare some basic facts from the book. Manin looked surprised and quickly figured out that I was a second-year student who basically knew nothing. Nevertheless, with his confidence-inspiring intellectual generosity which I have experienced ever since, Manin gently allowed me to try and added that Misha Tsfasman, a graduate student, should support me in this endeavor.

This was a magical moment for me, and it basically determined my entire life. Two weeks later, I gave a somewhat lousy talk (without the help of Misha it would have been much worse) and became a steady participant in Professor Manin's seminars.

Yuri Ivanovich and I became closer by 1980 when he visited Bulgaria and I happened to be his local guide. Over many years, we loved to remember our visit to Plovdiv, together with Yuri Ivanovich's wife Xenia Glebovna, where we ended up spending several days more than planned because my Soviet-made car broke down and could not be repaired. As a result, we had the gift of three joyful days together in a fascinating town. Later on, in 1993 I had the great honor of Yuri Ivanovich visiting me at the University of California at Riverside, where I had just received tenure. At that time, my home was in Lake Arrowhead, in the mountains of Southern California. Yuri Ivanovich and Xenia Glebovna were of course invited to spend a day in the mountains, and I was the driver. On the way up, Yuri Ivanovich started to get uncomfortable and asked Xenia Glebovna to pass him some books and folders from the back seat so that he could sit on them. I could not understand what was the matter, until I realized that Yuri Ivanovich was sitting on a heated seat, turned on to the maximum, on a warm California day. We all laughed loudly when we figured out what had happened: in the early morning I had driven my son to school on the mountain, and he had turned on the passenger seat heater. Driving back, I could not see whether the passenger seat was on or off, while Yuri Ivanovich did not realize that a seat in a car could be heated! We made it to Lake Arrowhead in a good mood, and I still remember this incident.

We, the students in the Mekh-Mat of the late 70s and early 80s were spoiled in the sense that there was a cohort of superb professors whom we could choose as advisors: A. A. Kirillov, V. A. Arnold, S. P. Novikov, etc., and of course Manin. Choosing a scientific advisor can be like falling in love. When this happens, one star in the sky starts shining brighter and more enticingly, as if it were the only star in the sky. From the day of our first meeting in September 1978 until our last conversation on Skype around December 17, 2022, Manin was this special star on my scientific and, more generally, intellectual sky. Manin had the ability to confidently guide his students and collaborators through mathematical landscapes and to gently demonstrate by example how to unravel mathematical truths. Having been influenced by Grothendieck, he looked for the "abstract core" in every mathematical theme. He would then study this, possibly new, mathematical structure with all available tools. Very much like turning on a light!

I observed this in person when Manin introduced supergrassmannians and flag supervarieties. At that time, around 1980, supergeometry was a hot topic and many key structures had not yet emerged. With the confidence of a master, Manin defined complex supergrassmannians as supervarieties representing certain natural functors and proved their existence and smoothness. Then we, his students, started carrying out detailed studies of the structure of supergrassmannians and flag supermanifolds and eventually built a somewhat incomplete Bott–Borel–Weil theory for flag supermanifolds. The problem of computing all cohomology groups of all line bundles of flag supermanifolds is still open today.

One may talk about the "chemistry" in a relationship. This applies in particular to the relationship of a scientific advisor with his or her students. In my case, this perfect chemistry was the feeling of constant support from Manin, which unlocked my self-confidence and eventually turned my mathematical dreams into mathematical reality. My (totally subjective) feeling was that we had similar mathematical dreams, and the process of collaboration in the only three joint papers that we wrote, was the process of sharing our dreams. My advisor's acknowledgment of my modest mathematical thoughts was absolutely crucial for my becoming a mathematician. Another fantastic privilege of being Manin's student was meeting some of the most prominent mathematicians of my generation. It suffices to mention Alexander Beilinson and Vladimir Drinfeld. Seeing them "in action" in the Manin seminars was a life-shaping experience. Early on I knew that meeting Manin in 1978 was the blessing of my life, and this has carried me through life ever since.

Ivan Penkov

Vadim Schechtman

My brightest memories of Yuri Ivanovich Manin belong to the blessed time at the end of the 1980s when the miraculous breath of freedom came to our country. I remember the remarkable beginning of his advisor Igor Rostislavovich Shafarevich's talk at the meeting of the Moscow Mathematical Society in 1987 devoted to the 50th anniversary of Yuri Ivanovich. Shafarevich had mentioned five great mathematicians: Golod, Anosov, Novikov, Arnold, and Manin who all had been born around 1937 and entered Moscow University in 1953. I remember meeting him and his good lady Xenia Glebovna in their small apartment on Vernadsky Ave. I remember our collaboration with Yuri Ivanovich on higher Bruhat orders which originated from the Zamolodchikov's string analogs of the Yang–Baxter equations. I remember sitting with Sasha Beilinson and Yuri Ivanovich writing the foreword to the *K-theory in Moscow* collection (Lecture Notes in Math. 1289); the other contributors to this volume were Boris Feigin, Boris Tsygan, Vladimir Hinich, and Mariusz Wodzicky. We ended our foreword with a joking citation from *Eugene Onegin*. Yuri Ivanovich was a man of a universal culture. Here is an example of what I have learned from him. The first phrase of his article on quantum groups sounded like "The aim here is to resist the charismatic influence of Drinfeld." At the time, I did not know the meaning of the word "charismatic;" so my interpretation was that Drinfeld was mischievously trying to badly influence his teacher Manin. I was a bit mystified, and this gave me an occasion to learn the true meaning of this word. Sitting with Yuri Ivanovich and Xenia Glebovna on the balcony of their apartment in

Vadim Schechtman is a professor of mathematics at Université Paul Sabatier in Toulouse. His email address is schechtman@math.ups-tlse.fr.

the charming city of Bonn, viewing the majestic Rhein was a true blessing. …und ruhig fließt der Rhein.

Vadim Schechtman

Alexei Skorobogatov

I became Manin's student in 1980 after Andrey Levin insisted that I should attend Manin's seminar. At that time, the overall scenery of the Department of Mechanics and Mathematics at the Moscow University was pretty drab: one would not normally bump into any luminaries who were around but not visible. Almost none of them were involved in undergraduate teaching. What one saw was a mass of compliant mathematical functionaries in indistinguishable grey suits, some of them prominent in notorious entrance examinations designed to stop Jewish students from being admitted to the University. The atmosphere was not very inspiring.

The experience of Manin's seminar was overwhelming, it was akin to going abroad. People just looked different, I could not understand a word they said, and it was all immensely attractive. Immediately after Manin accepted Andrey and me, alongside Mikhail Kapranov and Vera Serganova, he told us to read Sasha Beilinson's famous paper "Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra." I fondly remember the study group that Sasha organized for us where he carefully explained the notion of a derived category. This was all very nice, but I hardly knew what a sheaf was. The mathematics of Manin's seminar was amazing, but it was an expanding universe! It made things worse for me that at the time Manin had a keen interest in physics and was moving away from his earlier work in arithmetic geometry. I decided that mathematics was hard enough and if I wanted to survive I had to leave physics alone. At the time, my discussions with Manin touched on Hitchin's work on twistor spaces, the Calabi conjecture recently proved by Yau, and the beginnings of hyper-Kähler geometry. Although they did not lead me directly to do any research by myself, soon I found a problem that I was able to solve. The result was a two-page note in which I proved that the Kuga–Satake abelian variety of a Kummer surface attached to an abelian surface A is isogenous to a power of A. I remember a useful discussion with Misha Kapranov in a university canteen. Only later, I realized that it was always like this: Manin eagerly discussed mathematics with his students but did not give them problems to solve. Unless one was fortunate enough to do research on a subject of Manin's current interest, one was left to learn from preprints and other students. Crucial for my PhD was my joint work with Mikhail Tsfasman. I started working with Jean-Louis Colliot-Thélène during his visit to Moscow in 1986.

Manin was the epitome of cool. He dressed impeccably, and his mathematics was elegant. Mathematics was always at the center: what mattered was a "correct" approach to a problem—a cleaner, more transparent and instructive proof based on an idea, a proof that would allow us to see farther and clearer. It did not matter whose proof it was; mathematics was never an exercise of one's power or a strife for recognition. If I had to come up with one quote from Manin it would be this: "a good proof is a proof that teaches us something." I remember how he called me once to tell me that he had read my paper and thought that it contained the "right" way to prove a certain result on which he had worked earlier. I would even say that mathematics for Manin was not about solving problems: this would happen almost automatically if we achieved deeper understanding and asked the right questions. Manin had a unique ability to see a far-reaching theoretical potential in mathematical facts and observations that could be perceived by others as specific and isolated, and he was happy to share these visions. His research papers are scattered with open questions and suggestions; the same was true of his courses and seminars.

It was always a great pleasure for me and my wife Anna to visit Yuri Ivanovich and his beloved wife Xenia Glebovna in their appartment in Bonn. We talked about poetry, philosophy, and all sorts of things. I will finish by saying that besides being a towering figure in mathematics, Yuri Ivanovich Manin was a very good person, honest, sincere, gentle, and sensitive, with a wonderful sense of humor. All who had the good fortune to know him will miss him a lot.

Alexei Skorobogatov

Alexei Skorobogatov is a professor of mathematics at Imperial College London. His email address is `a.skorobogatov@imperial.ac.uk`.

Vyacheslav V. Shokurov

My recollections about Manin start with the Evening Mathematical School (EMS) meetings, an activity for schoolchildren organized by E. B. Dynkin. Starting in the fall of 1963, Dynkin invited several young bright stars of the faculty of mechanics and mathematics of Moscow State University to EMS. One of them was Yuri Ivanovich Manin who was only 26 years old, and just recently got his doctoral degree, the habilitation. He lectured not only on mathematics but also on linguistics: how to guess the meaning of a Serbian word. I understood that Manin was not only a mathematician but also had an interest in humanities. Later I found out that he was much broader than I could imagine.

Over time, lectures by professors of Moscow State University became a regular activity of EMS during the usual daytime classes of school No. 2, where both activities were held. During my senior year 1965–1967 in school No. 2, they were delivered by Y. I. Manin and E. B. Vinberg. Both were wonderful lecturers, and the schoolchildren liked them very much. Both were algebraists, but their styles were slightly different and complementary: Manin was more functorial, while Vinberg was more concrete. These lectures laid down a foundation of mathematical education for me and my classmates (B. Dubrovin, I. Cherednik, and S. Dobrokhotov).

Naturally, at MGU I started attending Manin's special courses and seminars. This was a remarkable period of the Moscow mathematical school and perhaps even of all of Soviet mathematics. Manin was one of the prominent figures of the time. Of course, in the late 1960s and early 1970s there were more famous and established mathematicians in Moscow, e.g., Kolmogorov, Pontryagin, Markov, Gelfand, Shafarevich. Manin belonged to the next generation of brilliant mathematicians. He was a very versatile person, and not only in mathematics. His lectures on recent mathematical advances, e.g., on Matiyasevich's solution of the Hilbert 10th problem, were accessible to a wide mathematical audience and at the same time deep and insightful.

In 1968, I became Manin's student, first as an undergraduate and later as a graduate student, until 1975. My PhD thesis was about modular symbols, Kuga's modular varieties and Shimura's integrals. However, later my research shifted to a previous interest of Manin and his former student V.A. Iskovskikh—birational geometry.

In the fall of 1968, Manin gave me some notes about the geometry of canonical embeddings of algebraic curves. It turned into my first publication, concerning a theorem of Noether–Enriques on canonical curves. Actually, I had started to work on this paper only at the end of 1969. Manin's notes were based on a paper by Babbage from 1939. In school, I had studied English and a little bit of French, but not enough even for mathematical literature. Manin proposed that I learn not only English and French, but also German. Another more important obstacle was my lack of knowledge of modern techniques in algebraic geometry: sheaves and their cohomologies. Fortunately, Manin already lectured on this subject in his courses and corresponding literature already appeared in mathematical bookstores of Moscow.

By the summer of 1970, my paper was finished and I came to show Manin the final version. Usually in the summer, Manin leased a cottage near Moscow, and it took some time to get there. Since I was young, I was neither tired nor hungry after the trip. Nonetheless, according to Russian tradition, Manin fed me and only after that we started to discuss the paper. Soon the paper was approved by him with a few minor remarks. One of which was about the acknowledgment. In the original version the author thanks Manin for help in writing the paper but he preferred a more modest role: "for posing the problem." Actually, the paper and Manin's guidance became important in my subsequent research because canonical curves are curve sections of K3 surfaces and Fano 3-folds.

After that, Manin suggested I investigate moduli of curves of general type. These moduli were just rigorously introduced by Deligne and Mumford. However, after Manin turned to modular forms and I finished my PhD thesis, my interest deviated from Manin to Iskovskikh—to birational geometry. Only recently, I got interested in moduli but mainly not of curves and not only of general type.

In the fall of 2017, I visited the Max Planck Institute in Bonn to congratulate Manin on his 80th birthday. I had a few meetings with him and delivered a talk in his seminar about complements and recent advances of Caucher Birkar. I had the feeling that these would be our last personal meetings. Once he told me during this visit that he was thinking about noncommutative modular symbols. I am not aware of his results in this direction but certainly this could be interesting to new generations of mathematicians.

Vyacheslav V. Shokurov

Vyacheslav V. Shokurov is a professor of mathematics at Johns Hopkins University. His email address is vshokur1@jhu.edu.

Michael Tsfasman

I first met Manin as a sophomore, when I started to attend his courses and seminars, understanding as yet almost nothing. My impression was mostly of how he lectured. In his hand he had a sheet of paper on which he had the plan of what to write on the blackboard and where. Then he took the second sheet, almost empty except for a rectangle to write on the erased piece of the board. Usually, he gave two courses and led two seminars. Every year the topic of his courses changed. Later he would tell me that every seven years one should completely change the subject in order not to be bored. For a long time, these subjects were mostly centered in between algebraic geometry and number theory, then crept towards mathematical physics. The sportive side of mathematics was quite foreign for him, his goal almost never was to solve a difficult problem, though sometimes he did it, but rather to understand the subject. I would even dare to say that to understand the object, he took an important piece of mathematical reality and tried to observe and study it from different angles. For Manin, mathematics does not stand alone, it is a part of science, and science is a part of culture. He was a man of culture, rather a scholar than just a scientist. His interests were as wide as we see only in the time of Ancient Greece or the Renaissance. Besides mathematics and physics—to mention just a little part—he was interested in biology, in the origin of human speech, in medieval French, in poetry. His poems and poetry translations are not many, but they are of extremely high quality.

Yuri Ivanovich was an illustrious teacher. I asked him to become my PhD advisor when I was 25, rather late by Moscow standards, and having already a published paper. I had just read his marvelous book *Cubic Forms*, was fascinated by its subject, and wanted to choose it as my subject. He said he would be honored to have me as a student (you can imagine how pleased I was) and warned me that since he had already shifted to another subject, it would be more difficult for me. Then I told him that I was on a kind of a blacklist of the Soviet regime, and it could be difficult for me to enter the doctoral school of Moscow State. He said we should try anyway and helped me to overcome the difficulties. A year and a half later, I was desperate, half of my PhD time had elapsed with no interesting result obtained. I dared to ask him how he would define what is a good thesis. After a short reflection he answered, "A good thesis is what a good PhD student does in three years." I was soothed and soon started getting some new results in arithmetic of rational surfaces. A year later at his seminar he asked a question about curves over a finite field, I managed to answer it, though I had never before worked over a finite field. He told me that the question is related to coding theory and gave me a book on it. In a while we wrote our first paper on algebraic geometry codes, learning the definition of a code on the way. This was by no means my best paper, but definitely my most cited. After that, algebraic geometry over finite fields and global fields became the center of my interest. My previous activity was crowned by a large survey on the arithmetic of rational varieties that I wrote with Manin.

At some point, Yuri Ivanovich told me that there is something we lack in the professor-student relationship. In the nineteenth century, professors had a so-called "*jour fixe*:" once a week every student could visit his professor on social grounds. And the series of Manin's *jours fixes* started. Every Friday his students, in many cases with their wives, came to his small one-bedroom in the southwest of Moscow. It was forbidden to discuss mathematics. Each time he proposed a subject to start with, it could be some development in science other than mathematics, the possible future influence of computers on social life, the origin of speech, etc. Once I was late and Manin explained to me that they were discussing why it was so easy for us to do mathematics, and so difficult to communicate with people. My reaction was that it was difficult for me to do math and quite easy to communicate. Another impression of these *jours fixes* was the hospitality and active participation of his wife. For my wife and me, these meetings were very important. One of these Fridays fell on Manin's fiftieth birthday. The situation in the country had drastically changed with perestroika, the iron curtain starting to fall, and the air of freedom sweeping out the suffocating air of the Soviet regime. His reaction to these changes was that of pure joy. It reminded him of his youth. One of his reminiscences was that he first came to Moscow at the end of Stalin's rule, and next soon after Stalin's death. The visible difference was that the first time the whole of Moscow was full of solid fences, and the next time they somewhat disappeared.


Michael Tsfasman

New Russia happened to be far away from his dreams, and he became pessimistic. I would have liked him to be wrong.... Once he said to me that the experience of his generation is totally negative and should be erased from the memory of younger generations. Manin spent the last three decades partly in the USA and mostly in Germany. He used to come to Moscow almost every year and I cherish our rare meetings.

Michael Tsfasman is the head of the Algebra and Number Theory lab at the Institute for Information Transmission Problems, vice president of the Independent University of Moscow, and emeritus research director of CNRS. His email address is mtsfasman@yandex.ru.

Alexander Voronov

Yuri Ivanovich Manin was a remarkable thinker, mathematician, and advisor. I am glad that we, his students, carry his legacy in the way we do mathematics, teach, and work with our own students.

Manin had scores of students back in the day and at times was even overwhelmed by them. When I approached him for the first time and asked if he would take me as a student, he told me he was not accepting any new students, as he already had too many (in fact, he had 11 just within the class of students one year more senior than I). Since my facial expression apparently showed that my whole world of hopes had just collapsed, he nevertheless gave me a chance. No wonder that the admission interview promised to be tough.

Manin asked me what mathematics I had been learning. Among other things, I told him that I was attending a topics course on the "Winnie-the-Pooh conjecture," given by Kostrikin. Manin's eyes sparked with curiosity. So, he asked me to describe the problem and explain the unusual name. The name is based on a verse from Milne's classic *The House at Pooh Corner*.

And the cuckoo isn't cooing,
But he's cucking and he's ooing,
And a Pooh is simply poohing
Like a bird.

In a Russian translation:

Возьмём это самое слово «опять».
Зачем мы его произносим,
Когда мы свободно могли бы сказать
«Ошесть», и «осемь», и «овосемь»?

The name is a pun because the words "опять, ошесть, осемь," and "овосемь" are pronounced the same way one would in Russian pronounce A_5, A_6, A_7, and A_8, which refer to the classical Lie algebra series A_n. The original English verse is also a pun, but the reference to the series A_n is totally "lost in translation:"

The conjecture was related to the fact that the existence of an orthogonal decomposition for a complex simple Lie algebra of type A_n had been settled for A_6, A_7, and A_8, but not for A_5 (and still has not). I successfully described the problem and recited the verse to Manin, but the real test came later: he asked me whose Russian translation it was. I started digging into the deep corners of my memory, but could not find anything. So, I came up with a wild guess: Boris Pasternak. I knew that Pasternak had made numerous Russian translations of English poetry (Shakespeare, Byron, Keats, to name a few), and it sounded to me like a reasonable guess. Yuri Ivanovich, in his gentle manner suggested that the translator was "probably" Boris Zakhoder and nevertheless immediately told me that he was admitting me as a student. My initial shame was quickly superseded by the feeling of happiness….

When I was Manin's student, he did not have regularly scheduled meetings with his students. They happened on an ad hoc basis, initiated by Manin. Sometimes, he would just see me in his seminar or call me up and say "Sasha I have an idea of a project that might be interesting for you. Would you like to come over to my place next Wednesday at noon?" That was exciting as I anticipated Manin outlining a beautiful piece of mathematics in front of me.

Manin held the plank of general culture very high. He was well versed in many languages, which was rather unusual for the time of isolation of the Soviet Union from the rest of the world, known as the Iron Curtain, and the dominance of one language on the vast lands of the Soviet empire. Once, when I only started to attend his seminars, still eyeing him as a prospective advisor, I arrived at a seminar a few minutes late, just to hear a tape recorded voice speaking German, while everybody including Manin, sat in total silence. This lasted for some long minutes with very few people understanding a word. At the end of the recording, Manin, who was obviously inspired by it, explained that it was Hilbert speaking on his own twenty-first problem on what had become known as the Riemann–Hilbert correspondence. That seminar was actually a joint venture with Sergei Gelfand, which resulted in them writing their famous book on Homological Algebra, which ends in a beautiful chapter on D-modules.

Another time, at one of the student-advisor meetings at Manin's home, Manin said: "I suggest that you read a modern French course on algebraic surfaces, such as this one,"—and handed me over Beauville's *Astérisque* volume on complex algebraic surfaces. When I exclaimed: "But it is in French!"—Manin looked at me with genuine surprise: "Oh, Sasha! Do not you read French?" It was said in an unassuming way, but I realized that if I wanted to be in this intellectual circle in which reading mathematics in French is the norm, I'd better do it. So, within a month, I was reading Beauville's text with great interest and secret pride.

At some point, Manin told me that he was turning 50 and decided it was time for him (as an old man, as I thought back then) to connect to the younger generation. This is how he and his wife Xenia Glebovna started those Friday gatherings, later known as *jours fixes*, in their apartment with numerous students and their significant others. Xenia Glebovna played a wonderful host role in these. Her warmth, energy, and wit often helped break the ice and

Alexander Voronov is a professor of mathematics at the University of Minnesota. His email address is `voronov@umn.edu`.

engage some of us less-sociable mathematicians in interesting nonmath discussions initiated by Yuri Ivanovich.

In retrospect, the six years I spent being Manin's undergraduate and then graduate student, were packed with mathematical events. There was amazing mathematics happening in front of my eyes, as Manin created new mathematics for his weekly lectures in various topics courses, and you would be among the first people on Earth to learn about it.

Manin's 1986 "critical dimensions" paper is especially dear to my heart. The paper pointed at the critical dimension 26 showing up in the Mumford isomorphism $\lambda_2 \cong \lambda_1^{\otimes 13}$ of line bundles over the moduli space of algebraic curves, directly related to the Polyakov string measure. I took it as a challenge to look for a similar statement in supergeometry, which led to proving the super Mumford isomorphism $\lambda_{3/2} = \lambda_{1/2}^{\otimes 5}$ in my PhD thesis.

These are just a few random flashes of memory I recollect from these years, which made an enormous impression on my future life, both professional and cultural. And now, many years later, I feel lucky to be a member of Manin's school of Enlightenment and carry on his wisdom, philosophy, and style, and pass it on to my students.

Alexander Voronov

Don Zagier

My first meeting with Yuri Ivanovich Manin was in Moscow in 1987, where I spent two months (one of about a dozen visits that I made to Russia in the Soviet and post-Soviet periods, but the only one that was more than a few days) that happened to include Yuri's 50th birthday. Of course at that time he was not yet "Yuri" to me, but "Professor Manin" or (since we were then mostly talking in Russian, mine being rather primitive) "Yuri Ivanovich." To my great surprise, since we had only just met, he invited me to one of the two birthday parties that he and his wife Xenia Glebovna gave in their Moscow apartment. One of these was a more private affair, for his students and intimate friends, and included poetry and maybe even some dancing, while the other, the one I attended, was for further colleagues and friends, including new ones. The experience was unforgettable, in particular because his apartment was like a beacon of culture in the midst of all the drabness of Soviet Moscow, overflowing with books in a multitude of languages and on a multitude of subjects (including, to my great delight, even several frivolous novels in English and other languages, not just books on intellectual subjects), and the conversations with him and the others were of a similar level. Altogether it was more like a salon in Paris in some earlier epoch than like anything I had ever experienced in the United States or in Europe.

My two months in Moscow were one of the high points of my life, not just because of meeting with Manin, but also because of the way mathematics was done there, which I had never seen before and which perhaps has never existed anywhere else, before or since. Almost none of the many people I met there worked in their university or institute office, and indeed many (like Sasha Beilinson) did not even have an affiliation with a mathematical institution. Instead, mathematical discussions took place in people's homes, often in their kitchens, with people dropping in unexpectedly and unannounced, and with an unheard-of level of intensity. It seemed to me then, and I still feel now, that this high level was possible, not despite but precisely because of the difficult external conditions and the fact that one was in no danger of being distracted by frivolities like shopping or going to restaurants (two activities that were more or less nonexistent). For most of the people I met, many of whom became friends for life, a normal academic life with normal academic duties was impossible, as was foreign travel and many of the other things that scientists in the West took for granted. The result was a hugely increased focus on the two things that mattered: friends and mathematics.

I have one other striking memory from this visit. On one occasion I attended a meeting of the Moscow Mathematical Society. There were two lectures as far as I can remember, a survey talk on Diophantine equations that I gave and a talk by Shafarevich on Yuri's mathematical work in honor of his 50th birthday. At the end of that talk somebody stood up and attacked Manin viciously, saying that honoring him in this way was a farce, that it was well known that he was a mediocre mathematician and that his most famous results were either wrong or due in essence to others. I was horrified and wanted to sink through the floor, but Yuri simply ignored the attack as if it had never happened. This was of course the only sensible way to deal with it, but not one that a lesser person could have carried out, and I was hugely impressed to witness it.

Don Zagier is a retired director at the Max Planck Institute for Mathematics in Bonn. His email address is dbz@mpim-bonn.mpg.de.

Only a very few years later there came a turn of events that neither I nor anybody else could have predicted: the great thaw of the perestroika and glasnost years and the possibility for Soviet mathematicians to emigrate. I am pretty sure that the idea that Yuri might be tempted to come to Bonn and to the Max Planck Institute for Mathematics, despite the innumerable offers he received from the United States, came originally from me, since I had often talked with my wife Silke about my meeting with him and we hoped that he could be tempted by a city that was nearer physically and culturally to the world he would be leaving behind. In fact, both we and, separately, Friedrich Hirzebruch met with him and Xenia in the United States to "woo" him, and the courtship was eventually successful. Xenia tells the anecdote that at that time he was hesitating about whether or not to sign a contract that MIT had offered him and that Bob MacPherson, whom he knew well from many meetings in Moscow, told him "You know, Yuri, a contract here is not a prison—even if you sign it you are still free!" He eventually did sign it but nevertheless ended up coming to Bonn, to my lasting joy and that of Hirzebruch and many other colleagues. For several years, until the horrendous bureaucracy of the post-9/11 years forced him to stop, he also had an affiliation with Northwestern University in Illinois and spent several months there every year, always, I believe, staying in the same small hotel where they knew him and where he felt at home. But his main home, physically and intellectually, was the MPI in Bonn where he worked for 30 years and to which he contributed more than any other mathematician through his presence, his works, and his seminars.

During those years we became close friends as well as colleagues. We had innumerable mathematical discussions, though in the end we wrote only two joint papers, each with an additional coauthor (Ralph Kaufmann and Paula Cohen). I had hoped that there would be more, in particular in connection with periods of modular forms, which was a subject to which he had made crucial contributions and in which my own work (with John Lewis and others) was also of special interest to him, but this never happened.

Working with Yuri made me aware in a way that I never had been before of the two types of mathematicians, what Yuri's friend Freeman Dyson called the "frogs" and the "eagles": those who look at mathematics from a position in the grass and perceive everything from below, and those who look from a position in the sky and see everything from above. Needless to say, I belonged to the first category and Yuri to the second, but the new insight was that a collaboration between an eagle and a frog was not only possible, but in many ways even more fruitful than one between members of the same species. This became particularly evident in the paper that we wrote about modular forms and pseudodifferential operators (the third collaborator, Paula Cohen, joined only at the end to add a "super" version that Yuri was very keen to have). Unlike other joint papers that I have written in which the two authors sat repeatedly at the same table and worked out pieces of mathematics and pieces of text together, we worked independently and then met at frequent intervals to discuss our progress. What I remember is that on several occasions Yuri told me excitedly "Don, I have great news to report. The result that you showed me last week with a complicated computational proof I can now understand purely conceptually, with no need for computations at all!" On other occasions, I would report with equal enthusiasm "Yuri, I also have great news to report. The result that you showed me last week with a difficult conceptual proof I can now prove by direct computation that anybody can understand, with no need for any thinking at all!" And, rather amazingly, this worked out well: what he thought of as progress and what I thought of as progress were exactly opposite, but somehow everything converged in the end to results that were pleasing to both of us.

Another thing that I remember with fondness from our collaboration was that Yuri once told me that he had a private tradition of proving a "Christmas theorem" as a Christmas present to himself every year if he could find one, which of course he almost always could. I loved this notion and proved my own Christmas theorem that year (I think it was 1993), the proof of a complicated combinatorial identity for our joint paper that I had discovered experimentally some time before. But I am afraid that I have not kept up the tradition since. Perhaps I can try again in the future, as a tribute to him.

As a mathematician, as an intellectual, and as a person, Yuri Manin was one of the few truly great people that I have ever met. Like so many others, I would like to thank him for having enriched my life and shown me the level that a human being can achieve.

Don Zagier

MEMORIAL TRIBUTE

Yuri G. Zarhin

The first time I saw Yuri Ivanovich was at a meeting of the Moscow Mathematical Society in Spring 1969 where John Tate gave a talk in English. Manin translated, and was elegant and charming.

The following semester I started to attend his seminar, and in Fall 1970 I became one of the *Maniniacs*—that is how fellow students called Manin's advisees, because the latter were so impressed by their advisor that they talked about him to everybody. I still remember the feeling of a forthcoming feast on Wednesdays and Thursdays—the days of Manin's seminars and special courses at Mekhmat. Besides official courses and seminars, we usually met every other week. I was told that I should ask questions and I tried to do my best. Usually, my questions resulted in a half-hour lecture by Yuri Ivanovich that I tried to comprehend in the following days and weeks. Once, trying to invent a good question, I asked whether two abelian varieties A and B over a number field K are isogenous if their ranks over every finite field extension of K coincide. (I had noticed that the equality of ranks means that the Galois modules of all algebraic points of A and B modulo torsion are isomorphic.)[7] After this question, Yuri Ivanovich decided I should learn *heights* and I started to read about Weil's heights and distributions, Néron–Tate heights, Néron pairings and their generalizations that were introduced around that time by Manin. In 1972, we published a joint paper that provided explicit upper bounds for the absolute value of the difference between the heights of Weil and Néron–Tate on abelian varieties in terms of the corresponding theta constants.

Since I never attended any graduate school, it was a difficult problem to find a place in the USSR where I could defend my PhD thesis (get a kandidat degree) and even to pass qualifiers (kandidat minimum). However, enormous joint efforts of Yuri Ivanovich, Solomon Grigor'evich Mikhlin, Dmitri Konstantinovich Faddeev, and Georgiy Ivanovich Petrashen' helped solve this problem in 1975 and I defended my thesis at the St.Petersburg (then Leningrad) branch of the Steklov Institute.

Later, we did not meet as often as in my student years but we started to talk about things not directly related to mathematics. In the middle of the 1980s, I heard from Yuri Ivanovich that he was again urged by the authorities to repent for signing *the letter of the 99* (in defense of Aleksander Yesenin-Volpin). "Yuri Ivanovich, you published a textbook in logic. Please do it in such a way that suits you." Manin's answer was that during the previous 17 years he had not given any reasons to believe that his point of view on the subject has changed.

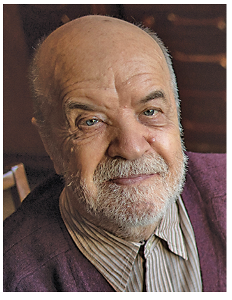
I vividly remember the celebration of Yuri Ivanovich's 50th birthday that took place in February 1987 in his Moscow apartment. I was siting next to Andrey Tyurin; our vis-á-vis were Don Zagier and Miles Reid. I. R. Shafarevich and Vjacheslav Vsevolodovich Ivanov tried to recall the last time they had seen each other: it turns out that it was in 1960, at the funeral of Boris Pasternak. Yuri Ivanovich recited his own translations into Russian of verses of Rudyard Kipling. Now Manin's translations and his own verses are published in his book *Mathematics as Metaphor*.

The last time I saw Manin was in May 2022, in his Bonn apartment. Yuri Ivanovich, Xenia Glebovna, my wife, and I were sitting on the balcony with a view of the Rhine river, drinking tea with apple pie, talking about everything but trying not to mention the war. I hoped to see him this year again. But it will not happen anymore. It is hard for me to accept that my teacher is no longer with us.

Yuri G. Zarhin is a professor of mathematics at Penn State University. His email address is `zarhin@math.psu.edu`.

[7] *Despite a certain progress in the case of elliptic curves over the rationals, this half-century old question remains unsettled.*

Yuri G. Zarhin



Fedor Bogomolov

Yuri Tschinkel

Credits

Figure 1 is courtesy of Yuri Tschinkel.

Photo of Alexander Beilinson is courtesy of Alexander Beilinson.

Photo of Vladimir Berkovich is courtesy of Daniel Berkovich.

Photo of Fedor Bogomolov is courtesy of Mazaeva Elisabeth.

Photo of Jean-Louis Colliot-Thélène is courtesy of Jean-Louis Colliot-Thélène.

Photo of Vladimir Drinfeld is courtesy of Tatyana Nesterova.

Photo of Alexander Goncharov is courtesy of Alexander Goncharov.
Photo of Michael Harris is courtesy of Jessica Wynne.
Photo of Nicholas M. Katz is courtesy of C. J. Mozzoch.
Photo of Ralph Kaufmann is courtesy of Ralph Kaufmann.
Photo of Matilde Marcolli is courtesy of Matilde Marcolli.
Photo of Ivan Penkov is courtesy of Ivan Penkov.
Photo of Vadim Schechtman is courtesy of Catherine Vidal.
Photo of Vyacheslav V. Shokurov is courtesy of S. I. Shokurova.
Photo of Alexei Skorobogatov is courtesy of Maxim Braverman.
Photo of Yuri Tschinkel is courtesy of the Simons Foundation.
Photo of Michael Tsfasman is courtesy of Michael Tsfasman.
Photo of Alexander Voronov is courtesy of the University of Minnesota.
Photo of Don Zagier is courtesy of the Max Planck Institute for Mathematics, Bonn.
Photo of Yuri G. Zarhin is courtesy of Tatiana Bandman.


MC of A
Mathematical Council of the Americas
AMS
AMERICAN
MATHEMATICAL
SOCIETY
IMSA
The Mathematical Congress of the Americas 2025
July 21-25, 2025
InterContinental Hotel, Miami
MCA
2025
MCA 25 is authorized by the Mathematical Council of the Americas and organized through
the Institute of the Mathematical Sciences of the Americas (IMSA) at the University of Miami
https://www.mca2025.org

WHAT IS...?

an Inductive Mean?

Frank Nielsen

Notions of means. The notion of means [10] is central to mathematics and statistics, and plays a key role in machine learning and data analytics. The three classical Pythagorean means of two positive reals x and y are the arithmetic (A), geometric (G), and harmonic (H) means, given respectively by

$$A(x,y)=\frac{x+y}{2},\quad G(x,y)=\sqrt{xy},\quad H(x,y)=\frac{2xy}{x+y}.$$

These Pythagorean means were originally geometrically studied to define proportions, and the harmonic mean led to a beautiful connection between mathematics and music. The Pythagorean means enjoy the following inequalities:

$$\min(x,y)\leq H(x,y)\leq G(x,y)\leq A(x,y)\leq \max(x,y),$$

with equality if and only if $x=y$. These Pythagorean means belong to a broader parametric family of means, the power means $M_p(x,y)=(x^p+y^p)^{\frac{1}{p}}$ defined for $p\in\mathbb{R}\backslash\{0\}$. We have $A(x,y)=M_1(x,y)$, $H(x,y)=M_{-1}(x,y)$ and in the limits: $G(x,y)=\lim_{p\to 0}M_p(x,y)$, $\max(x,y)=\lim_{p\to+\infty}M_p(x,y)$, and $\min(x,y)=\lim_{p\to-\infty}M_p(x,y)$. Power means are also called binomial, Minkowski, or Hölder means in the literature.

There are many ways to define and axiomatize means with a rich literature [8]. An important class of means are the quasi-arithmetic means induced by strictly increasing and differentiable real-valued functional generators $f(u)$:

$$M_f(x,y)=f^{-1}\left(\frac{f(x)+f(y)}{2}\right). \tag{1}$$

Frank Nielsen is a fellow at Sony Computer Science Laboratories Inc., Tokyo, Japan. His email address is `frank.nielsen.x@gmail.com`.

Additional material is available online at `https://franknielsen.github.io/MatrixGeometricMean/`.

Communicated by Notices Associate Editor Richard Levine.

DOI: https://doi.org/10.1090/noti2832

Quasi-arithmetic means satisfy the in-betweenness property of means: $\min(x,y)\leq M_f(x,y)\leq\max(x,y)$, and are called so because $f(M_f(x,y))=\frac{f(x)+f(y)}{2}=A(f(x),f(y))$ is the arithmetic mean on the f-representation of numbers.

The power means are quasi-arithmetic means, $M_p=M_{f_p}$, obtained for the following continuous family of generators:

$$f_p(u)=\begin{cases}\frac{u^p-1}{p}, & p\in\mathbb{R}\backslash\{0\},\\ \log(u), & p=0.\end{cases},$$

$$f_p^{-1}(u)=\begin{cases}(1+up)^{\frac{1}{p}}, & p\in\mathbb{R}\backslash\{0\},\\ \exp(u), & p=0.\end{cases}.$$

Power means are the only homogeneous quasi-arithmetic means, where a mean $M(x,y)$ is said to be homogeneous when $M(\lambda x,\lambda y)=\lambda M(x,y)$ for any $\lambda>0$.

Quasi-arithmetic means can also be defined for n-variable means (i.e., $M_f(x_1,\dots,x_n)=f^{-1}(\frac{1}{n}\sum_{i=1}^n f(x_i))$), and more generally for calculating expected values of random variables [10]: We denote by $\mathbb{E}_f[X]=f^{-1}(\mathbb{E}[f(X)])$ the quasi-arithmetic expected value of a random variable X induced by a strictly monotone and differentiable function $f(u)$. For example, the geometric and harmonic expected values of X are defined by $\mathbb{E}^G[X]=\mathbb{E}_{\log x}[X]=\exp(\mathbb{E}[\log X])$ and $\mathbb{E}^H[X]=\mathbb{E}_{x^{-1}}[X]=\frac{1}{\mathbb{E}[1/X]}$, respectively. The ordinary expectation is recovered for $f(u)=u$: $\mathbb{E}^A[X]=\mathbb{E}_x[X]=\mathbb{E}[X]$. The quasi-arithmetic expected values satisfy a strong law of large numbers and a central limit theorem ([10], Theorem 1): Let $X_1,\dots,X_n$ be independent and identically distributed (i.i.d.) with finite variance $\mathbb{V}[f(X)]<\infty$ and derivative $f'(\mathbb{E}_f[X])\neq 0$ at $x=\mathbb{E}_f[X]$.

Then we have

$$M_f(X_1, \dots, X_n) \xrightarrow{a.s.} \mathbb{E}_f[X]$$

$$\sqrt{n}\left(M_f(X_1, \dots, X_n) - \mathbb{E}_f[X]\right) \xrightarrow{d} N\left(0, \frac{\mathbb{V}[f(X)]}{\left(f'(\mathbb{E}_f[X])\right)^2}\right)$$

as $n \to \infty$, where $N(\mu, \sigma^2)$ denotes a normal distribution of expectation μ and variance σ^2.

Inductive means. An inductive mean is a mean defined as a limit of a convergence sequence of other means [15]. The notion of inductive means defined as limits of sequences was pioneered independently by Lagrange and Gauss [7] who studied the following double sequence of iterations:

$$a_{t+1} = A(a_t, g_t) = \frac{a_t + g_t}{2},$$
$$g_{t+1} = G(a_t, g_t) = \sqrt{a_t g_t},$$

initialized with $a_0 = x > 0$ and $g_0 = y > 0$. We have

$$g_0 \le \dots \le g_t \le \mathrm{AGM}(x, y) \le a_t \le \dots \le a_0,$$

where the homogeneous arithmetic-geometric mean (AGM) is obtained in the limit:

$$\mathrm{AGM}(x, y) = \lim_{t\to\infty} a_t = \lim_{t\to\infty} g_t.$$

There is no closed-form formula for the AGM in terms of elementary functions as this induced mean is related to the complete elliptic integral of the first kind $K(\cdot)$ [7]:

$$\mathrm{AGM}(x, y) = \frac{\pi}{4} \frac{x + y}{K\left(\frac{x-y}{x+y}\right)},$$

where $K(u) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-u^2\sin^2(\theta)}}$ is the elliptic integral. The fast quadratic convergence [11] of the AGM iterations makes it computationally attractive, and the AGM iterations have been used to numerically calculate digits of π or approximate the perimeters of ellipses among others [7].

Some inductive means admit closed-form formulas: For example, the arithmetic-harmonic mean $\mathrm{AHM}(x, y)$ obtained as the limit of the double sequence

$$a_{t+1} = A(a_t, h_t) = \frac{a_t + g_t}{2},$$
$$h_{t+1} = H(a_t, h_t) = \frac{2a_t h_t}{a_t + h_t},$$

initialized with $a_0 = x > 0$ and $h_0 = y > 0$ converges to the geometric mean:

$$\mathrm{AHM}(x, y) = \lim_{t\to\infty} a_t = \lim_{t\to\infty} h_t = \sqrt{xy} = G(x, y).$$

In general, inductive means defined as the limits of double sequences with respect to two smooth symmetric means M_1 and M_2:

$$a_{t+1} = M_1(a_t, b_t),$$
$$b_{t+1} = M_2(a_t, b_t),$$

are proven to converge quadratically [11] to $\mathrm{DS}_{M_1,M_2}(a_0, b_0) = \lim_{t\to\infty} a_t = \lim_{t\to\infty} b_t$ (order-2 convergence).

Inductive means and matrix means. We have obtained so far three ways to get the geometric scalar mean $G(x, y) = \sqrt{xy}$ between positive reals x and y:

1. As an inductive mean with the arithmetic-harmonic double sequence: $G(x, y) = \mathrm{AHM}(x, y)$,
2. As a quasi-arithmetic mean obtained for the generator $f(u) = \log u$: $G(x, y) = M_{\log}(x, y)$, and
3. As the limit of power means: $G(x, y) = \lim_{p\to 0} M_p(x, y)$.

Let us now consider the geometric mean $G(X, Y)$ of two symmetric positive-definite (SPD) matrices X and Y of size $d \times d$. SPD matrices generalize positive reals. We shall investigate the three generalizations of the above approaches of the scalar geometric mean, and show that they yield different notions of matrix geometric means when $d > 1$.

First, the AHM iterations can be extended to SPD matrices instead of reals:

$$A_{t+1} = \frac{A_t + H_t}{2} = A(A_t, H_t),$$
$$H_{t+1} = 2(A_t^{-1} + H_t^{-1})^{-1} = H(A_t, H_t),$$

where the matrix arithmetic mean is $A(X, Y) = \frac{X+Y}{2}$ and the matrix harmonic mean is $H(X, Y) = 2(X^{-1} + Y^{-1})^{-1}$. The AHM iterations initialized with $A_0 = X$ and $H_0 = Y$ yield in the limit $t \to \infty$, the matrix arithmetic-harmonic mean [3, 14] (AHM):

$$\mathrm{AHM}(X, Y) = \lim_{t\to+\infty} A_t = \lim_{t\to+\infty} H_t.$$

Remarkably, the matrix AHM enjoys quadratic convergence to the following SPD matrix:

$$\mathrm{AHM}(X, Y) = X^{\frac{1}{2}}(X^{-\frac{1}{2}} Y X^{-\frac{1}{2}})^{\frac{1}{2}} X^{\frac{1}{2}} = G(X, Y).$$

When $X = x$ and $Y = y$ are positive reals, we recover $G(X, Y) = \sqrt{xy}$. When $X = I$, the identity matrix, we get $G(I, Y) = Y^{\frac{1}{2}} = \sqrt{Y}$, the positive square root of SPD matrix Y. Thus the matrix AHM iterations provide a fast method in practice to numerically approximate matrix square roots by bypassing the matrix eigendecomposition. When matrices X and Y commute (i.e., $XY = YX$), we have $G(X, Y) = \sqrt{XY}$. The geometric mean $G(A, B)$ is proven to be the unique solution to the matrix Riccati equation $XA^{-1}X = B$, is invariant under inversion (i.e., $G(A, B) = G(A^{-1}, B^{-1})^{-1}$), and satisfies the determinant property $\det(G(A, B)) = \sqrt{\det(A)\det(B)}$.

Let $\mathbb{P}$ denote the set of symmetric positive-definite $d \times d$ matrices. The matrix geometric mean can be interpreted using a Riemannian geometry [5] of the cone $\mathbb{P}$: Equip $\mathbb{P}$ with the trace metric tensor, i.e., a collection of smoothly varying inner products g_P for $P \in \mathbb{P}$ defined by

$$g_P(S_1, S_2) = \operatorname{tr}\left(P^{-1}S_1P^{-1}S_2\right),$$

where S_1 and S_2 are matrices belonging to the vector space of symmetric $d \times d$ matrices (i.e., S_1 and S_2 are geometrically vectors of the tangent plane T_P of $P \in \mathbb{P}$). The geodesic length distance on the Riemannian manifold $(\mathbb{P}, g)$ is

$$\rho(P_1, P_2) = \left\|\log\left(P_1^{-\frac{1}{2}}P_2P_1^{-\frac{1}{2}}\right)\right\|_F = \sqrt{\sum_{i=1}^{d}\log^2\lambda_i\left(P_1^{-\frac{1}{2}}P_2P_1^{-\frac{1}{2}}\right)},$$

where $\lambda_i(M)$ denotes the i-th largest real eigenvalue of a symmetric matrix M, $\|\cdot\|_F$ denotes the Frobenius norm, and $\log P$ is the unique matrix logarithm of a SPD matrix P. Interestingly, the matrix geometric mean $G(X,Y) = \mathrm{AHM}(X,Y)$ can also be interpreted as the Riemannian center of mass of X and Y:

$$G(X,Y) = \arg\min_{P\in\mathbb{P}} \frac{1}{2}\rho^2(X,P) + \frac{1}{2}\rho^2(Y,P).$$

This Riemannian least squares mean is also called the Cartan, Kärcher, or Fréchet mean in the literature. More generally, the Riemannian geodesic $\gamma(X,Y;t) = X\#_tY$ between X and Y of $(\mathbb{P}, g)$ for $t \in [0,1]$ is expressed using the weighted matrix geometric mean $G(X,Y;1-t,t) = X\#_tY$ minimizing

$$(1-t)\rho^2(X,P) + t\rho^2(Y,P).$$

This Riemannian barycenter can be solved as

$$X\#_tY = X^{\frac{1}{2}}\left(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)^t X^{\frac{1}{2}},$$

with $G(X,Y) = X\#_{\frac{1}{2}}Y$, $X\#_tY = Y\#_{1-t}X$, and $\rho(X\#_tY, X) = t\rho(X,Y)$, i.e., t is the arc length parameterization of the constant speed geodesic $\gamma(X,Y;t)$. When matrices X and Y commute, we have $X\#_tY = X^{1-t}Y^t$. We thus interpret the matrix geometric mean $G(X,Y) = X\#Y = X\#_{\frac{1}{2}}Y$ as the Riemannian geodesic midpoint.

Second, let us consider the matrix geometric mean as the limit of matrix quasi-arithmetic power means which can be defined [13] as $Q_p(X,Y) = (X^p + Y^p)^{\frac{1}{p}}$ for $p \in \mathbb{R}, p \neq 0$, with $Q_1(X,Y) = A(X,Y)$ and $Q_{-1}(X,Y) = H(X,Y)$. We get $\lim_{p\to 0} Q_p(X,Y) = \mathrm{LE}(X,Y)$, the log-Euclidean matrix mean defined by

$$\mathrm{LE}(X,Y) = \exp\left(\frac{\log X + \log Y}{2}\right),$$

where exp and log denote the matrix exponential and the matrix logarithm, respectively. We have $\mathrm{LE}(X,Y) \neq G(X,Y)$. Consider the Loewner partial order $\preceq$ on the cone $\mathbb{P}$: $P \preceq Q$ if and only if $Q - P$ is positive semi-definite. A mean $M(X,Y)$ is said operator monotone [5] if for $X' \preceq X$ and $Y' \preceq Y$, we have $M(X',Y') \preceq M(X,Y)$. The log-Euclidean mean $\mathrm{LE}(X,Y)$ is not operator monotone but the Riemannian geometric matrix mean $G(X,Y)$ is operator monotone.

Third, we can define matrix power means $M_p(X,Y)$ for $p \in (0,1]$ by uniquely solving the following matrix equation [13]:

$$M = \frac{1}{2}M\#_pX + \frac{1}{2}M\#_pY. \tag{2}$$

Let $M_p(X,Y) = M$ denote the unique solution of Eq. 2. This equation is the matrix analogue of the scalar equation $m = \frac{1}{2}m^{1-p}x^p + \frac{1}{2}m^{1-p}y^p$ which can be solved as $m = \left(\frac{1}{2}x^p + \frac{1}{2}y^p\right)^{\frac{1}{p}} = M_p(x,y)$, i.e., the scalar p-power mean. In the limit case $p \to 0$, this matrix power mean M_p yields the matrix geometric/Riemannian mean [13]:

$$\lim_{p\to 0^+} M_p(X,Y) = G(X,Y).$$

In general, we get the following closed-form expression [13] of this matrix power mean for $p \in (0,1)$:

$$M_p(X,Y) = X\#_{\frac{1}{p}}\left(\frac{1}{2}X + \frac{1}{2}(X\#_pY)\right).$$

Inductive means, circumcenters, and medians of several matrices. To extend these various binary matrix means of two matrices to matrix means of n matrices $P_1, \ldots, P_n$ of $\mathbb{P}$, we can use induction sequences [9]. First, the n-variable matrix geometric mean $G(P_1, \ldots, P_n)$ can be defined as the unique Riemannian center of mass:

$$G(P_1, \ldots, P_n) = \arg\min_{P\in\mathbb{P}} \sum_{i=1}^{n} \frac{1}{n}\rho^2(P, P_i).$$

This geometric matrix mean $G = G(P_1, \ldots, P_n)$ can be characterized as the unique solution of $\sum_{i=1}^n \log\left(G^{-\frac{1}{2}}P_iG^{-\frac{1}{2}}\right) = 0$ (called the Kärcher equation), and is proven to satisfy the ten Ando–Li–Mathias properties [1] defining what should be a good matrix generalization of the scalar geometric mean.

Holbrook [12] proposed the following sequence of iterations to approximate $G(P_1, \ldots, P_n)$:

$$M_{t+1} = M_t\#_{\frac{1}{t+1}}P_t \mod n \tag{3}$$

with M_1 initialized to P_1. In the limit $t \to \infty$, we get the n-variable geometric mean: $\lim_{t\to\infty} M_t = G(P_1, \ldots, P_n)$. This deterministic inductive definition of the matrix geometric mean by Eq. 3 allows to prove that the geometric mean $G(P_1, \ldots, P_n)$ is monotone [12]: That is, if $P_1' \preceq P_1, \ldots, P_n' \preceq P_n$ then we have $G(P_1', \ldots, P_n') \preceq G(P_1, \ldots, P_n)$. The following matrix arithmetic-geometric-harmonic mean inequalities extends the scalar case:

$$\begin{aligned}
&H(X,Y;1-t,t)\\
&= ((1-t)X^{-1} + tY^{-1})^{-1}\\
&\preceq G(X,Y;1-t,t) \preceq A(X,Y;1-t,t) = (1-t)X + tY.
\end{aligned}$$

Now, if instead of taking cyclically the input matrices $P_1, \dots, P_n, P_1, \dots, P_n, \dots$, we choose at iteration t the farthest matrix in $P_1, \dots, P_n$ to M_t with respect to the Riemannian distance ρ, we get the Riemannian circumcenter [2] $C(P_1, \dots, P_n)$ which is the minimax minimizer:

$$C(P_1, \dots, P_n) = \arg\min_{C\in\mathbb{P}} \max_{i\in\{1,\dots,n\}} \rho(P_i, C).$$

The sequence of iterations

$$C_{t+1} = C_t \#_{\frac{1}{t+1}} P_{\text{farthest}(t)}, \tag{4}$$

where

$$\text{farthest}(t) = \arg\max_{i\in\{1,\dots,n\}} \rho(C_t, P_i),$$

initialized with $C_1 = P_1$ is such that

$$C(P_1, \dots, P_n) = \lim_{t\to\infty} C_t.$$

The uniqueness of the smallest enclosing ball and the proof of convergence of the iterations of Eq. 4 relies on the fact that the cone $\mathbb{P}$ is of nonpositive sectional curvatures [2]: $\mathbb{P}$ is a nonpositive curvature space or NPC space for short.

The Riemannian median minimizing $\arg\min_{P\in\mathbb{P}} \sum_{i=1}^n \frac{1}{n}\rho(P, P_i)$ is proven to be unique in Riemannian NPC spaces, and can be obtained as the limit of the following cyclic order sequence [4]:

$$\begin{aligned} X_{kn+1} &= X_{kn} \#_{t_{k,1}} P_1, \\ X_{kn+2} &= X_{kn+1} \#_{t_{k,2}} P_2, \\ \vdots &= \vdots \\ X_{kn+n} &= X_{kn+n-1} \#_{t_{k,n}} P_n, \end{aligned}$$

where $t_{k,n} = \min\left\{1, \frac{\lambda_k}{n\rho(P_n, X_{kn+n-1})}\right\}$ with the positive real sequence (λ_k) such that $\sum_{k=0}^\infty \lambda_k = \infty$ and $\sum_{k=0}^\infty \lambda_k^2 < \infty$ (e.g., $\lambda_k = \frac{1}{k+1}$).

Finally, let us mention that Bini, Meini, and Poloni [6] proposed a class of recursive geometric matrix means $G_{s_1,\dots,s_{n-1}}(P_1, \dots, P_n)$ parameterized by $(n-1)$-tuple of scalar parameters, and defined recursively as the common limit of the following sequences:

$$\begin{aligned} &P_i^{(r+1)} \\ &= P_i^{(r)} \#_{s_1} G_{s_2,\dots,s_{n-1}}(P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n), \quad i \in \{1, \dots, n\}. \end{aligned}$$

In particular, these matrix means exhibit a unique $(n-1)$-tuple for which the recursive mean $G_{\frac{n-1}{n}, \frac{n-2}{n-1}, \dots, \frac{1}{2}}(P_1, \dots, P_n)$ converges fast in cubic order (order-3 convergence). This geometric matrix mean is called the BMP mean in the literature. Furthermore, the mean $G_{1,1,\dots,1,\frac{1}{2}}(P_1, \dots, P_n)$ coincides with the Ando–Li–Mathias geometric mean [1] (ALM) which converges linearly.

Random variables, expectations, and the law of large numbers. Although inductive means as limits of sequences have been considered since the 18th century (AGM by Lagrange and Gauss), this term was only recently coined by Karl-Theodor Sturm in 2003 (see Definition 4.6 in [15]), who considered inductive sequences to calculate probability expectations of random variables on nonpositive curvature complete metric spaces. For example, let $\mathcal{P}(\mathbb{P})$ denote the set of probability measures on $\mathbb{P}$ with bounded support [15]. Let $X : \Omega \to \mathbb{P}$ be a SPD-valued random variable with probability density function p_X expressed with respect to the canonical Riemannian volume measure $\mathrm{d}\omega(P) = \sqrt{\det(g_P)}$. The expectation $\mathbb{E}[X]$ and the variance $\mathbb{V}[X]$ of a random variable $X \sim p_X$ are defined respectively as the unique minimizer of $C \mapsto \mathbb{E}[\rho^2(X, C)] = \int_{\mathbb{P}} \rho^2(C, P) p_X(P) \mathrm{d}\omega(P)$ and $\inf_{P\in\mathbb{P}} \mathbb{E}[\rho^2(X, P)]$. Consider $(X_i)_{i\in\mathbb{N}}$ to be an independent sequence of measurable maps $X_i : \Omega \to \mathbb{P}$ with identical distributions $p_{X_i} = p_X$, and let $p_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i} \in \mathcal{P}(\mathbb{P})$ denote the empirical distribution. Then the following empirical law of large numbers holds as $n \to \infty$:

$$G(X_1, \dots, X_n) \to \mathbb{E}[X].$$

Several proofs are reported in the literature (e.g., Proposition 6.6 of [15], Theorem 1 of [9], or Theorem 5.1 of [4]). Thus the expectation $\mathbb{E}[X]$ of a SPD-valued random variable can be estimated incrementally by considering increasing sequences $(X_i)_{i\in\mathbb{N}}$ of i.i.d. random vectors, and incrementally computing their Riemannian means. Experiments demonstrating convergence to various probability law expectations p_X are reported in [9].

Closing remarks. The AHM double sequence yielding the matrix geometric mean can further be generalized to define self-dual operators on convex functionals in Hilbert spaces [3] based on the Legendre–Fenchel transformation (called convex geometric mean functionals). For example, the AHM iterations initialized on a pair of nonzero complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ expressed in polar forms is proven to converge to $\mathrm{AHM}(z_1, z_2) = \sqrt{r_1 r_2} e^{i\frac{\theta_1+\theta_2}{2}}$ which involves both the scalar arithmetic mean $A(\theta_1, \theta_2)$ and the scalar geometric mean $G(r_1, r_2)$.

To conclude, let us say that not only is it important to consider which mean we mean [10] but it is also essential to state which matrix geometric mean we mean!

References

[1] T. Ando, Chi-Kwong Li, and Roy Mathias, *Geometric means*, Linear Algebra Appl. **385** (2004), 305–334, DOI 10.1016/j.laa.2003.11.019. MR2063358

[2] Marc Arnaudon and Frank Nielsen, *On approximating the Riemannian 1-center*, Comput. Geom. **46** (2013), no. 1, 93–104, DOI 10.1016/j.comgeo.2012.04.007. MR2949613

[3] Marc Atteia and Mustapha Raïssouli, *Self dual operators on convex functionals; geometric mean and square root of convex functionals*, J. Convex Anal. **8** (2001), no. 1, 223–240. MR1829063

[4] Miroslav Bačák, *Computing medians and means in Hadamard spaces*, SIAM J. Optim. **24** (2014), no. 3, 1542–1566, DOI 10.1137/140953393. MR3264572

[5] Rajendra Bhatia and John Holbrook, *Riemannian geometry and matrix geometric means*, Linear Algebra Appl. **413** (2006), no. 2-3, 594–618, DOI 10.1016/j.laa.2005.08.025. MR2198952

[6] Dario A. Bini, Beatrice Meini, and Federico Poloni, *An effective matrix geometric mean satisfying the Ando-Li-Mathias properties*, Math. Comp. **79** (2010), no. 269, 437–452, DOI 10.1090/S0025-5718-09-02261-3. MR2552234

[7] Jonathan M. Borwein and Peter B. Borwein, *Pi and the AGM*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1987. A study in analytic number theory and computational complexity; A Wiley-Interscience Publication. MR877728

[8] P. S. Bullen, *Handbook of means and their inequalities*, Mathematics and its Applications, vol. 560, Kluwer Academic Publishers Group, Dordrecht, 2003. Revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, Reidel, Dordrecht; MR0947142], DOI 10.1007/978-94-017-0399-4. MR2024343

[9] Guang Cheng, Jeffrey Ho, Hesamoddin Salehian, and Baba C. Vemuri, *Recursive computation of the Fréchet mean on non-positively curved Riemannian manifolds with applications*, Riemannian computing in computer vision, Springer, Cham, 2016, pp. 21–43. MR3444345

[10] Miguel de Carvalho, *Mean, what do you mean?*, Amer. Statist. **70** (2016), no. 3, 270–274, DOI 10.1080/00031305.2016.1148632. MR3535513

[11] D. M. E. Foster and G. M. Phillips, *A generalization of the Archimedean double sequence*, J. Math. Anal. Appl. **101** (1984), no. 2, 575–581, DOI 10.1016/0022-247X(84)90121-5. MR748590

[12] John Holbrook, *No dice: a deterministic approach to the Cartan centroid*, J. Ramanujan Math. Soc. **27** (2012), no. 4, 509–521. MR3027448

[13] Yongdo Lim and Miklós Pálfia, *Matrix power means and the Karcher mean*, J. Funct. Anal. **262** (2012), no. 4, 1498–1514, DOI 10.1016/j.jfa.2011.11.012. MR2873848

[14] Yoshimasa Nakamura, *Algorithms associated with arithmetic, geometric and harmonic means and integrable systems*, J. Comput. Appl. Math. **131** (2001), no. 1-2, 161–174, DOI 10.1016/S0377-0427(00)00316-2. MR1835710

[15] Karl-Theodor Sturm, *Probability measures on metric spaces of nonpositive curvature*, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 357–390, DOI 10.1090/conm/338/06080. MR2039961

Frank Nielsen

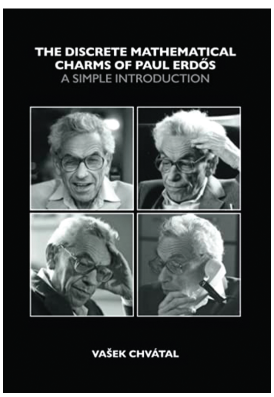
Credits

Photo of Frank Nielsen is courtesy of Maryse Beaumont.

BOOK REVIEW

The Discrete Mathematical Charms of Paul Erdős: A Simple Introduction

Reviewed by Ranjan Rohatgi



The Discrete Mathematical Charms of Paul Erdős
A Simple Introduction
By Vašek Chvátal. Cambridge University Press, 2021, 248 pp.

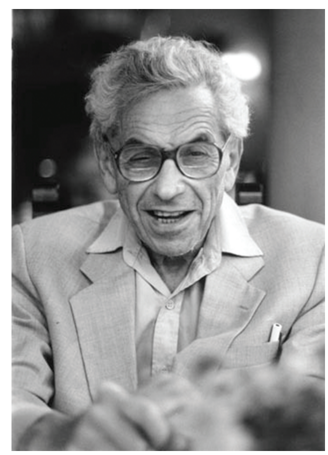
In *The Discrete Mathematical Charms of Paul Erdős: A Simple Introduction*, Vašek Chvátal tells the story of many discrete mathematical ideas, centered around the work of Paul Erdős. The book covers a wide range of topics in which Erdős left an indelible impression, including Ramsey theory, graph coloring, extremal graph and set theory, and discrete geometry. The foundation of this book is Chvátal's lecture notes from a graduate-level discrete mathematics class he taught at Concordia University in the late 2000s, just over a decade after Erdős' death. Chvátal does an exceptional job of putting the mathematical discoveries in their historical context and combining them with photos, personal anecdotes about his time spent with Erdős, and letters between the two. This context, along with the topics chosen and the clear writing, make this book exceptional.

Ranjan Rohatgi is an associate professor of mathematics at Saint Mary's College. His email address is rrohatgi@saintmarys.edu.

Communicated by Notices *Book Review Editor Emily Olson.*

DOI: https://doi.org/10.1090/noti2815

(1) Vašek Chvátal, 1972.



(2) Paul Erdős.

The Mathematics

Early in each chapter the reader is presented with a definition, conjecture, or theorem introducing the topic. After a proof, Chvátal shows the development of the topic over time, including extensions, generalizations, and related results, many with proofs. There are no exercises in the book; the chapters end with stories and photos. The nearly 400 references in this book are comprehensive, allowing interested readers to delve deeper into any topic they choose.

As an example, the first chapter, entitled *A glorious beginning: Bertrand's postulate*, begins with teenage Erdős' brilliant elementary proof of Bertrand's postulate that for every positive integer n, there is at least one prime number greater than n and at most $2n$. Following this proof are sketches of earlier proofs of Bertrand's postulate by

Chebyshev, Landau, and Ramanujan. The chapter concludes with other results and open problems concerning prime numbers, with recent work on small gaps between consecutive primes and conjectures of Erdős on large gaps between consecutive primes and primes in arithmetic progressions. Chvátal provides a chronology of the results in each of these areas and even includes the amounts of money Erdős promised to those who proved his conjectures.

The remaining ten chapters cover

- discrete geometry, including several proofs of the De Bruijn–Erdős Theorem;
- Ramsey numbers, ending with a proof of Ramsey's Theorem in full generality;
- delta-systems, introduced by Erdős and Rado in 1960;
- extremal set theory, including Sperner's Theorem and the Erdős–Ko–Rado Theorem;
- van der Waerden's Theorem and van der Waerden numbers;
- extremal graph theory, beginning with Turán's Theorem;
- the Friendship Theorem and strongly regular graphs;
- chromatic numbers, with a focus on graphs with large chromatic number;
- random graphs and finite probability theory; and
- Hamilton graphs, including a discussion of one of three joint papers Chvátal wrote with Erdős.

Chvátal writes that when teaching the class from which this book emerged, he generally covered nine of the eleven chapters over twelve 90-minute sessions. Given the breadth of topics and the depth of each chapter, the audience for this book should have at least the mathematical maturity of a beginning graduate student studying discrete mathematics.

Chvátal attempts to make the work accessible, perhaps to an advanced undergraduate audience, in two ways. First, he includes two appendices on prerequisite material. One is titled "A few tricks of the trade" and presents some commonly used methods in discrete mathematics while the other (shorter) appendix contains definitions, terminology, and notation. Secondly, and in keeping with a quintessential trait of Erdős' approach to mathematics by starting with simply stated questions, the entry point of every chapter can be understood by an undergraduate student with one course in discrete mathematics.

- Chapter 1 begins with a statement of Bertrand's postulate, given above, and Erdős' elementary proof.
- Chapter 2 begins with a theorem of Esther Klein that given any set of five points in the plane no three of which are collinear, it is always possible to select a fourth point such that the four points are the vertices of a convex quadrilateral.
- Chapter 3 begins with two versions of the same problem that appeared in the Eötvös Mathematics Competition in 1947 and the Putnam Mathematical Competition in 1953. In the former, it was posed as follows: "Prove that in any group of six people, either there are three people who know one another or three people who do not know one another. Assume 'knowing' is a symmetric relation" (p. 36).
- Chapter 4 begins with a definition of a delta-system, a family of sets such that the intersection of any two sets in the family is always the same.
- Chapter 5 begins with Sperner's Theorem: Let V be a set with n elements and E a family of subsets such that for any two subsets in E neither is contained in the other. Then E contains at most $\binom{n}{\lfloor n/2 \rfloor}$ elements with equality if and only if E consists of all subsets of V with $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ elements.
- Chapter 6 begins with a two-color version of van der Waerden's Theorem: for every positive integer k there is a number N such that if the integers $1, 2, \ldots, N$ were colored either red or white there must be an arithmetic progression of k distinct terms which are all the same color.
- Chapter 7 begins with the statement of Turán's Theorem that the graph with the most number of edges having no clique of size r is the complete $(r-1)$-partite graph whose $r-1$ parts have sizes as close to equal as possible.
- Chapter 8 begins with the Friendship Theorem of Erdős, Rényi, and Sós. The theorem says that if every two vertices in a finite graph G have exactly one common neighbor, then some vertex of G is adjacent to every vertex of G except itself.
- Chapter 9 begins with an introduction to the chromatic number of a graph and graph planarity via a brief history of what is now the Four Color Theorem.
- Chapter 10 begins by defining the "probability that a random graph with n vertices and m edges has some property" as the ratio between the number of such graphs with the property and the total number of graphs with n vertices and m edges, which is $\binom{\binom{n}{2}}{m}$ (p. 151).
- Chapter 11 begins again with Turán's Theorem, though stated differently.

The subtitle of the book is *A Simple Introduction*. I agree that each chapter does indeed have a simple introduction, given the graduate-level writing of this book. However, it quickly becomes out-of-reach for all but the best undergraduates. Upon finishing the book, I found myself wishing Chvátal had also written an undergraduate version, complete with the stories and photos, to inspire students to study discrete mathematics!

The Man Behind the Mathematics

The mathematics, though, is only part of the story. The photos and stories of Erdős transform this work from a textbook about a man's work to a tribute to the man himself. In no way do the stories presented comprise a full biography of Erdős; the final appendix includes lists of articles, books, films, and websites on Erdős and his mathematical oeuvre, including a link to a declassified FBI file. Chvátal's anecdotes paint a picture of Erdős only he could make. As he writes in the preface, "I have been one of the blind men holding onto an elephant, and these vignettes form my report on what I felt." These shared memories show how Erdős worked, his charm, his humility, his eccentricities, his political views, and his humor.

As an undergraduate, I spent one semester in Hungary, studying in the Budapest Semesters in Mathematics (BSM) program, which Erdős helped get off the ground in the 1980s. One of my favorite classes was Advanced Combinatorics taught by András Gyárfás, a frequent collaborator of Erdős. This course was my true introduction to the work of Paul Erdős. Of course, we covered many of the topics found in this book at an undergraduate level. But what I remember most about the class was a kind of excitement that I had never felt before in any classroom. I recall a passionate yet patient teacher, and classmates who became collaborators as we put forth efforts to discover theorems on our own.

Something about my understanding of mathematics changed during my time in Budapest. Prior to attending this program, I had a misguided ambition to be the best mathematician among my peers, which had taken some of the enjoyment of mathematics away. I was chasing recognition rather than results. But when I arrived in Budapest and began taking classes I realized two things. First, I simply was not going to be the best student in the program, however one defines "best". But more importantly, I found mathematics could be fun again. All I needed to do was replace competition with collaboration.

The Discrete Mathematical Charms of Paul Erdős quickly took me back to my days studying in the Budapest Semesters in Mathematics program. Of Erdős' character, Chvátal writes: "He empathized with people, he was generous with his money, he was generous with his ideas" (p. 49). Naturally, such a man had a welcoming take on doing mathematics. In Chvátal's words, "He did not rank people by their mathematical achievement. He was not a snob. What is more, he taught us by example that cooperation instead of rivalry is what makes mathematics gratifying, that collaboration with friends is enjoyable, and that new friends can be made through collaboration. This attitude of his was infectious" (p. 16).

This attitude of Erdős is, fittingly, evident in this book. The sheer number of references in each chapter is proof of this. But Erdős' approach is now widespread in both research and teaching. Many mathematicians I know pursue few single-author works and instead see mathematics as a cooperative pursuit. Further, they mimic this approach in their classrooms by providing ample time for collaboration.

Many works about Erdős dedicate space to his eccentricities and aggravating behaviors. These are documented, too, in Chvátal's stories, from Erdős' needing a host almost wherever he traveled because he could not be bothered with many mundane tasks, to Chvátal and his girlfriend being woken up several nights in a row by Erdős using the radio they had gifted him for Christmas.

Common, too, are stories of his brilliance. Chvátal writes, "But most people eventually climb to a level where a rude awakening lurks: suddenly the natural order of the universe breaks down and you are no longer the smartest kid on the block. The later disenchantment comes, the more it hurts. Paul Erdős never suffered this shock" (p. 16). On a more personal level, Chvátal amusingly recalls the end of a short-lived romantic relationship. Erdős was in town and Chvátal chose to spend the evening with him rather than his girlfriend. This choice led not only to the end of Chvátal's relationship with his girlfriend, but also to his second joint paper with Erdős.

The stories also aim to show Erdős' humanity. Chvátal recounts how Erdős' collaborative spirit went out the window when a ping-pong table was around. How the loss of his mother profoundly affected him. And how Erdős once asked him, while he was driving over 100 miles per hour, why he was going so slowly. Chvátal takes aim at the title (but not content) of a best-selling, award-winning biography of Erdős by Paul Hoffman called *The Man Who Loved Only Numbers* [1], going so far as to say "[t]he title of that book is clear libel" (p. 123).

Erdős has inspired countless works, including research articles, tributes, books, and movies, undoubtedly due in part to his gregarious approach to mathematics. He wrote over 1500 papers, had more than 500 collaborators, and was a titan of 20th-century mathematics. The two most well-known biographies of Erdős are Hoffman's *The Man Who Loved Only Numbers* [1] and Bruce Schechter's *My Brain is Open: The Mathematical Journeys of Paul Erdős* [2], both written for a general audience. So too are the George Csicsery films *N is a Number: A Portrait of Paul Erdős* [3] and *Erdős 100 Plus* [4]. Chvátal never attempts to recount Erdős' life in this book. He instead chronicles some of Erdős' most important contributions to discrete mathematics. There are many touching short tributes to Erdős written by other collaborators (for instance, the moving *Reminiscences of Paul Erdős* written by Melvin Henriksen is freely-available at `https://www.maa.org/reminiscences-of-paul-erdos`). Chvátal's book, mathematics removed, shows the deep reverence to Erdős apparent in these writings. Fan Chung and Ronald Graham's *Erdős on Graphs: His Legacy of Unsolved Problems* [5] is perhaps most similar to this book, as it is also geared towards researchers and ends with a chapter of stories on Erdős. However, I should emphasize that in no way is Chvátal rewriting Chung and Graham's book; they each have a different mathematical focus and, of course, different Erdős anecdotes.

In the introduction, Chvátal mentions that he has left out the Lovász Local Lemma and Szemerédi's Regular Partition Lemma (p. 2). He also makes choices in each chapter about which results and proofs to include or omit. With the eleven chapters spanning fewer than 200 pages, these omissions are understandable, especially in light of its origins as a series of lecture notes. The book can, of course, be used as a graduate-level textbook. But it is far more than that. *The Discrete Mathematical Charms of Paul Erdős* is a phenomenal reference for discrete mathematicians and a "simple introduction" to discrete mathematics for any mathematician with the desire to learn about the work and life of Paul Erdős.

References

[1] Paul Hoffman, *The man who loved only numbers*: *The story of Paul Erdős and the search for mathematical truth*, Hyperion Books, New York, 1998. MR1666054

[2] Bruce Schechter, *My brain is open*: *The mathematical journeys of Paul Erdős*, Simon & Schuster, New York, 1998. MR1638921

[3] George Paul Csicsery, *N is a number*: *A portrait of Paul Erdős*, George Paul Csicsery, Oakland, CA; distributed by A K Peters, Ltd., Natick, MA, 1993. MR1660995

[4] G. Csicsery, Director, *Erdős 100 Plus* [Film], USA: Zala Films, 2013.

[5] Fan Chung and Ron Graham, *Erdős on graphs*: *His legacy of unsolved problems*, A K Peters, Ltd., Wellesley, MA, 1998. MR1601954

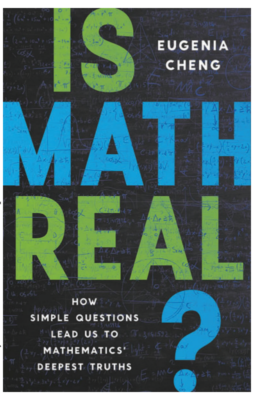
Ranjan Rohatgi

Credits

BOOKSHELF

New and Noteworthy Titles on our Bookshelf
December 2023

Note from the editor: I couldn't help myself! Given the similarity in titles, I felt compelled to review both of these books in the same month. Each book serves a different purpose and covers different topics, as you will see. This month, we ponder the realness (or not) of math!



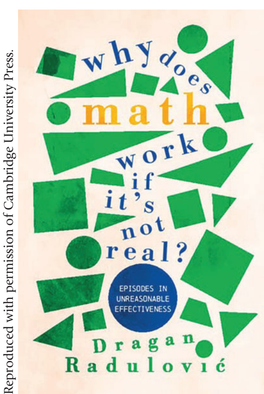
Courtesy of Hachette Book Group.

Is Math Real?
How Simple Questions Lead Us to Mathematics' Deepest Truths
By Eugenia Cheng.
Basic Books, 2023, 336 pp.

As I perceive it, the goal of Eugenia Cheng's latest book is to provide examples of why seemingly simple components of a typical math education are actually much deeper. She describes *Is Math Real?* as a "math emotions book," and her audience is both math-lovers and math-skeptics. While reading, I was reminded of a former student who liked to jokingly ask me, "Have mathematicians figured out how to divide by zero yet?" This book provides a compelling explanation to this question that even math-adverse folks could appreciate.

It comes as no surprise that in a typical math education, there is too much emphasis on having students *answer* questions rather than learning how to *ask* them. Cheng would like to see students learn "how questions lead us on a journey, about where the journey is leading, and about why we might want to go on that journey, and what we see on the way." She also advocates that math education should emphasize not only math's direct usefulness and how it is a basis for studying other related fields, but also that math is a powerful way of thinking that is very transferable. She addresses popular memes, such as one which asks, "How many holes does a straw have?" Cheng's response is: it depends on how you define a "hole."

The mathematics presented is approachable and Cheng's prose is conversational in tone. Readers who like to consider how the public views mathematics education would enjoy this book. There are some nuggets of wisdom that could affect your teaching philosophy as well, such as why to promote internalizing over memorizing and why you should relish the innocent questions students ask. Aimed more at the general public than mathematicians, this book would make a great gift for a K–12 educator or an intrigued but skeptical family member.



Reproduced with permission of Cambridge University Press.

Why Does Math Work... If It's Not Real?
Episodes in Unreasonable Effectiveness
By Dragan Radulović.
Cambridge University Press,
2023, 166 pp.

To most students, trigonometry seems self-contained. However, these functions model musical sound, ocean surges, and UV rays. While $\sin(x)$ and its siblings were used for measurements for thousands of years before mathematicians discovered their effectiveness at modeling natural phenomena, we are left wondering, "Why?"

Radulović takes on this assignment: provide examples of the "unreasonable effectiveness" of mathematics and pose insightful questions regarding this miracle. Throughout the book, the reader can expect forays into physics, set theory, geometry, and probability. We learn about the carefully balanced complexity of John Conway's Game of Life, which serves as an example of the difficulty of building a self-contained and interesting universe. And axioms... there is much to say about axioms!

If you have wondered about the philosophical underpinnings of mathematics, this book is for you. It contains insightful queries for a mathematician to ponder and could definitely be the start of some enlightening conversations, perhaps in a departmental book club or seminar course. I found myself enjoying the many tangents (pun intended!) and digressions in this wonderfully unique and well-articulated book.

This Bookshelf was prepared by Notices *Associate Editor Emily J. Olson.*

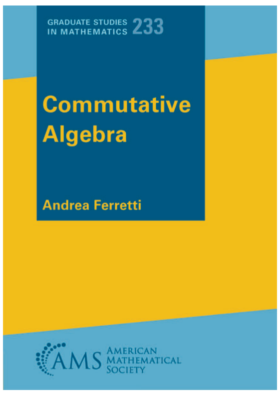
Appearance of a book in the Notices *Bookshelf does not represent an endorsement by the* Notices *or by the AMS.*

Suggestions for the Bookshelf can be sent to notices-booklist@ams.org.

DOI: https://doi.org/10.1090/noti2818

AMS BOOKSHELF

The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world's leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit **bookstore.ams.org** to explore the entire collection of AMS titles.



Commutative Algebra
By Andrea Ferretti.
GSM/233, 2023, 373 pp.

Commutative algebra is a foundational subject that must be mastered by any student wishing to pursue research in number theory, algebraic geometry, representation theory, or a number of other areas. Among the many resources available for beginning students, perhaps the three most widely known are the classic textbooks by Atiyah–Macdonald, by Matsumura, and by Eisenbud. A lively discussion of the relative merits of these and other works can be found in the comments section of a MathOverflow post by Andrea Ferretti from 2010. In this post, Ferretti was seeking suggestions for a textbook on commutative algebra that was "(i) more comprehensive than Atiyah–Macdonald; (ii) more readable than Matsumura; (iii) less thick than Eisenbud."

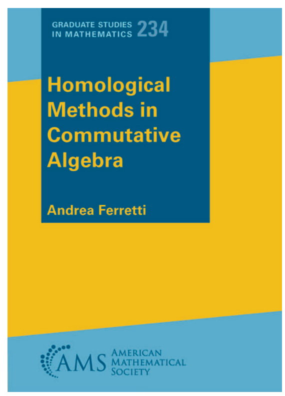
Ultimately, Ferretti decided to answer his own question by writing the two-volume set under review. In the introduction to Volume 1, Ferretti says that his intent is to "expand on the material in [Atiyah–Macdonald]." (Advanced topics from Matsumura or Eisenbud are covered in Volume 2; more on that later.)

Ferretti achieves this in two ways. First, the core topics from Atiyah–Macdonald (including localization, primary decomposition, and the going-up and going-down theorems) are covered at about half the pace of Atiyah–Macdonald, occupying roughly 300 pages of this 450-page book. In large part, the extra length is due to Ferretti's extended discussions of important special cases and examples, which serve as motivation for abstract definitions.

Pramod Achar is Shirley Blue Barton Professor of Mathematics at the Louisiana State University. His email address is `pramod.achar@math.lsu.edu`.

DOI: https://doi.org/10.1090/noti2816

Second, Ferretti's book covers several topics that are beyond the scope of Atiyah–Macdonald. For instance, there is a whole chapter devoted to "Computational methods," including Gröbner bases. This book covers a number of concepts of particular importance in number theory, such as Witt vectors and ramification. In my opinion, Volume 1 is an excellent choice for a one-semester introductory course on commutative algebra.



Homological Methods in Commutative Algebra
By Andrea Ferretti.
GSM/234, expected 2023.

Volume 2 is structured as two mini-books in one: the first four chapters are "pure" homological algebra (abelian categories, derived functors, spectral sequences); while the last six chapters are on topics in commutative algebra that require a background in homological methods.

The homological algebra part of this book is much more extensive than the corresponding parts of Matsumura or Eisenbud. Notably, the book includes a full proof of the Freyd–Mitchell embedding theorem. These chapters assume no prior contact with homological algebra, and they provide more than adequate preparation for the latter half of the book. They aren't quite enough for a self-contained course on modern homological algebra but they can serve as a warm-up for another text that does, such as that of Gelfand–Manin.

The second part of Volume 2 covers an array of topics including flatness, Koszul complexes, Cohen–Macaulay and Gorenstein rings, and local cohomology. In my opinion, these chapters are the crowning gem of the whole two-volume set. They include proofs of some of the most significant theorems in the field, including the Quillen–Suslin theorem and Kunz's theorem.

Together, this two-volume set provides an engaging and friendly introduction to the subject, and is a welcome addition to the literature.

The Heidelberg Laureate Forum: Bringing Together Some of the Brightest Minds in Mathematics

Marianne Freiberger and Rachel Thomas

How do you inspire the next generation of mathematicians? One way is to bring them together with the very best in the field—people whose work has been foundational, whose thoughts spark new ideas, and whose advice (and anecdotes) can be invaluable.

Every year the Heidelberg Laureate Forum (HLF) offers just such an opportunity for generational exchange between young researchers from around the globe and laureates of the main prizes in mathematics and computer science. Whether you are a young researcher, someone seeking to encourage the early career researchers you work with, or even a laureate yourself, we would like to introduce you to the HLF and invite you to apply, encourage applications, and take part.

Sparking Interactions

The HLF is organized by the Heidelberg Laureate Forum Foundation (HLFF) and takes place every year in September in the German university town of Heidelberg. Over the course of a week 200 young researchers, from undergraduate to postdoctoral level, mix and mingle with recipients of the Fields Medal and the Abel Prize in the field of mathematics, the ACM A. M. Turing Award and the ACM Prize in Computing in the field of computer science, and the IMU Abacus Medal and Nevanlinna Prize representing the overlap between the two fields.

Figure 1. Robert Endre Tarjan (1982 Nevanlinna Prize, 1986 ACM A. M. Turing Award) talking to young researchers at the 9th HLF 2022.

The HLF offers the scientific lectures, workshops, and poster sessions you would recognize from any scientific meeting. But in addition there are unique and welcoming opportunities for all participants to interact: panel discussions, speed networking sessions, discussions in small groups, and coffee breaks and meals allowing ample time for talk. There is also a busy social program including a boat trip down the river Neckar (with opportunities for dancing), a Bavarian beer fest, and dinner at the Heidelberg castle. At all events the laureates mix with the young researchers, formally and informally, and in return enjoy a

Marianne Freiberger and Rachel Thomas are the editors of `plus.maths.org`*, part of the Millennium Mathematics Project at the University of Cambridge. Their email address is* `plus@maths.cam.ac.uk`.

DOI: https://doi.org/10.1090/noti2834

chance to see old friends and meet a diverse group of brilliant young mathematicians and computer scientists who will lead future research.

"One of the goals of the HLF is sparking scientific interactions between the laureates and the young researchers," says Anna Wienhard, scientific chair of the Heidelberg Laureate Forum Foundation. "And I say sparking because that scientific interaction may start in a lecture or a panel discussion, but then continue in the coffee break afterwards, or in a discussion over dinner. The other goal is to get mathematicians and computer scientists together to exchange and discuss the challenges within these fields and at their interface, and also to go beyond and see what is the role that mathematics and computer science plays in our society."

Young Researchers from Around the World

The HLF is now in its tenth year. It was initiated, and is funded, by the German foundation Klaus Tschira Stiftung,[1] which promotes natural sciences, mathematics, and computer science. The HLF was inspired by the Lindau Nobel Laureate Meetings, which support exchange between Nobel Laureates and young scientists.

Figure 2. *"I enjoyed meeting the laureates a lot—asking questions and listening to their advice. I also had the possibility to network with young researchers and I think that many collaborations will arise from the conference."* Marithania Silvero, Young Researcher 8th HLF 2021, Spain.

The organizations which award the prizes that are being represented—the International Mathematical Union (IMU), the Norwegian Academy of Science and Letters (DNVA), and the Association for Computing Machinery (ACM)—are partners of the HLF. It's these organizations that choose the young researchers, who arrive from more than 50 countries each year. "The HLF provides the infrastructure for the applications by the young researchers, but the selection is in the hands of the award granting institutions," says Wienhard.

Applications from young researchers are invited between November and February each year. The 200 successful applicants are chosen in equal parts from mathematics and computer science, balancing the overlap between the fields, in a way that ensures geographical, gender, and social diversity and equity.

[1]https://www.klaus-tschira-stiftung.de/

"It's not just young researchers from top universities that come to the HLF," says Wienhard. "The group is much more diverse than that, so laureates can interact with people they may not meet at ordinary scientific conferences."

Figure 3. *"The HLF not only broadened my horizons by showcasing ground-breaking research but also provided a stimulating environment to exchange ideas, learn from laureates, and cultivate lifelong connections."* Oluwatosin Babasola, Young Researcher 9th HLF 2022, Nigeria.

To ensure a balanced choice, each application is read by three reviewers from a team chosen by the award granting institutions. The team is selected to ensure they have an understanding and experience of the geographical regions applicants come from, as well as the fields they are studying. And while quantitative measures, such as exam grades, are important, much weight is given to letters of recommendation and, crucially, the applicants' own letter of motivation. "The ability to ask good questions is vital," says Sergei Tabachnikov, professor of mathematics at Pennsylvania State University, who is involved in selecting applicants. "It's also important that they will be able to make the most of their opportunity to interact with other HLF participants when they are here."

Figure 4. *"The HLF really transformed my thinking. Seeing all these challenges in maths and computer science gives me a lot of inspiration for the application of AI in health systems, especially in Africa."* Jimoh Abdulganiyu, Young Researcher 9th HLF 2022, Nigeria.

The excitement among the young researchers each year is palpable. Laureate lectures and panel discussions themselves present rare opportunities, but being able to present lightning talks and posters to this eminent audience, and to have informal chats or even a little dance with a laureate, goes far beyond what most will have experienced in their academic life so far. And it creates a lasting impact. "There is a certain exposure to scientific ideas, topics, and discussions, which has an influence on the directions young researchers choose in the future," says Wienhard.

Figure 5. *"After the HLF keep in contact with each other. . . Today I am still working with people I met in 2019, we're still collaborating on papers and research work."* Jie Li, Young Researcher 7th HLF 2019 and MC 9th HLF 2022, China.

Equally important as interactions with laureates is the opportunity to connect with a select and interesting group of other young researchers. "One important goal is to create networks between young people, especially if they are geographically remote," says Tabachnikov. The HLF offers several projects to make sure the connections that are made within a week in September last far into the future, and that young researchers are supported beyond the event itself.

"Our network for young researchers to stay connected after the event is called AlumNode,[2] a cooperation with the Klaus Tschira Stiftung and the German Scholars Organisation," says Sarah MacLeod, head of Young Researcher Relations for the HLF. "It's an interdisciplinary alumni network the young researchers can join. They can apply for joint project funding and we offer them workshops, peer mentoring, and regular get togethers. We also have the HLFF Spotlight series[3] where we highlight the happenings in the lives of our alumni." Additionally, in 2023 the HLFF Inspiring Minds project[4] was launched, which provides expertise and guidance through an expert mentoring program and a range of engaging digital formats designed to support alumni in developing their personal career path.

Bringing Together the Brightest Minds

The content of the scientific program is planned by the HLF scientific committee, which comprises international experts in mathematics and computer science and representatives of the award granting organizations and the Heidelberg Laureate Forum Foundation. All laureates are welcome to contribute to the program. The exact nature of laureates' contributions is discussed in advance with the

Figure 6. A conversation between Ingrid Daubechies, Maryna Viazovska (2022 Fields Medal), Anna Wienhard, and Karen Uhlenbeck (2019 Abel Prize) at the 10th HLF, 2023.

committee open to all sorts of propositions—from a traditional talk on a topic of a laureate's choice, to discussion groups, master classes, and visits to local schools.

Apart from sessions involving laureates, there are also activities focused on particular areas chosen by the committee. For example, each year a *Hot Topic* is examined through talks and discussions involving experts in the area: quantum computing, epidemic modelling, and deep learning are examples of fields that have featured as Hot Topics over recent years.

Those participants who are not young researchers, and in particular the laureates, are attracted to the HLF by more than just the excellent food and accommodation, beautiful surroundings, and desire to support the next generation.

"The HLF is a chance to meet experts in mathematics and computer science, as well as very bright young people," says Efim Zelmanov who won a Fields Medal in 1994 and has attended almost every year. "It is also a fertile meeting ground of mathematics and computer science. The organizers' warm hospitality is another definite plus."

The interaction between mathematicians and computer scientists at all levels is indeed a key attraction of the HLF. "Very interesting things happen at the interface between maths and computer science and it's important to bring researchers together," says Wienhard.

Vinton Cerf, who received the ACM A. M. Turing Award in 2004 and has been involved with the HLF from the beginning, also enjoys the open invitation to all laureates of the prizes involved: "I got to see some old friends I hadn't seen in a long time and I keep getting to see them every year." In recent years new types of sessions, such as Laureate Discussions, have allowed participants to share ideas and insights in different ways, aiming, for example, to identify fruitful interactions between mathematics and computer science, and examining topics such as the role

[2] https://alumnode.org/

[3] https://www.heidelberg-laureate-forum.org/young-researchers/hlff-spotlight/hlff-spotlight-alumni-in-action.html

[4] https://www.heidelberg-laureate-forum.org/young-researchers/alumni/hlff-inspiring-minds.html

Figure 7. Every year the HLF is accompanied by a public exhibition. This photo, taken at the 7th HLF in 2020, is from the exhibition "La La Lab—The Mathematics of Music," which was developed by IMAGINARY. It shows Andreas Matt, director of IMAGINARY, and musician and mathematician Charles Gray.

Figure 8. Bavarian evening at the 9th HLF 2022.

of mathematical proof in computer science. The Laureate Discussions are designed with the laureates' input.

But it's not just the interaction with peers that laureates and other experts find invigorating. The young researchers who attend are not only among the brightest, they also come from regions of the world and social backgrounds that are not always represented at scientific conferences. "They bring problems to the table that are not easy to solve and even the laureates may not know the answers," Cerf said at the 9th HLF 2022. "I have problems with the systems I'm working on now and I love the opportunity to tell the students: here's the problem—do you have any good ideas for solving it? So this is very much a two way street."

Finally, the HLF provides an opportunity for the research community to celebrate those important prizes and so celebrate mathematics and computer science themselves.

Inspiring the Next Generation

This year in September the HLF celebrated its 10th birthday. Recognizing the growing impact of artificial intelligence on all our lives, the opportunity to bring together mathematicians and computer scientists was timely. Young researchers were particularly engaged with the sessions exploring AI and its role in society. This session sat alongside the eagerly anticipated lecture by Hugo Duminil-Copin whose work was recognized by a 2022 Fields Medal.

To mark the 10th HLF several new types of sessions were offered. Lightning Talks were one-slide presentations by laureates looking back at the last ten years in research, some even risking a glance into the future. In Spark Sessions laureates explored current research in short talks, while master classes explored specialized topics such as randomness (with Abel laureate Avi Wigderson) and algebraic structures (with Efim Zelmanov). Twenty HLF alumni were also invited, with some moderating discussion rounds and even running a workshop on improvised theatre.

A total of 12 mathematics laureates attended the 10th HLF, including Fields Medallist Maryna Viazovska and Abel laureate Karen Uhlenbeck. With Viazovska attending in person and Uhlenbeck remotely, it was significant to have the only two living female laureates in mathematics speaking at the Forum. In an interview for the HLF Vlog[5] Abel laureate László Lovász explained some of the benefits of the HLF for the laureates: "You can see that all over the world there are people who love your subject, who are devoted to science. So the future looks brighter than before meeting them."

As the HLF enters its second decade, we would like to invite young mathematicians from around the world, as well as laureates, to apply and attend. As young researcher Narinder Sing Punn said at the 9th HLF 2022, "Apply for the HLF. It is a life changing experience—don't miss this opportunity."

All students and early career researchers in mathematics and computer science up to postdoctoral level can apply. The application period for Young Researchers runs from November 9, 2023 to February 9, 2024. To apply, please visit `https://application.heidelberg-laureate-forum.org`.

[5]`https://youtu.be/aMCnoM0Q-bo?si=92CFKos13cWHiAGv`


Marianne Freiberger


Rachel Thomas

Credits

Figures 1 and 6–8 are courtesy of Heidelberg Laureate Forum Foundation.

Figure 2 is courtesy of Marithania Silvero.

Figure 3 is courtesy of Tosin Babasola.

Figure 4 is courtesy of Jimoh Abdulganiyu.

Figure 5 is courtesy of Yuxuan Feng from Light Memory Studio: `https://www.lightmemorystudio.com`.

Author photos are courtesy of Marianne Freiberger and Rachel Thomas.

FROM THE AMS SECRETARY

Calls for Nominations & Applications

AMS Prizes and Awards

I. Martin Isaacs Prize for Excellence in Mathematical Writing

The I. Martin Isaacs Prize is awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years.

About this Prize

The prize focuses on the attributes of excellent writing, including clarity, grace, and accessibility; the quality of the research is implied by the article's publication in *Communications of the AMS*, *Journal of the AMS*, *Mathematics of Computation*, *Memoirs*, *Proceedings of the AMS*, or *Transactions of the AMS*, and is therefore not a prize selection criterion.

Professor Isaacs is the author of several graduate-level textbooks and of about 200 research papers on finite groups and their characters, with special emphasis on groups—such as solvable groups—that have an abundance of normal subgroups. He is a Fellow of the American Mathematical Society, and received teaching awards from the University of Wisconsin and from the School of Engineering at the University of Wisconsin. He is especially proud of his 29 successful PhD students.

Next Prize: January 2025

Nomination Period: The deadline is March 31, 2024.

Nomination Procedure: `www.ams.org/isaacs-prize`

Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination, and a complete bibliographic citation for the article being nominated.

Joint Prizes and Awards

2024 MOS–AMS Fulkerson Prize

The Fulkerson Prize Committee invites nominations for the Delbert Ray Fulkerson Prize, sponsored jointly by the Mathematical Optimization Society (MOS) and the American Mathematical Society (AMS). Up to three awards of US$1,500 each are presented at each (triennial) International Symposium of the MOS. The Fulkerson Prize is for outstanding papers in the area of discrete mathematics. The prize will be awarded at the 25th International Symposium on Mathematical Programming to be held in Montreal, Canada, in the summer of 2024.

Eligible papers should represent the final publication of the main result(s) and should have been published in a recognized journal or in a comparable, well-refereed volume intended to publish final publications only, during the six calendar years preceding the year of the Symposium (thus, from January 2018 through December 2023). The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided.

The term "discrete mathematics" is interpreted broadly and is intended to include graph theory, networks, mathematical programming, applied combinatorics, applications of discrete mathematics to computer science, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will be based only on their mathematical quality and significance.

Previous winners of the Fulkerson Prize are listed here: `www.mathopt.org/?nav=fulkerson#winners`.

Further information about the Fulkerson Prize can be found at `www.mathopt.org/?nav=fulkerson` and `https://www.ams.org/fulkerson-prize`.

The Fulkerson Prize Committee consists of

- Julia Böttcher (London School of Economics), MOS Representative
- Rosa Orellana (Dartmouth College), AMS Representative
- Dan Spielman (Yale University), Chair and MOS Representative

Please send your nominations (including reference to the nominated article and an evaluation of the work) by February 15, 2024 to the chair of the committee:
Professor Daniel Spielman
Email: `daniel.spielman@yale.edu`

Catching Up with AMS Congressional Fellows

Elaine Beebe

Since 2005, the American Mathematical Society has sponsored 19 mathematicians to work in Congress through the American Association for the Advancement of Science (AAAS) Science and Technology Policy Fellowships program.[1]

Of these 19 Fellows, 13 served in the offices of US senators from around the country. Other Fellows worked for members of the House of Representatives, for the House Science Committee, and the Senate Homeland Security and Government Affairs Committee. All were embedded in governmental operations.

AMS Congressional Fellows—most of whom enter the fellowship from academia—lend expertise to legislators as they develop policies, explained Karen Saxe, AMS associate executive director and director of government relations.

"Fellows learn how mathematics and mathematical thinking can help make change at the federal level," said Saxe, a former AMS Congressional Fellow herself. "They take away improved communications skills and a new view of our government that can help them move to careers in government or industry." Fellows who returned to academia found that the knowledge and skills helped them be a strong department or campus leader.

"The experience is a win for the individual Fellow, a win for the math community, and a win for the government," Saxe said.

Some AMS Congressional Fellows were so captivated by their year in Washington that they continued their employment at the intersection of mathematics and government.

"At its heart, mathematics is about problems solving, and right now, government is working on solutions to some of the biggest problems in generations," said A.J. Stewart, the 2021–2022 AMS Congressional Fellow. "These are not clear-cut problems that have an easy solution, so having the ability to anticipate causal relations and find novel approaches is vital."

When he decided to apply for the AMS Congressional Fellowship, Stewart was a mathematics instructor at Seattle University, teaching undergraduate courses on financial and consumer mathematics and mathematical reasoning plus graduate courses on data science and discrete math for data analytics. While teaching, he developed a taste for policymaking through the university's academic assembly, which revised its voting system with Stewart's help.

Stewart served his AMS Fellowship in the office of Sen. Raphael Warnock (GA). With two other staffers, he covered the economic portfolio: housing, tax, trade, and financial services. "My role as a mathematician was always valued in whatever project I was working on, and I was allowed to apply my skills as I saw fit the situation best," he said.

After a year of working for Warnock, Stewart chose to apply for a second governmental fellowship; he currently is an AAAS Executive Branch Fellow working at the US Department of Treasury in the Office of Investment Security. ("My response is in a personal capacity and does not represent the views of AAAS, US Treasury, the office of Sen. Raphael Warnock, or any portion of the US Government," he noted.)

"During both my Congressional Fellowship in the Warnock office and my current fellowship at Treasury, I have learned that a lot of the skills that are developed in mathematics are applicable within government," Stewart said. His work at Treasury is related to the department's role as chair of the Committee on Foreign Investment in

Elaine Beebe is the communication and outreach content specialist at the American Mathematical Society. Her email address is exb@ams.org.

DOI: https://doi.org/10.1090/noti2835

[1] http://fellowships.aaas.org/index.shtml

the United States (CFIUS), which reviews the national security implications of foreign investments in the US.

"I was surprised that even in areas like housing or national security—which seem very removed from mathematics—I have been able to apply mathematical thinking in my position," said Stewart, who earned his PhD from the University of Oregon with a dissertation on algebraic geometry.

Mathematicians bring a distinct set of skills and insights to the process of government. "The legislative rhythm is fast and often unpredictable, requiring things like navigating knowledge gaps effectively and efficiently, synthesizing large quantities of information quickly, and learning by doing," said Lucia D. Simonelli, 2019–2020 Fellow, who earned her PhD from the University of Maryland working predominantly in dynamical systems. "One of the skills I used the most is the capacity to be comfortable and productive while in a constant state of learning."

Simonelli's fellowship year in the office of Sen. Sheldon Whitehouse (RI) focused on climate policy areas such as energy, carbon removal, and carbon pricing. This experience prepared her for her current position as a senior climate researcher at Giving Green, a guide that assists individuals and businesses with decisions about climate-oriented donations.

"I was fortunate to have a mentor who ensured that I experience as many aspects of legislative work as possible," said Simonelli, who wrote speeches, drafted legislation, met with constituents, staffed hearings, organized briefings, and even appeared on C-SPAN.

"Plunging into the policy world was a great adventure and every day I was faced with things I didn't know," she said, crediting the preparation of her mathematical training. "Mathematics influenced my approach to policy in a subtle but powerful way: Mathematics taught me humility.

"Being a mathematician has taught me to understand when I really know something, and of equal importance, when I don't."

In turn, the work of government has lessons to impart to mathematicians, Simonelli said. "The experience taught me the power and importance of creating a strong network—not for self-gain, but more as a collective that can work together in various ways to support the advancement of common goals and causes.

"The positive power of networking is something that I have continued to apply both personally and professionally. This comes in many forms, including the comfort to ask for help, the capacity to offer help, and the privilege of being at most one degree of separation from someone who certainly knows the answer."

In his 2018–2019 fellowship with Sen. Amy Klobuchar (MN), James Ricci drafted legislation and much more, working on the Health, Education, Labor and Pensions legislative team and the Commerce team in the senator's office. Focusing on education, data privacy, and science policy, he learned how to work under tight deadlines, and how to know "when something is good enough even if it is not perfect. As a mathematician and academic, it is hard to put out something you think might not be completely accurate or isn't as well researched as you would like," he said. "Many circumstances don't afford the time needed to vet everything, though, and some uncertainty is better than not providing any information at all."

Ricci, a number theorist with a PhD from Wesleyan University, found his fellowship to be life-changing. "I thought I was going to be a professor for the rest of my career, but this position opened up several new career paths for me and showed me how I could impact mathematical and scientific research in ways that I had never considered before," he said.

Ricci stepped down from a tenure-track position to move into private philanthropy and science funding as a director at Schmidt Futures in New York, which he calls "an extremely hard decision." And he too has used his fellowship skills on the job.

"I have found myself much more ready and confident to jump into new experiences," Ricci said. "I can speak more knowledgably about a much broader array of topics, and can navigate talking to a wide variety of stakeholders or present to different audiences at varying levels of technical sophistication. This combines my research background, teaching experience, and knowledge and skills from my fellowship experience," he said.

"Personally, I am also much more attuned to the political process in my everyday life than I was before," Ricci adds. "I get my news from different sources and can read into subtexts, posturing, or indications of real progress that I never was able to before.

"Being more attuned and knowledgeable about this also helps me be a better advocate for the issues I care about."

AMS Updates

CRM, PIMS, and AARMS Are Newest JMM Partners

The AMS welcomes a new Canadian partnership to the JMM.

The Centre de recherches mathématiques (CRM), Pacific Institute for the Mathematical Sciences (PIMS), and Atlantic Association for Research in Mathematical Sciences (AARMS) have signed a joint long-term partnership agreement with the American Mathematical Society (AMS) to organize a plenary address and an associated special session at the Joint Mathematics Meetings (JMM), beginning in 2024. They join fifteen other partner organizations to date, adding to a robust program.

The first CRM-PIMS-AARMS invited address will be "Fourier coefficients of modular forms," given by Henri Darmon of McGill University at JMM 2024 in San Francisco.

About CRM: Founded in 1968, the Centre de recherches mathématiques is dedicated to excellence in research and training in pure and applied mathematics, and in statistics, with all their ramifications for different domains of human endeavor, including physics, health sciences, and computer science. The CRM collaborates with other institutions and organizations, notably the CNRS (French National Centre for Scientific Research), which is present at the CRM through an International Research Laboratory (IRL).

About PIMS: The Pacific Institute for the Mathematical Sciences is a collaborative consortium of ten universities in western Canada and the Pacific Northwest. PIMS's mandate is to promote research in the mathematical sciences and their applications; to facilitate the training of highly qualified personnel; to create an equitable, diverse, and inclusive community; to enrich public awareness of and education in the mathematical sciences; and to create mathematical partnerships with similar organizations in other countries. PIMS also is a CNRS International Research Laboratory (IRL).

DOI: https://doi.org/10.1090/noti2837

About AARMS: The mission of the Atlantic Association for Research in Mathematical Sciences is to strengthen research and education in the mathematical sciences, with special focus on Atlantic Canada. AARMS fosters scientific collaborations, both within the Atlantic mathematical community and with colleagues across Canada and beyond. AARMS provides outstanding educational opportunities in order to build expertise and attract talent to the region. In addition, AARMS supports initiatives that raise interest and competence in mathematics among the public in general and schoolchildren in particular. Through its activities, AARMS aims to promote scholarly excellence and to maintain a strong and vibrant mathematical research community in Atlantic Canada.

—AMS Communications

Forty-six Awardees Receive Inaugural AMS-Simons Research Enhancement Grants for PUI Faculty

Forty-six mathematical scientists have been named inaugural recipients of AMS-Simons Research Enhancement Grants for Primarily Undergraduate Institution (PUI) Faculty. Awardees will receive $3,000 a year for three years to support research-related activities.

Recipients hail from 39 institutions across 22 states, including 11 minority-serving institutions (MSIs). Their research programs involve an exciting range of study in the mathematical sciences.

Launched in 2023, the AMS-Simons Research Enhancement Grants for PUI Faculty program is a new opportunity to foster and support research collaboration by full-time mid-career mathematicians at US institutions that do not offer a mathematics doctoral degree.

Under the grant, any activities that will further the awardee's research program are allowed. Expenses include but are not limited to conference participation, institute visits, collaboration travel (awardee or collaborator), computer equipment or software, family-care expenses, teaching assistants, publication expenses, stationery, supplies, books, and membership fees to professional organizations. Almost all awardees indicated that they will be using their grants to advance their research by traveling to conferences or meeting with collaborators.

Annual discretionary funds for an awardee's department and administrative funds for an awardee's institution will be available to institutions that administer the grant on behalf of the AMS. The grants are made possible through funding from the Simons Foundation and the AMS, as well as Eve, Kirsten, Lenore, and Ada of the Menger family.

Applications for the next cohort are anticipated to open on MathPrograms.org in January 2024. Note: Faculty who applied for but did not receive the 2023 awards are eligible to apply again.

—AMS Programs

AMS Advisory Group on Artificial Intelligence Seeks Input

Everywhere you look, developments in artificial intelligence (AI) affect mathematics. To focus on issues at the AI forefront, AMS President Bryna Kra has formed the ad hoc Advisory Group on Artificial Intelligence and the Mathematical Community. This group is charged with framing questions related to AI (broadly construed) that are of importance to the mathematical community, with the goal that existing committees in the AMS will then be able to study these questions and develop resources for the community.

The advisory group would like to hear from you at `https://www.ams.org/about-us/governance/committees/artificial-intelligence`.

Please share your thoughts on ways in which the AMS can support its members in navigating the various issues raised by a world of widely deployed automated reasoning, for example:

Publications: Evolving technologies offer new models for publication and communication.

Education: The nature of undergraduate and graduate education in mathematics is changing rapidly.

Research: AI will create new research directions and funding opportunities for mathematicians, and may influence the nature of basic mathematical research.

Community: The role of mathematicians in society, and the types and availability of mathematical work, may all change.

Other: Feel free to share other ways in which you believe the AMS can support our community in handling AI-related questions.

″AI already is affecting our working lives as mathematicians, and will continue to do so in ways we cannot yet foresee,″ said Akshay Venkatesh, chair of the committee. "With your help, the AMS can help us navigate this new landscape.″

—AMS Communications

Mathematics People

Lim Wins Vannevar Bush Fellowship

Lek-Heng Lim, University of Chicago professor, has received a Vannevar Bush Faculty Fellowship from the US Department of Defense.

An AMS member, Lim is one of 10 faculty scientists in the 2023 cohort; each will receive up to $3 million over the five-year fellowship term to pursue cutting-edge fundamental research projects. The goal of Lim's project is to build a complete picture of deep neural networks by piecing together current insights from algebraic, geometric, and topological studies.

More information about the Vannevar Bush Faculty Fellowship is available on the Basic Research Office website: `https://basicresearch.defense.gov/`.

—US Department of Defense

Yu, Zhou Named Davidson Fellows

The Davidson Fellows Scholarship Program has announced the 2023 scholarship winners. Among the honorees is 18-year-old Edward Yu of Bellevue, WA, who won a $50,000 scholarship for his project, Turán Problems for Mixed Graphs. He is one of only two students nationwide to be recognized as a Davidson Fellows Laureate and one of only 21 scholarship winners in the 2023 Fellows class. Also among the 21 scholarship winners is 18-year-old Ethan Zhou of Vienna, VA, who won a $25,000 scholarship for his project, Online Learning of Smooth Functions.

—Davidson Institute

DOI: https://doi.org/10.1090/noti2836

MoMath Awards Math Communication Honors to Students

The National Museum of Mathematics (MoMath) has awarded first place in the 2023 Steven H. Strogatz Prizes for Math Communication to the following high-school students: Alex Rosenzweig (art), Isabelle Schwartz (art), and Akilan Sankaran (social media). Runners-up included Griffin Hon (video), Jaemin Kim (video), Zoë Nadal (performance), and Anaya Willibus (writing). Honorable mention winners are Rohan Mehta (writing), Parth Patel (writing), and Sohil Wrath (writing).

Eligible submissions in the categories of art, audio, performance, social media, video, and writing included podcasts, articles, school newspaper columns, art exhibits, YouTube videos, websites, Instagram accounts, songs, plays, and other modes of public communication. Projects should be designed to be suitable to share with a general audience and should inspire an interest in or appreciation of mathematics. They are judged on content, creativity, and communication, and cash prizes are awarded. For more information or to see the winning entries, consult `https://momath.org/the-steven-h-strogatz-prize-for-math-communication/strogatz-prize-2023-winners/`.

—National Museum of Mathematics

Cadilhac Receives Zemánek Prize

The 2023 Jaroslav and Barbara Zemánek Prize in functional analysis with emphasis on operator theory was awarded to Léonard Cadilhac, Sorbonne University, France, for his fundamental contributions to noncommutative analysis centered around ergodic theory, harmonic analysis, and free probability.

The annual Zemánek Prize was founded in 2018 by the Institute of Mathematics of the Polish Academy of Sciences (IM PAN), Warsaw, to promote mathematicians younger than 35 years of age who have made important contributions to functional analysis, operator theory, and related topics. For more information, visit `https://www.impan.pl/en/events/awards/b-and-j-zemanek-prize`.

—Institute of Mathematics of the Polish Academy of Sciences

AWM Presents Service Awards to Franklin, Pershell

The Association for Women in Mathematics (AWM) has announced recipients of the 2024 AWM Service Awards.

Johanna Franklin, associate professor of mathematics, Hofstra University, is recognized for her exceptional leadership as coordinator of the AWM-MfA Student Essay Contest, as chair of the Essay Contest Committee, and as chair of the Education and Outreach Portfolio Committee.

Karoline Pershell, chief operating officer and director of strategy and evaluation for Service Robotics and Technologies Labs, is recognized for her service as AWM executive director, "for the meaningful AWM programs she initiated and improved; for creating a warm, welcoming, and inclusive environment; and for continuing to generously share her time, her energy, her enthusiasm, and her wisdom as a volunteer."

—Association for Women in Mathematics

Classified Advertising

Employment Opportunities

MASSACHUSETTS

Assistant Professor of Mathematics at Williams College

The Williams College Department of Mathematics and Statistics invites applications for a tenure-track position in applied mathematics, beginning fall 2024, at the rank of assistant professor (a more senior appointment is possible under special circumstances). The candidate should have a PhD in Applied Mathematics, Mathematics, or a closely related field by the time of appointment. We are seeking candidates who are committed to inclusive undergraduate education and show evidence and/or promise of excellence in teaching students from diverse backgrounds as well as a strong research program in applied mathematics that can engage undergraduate students. The candidate will join a department that actively supports interdisciplinary research and is expanding its existing applied curriculum. We welcome applications from all applied areas.

Our department offers a vibrant undergraduate program with majors in mathematics (including an applied mathematics emphasis) and statistics; for more information, see `https://math.williams.edu`. The multidisciplinary environment is a rich and collegial setting for student education and faculty research. Williams College provides the opportunity to apply for student research assistant support, an annual allocation of funds to support travel and research, a shared computer cluster for parallel computation, a grants office, and several internal research funding opportunities. In addition, Williams College offers faculty participation in the college's professional development program First3 and in the NCFDD Faculty Success Program, and support through the newly established Rice Center for Teaching.

Please submit your application via MathJobs: `https://www.mathjobs.org/jobs/list/22682`. In your application materials, we ask you to address how your teaching, scholarship, mentorship and/or community service might support our commitment to diversity and inclusion. Your application should include the following components:

1. A cover letter. This should provide a brief summary of your professional experience and future goals, and should address your interest in working at Williams College in particular.
2. A current CV.
3. A research statement.
4. A teaching statement. This should address your teaching philosophy and experience, ways in which you foster an inclusive learning environment, and other reflections or relevant information you would like to share.
5. Three recommendation letters, at least one of which addresses your teaching experience.

We encourage applications from members of underrepresented groups with respect to gender, race and ethnicity, religion, sexual orientation, disability status, socioeconomic background, and other axes of diversity. If you have questions about this position, contact the Chair of the Hiring Committee, Julie Blackwood (`jcb5@williams.edu`). Applications will be accepted until the position is filled, but all applications received by November 15, 2023 will be guaranteed full consideration. All offers of employment are contingent upon completion of a background check. Further information is available at `https://faculty.williams.edu/prospective-faculty/background-check-policy`.

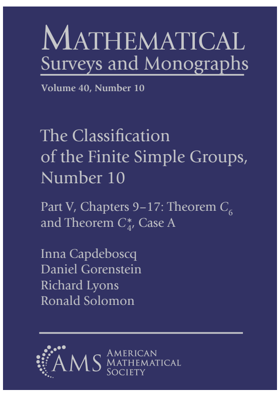
Williams College is a liberal arts institution located in the Berkshire Hills of western Massachusetts. The college has built its reputation on outstanding teaching and scholarship and on the academic excellence of its approximately 2,000 students. Please visit the Williams College website (`http://www.williams.edu`). Beyond meeting fully its legal obligations for non-discrimination, Williams College is committed to building a diverse and inclusive community where members from all backgrounds can live, learn, and thrive.

15


Introducing our new
SECTION EDITORS
LONDON MATHEMATICAL SOCIETY
EST. 1865
Srikath B. Iyengar
University of Utah
Algebra
Arend Bayer
University of Edinburgh
Geometry and Topology
Jack A. Thorne
University of Cambridge
Number Theory
Nadia S. Larsen
University of Oslo
Analysis
Alex Fink
Queen Mary University of London
Combinatorics, Discrete and Computational Mathematics and Logic
Amanda Turner
University of Leeds
Probability, Stochastic Analysis & Dynamical Systems
José A. Carrillo
University of Oxford
Partial Differential Equations and Geometric and Numerical Analysis
Bulletin of the
Journal of the
London Mathematical Society
• Prestigious, high-quality publications with international reach
• Editorial decisions made by a Section Editor whose subject is close to yours
• Providing grants and funding for mathematics and mathematicians
LEARN MORE AT lms.ac.uk/BJ
WILEY
lms.ac.uk/BLMS
lms.ac.uk/JLMS

New Books Offered by the AMS

Algebra and Algebraic Geometry



The Classification of the Finite Simple Groups, Number 10

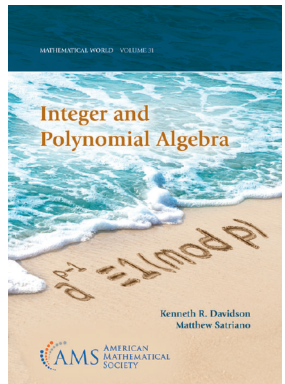
Part V, Chapters 9–17: Theorem C_6 and Theorem C_4^*, Case A

Inna Capdeboscq, *University of Warwick, Coventry, United Kingdom*, **Daniel Gorenstein**; **Richard Lyons**, *Rutgers University, Piscataway, NJ*, and **Ronald Solomon**, *The Ohio State University, Columbus, OH*

This book is the tenth in a series of volumes whose aim is to provide a complete proof of the classification theorem for the finite simple groups based on a fairly short and clearly enumerated set of background results. Specifically, this book completes our identification of the simple groups of bicharacteristic type begun in the ninth volume of the series (see SURV/40.9). This is a fascinating set of simple groups which have properties in common with matrix groups (or, more generally, groups of Lie type) defined both over fields of characteristic 2 and over fields of characteristic 3. This set includes 11 of the celebrated 26 sporadic simple groups along with several of their large simple subgroups. Together with SURV/40.9, this volume provides the first unified treatment of this class of simple groups.

Mathematical Surveys and Monographs, Volume 40
December 2023, 570 pages, Softcover, ISBN: 978-1-4704-7553-6, 2020 *Mathematics Subject Classification*: 20D05, 20D06, 20D08, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code SURV/40.10

bookstore.ams.org/surv-40-10



Integer and Polynomial Algebra

Kenneth R. Davidson, *University of Waterloo, ON, Canada*, and **Matthew Satriano**, *University of Waterloo, ON, Canada*

This book is a concrete introduction to abstract algebra and number theory. Starting from the basics, it develops the rich parallels between the integers and polynomials, covering topics such as Unique Factorization, arithmetic over quadratic number fields, the RSA encryption scheme, and finite fields.

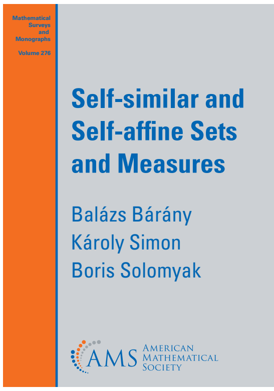
In addition to introducing students to the rigorous foundations of mathematical proofs, the authors cover several specialized topics, giving proofs of the Fundamental Theorem of Algebra, the transcendentality of *e*, and Quadratic Reciprocity Law. The book is aimed at incoming undergraduate students with a strong passion for mathematics.

This item will also be of interest to those working in number theory.

Mathematical World, Volume 31
January 2024, 185 pages, Softcover, ISBN: 978-1-4704-7332-7, 2020 *Mathematics Subject Classification*: 11–01, 12–01, 13–01, List US$65, **AMS members US$52**, **MAA members US$58.50**, Order code MAWRLD/31

bookstore.ams.org/mawrld-31

Analysis



Self-similar and Self-affine Sets and Measures

Balázs Bárány, *Budapest University of Technology and Economics, Hungary*, **Károly Simon**, *Budapest University of Technology and Economics, Hungary*, and **Boris Solomyak**, *Bar-Ilan University, Ramat Gan, Israel*

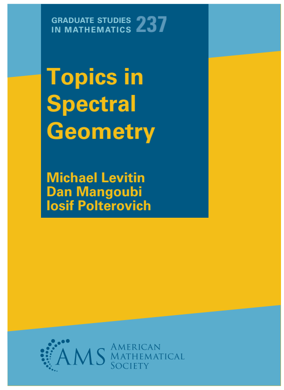
Although there is no precise definition of a "fractal", it is usually understood to be a set whose smaller parts, when magnified, resemble the whole. Self-similar and self-affine sets are those for which this resemblance is precise and given by a contracting similitude or affine transformation. The present book is devoted to this most basic class of fractal objects.

The book contains both introductory material for beginners and more advanced topics, which continue to be the focus of active research. Among the latter are self-similar sets and measures with overlaps, including the much-studied infinite Bernoulli convolutions. Self-affine systems pose additional challenges; their study is often based on ergodic theory and dynamical systems methods. In the last twenty years there have been many breakthroughs in these fields, and our aim is to give introduction to some of them, often in the simplest nontrivial cases.

The book is intended for a wide audience of mathematicians interested in fractal geometry, including students. Parts of the book can be used for graduate and even advanced undergraduate courses.

Mathematical Surveys and Monographs, Volume 276
November 2023, approximately 455 pages, Softcover, ISBN: 978-1-4704-7046-3, 2020 *Mathematics Subject Classification*: 28A80; 28A78, 28A75, 37D35, 42A38, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code SURV/276

bookstore.ams.org/surv-276



Topics in Spectral Geometry

Michael Levitin, *University of Reading, United Kingdom*, **Dan Mangoubi**, *The Hebrew University, Jerusalem, Israel*, and **Iosif Polterovich**, *Université de Montréal, QC, Canada*

It is remarkable that various distinct physical phenomena, such as wave propagation, heat diffusion, electron movement in quantum mechanics, oscillations of fluid in a container, can be described using the same differential operator, the Laplacian. Spectral data (i.e., eigenvalues and eigenfunctions) of the Laplacian depend in a subtle way on the geometry of the underlying object, e.g., a Euclidean domain or a Riemannian manifold, on which the operator is defined. This dependence, or, rather, the interplay between the geometry and the spectrum, is the main subject of *spectral geometry*. Its roots can be traced to Ernst Chladni's experiments with vibrating plates, Lord Rayleigh's theory of sound, and Mark Kac's celebrated question "*Can one hear the shape of a drum?*" In the second half of the twentieth century spectral geometry emerged as a separate branch of geometric analysis. Nowadays it is a rapidly developing area of mathematics, with close connections to other fields, such as differential geometry, mathematical physics, partial differential equations, number theory, dynamical systems, and numerical analysis.

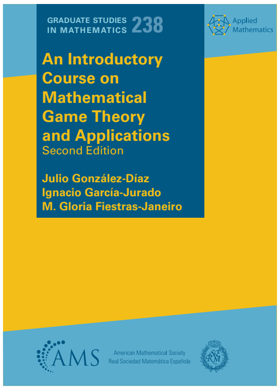
This book can be used for a graduate or an advanced undergraduate course on spectral geometry, starting from the basics but at the same time covering some of the exciting recent developments which can be explained without too many prerequisites.

This item will also be of interest to those working in differential equations and geometry and topology.

Graduate Studies in Mathematics, Volume 237
January 2024, 325 pages, Hardcover, ISBN: 978-1-4704-7525-3, LC 2023030372, 2020 *Mathematics Subject Classification*: 35Pxx; 47A75, 58C40, 58J50, 58J53, 65N25, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/237

bookstore.ams.org/gsm-237

Applications



An Introductory Course on Mathematical Game Theory and Applications

Second Edition

Julio González-Díaz, *Universidade de Santiago de Compostela, Spain*, **Ignacio García-Jurado**, *Universidade da Coruña, A Coruña, Spain*, and **M. Gloria Fiestras-Janeiro**, *Universidade de Vigo, Spain*

Game theory provides a mathematical setting for analyzing competition and cooperation in interactive situations. The theory has been famously applied in economics, but is relevant in many other sciences, such as psychology, computer science, artificial intelligence, biology, and political science. This book presents an introductory and up-to-date course on game theory addressed to mathematicians and economists, and to other scientists having a basic mathematical background. The book is self-contained, providing a formal description of the classic game-theoretic concepts together with rigorous proofs of the main results in the field. The theory is illustrated through abundant examples, applications, and exercises.

The style is distinctively concise, while offering motivations and interpretations of the theory to make the book accessible to a wide readership. The basic concepts and results of game theory are given a formal treatment, and the mathematical tools necessary to develop them are carefully presented.

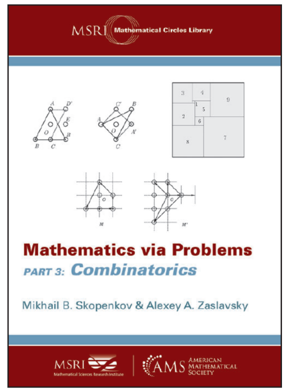
In this second edition, the content on cooperative games is considerably strengthened, with a new chapter on applications of cooperative games and operations research, including some material on computational aspects and applications outside academia.

This item will also be of interest to those working in discrete mathematics and combinatorics.

Graduate Studies in Mathematics, Volume 238
December 2023, 415 pages, Hardcover, ISBN: 978-1-4704-6796-8, 2020 *Mathematics Subject Classification*: 91–01; 91A10, 91A12, 91A18, 91A80, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/238

bookstore.ams.org/gsm-238

General Interest



Mathematics via Problems

Part 3: Combinatorics

Mikhail B. Skopenkov, *King Abdullah University of Science and Technology, Thuwal, Saudi Arabia, National Research University Higher School of Economics, Moscow, Russia, and Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, Russia*, and **Alexey A. Zaslavsky**, *Central Economical and Mathematical Institute, Moscow, Russia, and Moscow Power Energetic Institute, Russia*

This book is a translation from Russian of Part III of the book *Mathematics Through Problems: From Olympiads and Math Circles to Profession*. Part I, *Algebra*, and Part II, *Geometry*, have been published in the same series.

The main goal of this book is to develop important parts of mathematics through problems. The authors tried to put together sequences of problems that allow high school students (and some undergraduates) with strong interest in mathematics to discover such topics in combinatorics as counting, graphs, constructions and invariants in combinatorics, games and algorithms, probabilistic aspects of combinatorics, and combinatorial geometry.

Definitions and/or references for material that is not standard in the school curriculum are included. To help students that might be unfamiliar with new material, problems are carefully arranged to provide gradual introduction into each subject. Problems are often accompanied by hints and/or complete solutions.

The book is based on classes taught by the authors at different times at the Independent University of Moscow, at a number of Moscow schools and math circles, and at various summer schools. It can be used by high school students and undergraduates, their teachers, and organizers of summer camps and math circles.

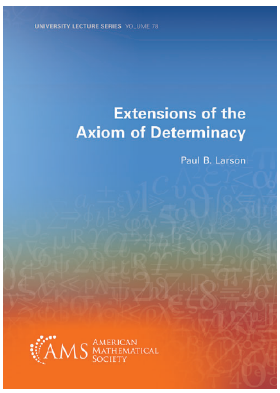
In the interest of fostering a greater awareness and appreciation of mathematics and its connections to other disciplines and everyday life, MSRI and the AMS are publishing books in the Mathematical Circles Library series as a service to young people, their parents and teachers, and the mathematics profession.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

MSRI Mathematical Circles Library, Volume 29
January 2024, approximately 210 pages, Softcover, ISBN: 978-1-4704-6010-5, 2020 *Mathematics Subject Classification*: 00–01, 00A07, 05–01, 52–01, 60–01, 94–01, 97K10, 97–01, List US$55, **AMS Individual member US$41.25**, AMS Institutional member US$44, **MAA members US$49.50**, Order code MCL/29

bookstore.ams.org/mcl-29

Logic and Foundations



Extensions of the Axiom of Determinacy

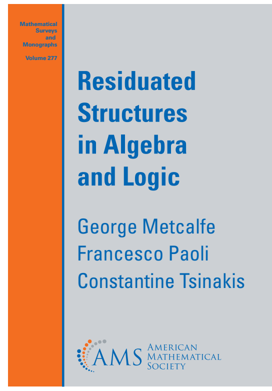
Paul B. Larson, *Miami University, Oxford, OH*

This is an expository account of work on strong forms of the Axiom of Determinacy (AD) by a group of set theorists in Southern California, in particular by W. Hugh Woodin. The first half of the book reviews necessary background material, including the Moschovakis Coding Lemma, the existence of strong partition cardinals, and the analysis of pointclasses in models of determinacy. The second half of the book introduces Woodin's axiom system **AD**$^+$ and presents his initial analysis of these axioms. These results include the consistency of **AD**$^+$ from the consistency of AD, and its local character and initial motivation. Proofs are given of fundamental results by Woodin, Martin, and Becker on the relationships among AD, **AD**$^+$, the Axiom of Real Determinacy, and the Suslin property. Many of these results are proved in print here for the first time. The book briefly discusses later work and fundamental questions which remain open. The study of models of **AD**$^+$ is an active area of contemporary research in set theory.

The presentation is aimed at readers with a background in basic set theory, including forcing and ultrapowers. Some familiarity with classical results on regularity properties for sets of reals under AD is also expected.

University Lecture Series, Volume 78
December 2023, 165 pages, Softcover, ISBN: 978-1-4704-7210-8, 2020 *Mathematics Subject Classification*: 03E60, 03E15, 03E25, 03E45, List US$69, **AMS members US$55.20**, **MAA members US$62.10**, Order code ULECT/78

bookstore.ams.org/ulect-78



Residuated Structures in Algebra and Logic

George Metcalfe, *University of Bern, Switzerland*, **Francesco Paoli**, *University of Cagliari, Italy*, and **Constantine Tsinakis**, *Vanderbilt University, Nashville, TN*

This book is an introduction to residuated structures, viewed as a common thread binding together algebra and logic. The framework includes well-studied structures from classical abstract algebra such as lattice-ordered groups and ideals of rings, as well as structures serving as algebraic semantics for substructural and other non-classical logics. Crucially, classes of these structures are studied both algebraically, yielding a rich structure theory along the lines of Conrad's program for lattice-ordered groups, and algorithmically, via analytic sequent or hypersequent calculi. These perspectives are related using a natural notion of equivalence for consequence relations that provides a bridge offering benefits to both sides. Algorithmic methods are used to establish properties like decidability, amalgamation, and generation by subclasses, while new insights into logical systems are obtained by studying associated classes of structures.

The book is designed to serve the purposes of novices and experts alike. The first three chapters provide a gentle introduction to the subject, while subsequent chapters provide a state-of-the-art account of recent developments in the field.

Mathematical Surveys and Monographs, Volume 277
December 2023, approximately 272 pages, Softcover, ISBN: 978-1-4704-6985-6, 2020 *Mathematics Subject Classification*: 03B47, 03C05, 03F52, 03G10, 06D35, 06F05, 06F15, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code SURV/277

bookstore.ams.org/surv-277

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Toric Periods and *p*-adic Families of Modular Forms of Half-Integral Weight

V. Vatsal, *University of British Columbia, Vancouver, BC, Canada*

Memoirs of the American Mathematical Society, Volume 289, Number 1438
September 2023, 95 pages, Softcover, ISBN: 978-1-4704-6550-6, 2020 *Mathematics Subject Classification*: 11F03, 11F27, 11F37, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/289/1438

`bookstore.ams.org/memo-289-1438`

Discrete Mathematics and Combinatorics

Hopf Monoids and Generalized Permutahedra

Marcelo Aguiar, *Cornell University, Ithaca, NY*, and **Federico Ardila**, *San Francisco State University, CA, and Universidad de Los Andes, Bogotá, Colombia*

This item will also be of interest to those working in geometry and topology and algebra and algebraic geometry.

Memoirs of the American Mathematical Society, Volume 289, Number 1437
September 2023, 119 pages, Softcover, ISBN: 978-1-4704-6708-1, 2020 *Mathematics Subject Classification*: 05A15, 16T30, 18M80, 52B05, 52B40, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/289/1437

`bookstore.ams.org/memo-289-1437`

Geometry and Topology

Fundamental Factorization of a GLSM Part I: Construction

Ionut Ciocan-Fontanine, *University of Minnesota, Minneapolis, MN, and Korea Institute for Advanced Study, Seoul, Republic of Korea*, **David Favero**, *University of Minnesota, Minneapolis, MN, and Korea Institute for Advanced Study, Seoul, Republic of Korea*, **Jérémy Guéré**, *Université Grenoble Alpes, France*, **Bumsig Kim**, *Korea Institute for Advanced Study, Seoul, Republic of Korea*, and **Mark Shoemaker**, *Colorado State University, Fort Collins, CO*

This item will also be of interest to those working in algebra and algebraic geometry.

Memoirs of the American Mathematical Society, Volume 289, Number 1435
September 2023, 96 pages, Softcover, ISBN: 978-1-4704-6543-8, 2020 *Mathematics Subject Classification*: 14N35, 14F08; 53D45, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/289/1435

`bookstore.ams.org/memo-289-1435`

Purity and Separation for Oriented Matroids

Pavel Galashin, *University of California, Los Angeles, CA, and Massachusetts Institute of Technology, Cambridge, MA*, and **Alexander Postnikov**, *Massachusetts Institute of Technology, Cambridge, MA*

This item will also be of interest to those working in discrete mathematics and combinatorics.

Memoirs of the American Mathematical Society, Volume 289, Number 1439
September 2023, 79 pages, Softcover, ISBN: 978-1-4704-6700-5, 2020 *Mathematics Subject Classification*: 52C40; 05E99, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/289/1439

`bookstore.ams.org/memo-289-1439`

Smooth Homotopy of Infinite-Dimensional C^∞-Manifolds

Hiroshi Kihara, *University of Aizu, Fukushima, Japan*

Memoirs of the American Mathematical Society, Volume 289, Number 1436
September 2023, 129 pages, Softcover, ISBN: 978-1-4704-6542-1, 2020 *Mathematics Subject Classification*: 58B05; 58A40, 18N40, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/289/1436

`bookstore.ams.org/memo-289-1436`

Probability and Statistics

Percolation on Triangulations: A Bijective Path to Liouville Quantum Gravity

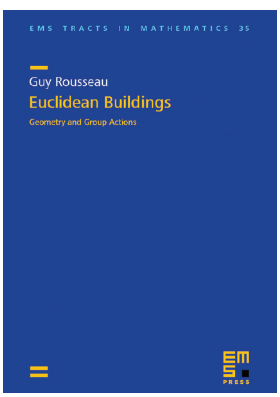
Olivier Bernardi, *Brandeis University, Waltham, MA*, **Nina Holden**, *New York University, NY*, and **Xin Sun**, *Peking University, Beijing, China*

Memoirs of the American Mathematical Society, Volume 289, Number 1440
September 2023, 176 pages, Softcover, ISBN: 978-1-4704-6699-2, 2020 *Mathematics Subject Classification*: 60F17, 05A19, 60C05, 60D05, 60G60, 60J67; 82B41, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/289/1440

bookstore.ams.org/memo-289-1440

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Euclidean Buildings

Geometry and Group Actions

Guy Rousseau, *Université de Lorraine, IECL, Vandoeuvre-lès-Nancy, France*

The theory of buildings lies at the interplay between geometry and group theory and is one of the main tools for studying the structure of many groups.

Actually, buildings were introduced by Jacques Tits in the 1950s to better understand and study a semi-simple algebraic group over a field. For a general field, its associated building is a spherical building, called its Tits building. It is a simplicial complex and, in this book, one considers a geometric realization called vectorial building. When the field is real valued, François Bruhat and Jacques Tits constructed another building taking into account the topology of the field. This Bruhat-Tits building is a polysimplicial complex and its usual geometric realization is an affine building.

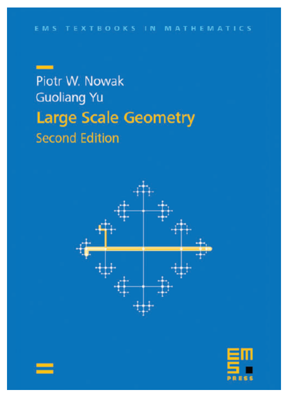
These vectorial or affine buildings are the main examples of Euclidean buildings. This book develops the general abstract theory of these Euclidean buildings (the buildings with Euclidean affine spaces as apartments). It is largely self contained and emphasizes the metric aspects of these objects, as CAT(0) spaces very similar to Riemannian symmetric spaces of non-compact type. The book studies their compactifications, their links with groups, many classical examples, and some applications (for example, to Hecke algebras).

This item will also be of interest to those working in geometry and topology and number theory.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Tracts in Mathematics, Volume 35
August 2023, 597 pages, Hardcover, ISBN: 978-3-98547-039-6, 2020 *Mathematics Subject Classification*: 51E24; 20E42, 20F55, 51F15, 51–02, List US$109, **AMS members US$87.20**, Order code EMSTM/35

bookstore.ams.org/emstm-35



Large Scale Geometry

Second Edition

Piotr W. Nowak, *Polish Academy of Sciences, Warsaw, Poland*, and **Guoliang Yu**, *Texas A&M University, College Station, Texas*

Large scale geometry is the study of geometric objects viewed from a great distance. The idea of large scale geometry can be traced back to Mostow's work on rigidity and the work of Švarc, Milnor, and Wolf on growth of groups and is greatly influenced by Gromov's work on geometric group theory. In the last decades, large scale geometry has found important applications in group theory, topology, geometry, higher index theory, computer science, and large data analysis.

This book provides a friendly approach to the basic theory of this exciting and fast growing subject and offers a glimpse of its applications to topology, geometry, and higher index theory. The authors have made a conscientious effort to make the book accessible to advanced undergraduate students, graduate students, and non-experts.

The second edition has been updated to cover recent developments involving constructions of groups and metric spaces with exotic properties as well as results charting new directions in index theory, and it also includes minor improvements in the presentation and an updated bibliography.

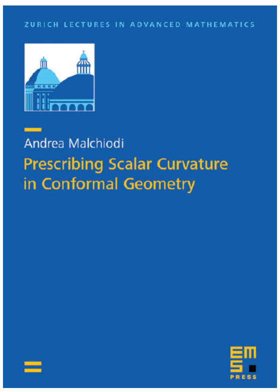
This item will also be of interest to those working in geometry and topology.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Textbooks in Mathematics, Volume 27
July 2023, 213 pages, Hardcover, ISBN: 978-3-98547-018-1, 2020 *Mathematics Subject Classification*: 51–01, 51F99, 20F69, 19K56, 57–01, 46L87, 58B34, 46L99, 53C20, List US$55, **AMS members US$44**, Order code EMSTEXT/27

bookstore.ams.org/emstext-27

Differential Equations



Prescribing Scalar Curvature in Conformal Geometry

Andrea Malchiodi, *Scuola Normale Superiore di Pisa, Italy*

This book treats the classical problem, posed by Kazdan and Warner, of prescribing a given function on a closed manifold as the scalar curvature of a metric within a conformal class. Since both critical equations and obstructions to the existence of solutions appear, the problem is particularly challenging.

The author's focus is to present a general approach for understanding the matter, particularly the issue of loss of compactness. The task of establishing the existence of solutions is attacked, combining several tools: the variational structure of the problem, Liouville-type theorems, blow-up analysis, elliptic regularity theory, and topological arguments.

Treating different aspects of the subject and containing several references to up-to-date research directions and perspectives, the book will be useful to graduate students and researchers interested in geometric analysis and partial differential equations.

This item will also be of interest to those working in geometry and topology.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Zurich Lectures in Advanced Mathematics, Volume 31
December 2023, 161 pages, Softcover, ISBN: 978-3-98547-052-5, 2020 *Mathematics Subject Classification*: 58J05; 35B33, 35J60, 35J20, 53C21, 58E05, List US$45, **AMS members US$36**, Order code EMSZLEC/31

bookstore.ams.org/emszlec-31

Meetings & Conferences of the AMS

December Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to `https://www.ams.org/meetings/meetings-general` for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: `stovall@math.wisc.edu`; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: `mamanes@hawaii.edu`; telephone: (808) 956-4679.

Meetings in this Issue

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at `https://www.ams.org/welcoming-environment-policy`.

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

San Francisco, California

Moscone North/South, Moscone Center

January 3–6, 2024

Wednesday – Saturday

Meeting #1192

Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 1

Deadlines

For organizers: Expired

For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/national.html`.

Joint Invited Addresses

Maria Chudnovsky, Princeton University, *What Makes a Problem Hard?* (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

Anne Schilling, University of California, Davis, *The Ubiquity of Crystal Bases* (AWM-AMS Noether Lecture).

Peter M Winkler, Dartmouth College, *Permutons* (AAAS-AMS Invited Address).

Kamuela E. Yong, University of Hawaii West Oahu, *When Mathematicians Don't Count* (MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture).

AMS Invited Addresses

Ruth Charney, Brandeis University, *From Braid Groups to Artin Groups* (AMS Retiring Presidential Address).

Daniel Erman, University of Hawaii, *From Hilbert to Mirror Symmetry*.

Suzanne Marie Lenhart, University of Tennessee, Knoxville, *Natural System Management: A Mathematician's Perspective* (AMS Josiah Willard Gibbs Lecture).

Ankur Moitra, Massachusetts Institute of Technology, *Learning From Dynamics* (von Neumann Lecture).

Kimberly Sellers, North Carolina State University, *Dispersed Methods for Handling Dispersed Count Data*.

Terence Tao, UCLA, *Machine Assisted Proof* (AMS Colloquium Lecture I - Terence Tao, University of California, Los Angeles).

Terence Tao, UCLA, *Translational Tilings of Euclidean Space* (AMS Colloquium Lecture II - Terence Tao, University of California, Los Angeles).

Terence Tao, UCLA, *Correlations of Multiplicative Functions* (AMS Colloquium Lecture III - Terence Tao, University of California, Los Angeles).

John Urschel, MIT, *From Moments to Matrices* (AMS Erdős Lecture for Students).

Suzanne L Weekes, SIAM, *Mathematics in (and for) the Real World* (AMS Lecture on Education).

Melanie Matchett Wood, Harvard University, *An Application of Probability Theory for Groups to 3-Dimensional Manifolds* (AMS Maryam Mirzakhani Lecture).

AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm`.

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

Advances in Analysis, PDE's and Related Applications, **Tepper L. Gill**, Howard University, **E. Kwessi**, Trinity University, and **Henok Mawi**, Howard University (Washington, DC, US).

Advances in Coding Theory, **Emily McMillon**, Virginia Tech, **Christine Ann Kelley** and **Tefjol Pllaha**, University of Nebraska - Lincoln, and **Mary Wootters**, Stanford.

Algebraic Approaches to Mathematical Biology, **Nicolette Meshkat**, Santa Clara University, **Cash Bortner**, California State University, Stanislaus, and **Anne Shiu**, Texas A&M University.

Algebraic Structures in Knot Theory, **V Sam Nelson**, Claremont McKenna College, and **Neslihan Gugumcu**, Izmir Institute of Technology in Turkey.

AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology, **Sarah Blackwell**, Max Planck Institute for Mathematics, **Luya Wang**, University of California, Berkeley, and **Nicole Magill**, Cornell University (AMS-AWM).

Analysis and Differential Equations at Undergraduate Institutions, **Evan Daniel Randles**, Colby College, and **Lisa Naples**, Fairfield University.

Applications of Extremal Graph Theory to Network Design, **Kelly Isham**, Colgate University, and **Laura Monroe**, Los Alamos National Laboratory.

Applications of Hypercomplex Analysis, **Mihaela B. Vajiac**, Chapman University, Orange, CA, **Daniel Alpay**, Chapman University, and **Paula Cerejeiras**, University of Aveiro, Portugal.

Applied Topology Beyond Persistence Diagrams, **Nikolas Schonsheck**, University of Delaware, **Lori Ziegelmeier**, Macalester College, **Gregory Henselman-Petrusek**, University of Oxford, and **Chad Giusti**, Oregon State University.

Applied Topology: Theory, Algorithms, and Applications, **Woojin Kim**, Duke University, **Johnathan Bush**, University of Florida, **Alex McCleary**, Ohio State University, **Sarah Percival**, Michigan State University, and **Iris H. R. Yoon**, University of Delaware.

Arithmetic Geometry with a View toward Computation, **David Lowry-Duda**, ICERM & Brown University, **Barinder Banwait**, Boston University, **Shiva Chidambaram**, Massachusetts Institute of Technology, **Juanita Duque-Rosero**, Boston University, **Brendan Hassett**, ICERM/Brown University, and **Ciaran Schembri**, Dartmouth College.

Bridging Applied and Quantitative Topology, **Henry Hugh Adams**, University of Florida, and **Ling Zhou**, Duke University.

Coding Theory for Modern Applications, **Rafael D'Oliveira**, Clemson University, **Hiram H. Lopez**, Cleveland State University, and **Allison Beemer**, University of Wisconsin-Eau Claire.

Combinatorial Insights into Algebraic Geometry, **Javier Gonzalez Anaya**, Harvey Mudd College.

Combinatorial Perspectives on Algebraic Curves and their Moduli, **Sam Payne**, UT Austin, **Melody Chan**, Brown University, **Hannah K. Larson**, Harvard University and UC Berkeley, and **Siddarth Kannan**, Brown University.

Combinatorics for Science, **Stephen J Young**, **Bill Kay**, and **Sinan Aksoy**, Pacific Northwest National Laboratory.

Commutative Algebra and Algebraic Geometry (associated with Invited Address by Daniel Erman), **Daniel Erman**, University of Hawaii, and **Aleksandra C Sobieska**, University of Wisconsin - Madison.

Complex Analysis, Operator Theory, and Real Algebraic Geometry, **J. E. Pascoe**, Drexel University, **Kelly Bickel**, Bucknell University, and **Ryan K. Tully-Doyle**, Cal Poly SLO.

Complex Social Systems (a Mathematics Research Communities session) I, **Ekaterina Landgren**, University of Colorado, Boulder, **Cara Sulyok**, Lewis University, **Casey Lynn Johnson**, UCLA, **Molly Lynch**, Hollins University, and **Rebecca Hardenbrook**, Dartmouth College.

Computable Mathematics: A Special Session Dedicated to Martin D. Davis, **Valentina S Harizanov**, George Washington University, **Alexandra Shlapentokh**, East Carolina University, and **Wesley Calvert**, Southern Illinois University.

Computational Biomedicine: Methods - Models - Applications, **Nektarios A. Valous**, Center for Quantitative Analysis of Molecular and Cellular Biosystems (Bioquant), Heidelberg University, Im Neuenheimer Feld 267, 69120, Heidelberg, Germany, **Anna Konstorum**, Center for Computing Sciences, Institute for Defense Analyses, 17100 Science Drive, Bowie, MD, 20715, USA, **Heiko Enderling**, Department of Integrated Mathematical Oncology, H. Lee Moffitt Cancer Center & Research Institute, Tampa, FL, 33647, USA, and **Dirk Jäger**, Department of Medical Oncology, National Center for Tumor Diseases (NCT), University Hospital Heidelberg (UKHD), Im Neuenheimer Feld 460, 69120, Heidelberg, Germany.

Computational Techniques to Study the Geometry of the Shape Space, **Shira Faigenbaum-Golovin**, Duke University, **Shan Shan**, University of Southern Denmark, and **Ingrid Daubechies**, Duke University.

Covering Systems of the Integers and Their Applications, **Joshua Harrington**, Cedar Crest College, **Tony Wing Hong Wong**, Kutztown University of Pennsylvania, and **Matthew Litman**, UC Davis.

Cryptography and Related Fields, **Ryann Cartor**, Clemson University, **Angela Robinson**, NIST, and **Daniel Everett Martin**, Clemson University.

Derived Categories, Arithmetic, and Geometry (a Mathematics Research Communities session) I, **Anirban Bhaduri**, University of South Carolina, **Gabriel Dorfsman-Hopkins**, St. Lawrence University, **Patrick Lank**, University of South Carlina, and **Peter McDonald**, University of Utah.

Developing Students' Technical Communication Skills through Mathematics Courses, **Michelle L. Ghrist**, Gonzaga University, **Timothy P Chartier**, Davidson College, **Maila B. Hallare**, US Air Force Academy, USAFA CO USA, and **Denise Taunton Reid**, Valdosta State University.

Diffusive Systems in the Natural Sciences, **Francesca Bernardi**, Worcester Polytechnic Institute, and **Owen L Lewis**, University of New Mexico.

Discrete Homotopy Theory, **Krzysztof R. Kapulkin**, University of Western Ontario, **Anton Dochtermann**, Texas State University, and **Antonio Rieser**, CONACYT-CIMAT.

Dynamical Systems Modeling for Biological and Social Systems, **Daniel Brendan Cooney**, University of Pennsylvania, **Chadi M Saad-Roy**, University of California, Berkeley, and **Chris M. Heggerud**, University of California, Davis.

Dynamics and Management in Disease or Ecological Models (associated with Gibbs Lecture by Suzanne Lenhart), **Suzanne Lenhart**, University of Tennessee, Knoxville, **Christina Edholm**, Scripps College, and **Wandi Ding**, Middle Tennessee State University.

Dynamics and Regularity of PDEs, **Zongyuan Li**, Rutgers University, **Zhiyuan Zhang**, Northeastern University, **Xueying Yu**, Oregon State University, and **Weinan Wang**, University of Oklahoma.

Epistemologies of the South and the Mathematics of Indigenous Peoples, **María Del Carmen Bonilla Tumialán**, National University of Education Enrique Guzman y Valle, **Wilfredo Vidal Alangui**, College of Science, University of the Philippines Baguio, and **Domingo Yojcom Rocché**, Center for Scientific and Cultural Research.

Ergodic Theory, Symbolic Dynamics, and Related Topics, **Andrew T Dykstra**, Hamilton College, and **Shrey Sanadhya**, Ben Gurion University of the Negev, Israel.

Ethics in the Mathematics Classroom, **Victor Piercey**, Ferris State University, and **Catherine Buell**, Fitchburg State University.

Explicit Computation with Stacks (a Mathematics Research Communities session) I, **Santiago Arango**, Emory University, **Jonathan Richard Love**, CRM Montreal, and **Sameera Vemulapalli**, Princeton University.

Exploring Spatial Ecology via Reaction Diffusion Models: New Insights and Solutions, **Jerome Goddard II**, Auburn University Montgomery, and **Ratnasingham Shivaji**, University of North Carolina Greensboro.

Extremal and Probabilistic Combinatorics, **Sam Spiro**, Rutgers University, and **Corrine Yap**, Georgia Institute of Technology.

Geometric Analysis in Several Complex Variables, **Ming Xiao**, University of California, San Diego, **Bernhard Lamel**, Texas A&M University At Qatar, and **Nordine Mir**, Texas A&M University at Qatar.

Geometric Group Theory (Associated with the AMS Retiring Presidential Address), **Kasia Jankiewicz**, University of California Santa Cruz, **Edgar A. Bering**, San José State University, **Marion Campisi**, San Jose State University, and **Tim Hsu** and **Giang Le**, San José State University.

Geometry and Symmetry in Differential Equations, Control, and Applications, **Taylor Joseph Klotz** and **George Wilkens**, University of Hawai'i.

Geometry and Topology of High-Dimensional Biomedical Data, **Smita Krishnaswamy**, Yale, **Dhananjay Bhaskar**, Yale University, **Bastian Rieck**, Technical University of Munich, and **Guy Wolf**, Université de Montréal.

Group Actions in Commutative Algebra, **Alessandra Costantini**, Oklahoma State University, **Alexandra Seceleanu**, University of Nebraska-Lincoln, and **Andras Cristian Lorincz**, University of Oklahoma.

Hamiltonian Systems and Celestial Mechanics, **Zhifu Xie**, The University of Southern Mississippi, and **Ernesto Perez-Chavela**, ITAM.

Harmonic Analysis, Geometry Measure Theory, and Fractals, **Kyle Hambrook**, San Jose State University, **Chun-Kit Lai**, San Francisco State University, and **Caleb Z Marshall**, University of British Columbia.

History of Mathematics, **Adrian Rice**, Randolph-Macon College, **Sloan Evans Despeaux**, Western Carolina University, **Deborah Kent**, University of St. Andrews, and **Jemma Lorenat**, Pitzer College.

Homological Techniques in Noncommutative Algebra, **Robert Won**, George Washington University, **Ellen E Kirkman**, Wake Forest University, and **James J. Zhang**, University of Washington.

Homotopy Theory, **Krzysztof R. Kapulkin**, University of Western Ontario, **Daniel K. Dugger**, University of Oregon, **Jonathan Beardsley**, University of Nevada, Reno, and **Thomas Brazelton**, University of Pennsylvania.

Ideal and Factorization Theory in Rings and Semigroups, **Scott Chapman**, Sam Houston State University, and **Alfred Geroldinger**, University of Graz.

Informal Learning, Identity, and Attitudes in Mathematics, **Sergey Grigorian**, **Mayra Ortiz**, **Xiaohui Wang**, and **Aaron T Wilson**, University of Texas Rio Grande Valley.

Integer Partitions, Arc Spaces and Vertex Operators, **Hussein Mourtada**, Université Paris Cité, and **Andrew R. Linshaw**, University of Denver.

Interplay Between Matrix Theory and Markov Systems: Applications to Queueing Systems and of Duality Theory, **Alan Krinik** and **Randall J. Swift**, California State Polytechnic University, Pomona.

Issues, Challenges and Innovations in Instruction of Linear Algebra, **Feroz Siddique**, University of Wisconsin-Eau Claire, and **Ashish K. Srivastava**, Saint Louis University.

Knots, Skein Modules, and Categorification, **Rhea Palak Bakshi**, ETH Institute for Theoretical Studies, Zurich, **Sujoy Mukherjee**, University of Denver, and **Jozef Henryk Przytycki**, George Washington University.

Large Random Permutations (affiliated with AAAS-AMS Invited Address by Peter Winkler), **Peter M Winkler**, Dartmouth College, and **Jacopo Borga**, Stanford University.

Loeb Measure after 50 Years, **Yeneng Sun**, National University of Singapore, **Robert M Anderson**, UC Berkeley, and **Matt Insall**, Missouri University of Science and Technology.

Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later, **Younhee Lee**, Southern Connecticut State University, **James Alvarez**, University of Texas Arlington, **Ekaterina Fuchs**, City College of San Francisco, **Tyler Kloefkorn**, American Mathematical Society, **Yvonne Lai**, University of Nebraska-Lincoln, and **Carl Olimb**, Augustana University.

Mathematical Modeling and Simulation of Biomolecular Systems, **Zhen Chao**, University of Michigan-Ann Arbor, and **Jiahui Chen**, University of Arkansas.

Mathematical Modeling of Nucleic Acid Structures, **Pengyu Liu**, University of California, Davis, **Van Pham**, University of South Florida, and **Svetlana Poznanovic**, Clemson University.

Mathematical Physics and Future Directions, **Shanna Dobson**, University of California, Riverside, **Tepper L. Gill**, Howard University, **Michael Anthony Maroun**, University of California, Riverside, CA, and **Lance Nielsen**, Creighton University.

Mathematics and Philosophy, **Tom Morley**, Georgia Tech, and **Bonnie Gold**, Monmouth University.

Mathematics and Quantum, **Kaifeng Bu** and **Arthur M. Jaffe**, Harvard, **Sui Tang**, UCSB, and **Jonathan Weitsman**, Northeastern University.

Mathematics and the Arts, **Karl M Kattchee**, University of Wisconsin-La Crosse, **Doug Norton**, Villanova University, and **Anil Venkatesh**, Adelphi University.

Mathematics of Computer Vision, **Timothy Duff** and **Max Lieblich**, University of Washington.

Mathematics of DNA and RNA, **Marek Kimmel**, Rice University, **Chris McCarthy**, BMCC, City University of New York, and **Johannes Familton**, Borough of Manhattan Community College, CUNY.

Metric Dimension of Graphs and Related Topics, **Briana Foster-Greenwood**, Cal Poly Pomona, and **Christine Uhl**, St. Bonaventure University.

Metric Geometry and Topology, **Christine M. Escher**, Oregon State University, and **Catherine Searle**, Wichita State University.

Mock Modular forms, Physics, and Applications, **Amanda Folsom**, Amherst College, **Terry Gannon**, University of Alberta, and **Larry Rolen**, Vanderbilt University.

Modeling Complex Adaptive Systems in Life and Social Sciences, **Yun Kang** and **Theophilus Kwofie**, Arizona State University, and **Sabrina H Streipert**, University of Pittsburgh.

Modeling to Motivate the Teaching of the Mathematics of Differential Equations, **Brian Winkel**, SIMIODE, **Kyle T Allaire**, Worcester State University, Worcester MA USA, **Maila B. Hallare**, US Air Force Academy, USAFA CO USA, **Yanping Ma**, Loyola Marymount University, Los Angeles CA USA, and **Lisa Naples**, Fairfield University.

Modelling with Copulas: Discrete vs Continuous Dependent Data, **Martial Longla**, University of Mississippi, and **Isidore Seraphin Ngongo**, University of Yaounde I.

Modern Developments in the Theory of Configuration Spaces, **Christin Bibby**, Louisiana State University, and **Nir Gadish**, University of Michigan.

Modular Tensor Categories and TQFTs beyond the Finite and Semisimple, **Colleen Delaney**, UC Berkeley, and **Nathan Geer**, Utah State University.

New Faces in Operator Theory and Function Theory, **Michael R Pilla**, Ball State University, and **William Thomas Ross**, University of Richmond.

Nonlinear Dynamics in Human Systems: Insights from Social and Biological Perspectives, **Armando Roldan**, University of Central Florida, and **Thomas Dombrowski**, Moffitt Cancer Center.

Number Theory in Memory of Kevin James, **Jim L. Brown**, Occidental College, and **Felice Manganiello**, Clemson University.

Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms, **Anastasiia Minenkova**, University of Hartford, and **Gamal Mograby**, University of Maryland.

Partition Theory and q-Series, **William Jonathan Keith**, Michigan Technological University, **Brandt Kronholm**, University of Texas Rio Grande Valley, and **Dennis Eichhorn**, University of California, Irvine.

Polymath Jr REU Student Research, **Steven Joel Miller**, Williams College, and **Alexandra Seceleanu**, University of Nebraska-Lincoln.

Principles, Spatial Reasoning, and Science in First-Year Calculus, **Yat Sun Poon** and **Catherine Lussier**, University of California, Riverside, and **Bryan Carrillo**, Saddleback College.

Quantitative Justice, **Ron Buckmire**, Occidental College, **Omayra Ortega**, Sonoma State University, and **Robin Wilson**, California State Polytechnic University, Pomona (NAM-SIAM-AMS).

Quaternions, **Chris McCarthy**, BMCC, City University of New York, **Johannes Familton**, Borough of Manhattan Community College, CUNY, and **Terrence Richard Blackman**, Medgar Evers Community College, CUNY.

Recent Advances in Mathematical Models of Diseases: Analysis and Computation, **Najat Ziyadi** and **Jemal S Mohammed-Awel**, Department of Mathematics, Morgan State University.

Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems, **Tuan A. Phan**, IMCI, University of Idaho, **Nhu N. Nguyen**, University of Rhode Island, and **Jianjun P. Tian**, New Mexico State University.

Recent Developments in Commutative Algebra, **Austyn Simpson** and **Alapan Mukhopadhyay**, University of Michigan, and **Thomas Marion Polstra**, University of Virginia.

Recent Developments in Numerical Methods for PDEs and Applications, **Chunmei Wang**, University of Florida, **Long Chen**, UC Irvine, **Shuhao Cao**, University of Missouri-Kansas City, and **Haizhao Yang**, University of Maryland College Park.

Recent Developments on Markoff Triples, **Elena Fuchs**, UC Davis, and **Daniel Everett Martin**, Clemson University.

Recent Progress in Inference and Sampling (Associated with AMS Invited Address by Ankur Moitra), **Ankur Moitra**, Massachusetts Institute of Technology, and **Sitan Chen**, Harvard University.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, **Darren A. Narayan**, Rochester Institute of Technology, **John C. Wierman**, Johns Hopkins University, **Mark Daniel Ward**, Purdue University, **Khang Duc Tran**, California State University, Fresno, and **Christopher O'Neill**, San Diego State University.

Research Presentations by Math Alliance Scholar Doctorates, **Theresa Martines**, University of Texas, Austin, and **David Goldberg**, Math Alliance/Purdue University.

Ricci Curvatures of Graphs and Applications to Data Science (a Mathematics Research Communities session) I, **Aleyah Dawkins**, George Mason University, **Xavier Ramos Olive**, Smith College, **Zhaiming Shen**, University of Georgia, **David Harry Richman**, University of Washington, and **Michael G Rawson**, PNNL.

Roots of Unity - Mathematics from Graduate Students in the Roots of Unity Program, **Allechar Serrano Lopez**, Harvard University, and **Patricia Klein**, University of Minnesota.

Serious Recreational Mathematics, **Erik Demaine**, Massachusetts Institute of Technology, **Robert A. Hearn**, Gathering 4 Gardner, and **Tomas Rokicki**, California.

Solvable Lattice Models and their Applications Associated with the Noether Lecture, **Anne Schilling**, University of California, Davis, **Amol Aggarwal**, Columbia, **Benjamin Brubaker**, University of Minnesota - Twin Cities, **Daniel Bump**, Stanford, **Andrew Hardt**, Stanford University, **Slava Naprienko**, University of North Carolina at Chapel Hill, **Leonid Petrov**, University of Virginia, and **Anne Schilling**, University of California, Davis.

Spectral Methods in Quantum Systems, **Matthew Powell**, Georgia Institute of Technology, and **Wencai Liu**, Texas A&M University.

Structure-preserving Algorithms, Analysis and Simulations for Differential Equations, **Brian E Moore**, University of Central Florida, and **Qin Sheng**, Baylor University.

The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors, **Quiyana Murphy**, Virginia Tech, **Sofia Rose Rose Martinez Alberga**, Purdue University, **Kelly Buch**, Austin Peay State University, and **Alexis Hardesty**, Texas Tech University.

The Mathematics of Decisions, Elections, and Games, **David McCune**, William Jewell College, **Michael A. Jones**, Mathematical Reviews | AMS, and **Jennifer M. Wilson**, Eugene Lang College, The New School.

Theoretical and Numerical Aspects of Nonlocal Models, **Nicole Buczkowski**, Worcester Polytechnic Institute, **Christian Alexander Glusa**, Sandia National Laboratories, and **Animesh Biswas**, University of Nebraska Lincoln.

Theta Correspondence, **Edmund Karasiewicz** and **Petar Bakic**, University of Utah.

The Teaching and Learning of Undergraduate Ordinary Differential Equations, **Viktoria Savatorova**, Central Connecticut State University, **Chris Goodrich**, The University of New South Wales, **Itai Seggev**, Wolfram Research, **Beverly H West**, Cornell University, and **Maila B. Hallare**, US Air Force Academy, USAFA CO USA.

Thresholds in Random Structures, **Will Perkins**, Georgia Tech.

Topics in Combinatorics and Graph Theory, **Cory Palmer** and **Anastasia Halfpap**, University of Montana, and **Neal Bushaw**, Virginia Commonwealth University.

Topics in Equivariant Algebra, **Ben Spitz**, University of California Los Angeles, and **Christy Hazel** and **Michael A. Hill**, UCLA.

Topological and Algebraic Approaches for Optimization, **Ali Mohammad Nezhad**, Carnegie Mellon University.

Using 3D-Printed and Other Digitally-Fabricated Objects in the Mathematics Classroom, **Shelby Stanhope**, U.S. Air Force Academy, **Paul E. Seeburger**, Monroe Community College, and **Stepan Paul**, North Carolina State University.

Water Waves, **Anastassiya Semenova** and **Bernard Deconinck**, University of Washington, **John D Carter**, Seattle University, and **Eleanor Devin Byrnes**, University of Washington.

Invited Addresses of Other JMM Partners

Julie Blackwood, Williams College, *The Role of Spatial Interactions in Managing Ecological Systems: Insights From Mathematical Models* (Spectra Lavender Lecture).

Henri Darmon, McGill University, *Fourier Coefficients of Modular Forms* (CRM-PIMS-AARMS Invited Address - Henri Darmon, McGill University).

Ranthony A C Edmonds, Duke University, *Quantitative Justice: Intersections of Mathematics and Society* (NAM Cox-Talbot Address).

Katherine Ensor, Rice University, *Celebrating Statistical Foundations Driving 21st -Century Innovation* (ASA Invited Address- Kathy Ensor, Rice University).

Stephan Ramon Garcia, Pomona College, *Fast Food for Thought: What Can Chicken Nuggets Tell Us About Linear Algebra?* (ILAS Invited Address).

Sylvester James Gates, Jr, Clark Leadership Chair in Science, University of Maryland; past president of American Physical Society, National Medal of Science, *What Challenges Does Data Science Present to Mathematics Education?* (TPSE Invited Address - Sylvester James Gates, Jr, Clark Leadership Chair in Science, University of Maryland).

Matthew Harrison-Trainor, University of Illinois Chicago, *The Complexity of Classifying Topological Spaces* (ASL Invited Address).

Åsa Hirvonen, University of Helsinki, *Games for Measuring Distances Between Metric Structures* (ASL Invited Address).

Trachette Jackson, University of Michigan, *Mobilizing Mathematics for the Fight Against Cancer* (PME Invited Address).

Shelly M Jones, Central Connecticut State University, *Choosing Hope: Teaching Culturally Relevant Mathematics as a Human Endeavor* (NAM Claytor-Woodard Lecture).

Yvonne Lai, University of Nebraska-Lincoln, *(Why) To Build Bridges in Mathematics Education* (MAA Lecture on Teaching and Learning).

Francois Loeser, Institut Universitaire de France, Sorbonne, *Model Theory and Non-Archimedean Geometry* (ASL Invited Address).

Toby Meadows, University of California, Irvine, *A Modest Foundational Argument for the Generic Multiverse* (ASL Invited Address).

Dima Svetosla Sinapova, Rutgers University, *Combinatorial Principles at Successors of Singular Cardinals* (ASL Invited Address).

Slawomir Solecki, Cornell University, *Descriptive Set Theory and Generic Measure Preserving Transformations* (ASL Invited Address).

Joni Teräväinen, University of Turku, *Uniformity of the Möbius Function in Short Intervals* (AIM Alexanderson Award Lecture - Joni Teräväinen).

Mariel Vazquez, University of California, Davis, *Topological Considerations in Genome Biology* (SIAM Invited Address).

Mariana Vicaria, University of California, Los Angeles, *Model Theory of Valued Fields* (ASL Invited Address).

Invited Addresses of Other Organizations

Arezoo Islami, San Francisco State University, *The Unreasonable Effectiveness of Mathematics: Dissolving Wigner's Applicability Problem* (Special Interest Group of the MAA on the Philosophy of Mathematics Guest Lecture and Discussion).

AIM Special Sessions

Equivariant Techniques in Stable Homotopy Theory, **Michael A. Hill**, UCLA, and **Anna Marie Bohmann**, Vanderbilt University.

Graphs and Matrices, **Mary Flagg**, University of St. Thomas, and **Bryan A Curtis**, Iowa State University.

Little School Dynamics: Cool Research by Researchers at PUIs, **Kimberly Ayers**, California State University, San Marcos, **Ami Radunskaya**, Pomona College, **Andy Parrish**, Eastern Illinois University, **David M. McClendon**, Ferris State University, and **Han Li**, Wesleyan University.

AIM-MAA Math Circle Activities as a Gateway Into Research, **Jeffrey Musyt**, Slippery Rock University, **Lauren L Rose**, Bard College, **Tom G. Stojsavljevic**, Beloit College, **Nick Rauh**, Julia Robinson Math Festivals, **Edward Charles Keppelmann**, University of Nevada Reno, **Allison Henrich**, Seattle University, **Violeta Vasilevska**, Utah Valley University, and **Gabriella A. Pinter**, University of Wisconsin, Milwaukee.

Multiplicative Number Theory and Additive Combinatorics, **Joni Teräväinen**, University of Turku, **Terence Tao**, UCLA, **Kasia Matomäki**, University of Turku, **Maksym Radziwill**, Northwestern University, and **Tamar Ziegler**, Hebrew University.

ASL Special Sessions

Descriptive Methods in Dynamics, Combinatorics, and Large Scale Geometry, **Jenna Zomback**, University of Maryland, College Park, and **Forte Shinko**, UCLA.

AWM Special Sessions

EvenQuads Live and in person: The honorees and the games, **sarah-marie belcastro**, Mathematical Staircase, Inc., **Sherli Koshy-Chenthittayil**, Touro University Nevada, **Oscar Vega**, California State University, Fresno, **Monica D. Morales-Hernandez**, Adelphi University, **Linda McGuire**, Muhlenberg College, and **Denise A. Rangel Tracy**, Fairleigh Dickinson University.

Mathematics in the Literary Arts and Pedagogy in Creative Settings, **Shanna Dobson**, University of California, Riverside, and **Claudia Maria Schmidt**, California State University.

Recent Developments in Harmonic Analysis, **Betsy Stovall**, University of Wisconsin-Madison, and **Sarah E Tammen**, UW-Madison.

Women in Mathematical Biology, **Christina Edholm**, Scripps College, **Lihong Zhao**, University of California, Merced, and **Lale Asik**, University of the Incarnate Word.

COMAP Special Sessions

Math Modeling Contests: What They Are, How They Benefit, What They Did – Discussions with the Students and Advisors, **Kayla Blyman**, Saint Martin's University.

ILAS Special Sessions

Generalized Numerical Ranges and Related Topics, **Tin-Yau Tam** and **Pan-Shun Lau**, University of Nevada, Reno.

Graphs and Matrices, **Jane Breen**, Ontario Tech University, and **Stephen Kirkland**, University of Manitoba.

Innovative and Effective Ways to Teach Linear Algebra, **David M. Strong**, Pepperdine University, **Sepideh Stewart**, University of Oklahoma, **Gil Strang**, MIT, and **Megan Wawro**, Virginia Tech.

Linear Algebra, Matrix theory, and its Applications, **Stephan Ramon Garcia** and **Konrad Aguilar**, Pomona College.

Sign-pattern Matrices and Their Applications, **Bryan L Shader**, University of Wyoming, and **Minerva Catral**, Xavier University.

Spectral and combinatorial problems for nonnegative matrices and their generalizations, **Pietro Paparella**, University of Washington Bothell, and **Michael J. Tsatsomeros**, Washington State University.

MAA Special Sessions

Navigating the Benefits and Challenges of Mentoring Students in Data-Driven Undergraduate Research Projects, **Vinodh Kumar Chellamuthu**, Utah Tech University, and **Xiaoxia Xie**, Idaho State University.

Undergraduate Research Activities in Mathematical and Computational Biology, **Timothy D Comar**, Benedictine University, and **Anne E. Yust**, University of Pittsburgh.

PMA Special Sessions

BSM Special Session: Mathematical Research in Budapest for Students and Faculty, **Kristina Cole Garrett**, St. Olaf College.

SIAM Minisymposium

SIAM ED Session on Artificial Intelligence and its Uses in Mathematical Education, Research, and Automation in the Industry, **Alvaro Alfredo Ortiz Lugo**, University of Cincinnati, **Kathleen Kavanagh**, Clarkson University, and **Sergio Molina**, University of Cincinnati.

SIAM Minisymposium on Computational Mathematics and the Power Grid, **Todd Munson**, Argonne National Laboratory.

SIAM Minisymposium on Current Advances in Modeling and Simulation to Uncover the Complexity of Disease Dynamics, **Naveen K. Vaidya**, San Diego State University, and **Elissa Schwartz**, Washington State University.

SIAM Minisymposium on Mathematical Methods in Computer Vision and Image Analysis, **Andreas Mang**, University of Houston.

SIAM Minisymposium on Mathematics of Bacterial Viruses: From Virus Discovery to Mathematical Principles, **Javier Arsuaga**, University of California, Davis, **Carme Calderer**, University of Minnesota, and **Ami Bhatt**, Stanford University.

SIAM Minisymposium on Recent Developments in the Analysis and Control of Partial Differential Equations Arising in Fluid and Fluid-Structure Interactive Dynamics, **George Avalos**, University of Nebraska-Lincoln, and **Pelin Guven Geredeli**, Clemson University.

SIAM Minisymposium on Scientific Machine Learning to Advance Modeling and Decision Support, **Erin Acquesta**, Sandia National Laboratories, **Timo Bremer**, Lawrence Livermore National Laboratories, and **Joseph Hart**, Sandia National Laboratories.

SIAM-USNCTAM Minisymposium on Mathematical Modeling of Complex Materials Systems, **Maria G Emelianenko**, George Mason University, and **Dmitry Golovaty**, The University of Akron.

SLMATH Special Sessions

African Diaspora Joint Mathematics Working Groups (ADJOINT), **Caleb Ashley**, Boston College, and **Anisah Nabilah Nu'Man**, Spelman College.

Summer Research in Mathematics (SRiM): Recent Trends in Nonlinear Boundary Value Problems, **Maya Chhetri**, UNC Greensboro, **Elliott Zachary Hollifield**, University of North Carolina at Pembroke, and **Nsoki Mavinga**, Swarthmore College.

The MSRI Undergraduate Program (MSRI-UP), **Maria Mercedes Franco**, Queensborough Community College-CUNY.

SPECTRA Special Sessions

Research by LGBTQ+ Mathematicians, **Devavrat Dabke**, Princeton University, **Joseph Nakao**, Swarthmore College, and **Michael A. Hill**, UCLA.

Other Special Sessions

Exploring Funding Opportunities in the Division of Mathematical Sciences, **Elizabeth Wilmer**, National Science Federation, and **Junping Wang**, National Science Foundation.

Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Sciences I, **Michael Ferrara**, Division of Undergraduate Education, National Science Foundation.

AMS Contributed Paper Sessions

AMS Contributed Paper Session, **Michelle Ann Manes**, American Institute of Mathematics.

AMS Contributed Paper Session on Combinatorics, **Michelle Ann Manes**, American Institute of Mathematics.

AMS Contributed Paper Session on Control Theory, Quantum Theory, and Related Topics, **Michelle Ann Manes**, American Institute of Mathematics.

AMS Contributed Paper Session on Difference Equations and Integral Equations, **Michelle Ann Manes**, American Institute of Mathematics.

AMS Contributed Paper Session on Harmonic Analysis, Probability Theory, and Related Topics, **Michelle Ann Manes**, American Institute of Mathematics.

AMS Contributed Paper Session on Mathematical Biology, **Michelle Ann Manes**, American Institute of Mathematics.

ASL Contributed Paper Sessions

ASL Contributed Paper Session, **David Reed Solomon**, University of Connecticut.

NAM Contributed Paper Sessions

NAM Haynes-Granville-Browne Session of Presentations by Recent Doctoral Recipients, **Aris Winger**, Georgia Gwinnett College, **Torina D. Lewis**, American Mathematical Society, and **Omayra Ortega**, Sonoma State University.

PME Contributed Paper Sessions

PME Contributed Session on Research by Undergraduates, **Thomas Philip Wakefield**, Youngstown State University, and **Jennifer Beineke**, Western New England University.

TPSE Contributed Paper Sessions

TPSE Contributed Paper Session on Using Institutional and National Data Sources to Recruit, Retain and Support a Diverse Population of Mathematics Students, **Rick Cleary**, Babson College, and **Mitchel T. Keller**, University of Wisconsin - Madison.

AMS Other Events

AMS Advocacy Session: Advocacy for Mathematics and Science Policy, **Karen Saxe**, American Mathematical Society.

AMS Committee on Education Panel Discussion, I - "Mathematics online: PDFs and issues regarding accessibility", **Tyler Kloefkorn**, American Mathematical Society, and **Terrence Blackman**, Medgar Evers College.

AMS Committee on Equity, Diversity and Inclusion Panel Discussion: Successful Programs that Support Equity, Diversity and Inclusion, **Sarah Greenwald,** Appalachian State University, **Lily Khadjavi,** Loyola Marymount University, and **Dennis Davenport,** Howard University.

AMS Committee on Science Policy Panel Discussion - "Artificial Intelligence in Mathematics, Science and Society", **Karen Saxe,** American Mathematical Society, and **Anita Benjamin**, American Mathematical Society.

AMS Committee on the Profession Panel Discussion: Building a Successful Research Career in Mathematics, **Edray Goins,** Pomona College, and **Pamela Harris**, Williams College.

AMS Current Events Bulletin, **David Eisenbud**, MSRI.

AMS Department Chairs and Leaders Workshop, **Timothy Flood,** Pittsburgh State University, **Emilie Lawrence**, University of San Francisco, and **Charles Moore**, Washington State University.

Focus Group for Directors of Graduate Studies Workshop, **Sarah Bryant,** American Mathematical Society.

Focus Group for Directors of Undergraduate Studies Workshop, **Sarah Bryant**, American Mathematical Society.

Open Access: AMS Journals and Federal Policies, **Karen Saxe,** American Mathematical Society, and **Robert Harrington,** American Mathematical Society.

AWM Other Events

AWM Panel: Celebrating Academic Pivots in Mathematics, **Michelle Ann Manes**, American Institute of Mathematics.

AWM Workshop: Mathematicians + Wikipedia – A Training Edit-a-thon, **Devavrat Dabke**, Princeton University, **Joseph Nakao**, Swarthmore College, and **Michael A. Hill**, UCLA.

AWM Workshop: Women in Operator Theory, **Catherine Anne Beneteau**, University of South Florida, and **Asuman Aksoy**, Claremont McKenna College.

JMM Other Events

JMM Panel: Cal-Bridge: Building Bridges and Diversifying Mathematics, **Catherine Anne Beneteau**, University of South Florida, and **Asuman Aksoy**, Claremont McKenna College.

JMM Panel: Decolonizing Mathematics, **David M. Strong**, Pepperdine University, **Sepideh Stewart**, University of Oklahoma, **Gil Strang**, MIT, and **Megan Wawro**, Virginia Tech.

JMM Panel: Regional Math Alliances: Activities and Formation of Regional Groups to Support the Goals of the National Math Alliance, **Todd Munson**, Argonne National Laboratory.

JMM Panel: The Future of Graduate Mathematics Textbooks, **Keegan Kang**, Bucknell University, **Rachel Perrier**, Franciscan University of Steubenville, and **Shuyi Weng**, Purdue University.

JMM Workshop on Building Conceptual Understanding of Multivariable Calculus using 3D Visualization in CalcPlot3D and 3D-Printed Surfaces, **David M. Strong**, Pepperdine University, **Sepideh Stewart**, University of Oklahoma, **Gil Strang**, MIT, and **Megan Wawro**, Virginia Tech.

JMM Workshop on Leveraging Research-Based Instruction in Introductory Proofs Courses, **Michelle Ann Manes**, American Institute of Mathematics.

JMM Workshop on Teaching Student-Centered Mathematics: Active Learning & the Learning Assistant Model, **David M. Strong**, Pepperdine University, **Sepideh Stewart**, University of Oklahoma, **Gil Strang**, MIT, and **Megan Wawro**, Virginia Tech.

Joint Committee on Women Panel: Financial Empowerment for Mathematicians, **Elizabeth Wilmer**, National Science Federation, and **Junping Wang**, National Science Foundation.

MAA Other Events

MAA Project NExT Active Learning Strategies for a Large Class, **Hannah Burson**, University of Minnesota, **Paul Herstedt**, Macalester College, and **Richard Wong**, UCLA.

MAA Project NExT Classrooms Meet the Future: How Modern Technology Is Enhancing the Classroom Experience of Mathematics., **Keegan Kang**, Bucknell University, **Rachel Perrier**, Franciscan University of Steubenville, and **Shuyi Weng**, Purdue University.

MAA Project NExT Fostering a Growth Mindset in the Classroom, **Radmila Sazdanovic**, NC State University.

MAA Project NExT Making Student Thinking Visible with Team-Based Inquiry Learning, **Christina Duron**, Pepperdine University, **Erin Ellefsen**, Earlham College, and **Aaron Osgood-Zimmerman**, Bucknell University.

MAA Project NExT Panel Discussion on Diversity, Equity, and Inclusion Practices in an Undergraduate Math Class, **Maria Amarakristi Onyido**, Northern Illinois University, and **George Nasr**, Augustana University.

MAA Project NExT Setting a New Standard: Implementing Standards-Based Grading, **Daniel Graybill**, Fort Lewis College, **Alexis Hardesty**, Texas Woman's University, and **Margaret Regan**, College of the Holy Cross.

OTH Other Events

AMS - PME Undergraduate Student Poster Session, **Chad Awtrey** and **Frank Patane**, Samford University.

Association for Symbolic Logic Tutorial: Large Cardinals, Determinacy, and Inner Models, **David M. Strong**, Pepperdine University, **Sepideh Stewart**, University of Oklahoma, **Gil Strang**, MIT, and **Megan Wawro**, Virginia Tech.

AWM Workshop Poster Presentations, **Radmila Sazdanovic**, NC State University.

COMAP Panel on Math Modeling Contests: Trends, Topics, and Tips, **David Reed Solomon**, University of Connecticut.

COMAP Workshop on Modeling for Educators: Introducing Students to Modeling in Your Classroom, **Vinodh Kumar Chellamuthu**, Utah Tech University, and **Xiaoxia Xie**, Idaho State University.

Julia Robinson Math Festival, **Catherine Anne Beneteau**, University of South Florida, and **Asuman Aksoy**, Claremont McKenna College.

Mathematically Bent Theater, **Radmila Sazdanovic**, NC State University.

PME Panel: What Every Student Should Know about the JMM, **Keegan Kang**, Bucknell University, **Rachel Perrier**, Franciscan University of Steubenville, and **Shuyi Weng**, Purdue University.

SIAM Panel on Business-Industry-Government Careers for Mathematicians, **Aleyah Dawkins**, George Mason University, **Xavier Ramos Olive**, Smith College, **Zhaiming Shen**, University of Georgia, **David Harry Richman**, University of Washington, and **Michael G Rawson**, PNNL.

Spectra Workshop: Creating an Inclusive Undergraduate Mathematics Curriculum, **Thomas Philip Wakefield**, Youngstown State University, and **Jennifer Beineke**, Western New England University.

TPSE Panel on Developing Innovative Upper Division Pathways in Mathematics: Strategies for Enrollment and Inclusion, **Elizabeth Wilmer**, National Science Federation, and **Junping Wang**, National Science Foundation.

TPSE Panel on Grading for Active Learning & Department Change, **Thomas Philip Wakefield**, Youngstown State University, and **Jennifer Beineke**, Western New England University.

Tallahassee, Florida

Florida State University

March 23–24, 2024

Saturday – Sunday

Meeting #1193

Southeastern Section

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: To be announced
For abstracts: January 23, 2024

MEETINGS & CONFERENCES

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Wenjing Liao, Georgia Institute of Technology, *Title to be announced.*
Olivia Prosper, University of Tennessee, Knoxville, *Title to be announced.*
Jared Speck, Vanderbilt University, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advanced Numerical Methods for Partial Differential Equations and Their Applications (Code: SS 1A), **Seonghee Jeong**, Louisiana State University, **Sanghyun Lee**, Florida State University, and **Seulip Lee**, University of Georgia.

Advances in Financial Mathematics (Code: SS 2A), **Qi Feng**, **Alec N Kercheval**, and **Lingjiong Zhu**, Florida State University.

Advances in Shape and Topological Data Analysis (Code: SS 3A), **Emmanuel L Hartman**, **Eric Klassen**, and **Ethan Semrad**, Florida State University.

Algebraic Groups and Local-Global Principles (Code: SS 4A), **Suresh Venapally**, Emory University, and **Daniel Reuben Krashen**, University of Pennsylvania.

Bases and Frames in Hilbert spaces (Code: SS 5A), **Laura De Carli**, Florida International University, and **Azita Mayeli**, City University of New York.

Combinatorics in Geometry of Polynomials (Code: SS 6A), **Papri Dey**, Georgia Institute of Technology.

Control, Inverse Problems and Long Time Dynamics of Evolution Systems (Code: SS 7A), **Shitao Liu**, Clemson University, and **Louis Tebou**, Florida International University.

Data Integration and Identifiability in Ecological and Epidemiological Models (Code: SS 8A), **Omar Saucedo**, Virginia Tech, and **Olivia Prosper**, University of Tennessee/Knoxville.

Diversity in Mathematical Biology (Code: SS 9A), **Daniel Alejandro Cruz** and **Skylar Grey**, University of Florida.

Fluids: Analysis, Applications, and Beyond (Code: SS 10A), **Aseel Farhat** and **Anuj Kumar**, Florida State University.

Geometric Measure Theory and Partial Differential Equations (Code: SS 11A), **Alexander B. Reznikov**, **John Hoffman**, and **Richard Oberlin**, Florida State University.

Geometry and Symmetry in Data Science (Code: SS 12A), **Dustin G. Mixon**, The Ohio State University, and **Thomas Needham**, Florida State University.

Homotopy Theory and Category Theory in Interaction (Code: SS 13A), **Ettore Aldrovandi** and **Brandon Doherty**, Florida State University, and **Philip John Hackney**, University of Louisiana at Lafayette.

Human Behavior and Infectious Disease Dynamics (Code: SS 14A), **Bryce Morsky**, Florida State University.

Mathematical Advances in Scientific Machine Learning (Code: SS 15A), **Wenjing Liao**, Georgia Institute of Technology, and **Feng Bao** and **Zecheng Zhang**, Florida State University.

Mathematical Modeling and Simulation in Fluid Dynamics (Code: SS 16A), **Pejman Sanaei**, Georgia State University.

Mathematical Models for Population and Methods for Parameter Estimation in Epidemiology (Code: SS 17A), **Yang LI**, Georgia State University, and **Guihong Fan**, Columbus State University.

Moduli Spaces in Algebraic Geometry (Code: SS 18A), **Jeremy Usatine**, Florida State University, **Hulya Arguz** and **Pierrick Bousseau**, University of Georgia, and **Matthew Satriano**, University of Waterloo.

Nonlinear Evolution Partial Differential Equations in Physics and Geometry (Code: SS 19A), **Jared Speck** and **Leonardo Abbrescia**, Vanderbilt University.

Numerical Methods and Deep Learning for PDEs (Code: SS 20A), **Chunmei Wang**, University of Florida, and **Haizhao Yang**, University of Maryland College Park.

PDEs in Incompressible Fluid Mechanics (Code: SS 21A), **Wojciech S. Ozanski**, Florida State University, **Stanley Palasek**, UCLA, and **Alexis F Vasseur**, The University of Texas At Austin.

Recent Advances in Geometry and Topology (Code: SS 22A), **Thang Nguyen**, **Samuel Aaron Ballas**, **Philip L. Bowers**, and **Sergio Fenley**, Florida State University.

Recent Advances in Inverse Problems for Partial Differential Equations and Their Applications (Code: SS 23A), **Anh-Khoa Vo**, Florida A&M University, and **Thuy T. Le**, North Carolina State University.

Recent Development in Deterministic and Stochastic PDEs (Code: SS 24A), **Quyuan Lin**, Clemson University, and **Xin Liu**, Texas A&M University.

Recent Developments in Numerical Methods for Evolution Partial Differential Equations (Code: SS 25A), **Thi-Thao-Phuong Hoang**, **Yanzhao Cao**, and **Hans-Werner Van Wyk**, Auburn University.

Regularity Theory and Free Boundary Problems (Code: SS 27A), **Lei Zhang**, University of Florida, and **Eduardo V. Teixeira**, University of Central Florida.

Stochastic Analysis and Applications (Code: SS 28A), **Hakima Bessaih**, Florida International University, and **Oussama Landoulsi**, FLORIDA INTERNATIONAL UNIVERSITY.

Stochastic Differential Equations: Modeling, Estimation, and Applications (Code: SS 29A), **Sher B Chhetri**, University of South Carolina Sumter, **Hongwei Long**, Florida Atlantic University, and **Olusegun M. Otunuga**, Augusta University.

Theory of Nonlinear Waves (Code: SS 30A), **Nicholas James Ossi** and **Ziad H Musslimani**, Florida State University.

Topics in Graph Theory (Code: SS 31A), **Songling Shan**, Auburn University, and **Guantao Chen**, Georgia State University.

Topics in Stochastic Analysis/Rough Paths/SPDE and Applications in Machine Learning (Code: SS 32A), **Cheng Ouyang**, University of Illinois At Chicago, **Fabrice Baudoin**, University of Connecticut, and **Qi Feng**, Florida State University.

Topological Algorithms for Complex Data and Biology (Code: SS 33A), **Henry Hugh Adams**, **Johnathan Bush**, and **Hubert Wagner**, University of Florida.

Topological Interactions of Contact and Symplectic Manifolds (Code: SS 34A), **Angela Wu**, University College of London and Louisiana State University, and **Austin Christian**, Georgia Institute of Technology.

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday

Meeting #1194

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 13, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Ryan Charles Hynd, University of Pennsylvania, *Title to be announced*.

Jinyoung Park, Institute for Advanced Study, *Title to be announced*.

Jian Song, Rutgers, State University of New Jersey, *Title to be announced*.

Talitha M Washington, Clark Atlanta University & Atlanta University Center, *Title to be announced* (Einstein Public Lecture in Mathematics).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advanced Mathematical Methods in Naval Engineering Research (Code: SS 1A), **Michael Traweek**, Office of Naval Research, **Bourama Toni**, Howard University, and **Anthony Ruffa**, Emeritus Naval Undersea Warfare Center.

Algebraic and Enumerative Combinatorics (Code: SS 2A), **Samuel Francis Hopkins**, Howard University, **Joel Brewster Lewis**, George Washington University, and **Peter R. W McNamara**, Bucknell University.

Analysis of PDE in Inverse Problems and Control Theory (Code: SS 3A), **Matthias Eller**, Georgetown University, and **Justin Thomas Webster**, University of Maryland, Baltimore County.

Artificial Intelligence Emergent From Mathematics and Physics (Code: SS 4A), **Bourama Toni**, Howard University, and **Artan Sheshmani**, MIT IAiFi.

Automorphic Forms and Langlands Program (Code: SS 5A), **Baiying Liu** and **Freydoon Shahidi**, Purdue University.

Automorphic Forms and Trace Formulae (Code: SS 6A), **Yiannis Sakellaridis**, Johns Hopkins University, **Bao Chau Ngo**, University of Chicago, and **Spencer Leslie**, Boston College.

Coding Theory & Applications (Code: SS 7A), **Emily McMillon**, **Eduardo Camps**, and **Hiram H. Lopez**, Virginia Tech.

Commutative Algebra and its Applications (Code: SS 8A), **Hugh Geller**, West Virginia University, and **Rebecca R.G.**, George Mason University.

Complex Systems in the Life Sciences (Code: SS 9A), **Zhisheng Shuai**, University of Central Florida, **Junping Shi**, College of William & Mary, and **Seoyun Choe**, University of Central Florida.

Computability, Complexity, and Algebraic Structure (Code: SS 10A), **Valentina S Harizanov**, George Washington University, **Keshav Srinivasan**, The George Washington University, and **Philip White** and **Henry Klatt**, George Washington University.

Computational and Machine Learning Methods for Modeling Biological Systems (Code: SS 11A), **Christopher Kim**, **Vipul Periwal**, **Manu Aggarwal**, and **Xiaoyu Duan**, National Institutes of Health.

Control of Partial Differential Equations (Code: SS 12A), **Gisele Adelie Mophou**, Universite des Antilles en Guadeloupe, and **Mahamadi Warma**, George Mason University.

Culturally Responsive Mathematical Education in Minority Serving Institutions (Code: SS 13A), **Lucretia Glover**, **Lifoma Salaam**, and **Julie Lang**, Howard University.

Elementary Number Theory and Elliptic Curves (Code: SS 14A), **Sankar Sitaraman** and **Francois Ramaroson**, Howard University.

Fresh Researchers in Algebra, Combinatorics, and Topology (FRACTals) (Code: SS 15A), **Dwight Anderson Williams II**, Morgan State University, and **Saber Ahmed**, Hamilton College.

GranvilleFest 100: A Celebration of the Legacy of Evelyn Boyd Granville (Code: SS 16A), **Edray Herber Goins**, Pomona College, **Torina D. Lewis**, American Mathematical Society, **Talitha M Washington**, Clark Atlanta University & Atlanta University Center, and **Bourama Toni**, Howard University.

Interactions Between Analysis, Geometric Measure Theory, and Probability in Non-Smooth Spaces (Code: SS 17A), **Luca Capogna**, Smith College, **Jeremy Tyson**, University of Illinois at Urbana-Champaign, and **Nageswari Shanmugalingam**, University of Cincinnati.

Mathematical Modeling, Computation, and Data Analysis in Biological and Biomedical Applications (Code: SS 18A), **Maria G Emelianenko** and **Daniel M Anderson**, George Mason University.

Mathematical Modeling of Climate-Biosphere Interactions (Code: SS 19A), **Ivan Sudakow**, Department of Mathematics, Howard University.

Mathematical Modeling of Type 2 Diabetes and Its Clinical Studies (Code: SS 20A), **Joon Ha**, Howard University.

Mathematics of Infectious Diseases: A Session in Memory of Dr. Abdul-Aziz Yakubu (Code: SS 21A), **Abba Gumel**, University of Maryland, **Daniel Brendan Cooney**, University of Pennsylvania, and **Chadi M Saad-Roy**, University of California, Berkeley.

Modeling and Numerical Methods for Complex Dynamical Systems in Biology (Code: SS 22A), **Hye Won Kang** and **Bradford E. Peercy**, University of Maryland, Baltimore County.

Moduli Spaces in Geometry and Physics (Code: SS 23A), **Artan Sheshmani**, MIT IAiFi.

New Trends in Mathematical Physics (Code: SS 24A), **W. A. Zuniga-Galindo**, University of Texas Rio Grande Valley, and **Bourama Toni** and **Tristan Hubsch**, Howard University.

Nonlinear Hamiltonian PDEs (Code: SS 25A), **Benjamin Harrop-Griffiths**, Georgetown University, and **Maria Ntekoume**, Concordia University.

Optimization, Machine Learning, and Digital Twins (Code: SS 26A), **Harbir Antil**, **Rohit Khandelwal**, and **Sean Carney**, George Mason University.

Permutation Patterns (Code: SS 27A), **Juan B Gil**, Penn State Altoona, and **Alexander I. Burstein**, Howard University.

Post-Quantum Cryptography (Code: SS 28A), **Jason LeGrow**, Virginia Tech, **Veronika Kuchta**, Florida Atlantic University, **Travis Morrison**, Virginia Tech, and **Edoardo Persichetti**, Florida Atlantic University.

Qualitative Dynamics in Finite and Infinite Dynamical Systems (Code: SS 29A), **Roberto De Leo**, Howard University, and **Jim A Yorke**, University of Maryland.

Quantum Mathematics: Foundational Mathematics for Quantum Information Theory, Science and Communication (Code: SS 30A), **Tepper L. Gill** and **Bourama Toni**, Howard University.

Recent Advances in Harmonic Analysis and Their Applications to Partial Differential Equations (Code: SS 31A), **Guher Camliyurt** and **Jose Ramon Madrid Padilla**, Virginia Polytechnic Institute and State University.

Recent Advances in Optimal Transport and Applications (Code: SS 32A), **Henok Mawi**, Howard University (Washington, DC, US), and **Farhan Abedin**, Lafayette College.

Recent Advances on Machine Learning Methods for Forward and Inverse Problems (Code: SS 33A), **Haizhao Yang**, University of Maryland College Park, and **Ke Chen**, University of Maryland, College Park.

Recent Developments in Geometric Analysis (Code: SS 34A), **Yueh-Ju Lin**, Wichita State University, **Samuel Perez-Ayala**, Princeton University, and **Ayush Khaitan**, Rutgers University.

Recent Developments in Noncommutative Algebra and Tensor Categories (Code: SS 35A), **Kent B. Vashaw**, Massachusetts Institute of Technology, **Van C. Nguyen**, U.S. Naval Academy, **Xingting Wang**, Louisiana State University, and **Robert Won**, George Washington University.

Recent Developments in Nonlinear and Computational Dynamics (Code: SS 36A), **Emmanuel Fleurantin** and **Christopher K. R. T. Jones**, University of North Carolina.

Recent Developments in the Study of Free Boundary Problems in Fluid Mechanics (Code: SS 45A), **Huy Q. Nguyen**, University of Maryland, and **Ian Tice**, Carnegie Mellon University.

Recent Progress on Model-Based and Data-Driven Methods in Inverse Problems and Imaging (Code: SS 37A), **Yimin Zhong**, Auburn University, **Yang Yang**, Michigan State University, and **Junshan Lin**, Auburn University.

Recent Trends in Graph Theory (Code: SS 38A), **Katherine Perry**, Soka University of America, and **Adam Blumenthal**, Westminster College.

Riordan Arrays (Code: SS 39A), **Dennis Davenport** and **Lou Shapiro**, Howard University, and **Leon Woodson**, SPIRAL REU At Georgetown.

Skein Modules in Low Dimensional Topology (Code: SS 40A), **Jozef Henryk Przytycki**, George Washington University.

Spectral Theory and Quantum Systems (Code: SS 41A), **Laura Shou**, University of Maryland, and **Shiwen Zhang**, U Mass Lowell.

Stochastic Methods in Fluid Mechanics (Code: SS 42A), **Hussain Ibdah**, Univeristy of Maryland, **Theodore D. Drivas**, S, and **Kyle Liss**, Duke University.

Tensor Algebra & Networks (Code: SS 43A), **Giuseppe Cotardo**, **Gretchen Matthews**, and **Pedro Soto**, Virginia Tech.

Variational Problems with Lack of Compactness (Code: SS 44A), **Cheikh Birahim Ndiaye**, Howard University, and **Ali Maalaoui**, Clark University.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), **Steven H Weintraub**, Lehigh University.

Milwaukee, Wisconsin

University of Wisconsin-Milwaukee

April 20–21, 2024

Saturday – Sunday

Meeting #1195

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 20, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Mihaela Ifrim, University of Wisconsin-Madison, *Title To Be Announced*.
Lin Lin, University of California, Berkeley, *Title To Be Announced*.
Kevin Schreve, LSU, *Title To Be Announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic methods in graph theory and applications I (Code: SS 1A), **Tung T. Nguyen**, University of Chicago/ Western University, **Sunil K. Chebolu**, Illinois State University, and **Jan Minac**, Western University.

Algorithms, Number Theory, and Cryptography I (Code: SS 3A), **Jonathan P Sorenson**, Butler University, **Eric Bach**, University of Wisconsin at Madison, and **Jonathan Webster**, Butler University.

Applications of Algebra and Geometry I (Code: SS 8A), **Thomas Yahl**, University of Wisconsin - Madison, and **Jose Israel Rodriguez**, University of Wisconsin Madison.

Applications of Numerical Algebraic Geometry I (Code: SS 14A), **Emma R Cobian**, University of Notre Dame.

Artificial Intelligence in Mathematics I (Code: SS 9A), **Tony Shaska**, Oakland University, **Alessandro Arsie**, The University of Toledo, **Elira Curri**, Oakland University, Rochester Hills, MI, 48126, and **Mee Seong Im**, United States Naval Academy.

Automorphisms of Riemann Surfaces and Related Topics I (Code: SS 4A), **Aaron D. Wootton**, University of Portland, **Jennifer Paulhus**, Grinnell College, **Sean Allen Broughton**, Rose-Hulman Institute of Technology (emeritus), and **Tony Shaska**, Research Institute of Science and Technology.

Cluster algebras, Hall algebras and representation theory I (Code: SS 5A), **Xueqing Chen**, University of Wisconcin, Whitewater, and **Yiqiang Li**, SUNY At Buffalo.

Combinatorial and geometric themes in representation theory I (Code: SS 23A), **Jeb F. Willenbring**, UW-Milwaukee, and **Pamela E. Harris**, University of Wisconsin, Milwaukee.

Complex Dynamics and Related Areas I (Code: SS 16A), **James Waterman**, Stony Brook University, and **Alastair N Fletcher**, Northern Illinois University.

Computability Theory I (Code: SS 25A), **Matthew Harrison-Trainor**, University of Illinois Chicago, and **Steffen Lempp**, University of Wisconsin-Madison.

Connections between Commutative Algebra and Algebraic Combinatorics I (Code: SS 10A), **Alessandra Costantini**, Oklahoma State University, **Matthew James Weaver**, University of Notre Dame, and **Alexander T Yong**, University of Illinois at Urbana-Champaign.

Developments in hyperbolic-like geometry and dynamics I (Code: SS 11A), **Jonah Gaster**, University of Wisconsin-Milwaukee, **Andrew Zimmer**, University of Wisconsin-Madison, and **Chenxi Wu**, University of Wisconsin At Madison.

Geometric group theory I (Code: SS 28A), **G Christopher Hruska**, University of Wisconsin-Milwaukee, and **Emily Stark**, Wesleyan University.

Geometric Methods in Representation Theory I (Code: SS 2A), **Daniele Rosso**, Indiana University Northwest, and **Joshua Mundinger**, University of Wisconsin - Madison.

Harmonic Analysis and Incidence Geometry I (Code: SS 17A), **Sarah E Tammen** and **Terence L. J Harris**, UW Madison, and **Shengwen Gan**, Massachusetts Institute of Technology.

Mathematical aspects of cryptography and cybersecurity I (Code: SS 24A), **Lubjana Beshaj**, Army Cyber Institute.

Model Theory I (Code: SS 15A), **Uri Andrews**, University of Wisconsin-Madison, and **James Freitag**, University of Illinois Chicago.

New research and open problems in combinatorics I (Code: SS 12A), **Pamela Estephania Harris**, University of Wisconsin, Milwaukee, **Erik Insko**, Central College, and **Mohamed Omar**, York University.

Nonlinear waves I (Code: SS 22A), **Mihaela Ifrim**, University of Wisconsin-Madison, and **Daniel I Tataru**, UC Berkeley.

Nonstandard and Multigraded Commutative Algebra I (Code: SS 13A), **Mahrud Sayrafi**, University of Minnesota, Twin Cities, and **Maya Banks** and **Aleksandra C Sobieska**, University of Wisconsin - Madison.

Panorama of Holomorphic Dynamics I (Code: SS 21A), **Suzanne Lynch Boyd**, University of Wisconsin Milwaukee, and **Rodrigo Perez** and **Roland Roeder**, Indiana University - Purdue University Indianapolis.

Posets in algebraic and geometric combinatorics I (Code: SS 26A), **Martha Yip**, University of Kentucky, and **Rafael S. González D'León**, Loyola University Chicago.

Ramification in Algebraic and Arithmetic Geometry I (Code: SS 29A), **Charlotte Ure**, Illinois State University, and **Nick Rekuski**, Wayne State University.

Recent Advances in Nonlinear PDEs and Their Applications I (Code: SS 27A), **Xiang Wan**, Loyola University Chicago, **Rasika Mahawattege**, University of Maryland, Baltimore County, and **Madhumita Roy**, Graduate Student, University of Memphis.

Recent Advances in Numerical PDE Solvers by Deep Learning I (Code: SS 7A), **Dexuan Xie**, University of Wisconsin-Milwaukee, and **Zhen Chao**, University of Michigan-Ann Arbor.

Recent Developments in Harmonic Analysis I (Code: SS 6A), **Naga Manasa Vempati**, Louisiana State University, **Nathan A. Wagner**, Brown University, and **Bingyang Hu**, Auburn University.

Recent trends in nonlinear PDE I (Code: SS 19A), **Fernando Charro** and **Catherine Lebiedzik**, Wayne State University, and **Md Nurul Raihen**, Fontbonne University.

Stochastic Control and Related Fields: A Special Session in Honor of Professor Stockbridge's 70th Birthday I (Code: SS 18A), **Chao Zhu**, University of Wisconsin-Milwaukee, and **MoonJung Cho**, U.S. Bureau of Labor Statistics.

The Algebras and Special Functions around Association Schemes I (Code: SS 20A), **Paul M Terwilliger**, U. Wisconsin-Madison, **Sarah R Bockting-Conrad**, DePaul University, and **Jae-Ho Lee**, University of North Florida.

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday

Meeting #1196

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: Expired
For abstracts: March 12, 2024

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Antonio, Texas

University of Texas, San Antonio

September 14–15, 2024

Saturday – Sunday

Meeting #1198

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: February 13, 2024
For abstracts: July 23, 2024

Savannah, Georgia

Georgia Southern University, Savannah

October 5–6, 2024

Saturday – Sunday

Meeting #1199

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 5, 2024
For abstracts: August 13, 2024

Albany, New York

University at Albany

October 19–20, 2024

Saturday – Sunday

Meeting #1200

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 19, 2024
For abstracts: August 27, 2024

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Meeting #1201

Western Section

Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 26, 2024

For abstracts: September 3, 2024

Auckland, New Zealand

December 9–13, 2024

Monday – Friday

Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025

Wednesday – Saturday

Associate Secretary for the AMS: Brian Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Hartford, Connecticut

Hosted by University of Connecticut; taking place at the Connecticut Convention Center and Hartford Marriott Downtown

April 5–6, 2025

Saturday – Sunday

Eastern Section

Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

American Mathematical Society
Distribution Center

35 Monticello Place,
Pawtucket, RI 02861 USA

Get Ready For A New Year Of Math

Now Available

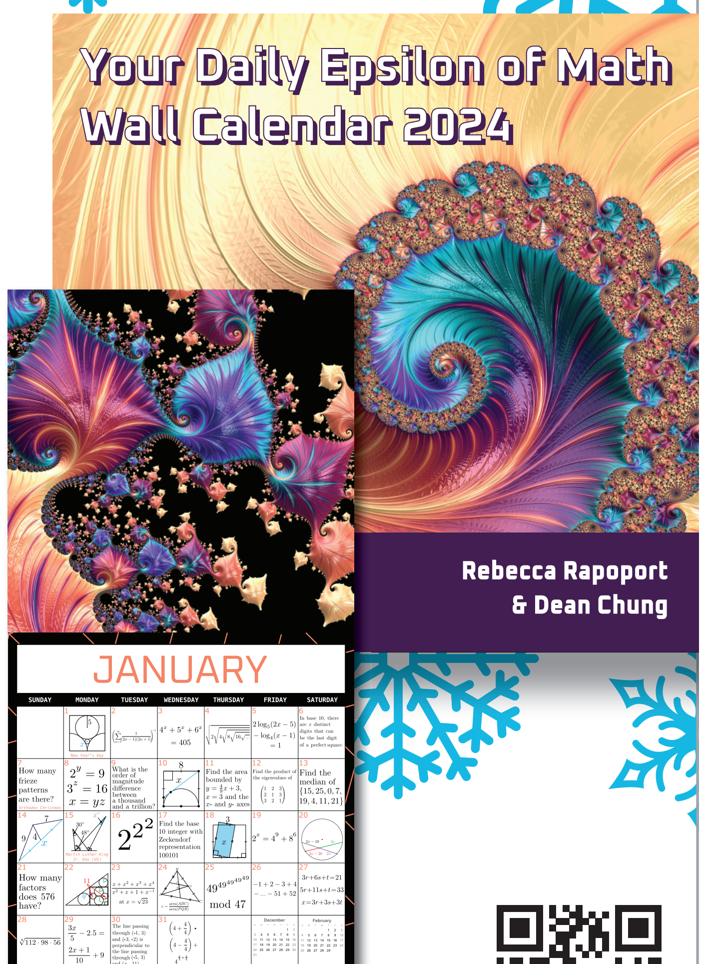
Your Daily Epsilon of Math Wall Calendar 2024

Rebecca Rapoport, *Harvard University, Cambridge, MA, and Michigan State University, East Lansing, MI*; and **Dean Chung**, *Harvard University, Cambridge, MA, and University of Michigan, Ann Arbor, MI*

Keep your mind sharp all year long with *Your Daily Epsilon of Math Wall Calendar 2024* featuring a new math problem every day and 13 beautiful math images! Let mathematicians Rebecca Rapoport and Dean Chung tickle the left side of your brain by providing you with a math challenge for every day of the year. The solution is always the date, but the fun lies in figuring out how to arrive at the answer.

End the year with more brains than you had when it began with *Your Daily Epsilon of Math Wall Calendar 2024*.

2023; ISBN: 978-1-4704-7423-2; List US$20; AMS members US$16; MAA members US$18; Order code MBK/147



AMS | BOOKSTORE

bookstore.ams.org/MBK/147