John Coates (1945–2022): Celebration of a Life

Barry Mazur, Stephen Lichtenbaum, Andrew Wiles, Leila Schneps, Sujatha, and Mahesh Kakde

1. Thoughts About John Coates

Barry Mazur

The broad span of John Coates's mathematical work, including his intellectual delights and his talent for inspiring and mentoring generations of mathematicians, was a gift to all of us. His energy and his generosity of thought, his appreciation of ideas, and his friendship were evident from the earliest days that I knew him. In 1969, he came to Harvard as a Benjamin Peirce Assistant Professor. There, as he wrote in his memorial to John Tate, the "very cramped quarters," its "physical smallness," and "tiny coffee room or corridors between the offices" offered him an environment that made it easy to interact with people. He was introduced by Tate to aspects of mathematics that would lead him to the area in which he would make some of his later great contributions.

As Steve Lichtenbaum recounts (§2), John Coates had started in the analytic number theoretic terrain of Kurt Mahler, algebraic approximation, and p-adic Thue, Roth and Baker methods, moving to the scheme theoretic terrain of Grothendieck's *Langage des Schémas*, before connecting with his true passion, the core of algebraic number theory and the construction of bridges between *analysis* and *arithmetic*: the Birch and Swinnerton-Dyer conjecture, and Iwasawa theory.

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Figure 1. John Coates.

The Coates–Wiles theorem (1977) is such a bridge and was a breakthrough moment in our understanding of the connection between purely arithmetic concepts—such as whether a CM elliptic curve E defined over the field of rational numbers has infinitely many rational points—and "corresponding" analytic concepts—such as whether its L-function $L(E,s)$ vanishes at the value $s = 1$. It follows from the Birch and Swinnerton-Dyer conjecture that if the first *arithmetic* event occurs, so does the second *analytic* event; this is the Coates–Wiles theorem (§3).

Iwasawa theory—a companion to the Birch and Swinnerton-Dyer conjecture—is anchored in the structure of cyclotomic fields (a subject that Serge Lang had called "the backbone of number theory"). For any prime p (say, $p > 2$) Iwasawa theory starts with that "backbone," the p-**cyclotomic tower**

$$K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset L := \cup_{n=1}^{\infty} K_n$$

of field extensions where K_n is the field generated over $\mathbb{Q}$ by the p^n-th roots of unity. The Galois group of that p-cyclotomic tower (i.e., of the field extension L/K) has an easily described topological generator: namely, the automorphism $L \xrightarrow{\gamma} L$ induced by the rule that sends any p^n-th root of unity ζ to ζ^{1+p}. Such a topological generator

γ plays the role in Iwasawa theory of a fundamental operator: it operates naturally on various arithmetic structures defined over the base field K when such a structure is "base-changed" to the extension field L. The operator also acts, for example, on the projective limit of the p-primary parts of the ideal class groups of the rungs K_n of the p-cyclotomic tower—this projective limit producing a finite dimensional vector space over the field of p-adic numbers, with γ operating as a linear automorphism on it.

Iwasawa had thought of this structure—i.e., an operator acting on a vector space defined using concepts in one branch of mathematics as having the (conjectured) property that its characteristic polynomial ties in with an object in another branch of mathematics—as metaphorically linked to André Weil's vision of the Frobenius operator (in the absolute Galois group of a finite field) acting on (then-conjectured, now-known) cohomology groups of varieties over those finite fields with the property that the resulting characteristic polynomials give the relevant L-functions—or, going further back,—as metaphorically linked to what is often referred to as the Hilbert–Polya dream: to express the Riemann zeta-function as the characteristic series of a Hermitian operator on some (perhaps: Hilbert) space.

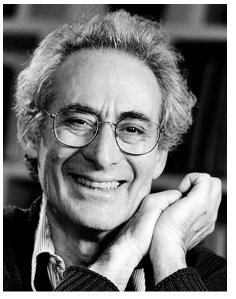
The "main conjecture" of Kenkichi Iwasawa was that the characteristic polynomial of this operator γ gives detailed information about relevant classical (and p-adic) L-functions. This connection between analytic number theory (L-functions) and arithmetic has resonance with the classical Dirichlet class number formula.

Iwasawa theory is now sometimes referred to as classical Iwasawa theory for even more extensive companion theories are being evolved (e.g., in the context of automorphic forms and even more recently in derived categories). John was one of the great contributors to that evolution,

- from his interest in motivic p-adic L-functions—thus studying—as did Ralph Greenberg—the arithmetic of general algebraic varieties as one ascends the rungs of a cyclotomic tower,
- to his shaping with collaborators the impressive beautiful noncommutative version of Iwasawa theory—where the role of a single operator is replaced by a more general p-adic Lie group arising from a Galois representation.

It was thrilling to hear John talk about his ideas about the Birch and Swinnerton-Dyer conjecture, his vision of Iwasawa theory, and about his joint work with Andrew Wiles (§3). I had that opportunity especially when he was based in Orsay (§4) and I was at the IHES, since the two of us would jog around the *bassin* in Bures-sur-Yvette—this is a reservoir, a catch-basin, for the overflow of water (usually dry). We would chat as we jogged. He would explain his latest mathematical thought, we would talk about our young children, and he would tell me about his other passion: Japanese and Chinese poetry and illuminating critical works about them.

What enormous influence he had in the development of mathematics, and as a teacher; the list of his students is incredibly impressive, both in terms of the great contributions to mathematics that they made, but also in view of the range of different interests (albeit within number theory) that they have. This is truly a gift to all of us.



Barry Mazur

2. Coates at Harvard (1969–1972)

Stephen Lichtenbaum

John Coates was born and raised on a farm in rural Australia. His extraordinary talent was noticed early in his life, and as a result he gained admittance to the Australian National University in Canberra. There he was mentored by the famous analytic number theorist Kurt Mahler. Coates wrote several papers directly inspired by his work with Mahler, and impressed Mahler so much that he arranged for Coates to study at the prestigious École Normale Supérieure in Paris. Led by Alexander Grothendieck and Jean-Pierre Serre, Paris had a very good claim to be the world center of modern abstract mathematics, especially in algebraic topology and algebraic geometry. But although he had learned much mathematics at the ANU, Coates had little background in subjects such as homological algebra and the theory of schemes that would be assumed known in Paris seminars. Nonetheless he volunteered, along with Olli Jussila, to take notes in a seminar given by Grothendieck on "Crystals and the de Rham cohomology of schemes". These became part of a book entitled *Dix exposés sur la cohomologie des schémas*, with Coates's

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chapter described as "notes by I. Coates and O. Jussila." It was quite remarkable that Coates was able to do this, and he did not find it a very pleasant experience. In any case, after a year in Paris, Coates decided that he was in the wrong place, and so he wrote to J. W. S. Cassels in Cambridge asking if he could be admitted to graduate study there, and Cassels gladly assented.

At Cambridge, Coates worked with Alan Baker, who would soon receive the Fields Medal. Under Baker's guidance, Coates wrote his doctoral thesis on "The effective solution of some Diophantine equations." He and Baker then wrote a paper together entitled "Integral points on curves of genus one" which appeared in *Inventiones*. Carl Ludwig Siegel had proved that any elliptic curve over a number field had only finitely many integral points, but his proof was not effective. Coates and Baker succeeded in giving a (rather enormous) bound for the size of the solutions in terms of the coefficients of the curve.

After Coates received his PhD, he left Cambridge to take up a three-year position as Benjamin Peirce Assistant Professor at Harvard. There he came under the influence of John Tate, and became part of the modern world of abstract mathematics that he had found so difficult to enter in Paris.

In the nineteenth century, Richard Dedekind had defined a zeta-function $\zeta_F(s)$ for any algebraic number field F, which yields the Riemann zeta-function $\zeta(s)$ when $F = \mathbb{Q}$. The Dedekind zeta-function has many of the same properties as the Riemann zeta-function. For example, it can be defined by a power series which converges for $\mathrm{Re}(s) > 1$ and then continued to a meromorphic function on the entire plane which is analytic except for a simple pole at $s = 1$, and satisfies a functional equation relating $\zeta_F(s)$ to $\zeta_F(1-s)$. It was known by work of Siegel that $\zeta_F(-1)$ is a rational number which is nonzero if and only if F is totally real: we have $\zeta_{\mathbb{Q}}(-1) = -1/12$. The zeta-function can also be defined for varieties over finite fields, where it can be completely described in terms of cohomology.

It is a very challenging, and so far completely unsolved, problem to give such a description for Dedekind zeta-functions, so any relation between Dedekind zeta-functions and cohomology is extremely interesting. Together with Bryan Birch of Oxford, Tate had made a cohomological conjecture about the value of $\zeta_F(-1)$, where F is a totally real number field. Let $\mathcal{O}_F$ denote the ring of integers in the number field F. The Birch–Tate conjecture (slightly modernized) says that if F is a totally real number field then the absolute value of the rational number $\zeta_F(-1)$ should be equal to $|K_2(\mathcal{O}_F)|/w_2(F)$, where $K_2(\mathcal{O}_F)$ is a certain group arising from algebraic K-theory and $w_2(F)$ is defined to be the number of roots of unity contained in the compositum of all quadratic extensions of F. Tate had shown that $|K_2(\mathbb{Z})| = 2$, and it is easy to see that $w_2(\mathbb{Q}) = 24$, so the Birch–Tate conjecture is true for $F = \mathbb{Q}$.

Projective nonsingular curves X over finite fields are often thought of as the geometric analog of rings of integers in algebraic number fields, and they also have their own zeta-functions, and in fact it is possible to state the Birch–Tate conjecture in such a way that it makes sense for this situation as well. We understand the zeta-functions of varieties over finite fields much better than we do the zeta-functions of algebraic number fields, and Tate was able to prove the Birch–Tate conjecture in that case.

Motivated by Tate's proof, Coates started working on the Birch–Tate conjecture in the original number field case. He had been studying the notes which Kenkichi Iwasawa had sent him for a course Iwasawa had given at the Institute for Advanced Study on p-adic L-functions. He then realized that a very natural conjecture describing the p-adic L-function as a characteristic polynomial could lead to a version of the Birch–Tate conjecture where $K_2(\mathcal{O}_F)$ could be described in terms of Galois cohomology. As it happens, I had been working on trying to understand $\zeta_F(-1)$, and I had attended Iwasawa's course on p-adic L-functions at the Institute. When I wrote to Tate telling him that I had a conjecture relating $\zeta_F(-1)$ to the orders of étale cohomology groups, which are fancy versions of Galois cohomology groups, he set to work trying to compare the two conjectures and to the work of Coates. Tate then succeeded in describing $K_2(\mathcal{O}_F)$ in terms of Galois cohomology, and so Coates's work showed that the conjecture on p-adic L-functions implied the Birch–Tate conjecture in the number field case. Tate also suggested that Coates and I ought to get to know each other, and so I invited Coates to give a talk at Cornell.

John drove from Harvard to Ithaca, bringing his wife Julie and his young son David with him. We each had already submitted a paper on K_2 and p-adic L-functions to the *Annals*, so it was too late to combine them as might have been desirable, but we realized that our work on special values of zeta-functions could be extended to L-functions, and in addition our earlier results could be improved. We then started to work together on these problems. In order to further this collaboration John invited me to visit him at Harvard, and I was glad to do so. I also brought my family, which at that time consisted of a wife and three small children. When we arrived, John told us that we should come to his apartment for a dinner which Julie had prepared for us before going to the hospital to have her second child. We were stunned.

Our collaboration eventually resulted in a third paper appearing in the *Annals*, and in our families becoming close. A few years later we spent a semester together at the Institut des Hautes Études in Bures-sur-Yvette, and then we saw each other every summer. I was visiting the IHES and John was living nearby because he had become a professor at the University of Paris at Orsay. Eventually these summertime meetings sadly came to an end as John left to go the Australian National University for one year and then went on to accept a well-deserved chair at Cambridge.

Stephen Lichtenbaum

3. Coates at Cambridge (1975–1978)

Andrew Wiles

Moving to Europe as a graduate student from his native Australia, and after a year spent in the Grothendieck seminar in Paris, John went to Cambridge to study under Alan Baker. This was his first brush with elliptic curves and together they proved a result bounding the size of integral points on cubic equations. There is a finality to this result which is very appealing, but already it was clear that the questions about rational points were much more exciting. There can be finitely many or there can be infinitely many. How do you tell which and how do you find them when they do exist?

After his thesis, John went to the US but returned to Cambridge in 1975, the same year that I started my thesis under his supervision. John only spent two years at Cambridge on this visit and it was part of a whirlwind tour which saw him taking permanent positions in Stanford (1972–75), Cambridge (1975–77), Canberra (1977–78), and Paris (1978–86) within the space of a few years. Nevertheless it was a very productive and exciting period in terms of research. A few months after John's arrival we started working together on elliptic curves and John would not leave this beloved topic for the rest of his career. I will try here to explain the fascination.

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Figure 2. Left to right: John Coates, Andrew Wiles, Ken Ribet, Karl Rubin.

For our purposes we can view an elliptic curve (defined over the rationals) as a cubic equation of the form

$$E = \left\{(x,y)\ :\ y^2 = x^3 + Ax + B\right\} \cup \{\infty\}, \qquad A, B \in \mathbb{Z}$$

with

$$\Delta_E := 4A^3 + 27B^3 \neq 0. \tag{1}$$

The last condition ensures that the cubic is irreducible and that the curve has genus 1. (If it does not hold then the curve has genus 0 and the question of rational points had been settled by the Greeks.) A point at ∞ is included with the curve to make it projective. Surprisingly perhaps, there is then an abelian group structure on the curve which has the property that the sum of any three points of the curve which lie on a line is zero. This holds over any field, for example over the rationals or over the complex numbers. In 1922, Mordell proved that over the rationals, this group (which we denote by $E(\mathbb{Q})$) is finitely generated. So as a group,

$$E(\mathbb{Q}) \simeq \mathbb{Z}^g \oplus T(E), \tag{2}$$

where $T(E)$ is a finite (abelian) group. The big questions are (i) how do we find g and (ii) how do we find generators for the group of points?

The form an answer should probably take had been given ten years earlier by Birch and Swinnerton-Dyer working in Cambridge just prior to the time that John was at Cambridge as a graduate student. It involves the L-function which we can define as follows. Let N_p denote the number of points on $E(\mathbb{F}_p)$, in other words the number of solutions mod p of the equation (1) including the point at ∞. Then set $a_p := 1 + p - N_p$ and define an L-series by

$$L(E,s) := \prod_{p \nmid \Delta_E} (1 - a_p p^{-s} + p^{1-2s})^{-1}. \tag{3}$$

Birch and Swinnerton-Dyer found heuristic reasons why the products $\prod_{p<N} \frac{N_p}{p}$ should tend to ∞ (as $N \to \infty$) if

and only if $E(\mathbb{Q})$ is infinite, and then formulated the conjecture:

Conjecture.

$$L(E,1) \neq 0 \Leftrightarrow E(\mathbb{Q}) \textit{ is finite}. \tag{4}$$

Now $L(E,s)$ is not absolutely convergent at $s = 1$ so this does not make any obvious sense. To overcome this we should also assume that $L(E,s)$ has an analytic continuation. The more refined version of their conjecture then gave the rank g as the order of vanishing of $L(E,s)$ at $s = 1$, and moreover a precise formulation of the leading term involved a regulator made from the generators of the group of points.

Since the work of Mordell, much of the arithmetic study of elliptic curves had made use of class field theory. The first step is to use that the complex points of E are given by

$$E(\mathbb{C}) \simeq \mathbb{C}/\Lambda, \tag{5}$$

where Λ is a lattice in $\mathbb{C}$ so isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. Then given any point P of $E(\mathbb{Q})$ we can consider the point $\frac{1}{2}P$. This will be defined over the complex numbers but if we assume that the 2-torsion points of E (i.e., the points Q satisfying $2Q = 0$) are defined over $\mathbb{Q}$ then $\frac{1}{2}P$ for any $P \in E(\mathbb{Q})$ will be defined over a quadratic extension of $\mathbb{Q}$. Such extensions are well understood and Mordell could use this to bound the number of generators. Now, in the nineteenth century it had been realised that there were some special elliptic curves, the ones with complex multiplication, where all the torsion points are defined over an abelian extension of an imaginary quadratic field. In the work of Kronecker and Weber, later completed by Fueter and Takagi, all the abelian extensions of an imaginary quadratic field were constructed explicitly in this way. As an example the curves $y^2 = x^3 + Ax$ (for any A) have complex multiplication by $\mathbb{Z}[i]$. This is because there is an automorphism of E given by $x \mapsto -x$, $y \mapsto iy$. The theory then gives an explicit construction of the abelian extensions of $\mathbb{Q}(i)$ using the torsion points of the elliptic curve.

The main result of our collaboration was the following, which we proved in the summer of 1976.

Theorem 1 ([8, Theorem 1]). *Suppose that E is an elliptic curve over the rationals with complex multiplication. Then*

$$L(E,1) \neq 0 \Rightarrow E(\mathbb{Q}) \ \textit{ is finite}. \tag{6}$$

This was the first general result on the Birch and Swinnerton-Dyer conjecture. The first thing to note is that the hypothesis of complex multiplication ensured that $L(E,s)$ has an analytic continuation by a theorem of Deuring. This theorem is now known without the hypothesis of complex multiplication but the reverse direction of (6) is still unknown.

The approach we took to proving this theorem was the reverse of the nineteenth-century programme in that we would use their explicit class field theory to study the elliptic curve. Let us assume that we are in the case of $y^2 = x^3 + Ax$ for simplicity. Let $K = \mathbb{Q}(i)$ and pick a prime that splits in K, $p = \pi\overline{\pi}$. Let $E_\pi = \{Q \in E(\mathbb{C}) : \pi Q = 0\}$ be the π-torsion points. Then by the theory of complex multiplication E_π is defined over an abelian extension of K which we write $K_0 = K(E_\pi)$. Suppose that P is a point of infinite order in $E(\mathbb{Q})$ and consider the point $\frac{1}{\pi}P \in E(\mathbb{C})$. This point lies in an abelian extension of K_0 and as always class field theory describes such abelian extensions of K_0 in terms of data coming from the field K_0 itself.

Now we need to see how $L(E,1)$ is related to all this. In the case of complex multiplication there is a canonical period on the elliptic curve denoted Ω which has the property that $L(E,1)/\Omega$ is rational. It turned out that we could relate this value to the explicit construction of units in the ring of integers of the field $K(E_\pi)$. Using this and class field theory we showed that

$$\frac{L(E,1)}{\Omega} \not\equiv 0 \textit{ mod } p \Rightarrow Gal(K_0/K) \textit{ is unramified}. \tag{7}$$

We suspected that the conclusion was false for most π but we could not prove it. However we realised after a while that using Iwasawa theory, a similar claim with π^n would also be true and in this case we could prove that the extension would be ramified for sufficiently large n. So given the existence of P we found that $\frac{L(E,1)}{\Omega} \equiv 0 \textit{ mod } p$ for infinitely many p, and hence was zero. This contradicts the hypothesis, so there was no point P of infinite order.

The influence of Iwasawa theory in the proof sketched above is only apparent at the end, but in fact it motivated the whole approach. John had used the Iwasawa theory of cyclotomic fields prominently with Lichtenbaum in his previous work on zeta values and Euler characteristics. From then on it would guide his approach to elliptic curves.

The idea in Iwasawa theory is to study not just $K_0 = K(E_\pi)$ but also the whole tower of fields $K_\infty = \cup_{n=1}^{\infty} K_n$, where $K_n = K(E_{\pi^{n+1}})$. Let M_n denote the maximal p-power abelian extension of K_n which is unramified outside the prime above π (π turns out to be totally ramified in K_n so this prime is unique), and set $M_\infty = \cup_{n=1}^{\infty} M_n$. Then set

$$X_\infty = Gal(M_\infty/K_\infty). \tag{8}$$

There is an action of $Gal(K_\infty/K)$ on this given by $\sigma : x \to \sigma x \sigma^{-1}$. Now $Gal(K_\infty/K)$ decomposes as

$$Gal(K_\infty/K) \simeq Gal(K_\infty/K_0) \times Gal(K_0/K). \tag{9}$$

We write $\Gamma = Gal(K_\infty/K_0) \simeq \mathbb{Z}_p$ and $\Delta = Gal(K_0/K) \simeq (\mathbb{Z}/(p-1)\mathbb{Z})$. Then we decompose X_∞ into eigenspaces for the tame action of Δ

$$X_\infty = \oplus X_\infty^{(\chi)}, \tag{10}$$

where $\chi : \Delta \to \mathbb{Z}_p^*$ runs through the characters of Δ. Then each $X_\infty^{(\chi)}$ is a Γ-module. Now such Γ-modules can be described in a more familiar way as $\mathbb{Z}_p[[T]]$-modules where the action of γ is the same as that of $1 + T$. Using results on the units in the fields K_n it can be shown that there is a homomorphism

$$X_\infty^{(\chi)} \to \oplus_{i=1}^{r} \mathbb{Z}_p[[T]]/(f_{i,\chi}) \tag{11}$$

with finite kernel and cokernel. We set $(f_\chi) = (\prod f_{i,\chi})$ and call this the characteristic power series of $X_\infty^{(\chi)}$. It is an invariant of the module.

At first encounter, this construction may seem artificial but it was suggested by Weil (in the cyclotomic case) and taken up by Iwasawa as an analog of a construction in the function field case of a module where the characteristic polynomial was related to the zeta function. So what corresponds to the zeta function in our case? This question is most interesting when $\chi = \kappa_0$ is the character giving the action on E_π, i.e., where

$$\kappa_0 Q = \zeta Q \quad \text{for } Q \in E_\pi, \quad \zeta \in \mu_{p-1} \subseteq \mathbb{Z}_p^*. \tag{12}$$

(Here $\zeta \equiv a \bmod p$ *for some* $a \in \mathbb{Z}$ and $\zeta Q = aQ$.) Similarly let $u = \kappa(\gamma) \in \mathbb{Z}_p^*$, where κ gives the action of Γ on E_{π^∞}. In this case the main conjecture (now a theorem due to Rubin) predicted

$$\big(f_\chi(T)\big) = \big(G_\chi(T)\big), \tag{13}$$

where

$$G_\chi(u^{k-1}) = \Omega_p^{1-k}\mu_k\Big(1 - \frac{\psi(\pi)^k}{p}\Big)L(k, \overline{\psi}^k), \tag{14}$$

where $\Omega_p \in \mathbb{C}_p$ is a p-adic period, ψ is the Grossencharakter associated to E, and

$$\mu_k = 12(-1)^{k-1}(k-1)!\,(\Omega/f)^{-k},$$

f being a generator of the conductor of ψ. For our purposes ψ is multiplicative on prime ideals of $\mathbb{Z}[i]$ and satisfies $\psi(\lambda) = \lambda$ for some choice of generator λ in a prime ideal (λ). One checks that $L(s, \psi) = L(s, E)$. The function on the right in (13) is called a p-adic L-function.

Although we were unable to prove the conjecture, we proved a related result that constructed the p-adic L-function from elliptic units. Let U_n denote the local units of the field K_n at the prime above π. We considered the elliptic units ξ_n as a subgroup of U_n and let $\overline{\xi}_n$ be the p-adic closure in U_n. We set

$$Y_\infty^{(\chi)} = (\varprojlim U_n)^{(\chi)}/(\varprojlim \overline{\xi}_n)^{(\chi)}. \tag{15}$$

Then we proved the following theorem.

Theorem 3 ([9]).

$$Y_\infty^{(\kappa_0)} \simeq \mathbb{Z}_p[[T]]/(G_{\kappa_0}(T)). \tag{16}$$

Now $Z_\infty^{(\chi)}$ and $Y_\infty^{(\chi)}$ have enough in common that we could give a slightly different proof of Theorem 1 using just one prime (see also [3]). But perhaps the more important part was the proof which involved attaching a canonical power series to elements of $\varprojlim U_n$ and then applying logarithmic derivatives. This was motivated by an explicit reciprocity law of Iwasawa. These power series were studied and generalised by Coleman and became important in the study of local fields.

The study of Iwasawa theory and in particular of p-adic L-functions of this kind were the main focus of John's work for many years after this. While elliptic curves were always his main interest he also tried to develop a theory of p-adic L-functions for motives (e.g., [2]) and studied particular cases such as the symmetric square of an elliptic curve (see [5]).

The study of p-adic L-functions in the context of elliptic curves had already begun with work of Mazur and Swinnerton-Dyer a few years before this, and p-adic versions of the Birch and Swinnerton-Dyer conjecture were formulated. It was hoped that these might be more tractable than the original complex version. However, the most important advances in the following decades were the work of Gross and Zagier on the analytic side and Kolyvagin on the algebraic side. This seemed to definitively move the focus of work on the conjecture to the study of modular forms, since that was the context of the seminal work of Gross and Zagier. Surprisingly, no one has been able to extend Theorem 1 even to the case of abelian surfaces with complex multiplication. On the other hand, the jury is still out on whether the p-adic versions are easier or harder than the original complex ones and whether the further study of the former will help in the study of the latter.

Andrew Wiles

4. Coates in Paris (1978–1986)

Leila Schneps

When John Coates came to Paris in 1978, it was not his first stay; as mentioned in the previous sections, he had already spent a year there as a graduate student, attending the famous SGA seminar of Alexander Grothendieck and trying to get started on a thesis before renouncing the idea due to a radical difference in style. Grothendieck speaks of Coates in his memoirs with a certain mea culpa:

> *...it happened that after a few weeks or months [he] found that my style didn't suit [him]. Actually, it seems to me now that it was a case of a mental block, which I too quickly interpreted as a sign of unsuitability for mathematical work. Today I would be much more prudent in making such a prediction. I had no hesitation in telling [him] about my impression, and advising [him] not to continue in a career which did not appear to me to correspond to [his] natural abilities. But I learned later that I was completely wrong—this young researcher went on to become well-known thanks to his work in very difficult subjects at the frontier of algebraic geometry and number theory.*

As was said earlier, Coates left Paris in 1966 for Cambridge where he completed his PhD with Alan Baker, followed by stints at Harvard (§2), then Stanford, then Cambridge again (§3), then Canberra. But in 1978, Georges Poitou of the École Normale Supérieur (ENS) in Paris organized a job offer for him at the University of Paris in Orsay. Poitou was gearing up to become director of the ENS, and with this aim in mind he wanted to wind down his activities of directing research, and thought that Coates would be just the person to fill the ensuing gap in Paris number theory.

Orsay is a small town located very near the IHES where Grothendieck's seminar had taken place in the 1960s, and the university there was and still is one of the most prestigious of the Paris math departments. Poitou's offer was interesting enough to make Coates decide once again to uproot himself. Naturally, the first thing he wanted to do when he came was to gather a group of graduate students to study questions arising from his recent work on the Birch and Swinnerton-Dyer conjecture with Wiles (§3), centered around p-adic L functions and Iwasawa theory. To start with, Rod Yager, a student of Coates from Australia, came to join him in Paris, and Poitou sent him three of his own students who had completed their third cycle theses and were about to embark on their thèse d'état[1]. In the spring of 1979, Coates taught a course in Orsay on elliptic curves with complex multiplication, to spread the word and to recruit more students. After that, he continued to teach research-level courses on recent work each year, generally gaining a further new graduate student or two each time.

The topics that John Coates gave his students mainly concerned generalizations of the recent results proven with Wiles in the case where E is an elliptic curve defined over a quadratic imaginary field K with complex multiplication by the ring of integers $\mathcal{O}_K$ (see §3). The more general situations he considered in Paris were firstly the case of CM elliptic curves defined over a finite extension F of K, and secondly the 2-variable p-adic L-functions, first defined by Katz in the case $F = K$, interpolating the values $L(k, \overline{\psi}^{k+j})$ instead of just $L(k, \overline{\psi}^{k})$. In each case, he proposed (a) showing that the values to be interpolated were algebraic, (b) constructing the p-adic L-function that interpolated them, (c) using it to state generalized versions of a number of conjectures, above all the "main conjecture" first given in his work with Wiles (see (13) of §3), and then various consequences of the main conjecture, considered as separate conjectures in their own right, (d) giving numerical evidence for the conjectures, and (e) proving any parts that could be proven directly. He viewed his advisees also as collaborators in this research program, and distributed problems among them in a purposeful and coherent way.

In the very first years of his stay in Paris, he worked with Catherine Goldstein on the case where E is defined over a finite extension F of K, and p a prime that splits as $p = \pi\overline{\pi}$ in K ([4]). They constructed the p-adic L-function $G_\pi(T)$ such that $G_\pi(u^s - 1)$ interpolates the values $L(k, \overline{\psi}^{k})$ for $k \equiv 1 \bmod p-1$ (suitably corrected for algebraicity as usual), formulated the main conjecture in that situation, and drew several consequences, for example:

A) if $E(F)$ contains a point of infinite order then $L(s, \overline{\psi})$ vanishes at $s = 1$,

B) if the subgroup $Ш_\pi$ of elements of the Tate–Shafarevitch group annihilated by a power of π contains a nontrivial divisible subgroup (so in particular is not finite), then $L(s, \overline{\psi})$ vanishes at $s = 1$,

C) under certain hypotheses on π, if $L(s, \overline{\psi})$ vanishes at $s = 1$, then either $E(F)$ contains a point of infinite order or $Ш_\pi$ contains a specific divisible subgroup, namely a copy of $K_\pi/\mathcal{O}_\pi$.

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[1] *At that time, the French system had two theses following the master's: the third cycle thesis that was expected to take about two years and produce a student's first original publication, and the thèse d'état that gave confirmed researchers the right to have students of their own.*


Figure 3. Left to right: John Coates, Kazuya Kato, Sujatha, Otmar Venjakob, Takako Fukaya.

In Dominique Bernardi, John was delighted to find a student who was also one of the rare mathematicians in France at that time who could handle a computer (back when most students still wrote up their work by hand and slipped it under their adviser's door…), and he grouped Dominique and Catherine together with a visiting former student of Bryan Birch's to work on numerical verifications of the conjectures for elliptic curves of the form $y^2 = x^3 - Dx$ ([1]). A former student of Coates at Stanford, Nicole Arthaud, gave a proof of consequence A) in the special case where F is an abelian extension of K and E is defined over K but has a point of infinite order defined over F. A particularly striking contribution was made by John's student Bernadette Perrin-Riou, who defined a p-adic height on $E(F)$ and a p-adic pairing (p-adic versions of the canonical height and the Néron–Tate pairing), and under certain hypotheses proved that the zero of $L_\pi(s, E)$ at $s = 1$ has multiplicity greater than or equal to the rank of $E(F)$ over $\mathcal{O}_K$, and that equality holds if and only if the π-primary component of the Tate–Shafarevitch group is finite and the pairing is nondegenerate ([12]). To Pierre Colmez, John proposed an algebraicity conjecture for special values of L-functions associated to Hecke characters not necessarily the Grossencharakter, in the case of a CM curve E defined over F ([10]).

He also had several students working on the 2-variable p-adic L-functions first defined by Katz in the case where E is defined over K. Rod Yager constructed an Iwasawa module associated to the elliptic curve whose characteristic power series was related to Katz's power series, stating and proving a 2-variable analog of Theorem 3 of §3 ([16]). Jacques Tilouine proved that if the normalized special value of the 2-variable L-function is $\not\equiv 0 \bmod p$, then the rank of the Mordell–Weil group is zero over the maximal $\mathbf{Z}_p^2$-extension of K ([14]). Later, Coates had me work together with Pierre Colmez on constructing the 2-variable p-adic L-function when E is defined over a finite extension F of K ([11]).

As time passed, Coates's style in choosing research topics changed. Being constantly in touch with researchers around the world and up-to-date on all the most recent results, he began to draw inspiration from interesting ideas that came his way – a conjecture stated in a lecture, or a preprint that landed on his desk. When Pierre Colmez first came to him in 1982 to do a master's thesis, he asked him to give a complete proof of Zagier's part of the yet-unpublished Gross–Zagier theorem based on a set of lecture notes he had taken. At the same time, he gave Bernadette Perrin-Riou the ambitious project of developing a p-adic analog of the Gross–Zagier theorem, which she did successfully in the case of an elliptic curve defined over the rationals with CM by a quadratic field K (under the condition that the discriminant of K is prime to the conductor N of the curve, and that all primes dividing N split in K), giving a striking formula valid for primes p where E has good ordinary reduction relating the first derivative of the p-adic L-function of E to the p-adic height of the Heegner point for K. As for me, when I arrived in the fall of 1983—his very last graduate student in Paris, as it turned out—John immediately handed me a preprint by Warren Sinnott, a former student of his at Stanford, giving a new proof of the Ferrero–Washington theorem, and asked if I could adapt the new argument to find a proof that the μ-invariant (the infimum of the p-adic valuations of the coefficients) of the p-adic L-function attached to a CM curve E defined over K was equal to zero ([13]). By that time, research in the whole subject was accelerating rapidly all over the world, and would lead in a few more years to the proof of the main conjecture by Karl Rubin ([9]).

Ken Ribet describes John as having a mission "to elevate young mathematicians around him—to promote their work and to invisibly improve it in various ways. He became an editor of Inventiones very early on in his career (in the late 1970s, I think) and would devote hours on end to rewriting and improving manuscripts that he wanted to publish but that he thought needed polishing. This was well before the era of laptops or even word processors; everything that he did was handwritten with a fountain pen." Certainly, John had a real flair for picking out topics that were simultaneously cutting-edge and approachable by students, and virtually all of our work ended up as published articles, to the quality of whose writing up he devoted particular attention.

When I first met him (in December 1982) to talk about pursuing my graduate studies in Paris, he was affable and welcoming, as he was to every student who wanted to work with him, but he warned me—perhaps from the memory of his own first unsuccessful experience there—that for foreigners, "while Paris was one huge math seminar, there was no actual campus and no student life, and it was easy to feel lost." Of course he was quite right, and of course I paid no attention—who would have? I came to Paris and plunged immediately into an intense and exciting period, filled with a perfect stream of new ideas and research results, lectures and courses on new papers, and summer conferences in Luminy in the South of France, where we would go hiking among the seaside cliffs and climb down to swim in the turquoise-blue calanques. There was also a constant, lively flux of short- and long-term visitors such as Norbert Schappacher, Gudrun Brattström, Peter Schneider, Leslie Federer, Nelson Stephens, Ralph Greenberg, Dick Gross, Barry Mazur, Andrew Wiles, Ken Ribet, John Tate, and many other younger and older researchers who came to give courses and lectures and sometimes to collaborate with his students—all people whose daily language spoke of elliptic curves, complex multiplication, Iwasawa theory, p-adic L-functions, Leopoldt transforms, Heegner points, Selmer groups and so on, while I was still struggling with French.

When John left Paris in 1986 to return to Cambridge, our tight-knit group scattered, and those who had not yet defended a thèse d'état or even completed their third cycle thesis were compelled to find substitute advisers (Michel Raynaud, Guy Henniart, and Jean-Marc Fontaine all stepped up to the plate). But the subject he had started continued to flower. By launching the arithmetic of elliptic curves and Iwasawa theory in Paris, John Coates left an unforgettable mark.

Leila Schneps

5. Iwasawa Theory in a Noncommutative Setting

Mahesh Kakde and Sujatha

5.1. Genesis of noncommutative Iwasawa theory. In the early 1990s, John Coates started exploring the formulation of the main conjecture of noncommutative Iwasawa theory. To systematically study the Iwasawa theory of elliptic curves and p-descent, it is natural to consider the extension of a number field obtained by adjoining all p-power torsion points of an elliptic curve. The algebraic study of such extensions was first taken up in the doctoral thesis of Michael Harris.

In the mid 1990s, John visited Ohio State University, where Sujatha was doing a postdoc. Their meeting led to John's visiting the Tata Institute of Fundamental Research (TIFR), where he gave some lectures on Iwasawa theory; following the TIFR tradition, these were subsequently expanded in a TIFR Lecture Series books entitled *Galois Cohomology of Elliptic Curves*. The meeting with Sujatha gave rise to a fruitful collaboration lasting several years.

By the time that Mahesh became a graduate student of John's at Cambridge in the early 2000s, he had developed an interest in the extension of Iwasawa theory to a noncommutative version, in which the main objects connected to the main conjecture needed to be defined.

For simplicity of exposition, we will only consider an elliptic curve E defined over the rational numbers $\mathbb{Q}$. Furthermore, we fix a prime number $p \geq 5$. Assume that E has good ordinary reduction at p. When E has complex multiplication by an order in an imaginary quadratic field K, we have already seen that the field generated by all p-power torsion points of E is an abelian extension of K. On

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the other hand, in the case when E does not have complex multiplication, a celebrated theorem of Serre says that the extension of $\mathbb{Q}$ generated by all p-power torsion points of E is a Galois extension of $\mathbb{Q}$ with Galois group given by an open subgroup of $GL_2(\mathbb{Z}_p)$.

Following the work of Venjakob on a good definition of *pseudo-null* modules over *Auslander regular rings* [15], Coates, Schneider, and Sujatha proved a structure theorem for finitely generated torsion modules over the Iwasawa algebra of a compact, *p-valued*, p-adic Lie group, much along the lines of the classical result [7]. However, it is unclear if this result can be used to define a characteristic element or even a characteristic ideal.

An important consequence of the main conjecture in classical Iwasawa theory is the formula for an Euler characteristic in terms of values of p-adic L-functions. The study of Euler characteristics of Selmer groups of elliptic curves without complex multiplication was undertaken by Coates with his graduate student Susan Howson. A noncommutative main conjecture would relate the Euler characteristic $\chi(G, Sel(E/\mathbb{Q}))$ with special values of the L-function of E, which would yield the p-adic Birch and Swinnerton-Dyer conjecture. Coates and Sujatha defined this refined G-Euler characteristic which was used to study the consequences of a noncommutative main conjecture.

The extension $\mathbb{Q}_\infty = \mathbb{Q}(E[p^\infty])$ contains the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\mathrm{cyc}}$ of $\mathbb{Q}$. Set $H = Gal(\mathbb{Q}_\infty/\mathbb{Q}_{\mathrm{cyc}})$ and $\Gamma = Gal(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \simeq G/H$. One of the major results of Coates in the joint work with Schneider and Sujatha [6] is the definition of the "Akashi series" (named after the Akashi chapter in *The Tale of Genji*, a favorite of John's). Let M be a $\Lambda(G)$-module that is finitely generated over $\Lambda(H)$. It is known that the homology groups

$$H_i(H, M) \qquad (i \geq 0)$$

are finitely generated torsion $\Lambda(\Gamma)$-modules and are 0 for $i \gg 0$. Let $g_i(M)$ be a characteristic element of $H_i(H, M)$. The Akashi series for M is defined by

$$f(M) = \prod_{i \geq 0} g_i(M)^{(-1)^i}.$$

These insights directly led to the definition of characteristic elements for an interesting class of modules over $\Lambda(G)$ in the habilitation thesis of Venkajob.

Venkajob observed that the right category to work with is $\mathfrak{M}_H(G)$, the category of finitely generated $\Lambda(G)$-modules X such that $X/X(p)$ is a finitely generated $\Lambda(H)$-module. This category was first defined in the paper of Coates–Schneider–Sujatha.

The $\mathfrak{M}_H(G)$ *conjecture* of Coates and Sujatha says that the Pontryagin dual of the Selmer group of E over $\mathbb{Q}(E[p^\infty])$ is in the category $\mathfrak{M}_H(G)$.

For simplicity, we make the additional assumption that G has no p-torsion. This ensures that all finitely generated $\Lambda(G)$-modules have a finite projective resolution. Then we can define a *characteristic element* for every module in $\mathfrak{M}_H(G)$. It is an element in a certain K_1-group.

The main conjecture for E can now be formulated as follows: There is a unique element $\mathcal{L}(E)$ such that

(i) $\mathcal{L}(E)$ is a characteristic element of the Pontryagin dual of the Selmer group of E.
(ii) at every Artin representation of G, the element $\mathcal{L}(E)$ interpolates the value at $s = 1$ of the L-function of E twisted by the representation (appropriately normalized).

The insight of relating noncommutative Iwasawa theory to commutative Iwasawa theory turns out to be useful in tackling all known cases of the noncommutative main conjecture. This study was initiated by Kato. Calculations of K_1-groups imply that a necessary condition for the existence of a noncommutative p-adic L-function is a congruence between commutative p-adic L-functions over extensions corresponding to abelian sub-quotients of G. These congruences are between p-adic L-functions over different number fields and seem very different from earlier congruences between L-functions (for example Kummer congruences).

John placed a great deal of faith in these two predictions coming from noncommutative Iwasawa theory—the congruences between p-adic L-functions and the $\mathfrak{M}_H(G)$ conjecture. Both of these remain wide open. What progress is made on these conjectures and what role they play in the arithmetic of elliptic curves, only time will tell.

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