WHAT IS...?

a Skew Brace?

Leandro Vendramin

A skew brace is a fascinating mathematical structure that involves a pair of compatible groups sharing the same underlying set. The notion of skew braces was inspired by a class of rings introduced by Jacobson in 1945, which we will discuss in more detail later. As algebraic structures, skew braces exhibit similarities to both groups and rings.

What makes skew braces particularly interesting is their ability to serve as an algebraic framework for exploring combinatorial solutions to the Yang–Baxter equation. By delving into the world of skew braces, we can uncover new insights and approaches to understanding this fundamental equation.

Let us begin by understanding what we mean by combinatorial solutions to the Yang–Baxter equation.

We are interested in pairs (X, r), where X is just a set and $r\colon X\times X \to X\times X$ is a bijective map that satisfies a specific equation in $X\times X\times X$, called the Yang–Baxter equation:

$$(r\times \mathrm{id})(\mathrm{id}\times r)(r\times \mathrm{id}) = (\mathrm{id}\times r)(r\times \mathrm{id})(\mathrm{id}\times r).$$

This equation may look a bit abstract, but there is a nice way to think about it:

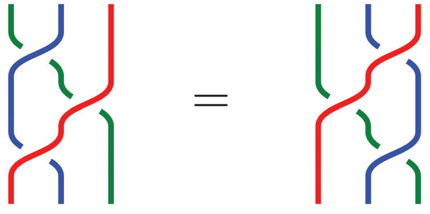

Figure 1. The Yang–Baxter (or braid) equation.

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The arrangements of strings, representing the Yang–Baxter equation, should be read from top to bottom, with the crossing symbolizing the application of the map r and the straight line representing the identity mapping. The picture itself is self-explanatory.

Because of this braiding-like behavior, the equation is also known as the *braid equation*.

The identity map on $X \times X$ satisfies the Yang–Baxter equation. However, without additional assumptions, the task of finding solutions becomes highly unpredictable. Therefore, we will focus on solutions that meet specific extra assumptions. Given the combinatorial nature of the problem and our intention to utilize group theory, it is compelling to investigate the following intriguing class of solutions. We say that a solution (X, r), where

$$r(x, y) = (\sigma_x(y), \tau_y(x)),$$

is *nondegenerate* if the maps $\sigma_x\colon X \to X$ and $\tau_y\colon X \to X$ are bijective, for all $x, y \in X$.

With the nondegeneracy assumption, exploring solutions to the Yang–Baxter equation becomes even more intriguing as we can now leverage groups that inherently act on our solutions. This opens up new avenues for investigation. By incorporating group actions, we gain a richer understanding of the equation's behavior and its connections to various mathematical structures.

We can find many examples of solutions:

(a) If $\sigma\colon X \to X$ and $\tau\colon X \to X$ are commuting bijections, then $r(x, y) = (\sigma(y), \tau(x))$ is a solution. In particular, the flip map $r(x, y) = (y, x)$ is a solution.

(b) If X is a group, then $r(x, y) = (y, y^{-1}xy)$ is a solution.

Definition 1. A *skew brace* is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are (not necessarily abelian) groups and

$$a\circ(b + c) = a\circ b - a + a\circ c$$

holds for all $a, b, c \in A$. The groups $(A, +)$ and $(A, \circ)$ are respectively the *additive* and *multiplicative* group of the skew brace A.

In one of the groups, we embrace the use of additive notation, even when our group is not necessarily assumed to be abelian.

Radical rings have their roots in Jacobson's work. Subsequently, Rump unearthed an algebraic structure encompassing Jacobson radical rings as examples and coined the term *braces*. This structure proposed by Rump was further expanded upon in [3], leading to what we currently know as *skew braces*.

Many familiar mathematical objects have skew brace structures. For example, groups trivially produce skew braces.

Example 2. If G is a group, the operations $x + y = xy$ and $x \circ y = xy$ define a skew brace structure on G.

Now let us explore Jacobson radical rings. Imagine we have a ring R. In this ring, we can define a new operation—called the *Jacobson circle operation*—that takes two elements, let us say x and y, and maps them to

$$x \circ y = x + xy + y.$$

What is surprising is that this operation is always associative with the zero of the ring being its neutral element. When $(R, \circ)$ is a group, we say that R is a *radical ring*. For instance, nilpotent rings, such as rings of strictly upper triangular matrices, are Jacobson radical rings.

Example 3. The subset

$$\left\{ \frac{2x}{2y+1} \,:\, x, y \in \mathbb{Z} \right\}$$

of the rational numbers is a radical ring with the usual addition of rational numbers and circle operation

$$u \circ v = u + uv + v.$$

Inverses of elements with respect to the circle operation are given by

$$\left(\frac{2x}{2y+1} \right)' = \frac{-2x}{2(x+y)+1}.$$

Now, here is the exciting part discovered by Rump: Jacobson radical rings are also examples of skew braces. When considering a radical ring, the combination of its addition with the Jacobson circle operation transforms the ring into a skew brace.

It is time to unveil the fascinating connection between skew braces and solutions to the Yang–Baxter equation.

Theorem 4. *Let A be a skew brace. Then (A, r_A), where*

$$r_A : A \times A \to A \times A,$$
$$r_A(x, y) = (-x + x \circ y, (-x + x \circ y)' \circ x \circ y),$$

where z' denotes the inverse of the element z with respect to the circle operation, is a solution to the Yang–Baxter equation. Moreover, $r_A^2 = \mathrm{id}_{A \times A}$ if and only if $(A, +)$ is abelian.

The reader is encouraged to explore the solutions obtained by applying Theorem 4 to the skew braces described in Examples 2 and 3.

For a solution (X, r), we define the *structure group* of (X, r) as the group $G(X, r)$ with generators X and relations

$$xy = uv$$

whenever $r(x, y) = (u, v)$.

In the upcoming example, we will express permutations as products of disjoint cycles. For instance, the symbol (123) denotes the bijective mapping from $\{1, 2, 3\}$ to $\{1, 2, 3\}$, where 1 is mapped to 2, 2 is mapped to 3, and 3 is mapped to 1.

Example 5. Let $X = \{1, 2, 3, 4\}$ and

$$r(x, y) = (\sigma_x(y), \tau_y(x)),$$

where

$$\begin{aligned} &\sigma_1 = (12), \quad &&\sigma_2 = (1324), \quad &&\sigma_3 = (34), \quad &&\sigma_4 = (1423), \\ &\tau_1 = (14), \quad &&\tau_2 = (1243), \quad &&\tau_3 = (23), \quad &&\tau_4 = (1342). \end{aligned}$$

Then (X, r) is a solution. The group $G(X, r)$ has generators x_1, x_2, x_3, x_4 and relations

$$\begin{aligned} &x_1^2 = x_2 x_4, \quad &&x_1 x_3 = x_3 x_1, \quad &&x_1 x_4 = x_4 x_3, \\ &x_2 x_1 = x_3 x_2, \quad &&x_2^2 = x_4^2, \quad &&x_3^2 = x_4 x_2. \end{aligned}$$

This group admits the following faithful linear representation:

$$x_1 \mapsto \begin{pmatrix} 0&1&0&0&1 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{pmatrix}, \qquad x_2 \mapsto \begin{pmatrix} 0&0&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \end{pmatrix},$$
$$x_3 \mapsto \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&1 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{pmatrix}, \qquad x_4 \mapsto \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \\ 1&0&0&0&1 \\ 0&0&0&0&1 \end{pmatrix}.$$

Each matrix's first principal 4×4 block contains a permutation matrix. For instance, in the matrix associated with x_1, the block contains the permutation matrix corresponding to the permutation σ_1.

The linear representation we observed in the previous example was first found by Etingof, Schedler, and Soloviev and can be now explained by the theory of skew braces.

Theorem 6. *Let (X, r) be a solution. Then there exists a unique skew brace structure over $G(X, r)$ such that $r_{G(X,r)}$ satisfies*

$$r_{G(X,r)}(\iota \times \iota) = (\iota \times \iota) r,$$

where $\iota \colon X \to G(X, r)$ is the canonical map. If $r^2 = \mathrm{id}_{X \times X}$, then the additive group of $G(X, r)$ is abelian and the map ι is injective.

The previous theorem reveals a profound connection between solutions and skew braces. It uncovers a hidden bridge that connects the world of combinatorial properties of solutions to the algebraic properties of skew braces and vice versa.

Example 7. Let us revisit Example 5. The map from X to $\mathbb{Z}^4$,

$$1 \mapsto \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad 2 \mapsto \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad 3 \mapsto \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad 4 \mapsto \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

can be extended to a bijection between $G(X,r)$ and $\mathbb{Z}^4$. This bijection is a 1-cocycle,[1] with the action of $G(X,r)$ on $\mathbb{Z}^4$ induced by the permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. The additive group structure of $G(X,r)$ is isomorphic to that of $\mathbb{Z}^4$.

It is remarkable how seemingly different realms of mathematics are intertwined, providing us with new opportunities to explore the intricate relationship between combinatorial and algebraic concepts.

Skew braces offer an appealing framework to explore various mathematical problems that may not initially seem connected to the solutions of the Yang–Baxter equation. Examples are the links between skew braces and Lie theory or Hopf–Galois structures. It is amazing how these apparently unrelated concepts can come together and inspire new insights in mathematics.

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[1] *If $\lambda : G \to \mathrm{Aut}(A)$ is a group homomorphism, a 1-cocycle is a map $\pi : G \to A$ such that $\pi(xy) = \pi(x) + \lambda(x)(\pi(y))$ for all $x, y \in G$.*

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