Tumbling Downhill Along a Given Curve

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Figure 1. The performance of an actual trajectoid rolling downhill (left-to-right). To demonstrate that trajectoids can be fabricated for rather general paths, we take as a prescribed path (black curve) the shape of “a boa constrictor digesting an elephant,” often mistaken to be simply “a hat,” from The Little Prince [dSE44], which is repeated periodically. The blue curve is followed by the actual fabricated trajectoid shown on the left, which has outer diameter of 4.128 cm. Black in-plane 1-cm scale bar is shown on the right. Deviations from the ideal black curve are attributed to the precision of the 3D printing, some inertia, and the errors of determining the location of the center of mass (CM) by the projection centroid method [SDT+23]. Note that this trajectoid is made of two identical pieces (green and pink), and returns to its original orientation after rolling along two periods (this double periodicity is denoted by alternating solid and empty black circles along the flat path). Adapted with permission from Fig. 4j of [SDT+23].

1. The Problem
A cylinder will roll down along a straight line. A cone will roll around a circle on that plane and then will stop rolling. We ask the inverse question: For which curves drawn on the inclined plane $\mathbb{R}^2$ can one carve a shape that will roll downhill following precisely this prescribed curve and its translationally repeated copies? See Fig. 1 for an example.

This simple question has a solution essentially always, but it turns out that for most curves, the shape will return to its initial orientation only after crossing a few copies of the curve—most often two copies will suffice, but some curves require an arbitrarily large number of copies.

2. Rolling Stones
There is an ample mathematical literature on rolling objects, but for our purpose, it may suffice to mention the study of rolling acrobatic apparatus [Seg21] and the rolling of balls on balls or Riemann surfaces [Lev93]. “Rolling” here means pure rolling motion without slipping or pivoting (that is, rotating around an axis perpendicular to the surface). With these rules of the game, a cylindrical stone will roll down along a straight line, indefinitely, in the projected direction of gravity.

But besides straight lines traced by cylinders, can one chisel objects that will follow more interesting planar paths—curved, convoluted, and self-crossing? This problem was posed and amply discussed in [SDT+23], and the aim of the current contribution is to go into more mathematical detail and add some new results about the general set of paths for which solutions exist.

A solution means that we have a way to sculpt such a wobbly stone, which we call “trajectoid” because it has the following property: Once placed at the starting point of the trajectory, appropriately oriented, and released, the trajectoid stone should roll along a pre-described infinite trajectory, and only along it. This periodic path is made by concatenating translated copies of an original finite path $P$, as shown in Fig. 1.

The principle of trajectoid design (whose realization in practice we will discuss later) is the following: We take a heavy ball and coat it with some lightweight material, which we assume is weightless. Thus, the rolling object is inhomogeneous, and we will see that this allows the
existence of trajectoids. The featherlight solid envelope is precisely chiseled (or “shaved”), such that it will lift the heavy ball whenever it tries to depart from the prescribed path. Gravity will therefore ensure that the object’s center of mass (CM) always stays at a constant height above the plane, and as a result, this trajectoid obediently follows the desired path. The concentration of all the mass in the heavy ball allows us define “following” simply as

**Definition 1.** A trajectoid follows a given path if a perpendicular projection of the ball’s center of mass traces the path, exactly.

We typically design and build such gadgets for rolling downhill on an inclined plane (as in Fig. 1). But from a mathematical point of view, it makes more sense to think about a slightly more general scenario where one puts the trajectoid on a horizontal plane into a certain starting position and orientation. One then starts rolling it carefully by hand, without ever sliding, or rotating the trajectoid around the axis perpendicular to the plane (“pivoting”), or ever lifting its CM, as in Fig. 2. In this scenario, “uphill” or self-crossing paths are allowed and feasible simply by changing the direction of the rolling hand.

We will discuss the mathematics of this problem, and note that the question is somewhat different from what is seen in ooids [Sch13], balls rolling on balls or Riemann surfaces [Lev93], or a general discussion of acrobatic rolling apparatus [Seg21]. In these cases, the objects are relatively simple, while for our inverse problem, each given path requires its own adapted wobbly stone, and complicated paths require extremely elaborate chiseling of the stone.

### 3. The Shaving Solution

To see how one can design a rolling stone as was done in [SDT+23], we consider first a simple polygonal curve $P$. The reader then might think of many other curves that are well-approximated by polygons with infinitesimal segments. We now repeat this curve periodically by adding successively identical copies to its end while maintaining the overall orientation in $\mathbb{R}^2$. More precisely, if $A$ and $\Omega$ are beginning and end of the path $P$ then we consider $P_\infty = \bigcup_{m \in \mathbb{Z}} (P + mA\Omega)$. We want to construct an object that will roll indefinitely along this infinite curve, either on an inclined plane, using gravity, or when we roll it by hand with the constraints described above. In our opinion, the beauty of this problem lies in it being a quite simple question with a relatively deep answer.

Our starting point is the very simple observation that a cylinder rolls along any piece of a straight line, with the axis of the cylinder perpendicular to that segment. Hence, by intersecting two cylinders, we can construct an object that rolls along two consecutive straight segments, as illustrated in Fig. 2.

![Figure 2. Top: The shaving. Right: After a second shaving, the object will turn and roll along the blue line, again keeping the CM at the same constant height of 1 cm. Bottom: The effect of rolling beyond the cut.](image)

Left: A green ball of radius 1 cm surrounded by a light-blue ball of radius 1.4 cm. A cylindrical surface is shaved from the blue envelope, across the violet line. The resulting object can roll along the red curve, which is a great arc on the inner green ball, while keeping its center of mass (CM) at fixed height (1 cm, the inner ball’s radius). Like a cylinder, the object will roll along a straight line, but its CM will be elevated as soon as it would try to roll beyond one of the ends of the red curve.

We now want to generalize this idea so we can follow a polygonal path with many segments. This path $P = A\Omega$ is parameterized by its arc-length $t \in [0, L]$, where $L$ is the overall length of $P$. For our purpose, it is essential to note that:

1. The planar path will always touch the inner, heavy ball, while the specifically-shaped weightless shell serves to stabilize the motion. We take the radius of the heavy ball to be $r$, and the maximum radius of the weightless shell to be $r_{\text{shell}}$.

2. Therefore, by the definition of “following” (Def. 1), the rolling motion can be decomposed into the pure translation of the heavy ball’s CM, at a height $r$ above the prescribed path $P$, and a pure rotation of the heavy ball around its CM.

Throughout, the radius of the heavy ball will be $r$, but for some figures and calculations, it will be convenient to use instead $\sigma = L/(2\pi r)$ where $L$ is the length of one (primitive) period of the path.

This allows us to formalize the shaving procedure by following the rolling motion in the frame of reference of
As the object keeps rolling, the contact point leaves a trace over the first repetition of the path—the gadget should be in exactly the same orientation as at the starting point. In more technical jargon, one then says that the holonomy is a pure translation.

In Fig. 1, we show a path and the trajecoid which was fabricated to follow this particular path “indefinitely.” The figure shows just four repetitions of the path, with the prescribed path shown in black and the actually followed path in blue, demonstrating the reachable quality of making the gadget by 3D-printing, as explained in Section 7.

The question we now ask is when such a construction is possible, and here, some deeper mathematics comes in. Before we go into details, one should note the scaling involved in the problem. Namely, if one has a solution, then by scaling the curve and the object by the same factor, one again has a solution. In the discussion below, we will therefore fix the curve and only adapt the radius $r$ of the ball until a certain mathematical condition is satisfied. If there exists such a radius $r$, we say that a trajecoid exists for the given curve, and if there is no such $r$, the curve has no solution.

Which curves have solutions?

Running over a polygon connecting $A$ to $\Omega$, the cumulative effect of rolling is just the product of a sequence of rotations of the shaved ball:

$$R_{A\Omega} = R_n \cdots R_2 R_1,$$

when there are $n$ segments in the polygon, with each rotation $R_j \in \text{SO}(3)$. To require that the orientation of the piece at the endpoint $\Omega$ of the trajectory returns to its initial orientation at the starting point $A$ is the same as requiring that the rotation product is $R_{A\Omega} = 1$, the identity.

In Fig. 3, we illustrate how a given path is actually mapped from the plane onto the ball. We denote the mapped objects by an overarc $\overset{\curvearrowright}{P}$. The condition $R_{A\Omega} = 1$ means that, when mapped onto the ball, the initial point...
A meets the final point $\Omega$, namely, the curve $\hat{P} = \hat{A}\hat{\Omega}$ on the ball (as in Fig. 3) must be \textit{closed}.

But we also need a condition to guarantee that the orientation of the ball is the same at A and at $\Omega$. For this, we need to understand the rotation on the surface of the ball, which involves the use of an index: Consider a path $P = A\Omega$, parameterized by its arc length $t \in [0, L]$, where $L$ is the length of $P$. Then we can describe $P$ by the normal $\mathbf{n}(t) = (\cos \psi(t), \sin \psi(t), 0) \in \mathbb{R}^3$ to the path (blue in Fig. 3) in the plane on which the ball rolls, and $\psi$ is the angle this normal forms (in $x, y$ coordinates on the plane). Equivalently, we can specify the path using its in-plane (geodesic) curvature $\kappa(t) = d\psi/dt$, when the initial angle $\psi(0)$ is given.

We consider for the path $P$ the integral over the curvature,

$$I_P = \frac{1}{2\pi} \int_P \kappa(t) dt = \frac{\Delta \psi}{2\pi}, \quad (2)$$

where we integrate over one period of $P$.

When we map $P$ onto the ball (Fig. 3), as $\hat{P} = \hat{A}\hat{\Omega}$, then, by isometry, the integral is also defined on the ball, and is equal to the integral of the planar curve: $I_{\hat{P}} = I_P$. This follows directly from the conservation of the geodesic curvature $\kappa(t)$ by the mapping.

Now, if the path on the ball is closed, then this integral is an index. For example, a path that does not self-intersect is of 0-index, $I_P = 0$, while each self-intersection adds a phase of $\Delta \psi = \pm 2\pi$ (with the sign reflecting left/right handedness), or $\pm 1$ to the index. Therefore, for the sake of simplicity, in the remainder of the paper, we consider enclosed areas modulo $2\pi r^2$ and indices modulo 1.

By the Gauss-Bonnet theorem, (see, e.g., the book [Kob21]) the surface enclosed by the path on the ball has area

$$S(r) = S = 2\pi r^2 (1 - I_{\hat{P}}).$$

(The Gauss-Bonnet theorem on the sphere relates the curvature of a closed curve to the enclosed area. This is in contrast to, say, plane triangles, where, giving the angles does not determine the size of the triangle. Harriot discovered in 1603 that for a triangle on a sphere with interior angles $\alpha_i, i = 1, 2, 3$, the area of a triangle is equal to the excess of the sum of the interior angles over $\pi$ (on a sphere of radius 1):

$$\text{area} = \sum_{i=1,2,3} \alpha_i - \pi.$$

This generalizes to polygons on the sphere, and, by continuous limits to curves on the sphere.) Therefore if we require $S = 2\pi r^2$, then $\hat{P}$ is of 0 index, $I_{\hat{P}} = 0$, and therefore also $I_P = \frac{1}{2\pi} \Delta \psi = 0$, implying that the orientation of the ball at the end $\Omega$ of $P$ is the same as at the beginning $A$.

We see that a \textit{trajectoid exists if the curve drawn on the ball encloses exactly half of its surface, namely $2\pi r^2$}. Note that for a given curve, there can be many radii $r$ for which this condition is fulfilled, as shown in Fig. 4 (B).

Before we discuss which paths have a trajectoid that fulfills the area condition, we generalize the problem somewhat.

5. \textit{n-Paths}

Definition 2. A path $P$ is called a $n$-path if any trajectoid for it reaches a pure translation after rolling over exactly $n$ copies of $P$, and never reaches pure translation when rolling over $k$ copies of $P$ with $k < n$ (where both numbers are natural, $k, n \in \mathbb{N}$).

The “experimental” situation seems to be as follows: In general, it seems that most path are not 1-paths. In other words, in this case there is no object that can recover its original orientation after tracing just 1 copy of the (irreducible) path. However, in stark contrast, it seems that “most” paths seem to be 2-paths, although we have no proof nor a good formulation for that (see the discussion...
in the next section). On the other hand, the following two theorems hold:

**Theorem 1.** Every path \( P \) is an \( n \)-path for some finite \( n \).

In other words, for any path \( P \) one can construct a trajectoid which will recover its orientation after passing through \( n \) copies of \( P \), for some finite \( n \).

Let \( C^{1, \beta} \) denote the space of differentiable functions \( \gamma : [0,1] \to \mathbb{R}^2 \) with the derivative \( \gamma' \in C^{\beta} \). By any path, we think of a finite \( C^{1, \beta} \) path in \( \mathbb{R}^2 \) with \( \beta > 0 \). The path may also contain corners and polygons, which may require some trivial reparameterization.

On the other hand, for arbitrary finite \( n \) one can easily construct paths which are \( j \)-paths with \( j > n \) as we show next:

**Theorem 2.** Consider the path \( P \) of Fig. 5 which consists of a subpath \( W \) and its copy rotated by some angle \( 0 < \beta < \pi / j \), where \( j \in \mathbb{N} \) and \( j \geq 2 \). If \( W \) is not a 1-path, and rolling over \( W \) is also for no \( r \) a pure rotation of the ball in the axis perpendicular to the plane of the curve, then \( P \) is an \( n \)-path with \( n > j \).

This means that for any \( j \) there are indeed paths for which any trajectoid must pass over more than \( j \) copies to recover its original orientation.

**Proof of Thm. 1.** We consider one traversal of a path \( P \). After the traversal, the rotation matrix can be written (for each fixed \( \sigma \)) as

\[
R(\sigma) = \begin{pmatrix}
\cos(\varphi(\sigma)) & -\sin(\varphi(\sigma)) & 0 \\
\sin(\varphi(\sigma)) & \cos(\varphi(\sigma)) & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where the rotation is around the \( z \)-axis of an adapted coordinate system by an angle \( \varphi(\sigma) \). The rotation angle is a function of the scaled inverse radius, \( \sigma = L/(2\pi r) \), with \( L \) the arclength of the path. (For a polygon, the motion of the ball is a sequence of rotations around the center of mass with a different axis of rotation for each segment. The concatenation of all these rotations is again a rotation, and this is, in some coordinate system, a rotation around the center of mass about the \( z \)-axis, and can therefore be written as in Eq. (3).)

To prove Thm. 1 we use a continuity argument. Here, it is convenient to use the parameter \( \sigma = L/(2\pi r) \) which we introduced before. In this case, when \( \sigma = 0 \) it suffices to note that \( \varphi(\sigma = 0) = 0 \). The trace of the rotation matrix

\[
\text{Tr}(R(\sigma)) = 1 + 2\cos(\varphi(\sigma))
\]

equals 3 for \( \varphi(\sigma) = 0 \mod 2\pi \), and this trace is continuous in \( \sigma \). If \( \text{Tr}(R(\sigma)) \) is constant in \( \sigma \), then \( P \) is a 1-path: The trace is equal to 3 for all \( \sigma \), and therefore every \( \sigma \) yields a trajectoid that reaches a pure translation after rolling over 1 period. We do not know how to characterize 1-paths. If \( \text{Tr}(R(\sigma)) \) is not constant in \( \sigma \), then, by continuity, there is a \( \sigma \) and a smallest \( n \) for which \( \text{Tr}(R(\sigma)) = 1 + 2\cos(2\pi/n) \). In other words, for this \( \sigma \), \( R(\sigma) \) is a rotation by \( \frac{2\pi}{n} \), and therefore \( R(\sigma)^n \) is the identity. We see that this path is an \( n \)-path.

**Proof of Thm. 2.** Let \( R(W, r) \) be the rotation matrix after rolling over the left part of the path \( W \) of Fig. 5. Then the full rotation matrix is \( R(P, r) = B^{-1}R_1(W, r)B R(W, r) \), where \( B \) is the rotation matrix of turning by the angle \( \beta \) around an axis \( n_1 = \hat{z} \) perpendicular to the plane of the drawing in Fig. 5. The matrix \( B \equiv R_1(W, r)B R(W, r) \) is simply a representation of the matrix \( B \) in a rotated coordinate system (where \( R(W, r) \) is the transformation). Therefore, the transformed \( B \) remains a rotation by the same angle \( \beta \), but around the rotated axis, the unit vector \( n_2(W, r) = R(W, r)n_1 \), which is different from \( n_1 \) unless \( R(W, r) = 1 \). The cosine of the net rotation angle \( \gamma \) corresponding to the overall rotation, \( R(P, r) = B^{-1}B \), is

\[
\cos \gamma = \cos^2 \beta + (n_1 \cdot n_2(W, r)) \sin^2 \beta,
\]

(usually called the Rodrigues formula, [Alt89][equ (19)], with some nice relations to quaternions). Because for the unit vectors \( n_1, n_2 \), one has \( |n_1 \cdot n_2| \leq 1 \), the value of \( \cos \gamma \)
We come back to the Gauss-Bonnet theorem, and will examine what happens after having crossed just one copy of the path $P$, the green curve in Fig. 7.

This curve is not closed, but we close it by the red geodesic arc between $M$ and $A$. (This shortest arc is unique unless $M$ and $\Omega$ are antipodal, in which case a slightly modified argument applies.) Consider then the quantity $S(P, r)$ corresponding to the green area of Fig. 7. In other words, we call $S_g$ the area spanned by the (green) curve, together with the red geodesic connecting $M$ with $\Omega$.

It turns out experimentally, that, for almost all polygonal paths $P$, we can find a radius $r_\ast$, for which the enclosed area is either zero, or exactly $S(P, r_\ast) = \pi r^2$. Assuming that we found such an $r_\ast$, then by the Gauss-Bonnet theorem, the index of the closed green + red curve equals one half, namely $I_g = \frac{1}{2\pi} \int d\kappa(t) = 1 - S_g/(2\pi r^2) = \frac{1}{2}$. However, the integral of the curvature vanishes along the great arc (because it is a geodesic), and along $\hat{\gamma} = AM$ (due to the periodicity of the planar curve $P$, $\Delta \psi = 0$). It follows, that the only contribution to the index comes from the corners $A$ and $M$ whose angles add up to $2\pi I_g = \pi$.

This facilitates the following construction, using Fig. 7: we rotate the green curve by $180^\circ$ about the midpoint of the red arc to form the blue curve. Because the two angles add up to $\pi$, the blue and green curves are connected smoothly, that is, without corners. The connected curve is therefore the mapping to the ball of a two-period repeat of $P$. Now, since the green area is $S_g = \pi r^2$ and is equal to the blue area $S_b$, the two areas add up to $2\pi r^2$. Thus, by doubling the nonclosed path, we achieved the condition that the enclosed area is that of a half-ball, and hence a trajectoid exists for $P$. It is easy to see that the same argument applies also to paths that begin at a sharp corner, simply by shifting the starting point to a smooth point on the curve.

But, we still have to find an $r_\ast$ for which the green area equals $\pi r^2$. We therefore consider the normalized green (or blue) area $S(P, r)/r^2$ as a function of given path $P$ and a ball of radius $r$. Unfortunately, the function $S(P, r)/r^2$ does not have nice monotonicity properties. On the other hand,

\[
\cos^2 \beta - \sin^2 \beta = \cos 2\beta \leq \cos \gamma \leq 1.
\]

Since $\beta < \pi/2$ by assumption, these bounds can be rewritten as $0 \leq \gamma \leq 2\beta$. Here, $\gamma$ is considered on the interval $[0, \pi]$ without any loss of generality. The path $P$ is a 1-path if $\gamma = 0$ for some finite $r$. But this occurs only if $n_2(W, r) = R(W, r)n_1 = n_1$, and therefore $R(W, r)$ is a rotation about the vertical axis $n_1$, thereby contradicting the theorem's assumption about the W path. Otherwise, the path $P$ is a $k$-path if and only if $\gamma = 2\pi/k$, which is only possible if $2\pi/k \leq 2\beta$ due to bounds above. But by assumption $\beta < \pi/j$, and therefore $P$ can only be an $n$-path for $n > j$. □

While Thm. 2 is quite general, it is perhaps a little unsatisfactory, as it does not guarantee that for any $n \geq 1$ there is a path which is an $n$-path. We have provided such an example for an explicit choice, as shown in Fig. 6. In this case, one can show, by an explicit multiplication of the 4 rotation matrices of the 4 segments of the path, that for every $n \geq 3$, there is a $\beta = \beta_n$ for which that path is an $n$-path. The calculation was done for $K = 1/\sqrt{2}$ and $\alpha = 3\pi/4$. It is important that $K$ is irrational, as otherwise, the left part of the figure is a 1-path, which we also had to exclude in the proof of Thm. 2.

6. Prevalence of 2-Paths
We come back to the Gauss-Bonnet theorem, and will explain, but not be able to prove, why almost any planar curve is a 2-path (what “almost” means is discussed later). Recall that any trajectoid is related to a closed curve on the ball, enclosing half of its surface, i.e., $2\pi r^2$. This is difficult to obtain with a primitive path, (a path which is not a multiple of some shorter one) but now consider 2-paths (the following discussion is adapted from [SDT+ 23]), and let us examine what happens after having crossed just one copy of the path $P$, the green curve in Fig. 7.

![Figure 6](image6.png)

**Figure 6.** A $\beta$-dependent family of paths: For each $n \geq 3$ there is a $\beta_n$ so that with $\beta = \beta_n$ the path is an $n$-path.

![Figure 7](image7.png)

**Figure 7.** Adding up the two areas of $\pi r^2$ (green and blue) by passing through 2 copies of the original (primitive) path leads to the desired area $2\pi r^2$, allowing the construction of a trajectoid. Adapted with permission from Fig. 3e of [SDT+ 23].
it proves relatively difficult to construct paths for which $S(P,r)/r^2 \neq n\pi$, $n \in \mathbb{Z}$ for all $r$, as we have seen in Thm. 2. (In Fig. 4 we use the more natural variable $\sigma = L/(2\pi r)$ for the dependence on the radius. This does not resolve the monotonicity problem.)

So at this point, we are left with the empirical result that for “most” paths $P$ we indeed find an $r$ for which a trajectoid exists when one passes two copies of the path $P$. Numerical experimentation shows that for randomly chosen paths $P$, one always seems to find an $r$ for which $S(P,r)/r^2 = \pi$. This motivates us to formulate this as a conjecture, but the reader should be aware that the space of finite curves on the plane is infinite-dimensional. In such contexts, notions like measure or genericity are delicate, and it can happen that sets of full measure only contain “uninteresting” examples.

A possible way out could be the piecewise affine interpolation of the curve, which is perhaps closest to the way trajectoids are constructed. Another possibility is to work in Fourier space, imposing conditions on the Fourier coefficients, see, e.g., [Kah68].

Let $C^{1,\beta}$ denote the space of differentiable functions $\gamma : [0,1] \rightarrow \mathbb{R}^2$ with the derivative $\dot{\gamma} \in C^\beta$.

**Conjecture 1.** The subset of piecewise $C^{1,\beta}$, $\beta \in (0,1]$ curves which are 2-paths, i.e., for which there is an $r$ where the enclosed area satisfies $S(P,r) = \pi r^2$, is dense in $C^{1,\beta}$.

The “piecewise” condition can be omitted if one allows for reparameterizations of the $\gamma$.

Still we can add the following density result: Assume that some path $P$ is a 2-path, i.e., $S(P,r)/r^2 = \pi \mod 2\pi$. In general, all paths in a small open $C^2$ neighborhood of $P$ are again 2-paths. Indeed, if the function defined as

$$S(P,\sigma) = \frac{S(P,r)}{r^2} \bigg|_{r=L/(2\pi \sigma)}$$

traverses $\pi \mod 2\pi$ transversely—which is the generic case—then, by continuity of the area, for curves near $P$ that transversality is maintained and we have a 2-path. One sees that the 2-path property is mostly an open condition in the space of paths. On the other hand, we conjecture above that close to any non-2-path, there is one which leads to a 2-path.

We illustrate the typical generic and nongeneric cases in Fig. 8. The generic situation appears when $\hat{S}(P,\sigma)$ crosses the level $\pi$ transversally. Two notable nongeneric cases appear if $\hat{S}(P,\sigma)$ is tangent to the level $\pi$, or if $\hat{S}(P,\sigma)$ jumps because for some $\sigma^*$ the points $A$ and $\Omega$ are antipodal.

Note that, for fixed $\beta_n$, the function $\text{Tr}(S(\beta_n,\sigma))$ oscillates (irregularly) in $[3-2\beta_n^2, 3]$ as a function of $\sigma$, and so, we can indeed see (cf. Fig. 4) that in general, there can be many trajectoids for a given curve (usually with decreasing radii $r$). It seems that the trajectoid with largest $r$ will roll most accurately along its path.

7. **Fabricating a Trajectoid and Further Remarks**

1. Readers can fabricate a trajectoid for a given curve of their choice in the following way. Prepare your curve as two columns of $(x,y)$ coordinates in a .csv (comma-separated values) file and load it into the online tool (Google Colab notebook, link below) and follow the instructions therein to obtain 3D-printer-ready files (.stl) for a trajectoid of your curve. For the quite common case of a trajectoid for a 2-path, you 3D-print two .stl files (one for each half). Insert the steel ball into the core cavity and glue the two halves together. The typical quality of the pieces we have tested is illustrated in Fig. 1. The gluing

![Figure 8](image1.png)

*Figure 8.* The normalized enclosed area function $\tilde{S}(P,\sigma)$, for three choices of $P$. The generic case is transversal crossing of $\pi$ (blue), whereas the nongeneric case are tangency to $\pi$ (orange) and a jump by $2\pi$ caused by antipodality of $A$ and $\Omega$ (green).

![Figure 9](image2.png)

*Figure 9.* An example of the fabrication of the two-period trajectoid from Fig. 1. The steel ball is 6.24 times more dense than the surrounding plastic (PLA) shell made of two identically shaped halves (magenta and green) 3D-printed and then glued together.
together of the steel ball and the “weightless” outer blue piece—which is made by a 3D-printer—is shown in Fig. 9. A link to a Google Colab notebook can be found on https://github.com/yaroslavsoobolev/trajectoids. We used stainless steel balls of diameter 1 in, weight 66.7g.

2. Because of the half-surface condition mentioned above, it turns out that the trajectoid for a 2-path is actually made of two identical pieces as shown in Fig. 1 and Fig. 9.

3. One can approximate any smooth curve by finer and finer polygons, shaving off each time a tiny piece for each infinitesimal segment.

4. We do not know precisely which primitive paths are 1-paths ("primitive" excludes paths that are just concatenations of two (or more) identical pieces). But any path which is a "V" (e.g., the lines connecting (−x, y), (0, 0), (x, y) with x > 0 and y > 0) is a 1-path. The problem probably needs a careful analysis of symmetries of the paths.

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References


Credits

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