ON S. BERNSTEIN'S THEOREM ON SURFACES $z(x, y)$ OF NONPOSITIVE CURVATURE

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The following classical theorem is due to S. Bernstein: If $z = z(x, y)$ is of class C'' in the whole x - y -plane and if

$$(1) \quad z_{xx}z_{yy} - z_{xy}^2 \leq 0, \quad z_{xx}z_{yy} - z_{xy}^2 \neq 0,$$

then $z(x, y)$ cannot be bounded.¹ The original proof was found to contain a gap of topological nature. It is the purpose of this note to bridge this gap and to prove a somewhat more general theorem.

THEOREM. *If $z(x, y)$ belongs to C'' in the whole x - y -plane and satisfies (1) then $z(x, y)$ cannot be $o(r)$ where r is the distance of (x, y) from an arbitrarily chosen fixed point.*

That this estimate of the order of magnitude at infinity cannot be essentially improved is shown by examples of the form $z = f(x) - g(y)$, $f'' > 0$, $g'' > 0$, where f and g can be chosen such that the order is just $O(r)$. A still open question is whether $z(x, y)$ can or cannot be $o(r)$ along a special sequence of radii $r_i \rightarrow \infty$.

In proving the theorem we shall, essentially, follow Bernstein's original arguments. For the sake of completeness the arguments will be repeated.

LEMMA 1 OF BERNSTEIN. *Let $z(x, y)$ be of class C'' in a bounded open set R and let $z_{xx}z_{yy} - z_{xy}^2 \leq 0$ in R . If z is continuous on the boundary B of R and if $z \leq 0$ on B , then $z \leq 0$ in the whole of R .*

PROOF (according to M. Shiffman). Let C be a circle in the plane $z=0$ whose interior contains $R+B$. Consider the parts $z \geq 0$ of all possible spheres which intersect the plane $z=0$ in C . They form a

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¹ S. Bernstein, *Ueber ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus*, Math. Zeit. vol. 26 (1927) pp. 151-158. This is the translation of a paper written in French (*Sur une thorme de gomtrie et son application aux quations aux drives partielles du type elliptique*) and published in 1914 by the Mathematical Society of Charkow.

Note added in proof. It had escaped the author's attention that Bernstein himself had already published stronger results, which include the generalized theorem. See his paper, *Renforcement de mon thorme sur les surfaces courbure ngative*, Bull. Acad. Sci. URSS. Sr. Math. (1942). His proof, however, contains the same gap as the original one. The argument fails if all four domains Ω spiral infinitely often around the origin.

monotonic family of surfaces $z = Z(x, y; \lambda)$ above $R+B$ with $\lambda = \max Z \geq 0$ taken on the sphere. We have $Z = 0$ for $\lambda = 0$, $Z > 0$ in $R+B$ for $\lambda > 0$, and $Z \rightarrow \infty$ uniformly in $R+B$ for $\lambda \rightarrow \infty$. Let λ^* be the greatest lower bound of all λ for which everywhere

$$(2) \quad Z(x, y; \lambda) \geq z(x, y) \quad \text{in } R+B.$$

It is to be proved that $\lambda^* = 0$. Suppose that $\lambda^* > 0$. For reasons of continuity (2) must hold also for $\lambda = \lambda^*$ and there must be some point x^*, y^* in $R+B$ where $=$ holds in (2), $\lambda = \lambda^*$. This point must lie in R because, according to hypothesis, $z \leq 0$ and $Z > 0$ ($\lambda > 0$) on B . In the space point x^*, y^* , $z^* = z(x^*, y^*)$ the surface must, according to (2), $\lambda = \lambda^*$, be internally tangent to a sphere. Evidently the point of tangency would be a point of positive Gaussian curvature for $z(x, y)$, in contradiction to the hypothesis.

LEMMA 2 (essentially due to Bernstein). Let $z(x, y)$ be of class C'' and let $z_{xx}z_{yy} - z_{xy}^2 \leq 0$ in a connected open set R . Let $z > 0$ in R and let z be continuous and equal to 0 on the boundary B of R . Suppose that R can be placed in an angle less than π . Consider the chord of this angle perpendicular to the line bisecting the angle and at distance u from the vertex. Then the maximum $M(u)$ of $z(x, y)$ on the $(R+B)$ -part of this chord is defined in an infinite u -interval and nowhere convex in this interval,

$$(3) \quad (u_3 - u_1)M(u_2) \leq (u_3 - u_2)M(u_1) + (u_2 - u_1)M(u_3), \\ u_1 < u_2 < u_3.$$

COROLLARY. Under the hypothesis of Lemma 2, there exists a positive constant c such that $M(u) > cu$ for all u sufficiently large.

PROOF. According to Lemma 1, R must be unbounded. Since R is connected, $M(u)$ must be defined for all $u \geq u_0$, and $M(u) \geq 0$, $M(u_0) = 0$. Suppose, now, that (3) were false for three fixed values $u_1 < u_2 < u_3$,

$$(4) \quad (u_3 - u_1)M(u_2) > (u_3 - u_2)M(u_1) + (u_2 - u_1)M(u_3).$$

As the linear function of x, y

$$l(x, y) = au + b \quad (a, b \text{ constants})$$

always satisfies the equality in (3), the chord maxima $M^*(u) = M(u) - (au + b)$ of the function

$$(5) \quad z^*(x, y) = z(x, y) - l(x, y)$$

again satisfy (4) for those fixed u_i ,

$$(6) \quad (u_3 - u_1)M^*(u_2) > (u_3 - u_2)M^*(u_1) + (u_2 - u_1)M^*(u_3).$$

If a, b are chosen such that

$$(7) \quad l(u_i) = M(u_i) \geq 0, \quad i = 1, 3,$$

we have $M^*(u_i) = 0$ for $i = 1, 3$ and, therefore,

$$(8) \quad z^* \leq 0 \quad \text{on } u = u_1 \text{ and } u = u_3.$$

(6) now implies that $M^*(u_2) > 0$ and that

$$(9) \quad z^* > 0 \quad \text{somewhere on } u = u_2.$$

From (7) we infer that $l \geq 0$ for $u_1 < u < u_3$. This shows, in conjunction with the hypothesis that $z = 0$ on B , that

$$(10) \quad z^* \leq 0 \quad \text{on } B \text{ for } u_1 \leq u \leq u_3.$$

If R' denotes the set common to R and to $u_1 < u < u_3$, (8) and (10) mean that $z^* \leq 0$ on the entire boundary of the bounded open set R' . Since

$$z_{xx}^* z_{yy}^* - (z_{xy}^*)^2 \leq 0$$

in R' , Lemma 1 shows that $z^* \leq 0$ in the whole of R' . This contradiction to (9) proves Lemma 2.

The corollary is an obvious consequence of the properties of nowhere convex functions $M(u) \geq 0$, $M(u_0) = 0$.

LEMMA 3. *Suppose that the hypotheses of Lemma 2 are fulfilled. Let $N(r)$ be the maximum of $z(x, y)$ on the $(R+B)$ -part of the circle of radius r about some fixed point. Then there exists a positive constant c' such that $N(r) > c'r$ for all r sufficiently large.*

PROOF. For all r sufficiently large the circle about the fixed point intersects each of the two rays of the angle just once. To each such value r_1 of r belongs a number $u = u_1$ defined as the maximum value of u at which the chord lies in the area $r \leq r_1$. It is geometrically obvious that

$$(11) \quad u_1/r_1 \geq c'' > 0 \quad (c'' \text{ constant})$$

for all sufficiently large values of r_1 . We further restrict r_1 to a range $r_1 \geq a$ where a is chosen such that both the chord u_1 and the circle r_1 always intersect $R+B$. Consider the region R_1 common to R and to $r < r_1$. On the part of the boundary of R_1 with $r < r_1$ we have, according to hypothesis, $z = 0$. On $r = r_1$, $z \leq N(r_1)$ and $N(r_1) \geq 0$. The function $z^* = z - N(r_1)$ has, therefore, values not greater than 0 on the whole boundary of the bounded open set R_1 . From Lemma 1 one in-

fers that $z^* \leq 0$ everywhere in R_1 and, in particular, on the chord $u = u_1$ which we know to lie in $r \leq r_1$. Therefore,

$$(12) \quad M(u_1) \leq N(r_1)$$

holds for all r_1 sufficiently large and the lemma evidently follows from (11), (12), and from the corollary of Lemma 2.

PROOF OF THE THEOREM. The points x, y where $z = z(x, y)$ has negative curvature form an open set. Among these points is surely one where $z_x = z_y = 0$ does not hold. We may suppose that

$$z = z_x = 0, \quad z_y = q_0 > 0, \\ z_{xx}z_{yy} - z_{xy}^2 < 0 \quad \text{at } x = y = 0.$$

The function $\xi = z - q_0 y$ satisfies $\xi_{xx}\xi_{yy} - \xi_{xy}^2 \leq 0$ and

$$(13) \quad \xi = \xi_x = \xi_y = 0, \quad \xi_{xx}\xi_{yy} - \xi_{xy}^2 < 0 \quad \text{at } x = y = 0.$$

Suppose, now, the theorem were false. There would exist a function $\epsilon(r)$, $\epsilon \rightarrow \infty$, such that

$$(14) \quad |z(x, y)| < r\epsilon(r),$$

where r is the distance of x, y from some fixed point. The statement is easily seen to be independent of the location of this initial point. We may suppose that $r^2 = x^2 + y^2$. We can also assume that ϵ is continuous and decreasing for $r \geq r_0$ and that

$$\epsilon(r_0) = q_0 \quad (r_0 > 0).$$

The region S defined by the inequalities $r < r_0$ and, for $r \geq r_0$,

$$(15) \quad |y| < \frac{r}{q_0} \epsilon(r)$$

is bounded by two continuous curves L^+ , L^- , defined respectively by

$$(16) \quad y = \pm \frac{r}{q_0} \epsilon(r), \quad r \geq r_0,$$

or, in polar coordinates, by $q_0 \sin \phi = \pm \epsilon(r)$. Each of these curves reaches towards $x = \pm \infty$. We also mention that $|y/r| \rightarrow 0$ along both curves as $|x| \rightarrow \infty$. It follows from (14), (15), and (16) that $\xi = z - q_0 y$ satisfies

$$(17) \quad \xi < 0 \text{ on } L^+, \quad \xi > 0 \text{ on } L^-,$$

and

$$(18) \quad |\xi| < 2r\epsilon(r) \text{ in } S, \quad r > r_0.$$

These properties of $\xi(x, y)$ together with the ones mentioned before are contradictory. We proceed to prove this.

(13) implies the existence of two straight line segments crossing each other in $(0, 0)$ such that, except at $(0, 0)$, $\xi > 0$ on one and $\xi < 0$ on the other segment. Two points on the same segment but on opposite sides of $(0, 0)$ can never be joined by a Jordan arc on which ξ does not change sign. Otherwise a bounded region would be enclosed in which $\xi \neq 0$ and on whose boundary $\xi = 0$, which contradicts Lemma 1. The open set where $\xi \neq 0$, therefore, contains exactly four components that contain, respectively, the four partial segments obtained by removing the point $(0, 0)$,

$$(19) \quad \xi > 0 \text{ in } \Omega_i^+, \quad \xi < 0 \text{ in } \Omega_i^-, \quad i = 1, 2.$$

$\xi = 0$ on the boundary of each of these four regions. Choose a $p > 0$ such that each of the four connected sets Ω contains a point in which

$$|\xi| > p.$$

The set where $|\xi| > p$ must contain four different components Ω' such that each of them contains one of those four points,

$$(20) \quad \Omega_i'^+ \subset \Omega_i^+, \quad \Omega_i'^- \subset \Omega_i^-.$$

Continuity of ξ implies that these inequalities remain true if the left-hand sets are replaced by their closures. On using these auxiliary regions Ω' we shall be able to bridge the gap mentioned in the beginning.

We prove that each of the regions Ω contains a Jordan curve $x(t)$, $y(t)$ that lies in the strip S and for which $x \rightarrow \infty$ as $t \rightarrow \infty$ and $x \rightarrow -\infty$ as $t \rightarrow -\infty$. This is trivial if Ω contains a point of one of the lines L because, according to (17) and to the definition of Ω , Ω would then contain all points of L . Suppose, now, that Ω contains no point of $L^+ + L^-$. Since $(0, 0)$ is an interior point of S and a boundary point of Ω the connected open set must be entirely contained in the strip S ,

$$(21) \quad \Omega \subset S.$$

Consider the component Ω' of the set $|\xi| > p$ whose closure is, according to (20), contained in Ω . The notion of closure was hitherto understood in the sense of adding all finite points of accumulation. We now add the two infinite points $x = \pm \infty$ to the boundary of the strip S . In the new sense of closure thus involved the situation is this. Ω' is a connected open subset of Ω . The boundaries (in the extended sense) of Ω and Ω' cannot have common points except $x = -\infty$

and $x = +\infty$. We now show that they must be common boundary points of Ω and Ω' . It is sufficient to show that $x = +\infty$ is a boundary point of Ω' . If this were false, Ω' would lie in some half-plane $x < x_0$. The common part of S and of this half-plane could be placed within an angle less than π that is bisected by the x -axis (even within an angle of arbitrarily small opening). The definition of Ω' implies that one of the functions $-\xi - p$, $\xi - p$ is greater than 0 in Ω' and equal to 0 on its boundary. Lemma 3 can, therefore, be applied to this function in Ω' (which region must in view of Lemma 1 be unbounded). The statement of this lemma is, however, incompatible with the property (18) of ξ , $\epsilon \rightarrow 0$ for $r \rightarrow \infty$. $x = \infty$ is, therefore, a boundary point of Ω' (and $x = -\infty$ as well). It now follows from the topological theorem proved in the preceding paper² (by mapping S on a bounded set it is seen that the theorem applies to our case) that each of the two boundary points $x = \pm \infty$ is accessible from Ω , which is precisely what we wanted to prove.³

The statement just proved, namely that each of the four regions Ω contains a Jordan curve in S that joins the two infinite boundary points $|x| = \infty$, is, however, contradictory. Let α_i be a Jordan curve that joins $(0, 0)$ (which is a boundary point of each Ω) through $S\Omega_i^+$ to $x = +\infty$. The "closed" curve $\alpha_1 + \alpha_2$ must enclose one of the regions Ω^- . $S\Omega^-$ clearly could not contain a Jordan curve running towards $x = -\infty$, which is in contradiction to the statement proved above. This finishes the proof of the theorem.

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² E. Hopf, *A theorem on the accessibility of boundary parts of an open point set*.

³ The error in the original proof of the theorem was, essentially, to infer the accessibility of $x = \infty$ from the mere fact that it is a boundary point of Ω .