

A THEOREM ON POLYNOMIAL IDENTITIES

J. LEVITZKI

1. **Introduction.** In a recent paper Kaplansky [4]¹ has emphasized the importance of rings which satisfy a polynomial identity (in short: PI-rings) and proved a number of interesting theorems concerning such rings. In the present note some further properties of PI-rings are derived. In the following, a PI-ring is said to be of degree d if d is the degree of a "minimal equation" (that is, a polynomial identity of minimal degree). In ascribing the term radical to the sum of all nilpotent ideals (notation: $N(S)$ = radical of S) we may express our main result as follows: The nilpotent elements of a PI-ring² are of bounded index modulo the radical. More precisely: The nilpotent elements of the quotient-ring $S/N(S)$ satisfy the identity $x^{[d/2]} = 0$. This is proved in §3 (Theorem 1), where also some immediate consequences of this theorem are listed. Extensive use is made of Kaplansky's Lemma 2 [4] which is restated in §2 of the present paper (Lemma 3) in a slightly generalized form.

In §4 we discuss in some detail an application to PI-nil-rings (in short: NPI-rings). Our result in this section is connected with the following construction due to Baer [1]: By means of the transfinite induction he defines the r th radical $N_r(S)$ as follows: (1) $N_1(S) = N(S)$. (2) If $r = t + 1$, then $N_r(S)$ is uniquely determined by the relations $N_r(S) \supseteq N_t(S)$, $N_r(S)/N_t(S) = N[S/N_t(S)]$. (3) If r is a limit-ordinal, then $N_r(S) = \bigcup_{t < r} N_t(S)$. There exists a smallest ordinal λ such that $N_\lambda(S) = N_{\lambda+1}(S)$. The ideal $N_\lambda(S)$ will be called the ultimate radical³ of S and denoted by $U(S)$. For this ideal $N[S/U(S)] = 0$, and if for an ideal A the relation $N(S/A) = 0$ holds, then $A \supseteq U(S)$. Since also the ideal $N^*(S)$, defined as the sum of all semi-nilpotent ideals, has the property $N[S/N^*(S)] = 0$ (see [5]), it follows that $N^*(S) \supseteq U(S)$, that is, the ultimate radical is semi-nilpotent.

A ring T which coincides with its ultimate radical will be called an L -ring. The smallest ordinal λ for which $N_\lambda(T) = N_{\lambda+1}(T)$ will be called the length of T (notation: $\lambda = \lambda(T) = \text{length of } T$). It is an immediate consequence of the definition that each L -ring is semi-nilpotent, and hence a nil-ring.

Presented to the Society, February 26, 1949; received by the editors January 4, 1949.

¹ Numbers in brackets refer to the bibliography at the end of the paper.

² As to the domain of coefficients of the polynomial identities, compare §2.

³ This ideal was termed by Baer the lower radical. For the purpose of the present note the term ultimate radical seems the more appropriate.

It has been proved by Kaplansky [4] that an NPI-ring is semi-nilpotent. In section 4 it is shown that an NPI-ring is an L -ring. It should be pointed out in this connection that whereas it is unknown whether or not nil-rings exist which are not semi-nilpotent, it can be deduced from an example due to Baer (see [1, §2]) that a semi-nilpotent ring need not be an L -ring.⁴ We further prove that the length $\lambda(S)$ of an NPI-ring S is finite, and that the degree $d(S)$ is an upper bound for $\lambda(S)$. More precisely: If $S \neq 0$, then $\lambda(S) \leq \log d(S)/\log 2$. (If $S=0$ then $\lambda=d=1$.) An immediate consequence of Theorem 1 is: If S is an NPI-ring, then the quotient-ring $S/N(S)$ is of bounded index. Moreover, the identity $x^{[d/2]}=0$ holds in $S/N(S)$.

2. Preliminary remarks. In the case of algebras the coefficients of the polynomial identities are chosen from the underlying field. In order to include this case in our deliberations, we assume that the coefficients $\alpha, \beta, \gamma, \dots$ of our polynomials belong to a domain D of operators having the following properties:

I. The domain D is a subset of the ring E of endomorphisms of the additive group defined by the given ring S .

II. For $\alpha \in D; s_1, s_2 \in S$, we have $\alpha(s_1 s_2) = (\alpha s_1) s_2 = s_1 (\alpha s_2)$.

III. If $\alpha \in D$, then either $\alpha S = 0$ (that is, α is the null-endomorphism) or α is an automorphism. In the latter case we assume that also α^{-1} belongs to D .

IV. The domain contains the endomorphisms $0, \pm 1$, and if $\alpha \in D$, then also $-\alpha \in D$.

Each ring possesses an operator domain with properties I-IV, for example, the set of the 3 numbers $0, \pm 1$.

A sub-ring A of S is called D -admissible, if for each α we have $\alpha S \subseteq A$.

LEMMA 1. *If $a \in S$, then Sa, aS , and SaS are D -admissible.*

PROOF. For $\alpha(Sa) = (\alpha S)a \subseteq Sa; \alpha(aS) = a(\alpha S) \subseteq aS; \alpha(SaS) = (\alpha S)aS \subseteq SaS$.

LEMMA 2. *If A is a D -admissible sub-ring of S and $0 \neq \alpha \in D$, then $\alpha A = A$.*

PROOF. For we have $\alpha A \subseteq A = \alpha(\alpha^{-1}A) \subseteq \alpha A$.

The polynomials which are considered in this paper are elements of the free algebra $E(x_1, x_2, \dots, x_n)$ generated (compare [4]) by the

⁴ This shows incidentally that the ring constructed by Baer furnishes an example of a nil-ring for which no polynomial identity can be found.

indeterminates x_1, \dots, x_n over the ring of endomorphisms E , but we restrict our attention only to such polynomials whose coefficients belong to D and whose constant coefficient is 0. If the polynomial identity

$$(1) \quad f(x_1, x_2, \dots, x_n) = 0$$

is not linear in one of the indeterminates, say x_1 , one performs (see [4, Lemma 2]) the transformation $g(u, t, x_2, \dots, x_n) = f(u+t, x_2, \dots, x_n) - f(u, x_2, \dots, x_n) - f(t, x_2, \dots, x_n)$. One obtains the identity $g(u, t, x_2, \dots, x_n) = 0$ which has a lower degree in u as well as in t than the degree of (1) in x_1 . The general degree of g is not higher than that of f , and if the coefficients of f belong to D , then this holds also for the coefficients of g . If, finally, an indeterminate, say x_1 , does not appear in at least one monomial with a nonzero coefficient, and if $h(x_2, \dots, x_n)$ is the sum of all such monomials, then evidently $h(x_2, \dots, x_n) = 0$ holds in S . As a consequence of these remarks we have the following lemma.

LEMMA 3. *If S is a PI-ring of degree d , then S satisfies a polynomial identity of the form*

$$(2) \quad f(x_1, x_2, \dots, x_d) = \sum_{(i) \in \Pi} \alpha_{(i)} x_{i_1} \cdots x_{i_d} = 0$$

where Π is a set of permutations (i) of the d numbers $1, 2, \dots, d$ and $\alpha_{(i)} \in D$.

3. The main theorem. We assume in this section that the ring S has an operator domain D satisfying conditions I-IV of §2 and that the coefficients of all polynomial identities belong to this domain.

THEOREM 1. *If S is a PI-ring of degree d and $N(S)$ the radical of S , then for each nilpotent element a of S we have*

$$(3) \quad a^{[d/2]} \in N(S).$$

PROOF. This is true for $a \in N(S)$. Now suppose that $a \notin N(S)$, and denote by n the index of a modulo $N(S)$, that is,

$$(4) \quad a^n \notin N(S), \quad a^{n+1} \in N(S).$$

The theorem will be proved if we show that $n < [d/2]$. To this end consider the following $2n+1$ sub-rings of S

$$(5) \quad \begin{aligned} A_{2j-1} &= a^{n-i+1} S a^{i-1}, & j &= 1, 2, \dots, n+1, \\ A_{2j} &= a^{n-i+1} S a^j, & j &= 1, 2, \dots, n, \end{aligned}$$

and form the products

$$(6) \quad B_i = A_1 \cdot A_2 \cdots A_i, \quad i = 1, 2, \dots, 2n + 1.$$

It follows that

$$(7) \quad \begin{aligned} B_{2j-1} &= (a^n S)^{2j-1} a^{j-1}, & j &= 1, 2, \dots, n + 1, \\ B_{2j} &= (a^n S)^{2j} a^j, & j &= 1, 2, \dots, n. \end{aligned}$$

From (5) we deduce that $A_s A_t \subseteq Sa^{n+1}S$ for $s > t$. Hence if for any integer r such that $r \leq 2n + 1$ one denotes by (i_1, \dots, i_r) a permutation of the integers $1, 2, \dots, r$, we have

$$(8) \quad A_{i_1} A_{i_2} \cdots A_{i_r} \subseteq Sa^{n+1}S \quad \text{if } (i_1 \cdots i_r) \neq (1, \dots, r)$$

or in other words

$$(9) \quad \begin{aligned} &a_{i_1} a_{i_2} \cdots a_{i_r} \in Sa^{n+1}S \\ &\text{for } a_k \in A_k, k = 1, \dots, r \text{ and } (i_1, \dots, i_r) \neq (1, \dots, r). \end{aligned}$$

We turn now to the given polynomial identity which according to Lemma 3 we may write in the form

$$(10) \quad \beta x_1 x_2 \cdots x_d = \sum_{(i) \in \Pi'} \beta_{(i)} x_{i_1} \cdots x_{i_d}, \quad \beta \neq 0,$$

where the permutations (i) of the set Π' are all different from the identical permutation. Suppose now that $n \geq [d/2]$; then we have $n \geq (d-1)/2$ or $d \leq 2n + 1$. Hence we may substitute in (9) the integer d for r , thus obtaining

$$(11) \quad a_{i_1} a_{i_2} \cdots a_{i_d} \in Sa^{n+1}S \quad \text{for } (i_1 \cdots i_d) \neq (1, \dots, d).$$

Since $\beta_{(i)} Sa^{n+1}S \subseteq Sa^{n+1}S$ (Lemma 1) we obtain in view of⁵ (10)

$$(12) \quad \beta a_1 a_2 \cdots a_d \in Sa^{n+1}S$$

or $\beta A_1 \cdots A_d = \beta B \subseteq Sa^{n+1}S$. By formulas (7) we now have

$$(13) \quad \begin{aligned} \beta B_d &= \beta (a^n S)^{2q-1} a^{q-1} \subseteq Sa^{n+1}S, & \text{if } d &= 2q - 1, \\ \beta B_d &= \beta (a^n S)^{2q} a^q \subseteq Sa^{n+1}S & \text{if } d &= 2q. \end{aligned}$$

Right multiplication of the first relation in (13) by $a^{n-q+1} S a^n S$, and

⁵ A slight simplification of the argument is possible by operating in the quotient-ring $S/Sa^{n+1}S$. We have then zero on the right side of formulas (8), (9), and (11)–(14). Passage to this quotient-ring is permissible since the ideal $Sa^{n+1}S$ is admissible. This ensures the survival of the polynomial identity and, in view of postulate III, also of its degree; each nonzero endomorphism of S becomes a nonzero endomorphism of $S/Sa^{n+1}S$.

of the second by $a^{n-q}S$, yields in both cases in view of $\beta(a^n S) = a^n S$ (Lemmas 1 and 2)

$$(14) \quad (a^n S)^{2q+1} \subseteq S a^{n+1} S.$$

Now we assumed in (4) that $a^{n+1} \in N(S)$, which implies that the ideal $S a^{n+1} S$ is nilpotent. For some integer t we have therefore $(a^n S)^{t(2q+1)} = 0$, which implies that $a^n \in N(S)$, contradicting (4). We were led to this contradiction by assuming that $n \geq [d/2]$. Hence $n < [d/2]$, q.e.d.

Consequences. (1) Let S denote a PI-ring of degree d , and T a D -homomorphic image of S . If $N(T) = 0$, then the nilpotent elements of T are of bounded index. Moreover, *these elements satisfy the identity* $x^{[d/2]} = 0$. Indeed, the polynomial identity satisfied by S as well as its degree are inherited by T , and our assertion is therefore a direct consequence of Theorem 1.

(2) If S is the ring of n by n matrices over a commutative or a noncommutative field F , and if d is the degree of a polynomial identity satisfied by S , then $n \leq [d/2]$. Indeed, this is a consequence of $N(S) = 0$ and of the fact that S contains nilpotent elements of index $n - 1$. This implies Kaplansky's Lemma 5 [4] which he applies in the proof of his Theorem 1 which states that each PI-primitive algebra in the sense of Jacobson [3] is finite-dimensional over its center. As to the problem raised by Kaplansky concerning an explicit upper bound for the order of the algebra over its center, our exponent $[d/2]$ yields a rather sharp estimate. By using Kaplansky's Lemma 3 [4] we have the following consequence.

(3) If k^2 is the order of a simple algebra over its center, and d is the degree of a polynomial identity, then we have

$$(15) \quad k^2 \leq [d/2]^2.$$

(4) In case $d = 1$ the situation is trivial ($S = 0$). Also the case $d = 2$, which is a slight generalization of the commutative case, yields nothing new. We have here $[d/2] = 1$, that is, all nilpotent elements belong to $N(S)$, and if S is an algebra of finite order, then $S/N(S)$ is in view of (3) a direct sum of commutative fields.

(5) In case $d = 3$ we have $[d/2] = 1$, that is, again all nilpotent elements of S belong to $N(S)$. It is of interest to note that just as in the commutative case we have also here the coincidence of all radical-like ideals which have been hitherto defined by using various types of nillicity. If S is an algebra of finite order it follows in view of (3) that $S/N(S)$ is the direct sum of commutative fields, a fact which was also mentioned by Kaplansky.

(6) In case $d=4$ or $d=5$ we have $[d/2]=2$. If S is a division algebra which satisfies a polynomial identity of degree 4 or 5, it follows in view of (3) that S is either a commutative field or a generalized quaternion algebra. This was previously proved by Hall [2, Theorem 6.2] for the case of a specific polynomial identity of degree 5. Compare also [4].

(7) A direct simple proof of statement (2) may be deduced as follows: Consider the n^2 matrices c_{ij} , $i, j=1, \dots, n$, satisfying the relations

$$c_{ij}c_{jk} = \begin{cases} 0 & \text{if } j \neq j', \\ c_{ik} & \text{if } j = j'. \end{cases}$$

Put $a_{2i-1}=c_{ii}$, $a_{2i}=c_{ii+1}$ and form the product $a_1a_2 \cdots a_r$ for each positive integer $r < 2n$. Suppose that $d(S) < 2n$ and write the polynomial identity in the form (10). By substituting a_i for x_i , $i=1, \dots, d$, one obtains zero on the right side of (10) and something not equal to 0 on the left. This contradiction is a consequence of $d < 2n$. Hence $d \geq 2n$, q.e.d.

4. Polynomial identities and L -rings. Also in this section we assume that all polynomial identities considered have their coefficients in a domain D as defined in §2.

THEOREM 2. *If S is an NPI-ring of degree d , and $N_r(S)$ is the r th radical of S , then the ring $S/N_r(S)$ satisfies the identity*

$$(16) \quad x^{[d/2]} = 0.$$

PROOF. For $r=1$ this is an immediate consequence of Theorem 1. For $r > 1$ this follows from $N_r(S) \supseteq N_1(S)$.

THEOREM 3. *If S is an NPI-ring and $S \supset 0$, then also $N(S) \supset 0$.⁶*

PROOF. If d is the degree of S , then for each $x \in S$ we have by Theorem 1 the relation $x^{[d/2]} \in N(S)$. From $S \neq 0$ it follows that $d \geq 2$. In case $2 \leq d \leq 3$, we have $[d/2]=1$, that is, $x \in N(S)$ or $S=N(S)$, and the theorem is proved. We consider now the case $d \geq 4$ and suppose that $N(S)=0$. It follows then that the identity $x^{[d/2]}=0$ whose degree is smaller than d holds in S , which is a contradiction. Hence $N(S) \neq 0$, q.e.d.

THEOREM 4. *Each NPI-ring is an L -ring.*

PROOF. By Theorem 2 the quotient-ring $S/N(S)$ satisfies an iden-

⁶ This theorem is in some respect a generalization of the fact that for each commutative nil-ring we have $N(S)=S$.

tity of the form $x^p=0$. Since $U(S)\supseteq N(S)$, this identity is satisfied also in $S/U(S)$. In view of $U[S/U(S)]=0$ we have also $N[S/U(S)]=0$ and hence it follows by Theorem 3 that $S/U(S)=0$, or $S=U(S)$, q.e.d.

COROLLARY. *Denote by S an arbitrary PI-ring. If $U(S)$ is the ultimate radical and $U'(S)$ is the upper radical (see Baer [1]), then: (1) $U(S)=U'(S)$ and (2) the quotient-ring $S/U(S)$ has no one-sided nil-ideals other than zero.*

Indeed, since $U(S)$ is D -admissible (see [1]) the polynomial identity, and in view of postulate III in §2 also its degree are inherited by $U'(S)/U(S)$. Since $N[U'(S)/U(S)]=0$ (compare [1]) it follows by Theorem 3 that $U'(S)/U(S)=0$, or $U'(S)=U(S)$. To prove the second part of the above statement, suppose that R is a right (left) nil-ideal in S . By Theorem 4 it follows that R is an L -ring, and hence semi-nilpotent. We have therefore $R\subseteq U^*(S)$, where $U^*(S)$ is the sum of all semi-nilpotent ideals of S . Since $U^*(S)\subseteq U'(S)=U(S)$, we have $R\subseteq U(S)$, which implies the second part of our statement.

THEOREM 5. *Let S be an NPI-ring of degree d_0 , $N_r(S)$ the r th radical of S , and d_r the degree of the quotient-ring $S_r=S/N_r(S)$. Then:*

- (1) *If $d_r > 1$ then $d_{r+1} \leq d_r/2$.*
- (2) *The ring S , which by Theorem 4 is an L -ring, has finite length.*
- (3) *If $\lambda = \lambda(S)$ denotes the length of S , then*

$$(17) \quad \lambda \leq \log d_0 / \log 2 \text{ for } d_0 > 1. \text{ If } d_0 = 1, \text{ then } \lambda = 1.$$

PROOF. (1) For $r=0$ this is an immediate consequence of Theorem 1. Now assume $r>0$. We know (for example, by Theorem 2) that S_r satisfies a polynomial identity. If d_r is the degree of S_r , then by Theorem 2 the ring $S_r/N(S_r)$ satisfies the identity

$$(18) \quad x^{[d_r/2]} = 0.$$

By definition $N(S_r) = N_{r+1}(S)/N_r(S)$ and therefore $S_r/N(S_r) = [S/N_r(S)]/[N_{r+1}(S)/N_r(S)] \cong S/N_{r+1}(S) = S_{r+1}$. This isomorphism implies that identity (18) is satisfied by S_{r+1} . If now d_{r+1} is the degree of S_{r+1} , we have therefore $d_{r+1} \leq [d_r/2] \leq d_r/2$, q.e.d.

(2) Suppose that $d_r > 1$ for each finite r ; then by (1) we would obtain an infinite sequence $d_1 > d_2 > \dots$, a contradiction. Hence there exists a finite index j so that $d_j = 1$. This means that the ring $S/N_j(S)$ satisfies an identity of degree 1, and hence $S/N_j(S) = 0$, or $S = N_j(S)$. By the definition of length we have therefore $\lambda \leq j$, q.e.d.

(3) If $d_0 = 1$, then $0 = S = N_1(S)$, or $\lambda = 1$ in which case (3) is proved. If $d_0 > 1$ and $\lambda = 1$, then (3) is also true. Now suppose that $\lambda > 1$, then evidently $d_r > 1$ for $r \leq \lambda - 1$ (otherwise we would have $S = N_r(S)$ for some r which is smaller than λ , contradicting the definition of length). This implies in view of (1) that $d_\lambda \leq d_0/2^\lambda$ or $2^\lambda \leq d_0/d_\lambda$, that is, $\lambda \log 2 \leq \log d_0 - \log d_\lambda \leq \log d_0$, q.e.d.

COROLLARY. *The length of an NPI-ring is smaller than the degree of the identity. Equality holds only in the trivial case $S = 0$.*

BIBLIOGRAPHY

1. R. Baer, *Radical ideals*, Amer. J. Math. vol. 65 (1943) pp. 537-568.
2. M. Hall, *Projective planes*, Trans. Amer. Math. Soc. vol. 54 (1943) p. 537.
3. N. Jacobson, *Structure theory for algebraic algebras of bounded degree*, Ann. of Math. (2) vol. 46 (1945) pp. 695-707.
4. I. Kaplansky, *Rings with a polynomial identity*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 575-580.
5. J. Levitzki, *On the radical of a general ring*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 461-466.

HEBREW UNIVERSITY