THE WEIERSTRASS THEOREM IN FIELDS WITH VALUATIONS

IRVING KAPLANSKY[1]

In [2, Theorem 32][2] the author showed that an analogue of the Weierstrass-Stone theorem holds in topological rings having ideal neighborhoods of 0. Earlier, Dieudonné [1] had proved the Weierstrass-Stone theorem for the field of p-adic numbers. Now the field of p-adic numbers has an open subring (the p-adic integers) with ideal neighborhoods of 0. It seems plausible, therefore, to expect that the method of [2] will apply, provided one has a supplementary device for "getting into" the p-adic integers. This is in fact the case, and the result is applicable to any division ring with a valuation of rank one.[3] The requisite lemma reads as follows:

LEMMA 1. *Let F be a division ring with a valuation of rank one, and B its valuation ring. Let a be any nonzero element in F, and K a compact subset of F. Then there exists a (non-commutative) polynomial f with coefficients in F, without a constant term, and satisfying $f(a)=1$, $f(K)\subset B$.*

PROOF. Let P denote the maximal ideal in B, that is, the set of all elements x with $V(x)>0$, where V denotes the valuation. Let K' denote the subset of K with values less than $V(a)$; K' will again be compact. For any c in K' there is a compact open subset of K' containing c and contained in $c(1+P)$. Take a finite covering of K' consisting of such neighborhoods: say $U_1, \cdots, U_r$ with U_i contained in $c_i(1+P)$. Suppose the c's numbered so that $V(c_i)\geqq V(c_{i+1})$. Now $1-c_i^{-1}U_i$ is a compact subset of P, and consequently the values of its elements have a positive lower bound α_i. Choose integers $n(1), \cdots, n(r)$ in succession large enough so that

$$V(a^{-1}c_i) + \sum_{j=1}^{i-1} n(j)V(c_j^{-1}c_i) + n(i)\alpha_i \geqq 0$$

for $i=1, \cdots, r$. Then the polynomial

$$f(x) = 1 - (1 - a^{-1}x)(1 - c_1^{-1}x)^{n(1)} \cdots (1 - c_r^{-1}x)^{n(r)}$$

satisfies the requirements of the lemma.

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[1] John Simon Guggenheim Memorial Fellow.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] That is to say, a valuation whose value group is a subgroup of the real numbers.

We shall restate [2, Theorem 32] in a slightly sharpened form.

LEMMA 2. *Let A be a topological ring with unit element and ideal neighborhoods of* 0, *let X be a totally disconnected locally compact Hausdorff space, and C the ring of all continuous functions from X to A vanishing at ∞. Topologize C by uniform convergence. Let D be a closed subring of C, containing for any distinct points $x, y \in X$ and any $a, b \in A$, a function f with $f(x)=a$, $f(y)=b$. Then $D=C$.*

PROOF. Let U be a fixed ideal neighborhood of 0, $1 \notin U+U$. Let K be a compact subset of X and x a point not in K. Then D contains a function f with $f(x)=1$ and $f(y)=0$ for a given y in K. The function f will take values in U in a suitable neighborhood of y. A finite number of these neighborhoods cover K; if g is the product of the corresponding f's, we have $g(x)=1$, $g(K) \subset U$.

We start again with an arbitrary $z \in X$ and an h in D with $h(z)=1$. There is a compact neighborhood L of z with $h(L) \subset 1+U$, and a larger compact set M such that h is in U in the complement of M. For any given w in $M-L$ we can, by the preceding paragraph, find a function p in D with $p(L) \subset U$, $p(w)=h(w)$. Then the function $h(h-p)$ has the following properties: in L its values lie in $1+U$, and it vanishes at w and accordingly lies in U in a neighborhood of w. A finite number of these neighborhoods cover the compact set $M-L$. The product of the corresponding elements $h(h-p)$ gives us an element q in D with $q(L) \subset 1+U$, $q(L') \subset U$, L' the complement of L. By combining such elements we can get "within U" of the characteristic function of any compact open set. Since D is closed, it actually contains all characteristic functions of compact open sets, from which it follows readily that $D=C$.

As an immediate consequence of Lemmas 1 and 2, we have the following generalization of Dieudonné's theorem:

THEOREM. *Let F be a division ring with a valuation of rank one, X a totally disconnected locally compact Hausdorff space, C the ring of all continuous functions from X to F vanishing at ∞. Topologize C by uniform convergence. Let D be a closed subring of C, admitting left-multiplication by the constant functions, and containing for any two distinct points $x, y \in X$ a function vanishing at x but not at y. Then $D=C$.*

BIBLIOGRAPHY

1. J. Dieudonné, *Sur les fonctions continues p-adiques*, Bull. Sci. Math. (2) vol. 68 (1944) pp. 79–95.

2. I. Kaplansky, *Topological rings*, Amer. J. Math. vol. 69 (1947) pp. 153–183.

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