$$\{\pi(x) - \pi(x^{1/2})\} \log x^{1/2} \leqq \theta(x) \leqq \pi(x) \log x,$$

Tschebyschef's theorem

$$\alpha \frac{x}{\log x} \leqq \pi(x) \leqq \gamma \frac{x}{\log x}$$

now follows at once.

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THE RADICAL OF A NON-ASSOCIATIVE RING

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In this paper a definition is proposed for the radical of a non-associative ring. Our results are somewhat similar to those given for algebras by Albert in [3],[1] but the difficulties that arose in the earlier theory from absolute divisors of zero have been overcome. With slight modifications, the present proofs are applicable to algebras.

A *non-associative ring* $\mathfrak{R}$ is an additive abelian group closed under a product operation with respect to which the two distributive laws hold. Multiplication on the right (left) by a fixed element $x\in\mathfrak{R}$ determines an endomorphism R_x (L_x) of $\mathfrak{R}$ as an additive group. For $x, y\in\mathfrak{R}$,

$$x\cdot y = xR_y = yL_x.$$

The R_x and L_y generate an associative ring $\mathfrak{A}$ called the *transformation ring* of $\mathfrak{R}$. Clearly $\mathfrak{R}$ can be construed as a representation space for $\mathfrak{A}$, and this representation is faithful. The two-sided ideals of $\mathfrak{R}$, which are defined as for associative rings, are exactly the $\mathfrak{A}$-subspaces of $\mathfrak{R}$. The theory of ring homomorphisms goes over intact to the non-associative case.

A nonzero element $a\in\mathfrak{R}$ is called an *absolute divisor of zero* if $a\cdot x=x\cdot a=0$ for all $x\in\mathfrak{R}$. If $\mathfrak{A}$ has a unit element I and $\mathfrak{R}$ contains no absolute divisors of zero, then the unit element I is the identity

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[1] Numbers in brackets refer to the references cited at the end of the paper.

mapping on $\mathfrak{R}$. For if $aI=b$ where $a, b\in\mathfrak{R}$, then $aI-b=0=(aI-b)I=(a-b)I$, $(a-b)\mathfrak{A}=0$ and so $a=b$.

We now assume that the minimum condition holds in $\mathfrak{A}$. It has been shown in [5, Theorems 3 and 14] that for the case of simple rings this is equivalent to the assumption that $\mathfrak{R}$ can be regarded as an algebra of finite dimension over a certain field.

THEOREM 1. *Let* $\mathfrak{A}$ *satisfy the minimum condition and* $\mathfrak{R}$ *contain no absolute divisors of zero. Then* $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_n$, *where the* $\mathfrak{R}_i$ *are simple if and only if* $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n$ *where the* $\mathfrak{A}_i$ *are simple, and conversely.*

PROOF. If $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_n$ where the $\mathfrak{R}_i$ are simple, then $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n$ where $\mathfrak{A}_i$ is the transformation ring for $\mathfrak{R}_i$. Since $\mathfrak{R}_i$ is an irreducible $\mathfrak{A}_i$-space, it follows that $\mathfrak{A}_i$ is simple. Conversely, suppose $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n$ where the $\mathfrak{A}_i$ are simple rings generated by pairwise orthogonal idempotents E_i in the center of $\mathfrak{A}$. Clearly $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_n$, where $\mathfrak{R}_i=\mathfrak{R}E_i$. It is easily seen that $\mathfrak{A}_i$ is the transformation ring for $\mathfrak{R}_i$. If $\mathfrak{R}_i$ has a proper ideal $\mathfrak{M}$, then $\mathfrak{R}_i=\mathfrak{M}\oplus\mathfrak{N}$, a direct sum of $\mathfrak{A}_i$-subspaces since $\mathfrak{A}_i$ is semisimple. Let $\mathfrak{S}$ be the ideal of $\mathfrak{A}_i$ which annihilates $\mathfrak{M}$. For every nonzero element x in $\mathfrak{N}$, R_x and L_x, at least one of which must be nonzero, are in $\mathfrak{S}$. Clearly E_i is not in $\mathfrak{S}$ and so $\mathfrak{S}$ is a proper ideal of $\mathfrak{A}_i$, which is impossible. Therefore $\mathfrak{R}_i$ is simple.

COROLLARY. *Under the assumptions of Theorem* 1, $\mathfrak{R}$ *is simple if and only if* $\mathfrak{A}$ *is simple.*

The ring $\mathfrak{R}$ is defined to be *semisimple* if the following conditions are satisfied:

(i) $\mathfrak{A}$ satisfies the minimum condition.

(ii) $\mathfrak{R}$ contains no absolute divisors of zero.

(iii) $\mathfrak{R}$ is a direct sum of simple rings.

We now eliminate the restriction that $\mathfrak{R}$ contain no absolute divisors of zero. Suppose $\mathfrak{R}$ has an ideal $\mathfrak{H}$. Let $\mathfrak{S}$ be the set consisting of all $S\in\mathfrak{A}$ such that $\mathfrak{R}S\subseteq\mathfrak{H}$. Clearly $\mathfrak{R}\mathfrak{A}\mathfrak{S}\subseteq\mathfrak{R}\mathfrak{S}\subseteq\mathfrak{H}$ and $\mathfrak{R}\mathfrak{S}\mathfrak{A}\subseteq\mathfrak{H}\mathfrak{A}\subseteq\mathfrak{H}$. Furthermore, $\mathfrak{S}$ is an additive subgroup and so an ideal of $\mathfrak{A}$. Let $[x]$ and $[y]$ be any two residue classes of $\mathfrak{R}-\mathfrak{H}$. Then $[x]\cdot[y]=[x\cdot y]=[xR_y]=[yL_x]$. If $[xU]=[xV]$ for all $[x]\in\mathfrak{R}-\mathfrak{H}$ where $U, V\in\mathfrak{A}$, then $U-V\in\mathfrak{S}$ and so right (left) multiplication by a fixed element of $\mathfrak{R}-\mathfrak{H}$ determines a unique residue class of $\mathfrak{A}-\mathfrak{S}$. Then it is easy to prove that $\mathfrak{A}-\mathfrak{S}$ is isomorphic to the transformation ring for $\mathfrak{R}-\mathfrak{H}$. It follows immediately that if $\mathfrak{A}$ satisfies the minimum condition, then so does the transformation ring for any difference ring of $\mathfrak{R}$.

Let $\mathfrak{M}_1$ be the ideal consisting of zero and all absolute divisors of zero in $\mathfrak{R}$. Continuing by induction, let $\mathfrak{M}_{i+1}$ be the set consisting of all $x \in \mathfrak{R}$ such that $a \cdot x$ and $x \cdot a \in \mathfrak{M}_i$ for all $a \in \mathfrak{R}$. The $\mathfrak{M}_i$ are ideals of $\mathfrak{R}$ and $\mathfrak{M}_i \subseteq \mathfrak{M}_{i+1}$.[2]

LEMMA 1. *$\mathfrak{M}_{j+1}$ is the set consisting of all $x \in \mathfrak{R}$ such that R_x and L_x are left-annihilators of $\mathfrak{A}^j$.*

PROOF. Suppose $x \in \mathfrak{M}_{j+1}$. Then aR_x and $aL_x \in \mathfrak{M}_j$ for all $a \in \mathfrak{R}$. Each succeeding application of transformation in $\mathfrak{A}$ gives rise to an element in the preceding $\mathfrak{M}_i$. Clearly $aR_xT_1T_2 \cdots T_j = aL_xT_1T_2 \cdots T_j = 0$ for all $a \in \mathfrak{R}$, where $T_1, T_2, \cdots, T_j$ are arbitrary elements of $\mathfrak{A}$. Therefore $R_xT_1T_2 \cdots T_j = L_xT_1T_2 \cdots T_j = 0$ and so R_x and L_x are left annihilators of $\mathfrak{A}^j$. Conversely, if $aR_xT_1T_2 \cdots T_j = aL_xT_1T_2 \cdots T_j = 0$ for all $a \in \mathfrak{R}$, where $T_1, T_2, \cdots, T_j$ are arbitrary elements of $\mathfrak{A}$, then $aR_xT_1T_2 \cdots T_{j-1}$ and $aL_xT_1T_2 \cdots T_{j-1}$ are in $\mathfrak{M}_1$ by definition of $\mathfrak{M}_1$. Continuing in this way, it is easily seen that aR_x and aL_x, that is, $a \cdot x$ and $x \cdot a$ are in $\mathfrak{M}_j$ for all $a \in \mathfrak{R}$ and so $x \in \mathfrak{M}_{j+1}$.

LEMMA 2. *If $\mathfrak{A}$ satisfies the minimum condition, then there exists a least integer l such that $\mathfrak{M}_l = \mathfrak{M}_{l+1}$.*

PROOF. Let k be the first integer for which $\mathfrak{A}^{k-1} = \mathfrak{A}^k$. The existence of k is insured by the minimum condition on $\mathfrak{A}$. Suppose x is an element of $\mathfrak{M}_{k+1}$ not in $\mathfrak{M}_k$. By Lemma 1, R_x and L_x would be left-annihilators of $\mathfrak{A}^k$ but not both of $\mathfrak{A}^{k-1}$. This is impossible, and so $\mathfrak{M}_{k+1} = \mathfrak{M}_k$. Therefore k is an upper bound for l and the lemma is proved.

In particular, if $\mathfrak{A}$ has a unit element, then $\mathfrak{M}_2 = \mathfrak{M}_1$, since $\mathfrak{A}$ can have no left-annihilators.

THEOREM 2. *Let $\mathfrak{A}$ satisfy the minimum condition. Then $\mathfrak{R} - \mathfrak{M}_l$ has no absolute divisors of zero. Furthermore, $\mathfrak{M}_l$ is contained in every ideal $\mathfrak{H}$ for which $\mathfrak{R} - \mathfrak{H}$ has no absolute divisors of zero.*

PROOF. If $a \cdot x$ and $x \cdot a \in \mathfrak{M}_l$ for all $a \in \mathfrak{R}$, then $x \in \mathfrak{M}_{l+1} = \mathfrak{M}_l$. This proves the first part. Suppose $\mathfrak{R} - \mathfrak{H}$ has no absolute divisors of zero. Clearly $\mathfrak{M}_1 \subseteq \mathfrak{H}$. Now assume $\mathfrak{M}_i \subseteq \mathfrak{H}$. If $x \in \mathfrak{M}_{i+1}$, then both $a \cdot x$ and $x \cdot a \in \mathfrak{M}_i \subseteq \mathfrak{H}$ for all $a \in \mathfrak{R}$ and so $x \in \mathfrak{H}$. Therefore $\mathfrak{M}_l \subseteq \mathfrak{H}$.

LEMMA 3. *If $\mathfrak{A}$ satisfies the minimum condition and $\mathfrak{R} - \mathfrak{H}$ is semi-*

[2] In the case of Lie rings, the $\mathfrak{M}_i$ constitute the upper central chain.

simple, then $\mathfrak{R}\mathfrak{N}\subseteq\mathfrak{H}$ *where* $\mathfrak{N}$ *is the radical of* $\mathfrak{A}$.

PROOF. The transformation ring for $\mathfrak{R}-\mathfrak{H}$ is isomorphic to $\mathfrak{A}-\mathfrak{S}$ where $\mathfrak{S}$ is the ideal consisting of all $S\in\mathfrak{A}$ such that $\mathfrak{R}S\subseteq\mathfrak{H}$. By Theorem 1, $\mathfrak{A}-\mathfrak{S}$ is semisimple. Therefore $\mathfrak{N}\subseteq\mathfrak{S}$, and so $\mathfrak{R}\mathfrak{N}\subseteq\mathfrak{R}\mathfrak{S}\subseteq\mathfrak{H}$.

We now consider the ideal $\mathfrak{M}_l$ for the ring $\mathfrak{R}-\mathfrak{R}\mathfrak{N}$. The existence of this ideal follows from Lemma 2 since the transformation ring for $\mathfrak{R}-\mathfrak{R}\mathfrak{N}$ satisfies the minimum condition. Let $\mathfrak{M}$ be the complete reciprocal image of $\mathfrak{M}_l$ under the natural homomorphism $\mathfrak{R}\rightarrow\mathfrak{R}-\mathfrak{R}\mathfrak{N}$. The ideal $\mathfrak{M}$ will be call d the *radical* of $\mathfrak{R}$. Since $\mathfrak{R}-\mathfrak{M}\cong(\mathfrak{R}-\mathfrak{R}\mathfrak{N})-\mathfrak{M}_l$, it follows from Theorem 2 that $\mathfrak{R}-\mathfrak{M}$ has no absolute divisors of zero. The transformation ring for $\mathfrak{R}-\mathfrak{M}$ is isomorphic to $\mathfrak{A}-\mathfrak{S}$, where $\mathfrak{S}$ is the ideal consisting of all $S\in\mathfrak{A}$ such that $\mathfrak{R}S\subseteq\mathfrak{M}$. Clearly $\mathfrak{N}\subseteq\mathfrak{S}$ and so $\mathfrak{A}-\mathfrak{S}$ is semisimple. Theorem 1 then implies

THEOREM 3. *Let* $\mathfrak{A}$ *satisfy the minimum condition. Then* $\mathfrak{R}-\mathfrak{M}$ *is semisimple.*

That $\mathfrak{M}$ is the minimal ideal having this property is shown by

THEOREM 4. *If* $\mathfrak{A}$ *satisfies the minimum condition and* $\mathfrak{R}-\mathfrak{H}$ *is semisimple for some ideal* $\mathfrak{H}$, *then* $\mathfrak{M}\subseteq\mathfrak{H}$.

PROOF. By Lemma 3, $\mathfrak{R}\mathfrak{N}\subseteq\mathfrak{H}$. Clearly $(\mathfrak{R}-\mathfrak{R}\mathfrak{N})-(\mathfrak{H}-\mathfrak{R}\mathfrak{N})\cong\mathfrak{R}-\mathfrak{H}$ and so by Theorem 2, $\mathfrak{M}_l\subseteq\mathfrak{H}-\mathfrak{R}\mathfrak{N}$, which implies $\mathfrak{M}\subseteq\mathfrak{H}$.

The writer is indebted to D. C. Murdoch for suggesting Lemma 1.

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