A REMARK ON A THEOREM OF MARSHALL HALL

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Marshall Hall [2][1] proved that the identity

$$x(yz - zy)^2 = (yz - zy)^2 x \tag{1}$$

characterizes quaternion algebras among associative but not commutative division rings. Our remark is that (1) characterizes Cayley-Dickson algebras among alternative but not associative division rings. This follows from Hall's proof and a result of A. A. Albert [1]. A theorem of R. D. Schafer [4, Theorem 4] permits us to conclude that (1) and Hall's Theorem L [2] (as universal) ensure that the coordinate ring of a projective plane is uniquely defined irrespective of the coordinate system.

It is easy to verify (1) in a Cayley-Dickson algebra. We should like to sketch a proof of the converse, which is independent of Albert's result in [1]. Let A be an alternative division ring and let F be the set of all elements $c \in A$ which satisfy $cx = xc$ for every $x \in A$. Then F is a field[2] and, when 3 is nonzero in A, F is the center of A [3; 5, Lemma 9]. If A satisfies (1), the proof of Hall's Lemma 1 [2, p. 262] yields $x^2 = t(x)x - n(x)$ for every $x \in A$ not in F, where $t(x), n(x) \in F$. Define $t(c) = 2c$, $n(c) = c^2$ for $c \in F$.

When 3 is zero in A and A satisfies (1), F is still the center of A as we shall now show.[3] Since 2 is a nonzero in A, for each $x \in A$ we have $(x+c)^2 \in F$ for some $c \in F$. If, then, c_1, c_2 are in F, we have $(c_1, c_2, (x+c)^2) = 0$. By use of some identities of Zorn [6, (1.6) and (1.8); 5, (1)] we find that $(c_1, c_2, (x+c)^2) = 2(x+c)(c_1, c_2, x+c) = 2(x+c) \cdot (c_1, c_2, x)$. We infer that $(c_1, c_2, x) = 0$ for every c_1, c_2 in F and every x in A. Now let $c \in F$ and $x, y \in A$. As before, $(x+c')^2 \in F$ for some $c' \in F$ and

$$0 = (c, (x + c')^2, y) = 2(x + c')(c, x + c', y) = 2(x + c')(c, x, y),$$

so that $(c, x, y) = 0$ as desired.

To complete the argument, we first prove (i) $t(x+y) = t(x) + t(y)$,

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] To verify that $c, c' \in F$ yields $cc' \in F$ even if $3 = 0$ in F, we may use the equations $(x, c, c') = (c', x, c) = (c, c', x)$, where $(x, y, z) = x(yz) - (xy)z$ is the *associator* of x, y, and z.

[3] We are indebted to the referee for his observation that our original proof was incomplete in this case and also for simplifying our amended version.

(ii) $t(cx)=ct(x)$, and (iii) $n(xy)=n(x)n(y)$ for every x, $y \in A$ and every $c \in F$, and then we apply the arguments of R. D. Schafer [4]. We indicate the proof of (i) and (iii) when $xy \neq yx$. We compute $(x+y)^2 = x^2+xy+yx+y^2 = t(x+y)(x+y)-n(x+y)$, and we obtain (i) by taking the commutator with x and then (iii) by multiplying by xy on the left.

Addendum (October 25, 1949). When $1+1=0$ in A, we cannot obtain our result from Albert's Theorem 1 of [7] which employs $1+1\neq 0$. In this case we must complete the proof of (i)–(iii) by using Lemma 2 of [2, p. 262] and some straightforward but rather lengthy calculations. In fact, we may regard our proof as establishing Albert's Theorem 1 even when $1+1=0$. First redefine *quadratic algebra* by deleting the coefficient 2 in $f(\xi, x)$. Then assume that A is quadratic and alternative and observe that A is a division algebra. Unless A is commutative and hence a field over F, one may use the arguments of Hall [2, pp. 262–263] to see that the center of A is F. For, if $xy \neq yx$, then $Q=F[1, x, y, xy]$ is a quaternion algebra over F, and the requirement that $c \in A$ commute and associate with the elements of Q implies that $c \in Q$ and hence that $c \in F$, since the center of Q is F. We are now in a position to apply the arguments of the present note.

References

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