A THEOREM ON THE EXTENSION OF A RECTANGULAR MATRIX OF CONTINUOUS FUNCTIONS TO BECOME A NONSINGULAR SQUARE MATRIX

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THEOREM. *If the elements $a_{ij}(t)$ ($i=1, \dots, p < q$; $j=1, \dots, q$) of a $p \times q$ rectangular matrix A are continuous functions of n variables¹ $t_1, \dots, t_n$ on a closed region R which is homeomorphic to a closed n -cell and A has maximum rank everywhere on R , then there exist polynomials $b_{kj}(t)$ ($k=p+1, \dots, q$) in t such that the matrix*

$$M = \begin{pmatrix} a_{ij}(t) \\ b_{kj}(t) \end{pmatrix}$$

is nonsingular everywhere on the closed region R .

There exists a homeomorphic mapping $u = \phi(t)$ ($u = u_1, \dots, u_n$) of R onto a closed n -dimensional cube K with one vertex at the origin and with the n coordinate axes as edges. The matrix $B = (a_{ij}(\phi^{-1}(u)) = \alpha_{ij}(u))$ has maximum rank everywhere on K and its elements are continuous on K . If the functions $\beta_1(u), \dots, \beta_q(u)$ are continuous solutions of the equations

$$(1) \quad \sum_{j=1}^q \alpha_{ij}(u) \beta_j(u) = 0 \quad (i = 1, \dots, p)$$

which do not vanish simultaneously on K , the matrix

$$C = \begin{pmatrix} \alpha_{ij}(u) \\ \beta_j(u) \end{pmatrix}$$

has maximum rank everywhere on K .

At any point $\bar{u}$ on K at least one of the $p \times p$ determinants of the matrix B is different from zero in a neighborhood of $\bar{u}$. Consequently the cube K can be divided into a finite number of cubes k_s , the length of whose edges are equal and whose edges are parallel to the n coordinate axes, so that for each cube k_s at least one $p \times p$ determinant of the matrix B does not vanish on the closed cube k_s . If the proper $q-p$ β 's are arbitrarily chosen on one of the cubes k_s so that they are continuous, the remaining β 's will be uniquely determined and con-

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¹ For the special case where $n=1$, see G. A. Bliss *The problem of Mayer with variable end-points*, Trans. Amer. Math. Soc. vol. 19 (1918) p. 312.

tinuous on k_i . At any point $\bar{u}$ on the cube K , more than one $p \times p$ determinant may be different from zero and corresponding to each of these determinants there is a different set of arbitrary β 's. If any one of these sets of arbitrary β 's vanish simultaneously at $\bar{u}$, all of the β 's must vanish at $\bar{u}$.

Starting with the small cube which has a vertex at the origin, denote this closed cube by k_1 and choose the arbitrary β 's for this cube so that they are continuous and do not vanish simultaneously on k_1 . Next choose any cube k_2 which has a common face with k_1 . By the preceding argument, the arbitrary β 's for k_2 can be extended as constants along straight lines through the origin. When the arbitrary β 's for k_2 have been extended in this manner, the β 's are uniquely determined, continuous, and do not vanish simultaneously on $k_1 + k_2$. This process can be continued by choosing the cubes in such an order that the distance of k_e from the origin is equal to or less than that for k_f where $e < f$. Then the arbitrary β 's for any cube can be extended as straight lines through the origin. Thus it is possible to find solutions of (1) which do not vanish simultaneously on K .

Since the matrix C has maximum rank everywhere on K , this process can be continued until a $q \times q$ matrix

$$D(u) = \begin{pmatrix} \alpha_{ij}(u) \\ \beta_{kj}(u) \end{pmatrix}$$

is obtained which is nonsingular everywhere on K . The matrix

$$D(\phi(t)) = \begin{pmatrix} a_{ij}(t) \\ \beta_{kj}(\phi(t)) \end{pmatrix}$$

is nonsingular everywhere on R and the functions $\beta_{kj}(\phi(t))$ are continuous on R . The functions $\beta_{kj}(\phi(t))$ can be approximated by polynomials $b_{kj}(t)$ so that the matrix M is nonsingular everywhere on R .