ON LACUNARY DIRICHLET SERIES

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The following theorem is suggested by a result of S. Mandelbrojt [2, p. 101][1] concerning lacunary Fourier series.

THEOREM 1. *Let* $f(s)=\sum_{k=1}^{\infty} a_k e^{-\lambda_k s}$, *where* $0<\lambda_1<\lambda_2<\lambda_3<\cdots$, $\lim_{k\to\infty}\lambda_k=\infty$. *We denote by* γ_c *the abscissa of convergence of this series and by* γ_a *the abscissa of analyticity. It is assumed that* $\gamma_c<\infty$, $\gamma_a>-\infty$. *Let* $\nu<1$ *be the exponent of convergence of the* λ_k's *and suppose that*

$$|f(s_a+\sigma)| = O\left[e^{-(1/\sigma)^{\mu}}\right] \qquad (\sigma \to 0+)$$

where $s_a=\gamma_a+i\tau_a$. *Then if* $\mu>\nu/(1-\nu)$, $f(s)\equiv 0$.

We shall in what follows prove a more general theorem including Theorem 1 as a special case. The methods of the present paper are closely related to and in part derived from the work of L. Schwartz [4]. However no appeal is made to other than standard theorems of analysis.

Let $m(0)=0$ and let $m(\sigma)$ be an increasing function defined for $0\leqq\sigma\leqq a$, $a>0$. A function $f(s)$ which we may suppose analytic in the half-plane $\sigma>\gamma_1$ is said to have a zero of modular order $m(\sigma)$ at $s_1=\gamma_1+i\tau_1$ if

$$|f(s_1+\sigma)| \leqq m(\sigma) \qquad (0<\sigma\leqq b)$$

for some $b>0$. Let us define σ as a function of ρ, $\sigma=\eta(\rho)$, by the equation $e^{-\rho\sigma}=m(\sigma)$. Since $m(\sigma)$ decreases as σ decreases to 0 this definition is effective. It is $\eta(\rho)$ which we shall use as the measure of the zero of $f(s)$. As an example, if $m(\sigma)=\exp[-(\sigma^{-\mu})]$ then $\eta(\rho)=\rho^{-1/(\mu+1)}$.

As the measure of the degree of lacunarity of our Dirichlet series we introduce

$$\zeta(\rho) = \sum_{k=1}^{\infty} \log\left(1+\frac{2\rho}{\lambda_k}\right).$$

If the sequence $\{\lambda_k\}_1^{\infty}$ has exponent of convergence ν, then by a standard theorem on integral functions [5, p. 251],

$$\zeta(\rho) = O(\rho^{\nu+\epsilon}) \qquad (\rho \to +\infty).$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

We may now state our principal theorem.

THEOREM 2. *Let* $0<\lambda_1<\lambda_2<\cdots, \sum_1^\infty \lambda_k^{-1}<\infty$, *and let* $\zeta(\rho)$ *be defined as above. Let*

$$f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s}$$

have abscissa of convergence $\gamma_c<\infty$ *and abscissa of analyticity* $\gamma_a>-\infty$, *and let* $\gamma_a+i\tau_a$ *be a zero of* $f(s)$ *of modular order* $m(\sigma)$. *Then if* $\eta(\rho)$ *is defined as above and if*

$$\liminf_{\rho\to\infty} \zeta(\rho) - \rho\eta(\rho) = -\infty, \tag{1}$$

we have $f(s)\equiv 0$.

In the case considered in Theorem 1 we have, as we have seen, $\zeta(\rho)=O(\rho^{\nu+\epsilon})$, $\eta(\rho)=\rho^{-1/(\mu+1)}$. The assumption $\mu>\nu/(1-\nu)$ shows that condition (1) is satisfied. Thus Theorem 2 does include Theorem 1.

It is clearly sufficient to prove our theorem for $\gamma_a=\tau_a=0$.

It is immediately verifiable that the function

$$\frac{1}{(1+z)} \prod_{k=1,k\neq n}^{\infty} \left(\frac{\lambda_k+\rho-z}{\lambda_k+\rho+z}\right) = k_n(z,\rho) \qquad (\rho>0)$$

is analytic for $x\geqq 0$, and that

$$\int_{-\infty}^{\infty} |k_n(x+iy,\rho)|^2 dy \leqq \pi \qquad (x\geqq 0).$$

By a simple extension of Plancherel's theorem, see [3, p. 8], we see that if

$$\phi(n,\rho,\sigma) = \underset{T\to\infty}{\overset{(2)}{\text{l.i.m.}}} \frac{1}{2\pi i}\int_{-iT}^{iT} \frac{k_n(z,\rho)}{k_n(\lambda_n+\rho,\rho)} e^{\sigma z} dz \qquad (0\leqq\sigma<\infty),$$

then

$$\int_0^\infty \phi(n,\rho,\sigma)e^{-\sigma z}d\sigma = \frac{k_n(z,\rho)}{k_n(\lambda_n+\rho,\rho)} \qquad (x>0). \tag{2}$$

Further

$$\|\phi(n,\rho,\sigma)\|_2 \equiv \left[\int_0^\infty |\phi(n,\rho,\sigma)|^2 d\sigma\right]^{1/2} \leqq [2^{1/2}k_n(\lambda_n+\rho,\rho)]^{-1}.$$

We assert that

(3) $$\limsup_{\rho\to\infty} \|\phi(n, \rho, \sigma)\|_2 e^{-\zeta(\rho)} < \infty.$$

We have

$$\|\phi(n, \rho, \sigma)\|_2 e^{-\zeta(\rho)}$$
$$= \left[\frac{1}{2^{1/2}} \prod_{k=1, k\neq n}^{\infty} \left(\frac{\lambda_n + \lambda_k + 2\rho}{\lambda_k - \lambda_n}\right)\right] [1 + \lambda_n + \rho] \left[\prod_{k=1}^{\infty}\left(\frac{\lambda_k}{\lambda_k + 2\rho}\right)\right]$$
$$= \frac{1}{2^{1/2}}\left[\prod_{k=1, k\neq n}^{\infty} \left(1 + \frac{\lambda_n}{\lambda_k + 2\rho}\right)\right]\left[\frac{1 + \lambda_n + \rho}{\lambda_n + 2\rho}\lambda_n\right]$$
$$\cdot\left[\prod_{k=1, k\neq n}^{\infty}\left(1 - \frac{\lambda_n}{\lambda_k}\right)\right]^{-1}$$
$$\sim \frac{\lambda_n}{2^{3/2}}\left[\prod_{k=1, k\neq n}^{\infty}\left(1 - \frac{\lambda_n}{\lambda_k}\right)\right]^{-1} \qquad (\rho \to \infty),$$

which proves our assertion.

We define

$$F(\rho, s) = e^{-\rho s} f(s).$$

We assert that if

$$\|F(\rho, \sigma)\|_2 \equiv \left[\int_0^{\infty} |F(\rho, \sigma)|^2 d\sigma\right]^{1/2},$$

then

(4) $$\limsup_{\rho\to\infty} \|F(\rho, \sigma)\|_2 e^{\rho\eta(\rho)} < \infty.$$

Let M be a bound for $f(\sigma)$ for $0 < \sigma < \infty$. We have

$$\|F(\rho, \sigma)\|_2 \leqq \left[\int_0^{\eta(\rho)} e^{-2\rho\sigma} |f(\sigma)|^2 d\sigma\right]^{1/2} + \left[\int_{\eta(\rho)}^{\infty} e^{-2\rho\sigma} |f(\sigma)|^2 d\sigma\right]^{1/2}$$
$$\leqq m(\eta(\rho)) + Me^{-\rho\eta(\rho)}$$
$$\leqq (M + 1)e^{-\rho\eta(\rho)},$$

which proves relation (4).

It follows from a well known theorem concerning restricted over-convergence of Dirichlet series, see [1, p. 141], that there exists an increasing sequence of integers l_k for which

$$\lim_{k\to\infty} \sum_{j=0}^{l_k} a_j e^{-\lambda_j \sigma} = f(\sigma)$$

uniformly for $\epsilon \leqq \sigma < \infty$ for any $\epsilon > 0$. Thus if $\epsilon > 0$,

$$\int_0^\infty \phi(n, \rho, \sigma)F(\sigma + \epsilon, \rho)d\sigma$$

$$= \lim_{k\to\infty} \int_0^\infty \phi(n, \rho, \sigma)e^{-\rho(\sigma+\epsilon)} \sum_{j=1}^{l_k} a_j e^{-\lambda_j(\sigma+\epsilon)}d\sigma.$$

By equation (2)

$$\int_0^\infty \phi(n, \rho, \sigma)e^{-\rho(\sigma+\epsilon)}a_j e^{-\lambda_j(\sigma+\epsilon)}d\sigma = \begin{cases} 0, & j \neq n, \\ e^{-\epsilon\rho-\epsilon\lambda_n}a_n, & j = n. \end{cases}$$

Hence

$$\int_0^\infty \phi(n, \rho, \sigma)F(\rho, \sigma + \epsilon)d\sigma = e^{-\epsilon\rho-\epsilon\lambda_n}a_n.$$

By Schwarz's inequality

$$|a_n| \leqq e^{\epsilon\rho+\epsilon\lambda_n}\|\phi(n, \rho, \sigma)\|_2\|F(\sigma + \epsilon, \rho)\|_2.$$

If ϵ is allowed to approach zero through positive values, we obtain

$$|a_n| \leqq \|\phi(n, \rho, \sigma)\|_2\|F(\sigma, \rho)\|_2.$$

Using equations (3) and (4) we see that

$$\log |a_n| \leqq \zeta(\rho) - \rho\eta(\rho) + C$$

where C is a constant which depends on n but not upon ρ. If we allow ρ to increase without limit, assumption (1) of Theorem 2 implies that

$$\log |a_n| = -\infty, \qquad \text{that is, } a_n = 0.$$

Since this holds for each $n = 1, 2, \cdots, f(s) \equiv 0$ and our theorem is proved.

We shall now prove that if assumption (1) is replaced by the weaker assumption

$$\liminf_{\rho\to\infty} c\zeta(\rho) - \rho\eta(\rho) = -\infty$$

where $c < 2^{-3/2}$, then Theorem 2 is false.

Let us take for the constants λ_k, $0 < a < b < 1 < 4 < 9 < \cdots$. As in Titchmarsh [5, p. 271] we find that

$$\log\left[\prod_1^\infty\left(1 + \frac{x}{k^2}\right)\right] \sim \pi x^{1/2} \qquad (x \to \infty). \tag{5}$$

It follows that

$$\zeta(\rho) = \log\left(1+\frac{2\rho}{a}\right) + \log\left(1+\frac{2\rho}{b}\right) + \sum_{k=1}^{\infty}\log\left(1+\frac{2\rho}{k^2}\right) \sim \pi(2\rho)^{1/2} \qquad (\rho\to\infty). \tag{6}$$

We define

$$f(s) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{sz}}{\left(1+\frac{z}{a}\right)\left(1+\frac{z}{b}\right)\prod_{k=1}^{\infty}\left(1+\frac{z}{k^2}\right)}\,dz. \tag{7}$$

Deforming the line of integration of this integral to $\mathrm{Rl}\, z = -(n+1/2)^2$ and allowing n to approach infinity we obtain for s real and positive an expansion of $f(s)$ into a Dirichlet series with exponents a, b, 1, 4, 9, $\cdots$ which then converges in the half-plane $\mathrm{Rl}\, s>0$. The details of this are left to the reader. Again deforming the line of integration, this time to $\mathrm{Rl}\, z=(\pi/2\sigma)^2$, and using equation (5), we see that if $\epsilon>0$ is fixed, then for $\sigma>0$ sufficiently small we have

$$|f(\sigma)| \leqq \exp\left[-\frac{1-2\epsilon}{4}\frac{\pi^2}{\sigma}\right]M(\sigma)$$

where

$$M(\sigma) = \frac{1}{2\pi}\int_{\mathrm{Rl}\, z=(\pi/2\sigma)^2}\left|\frac{dz}{\left(1+\frac{z}{a}\right)\left(1+\frac{z}{b}\right)}\right|$$

$$= o(1) \qquad (\sigma\to 0+).$$

Thus we may associate with $f(s)$ and the origin the modular order $m(\sigma)=\exp\,[-((1-2\epsilon)/4)(\pi^2/\sigma)]$. We immediately find that

$$\eta(\rho) = \frac{\pi}{2}(1-2\epsilon)^{1/2}\rho^{-1/2}. \tag{8}$$

Comparing equations (6) and (8), we deduce that Theorem 2 is false if $c<2^{-3/2}$, which is what we wished to show.

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ON THE ABSOLUTE CONVERGENCE OF TRIGONOMETRICAL SERIES

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1. THEOREM 1. *Suppose that a trigonometrical series*

$$\sum \rho_n \cos (nx - \alpha_n) \qquad (\rho_n \geqq 0,\ n = 1, 2, \cdots)$$

and its conjugate series

$$\sum \rho_n \sin (nx - \alpha_n)$$

are convergent absolutely at $x = x_0$ and $x = x_1$, respectively. If

$$x_1 - x_0 = p\pi/q \qquad (p/q \text{ irreducible})$$

where p is an integer positive, negative, or zero, and q is an odd integer, then

$$\sum \rho_n < \infty.$$

We shall see in Theorem 2 that the above theorem is no longer true if p/q is replaced by p'/q' with an even q' and $p' \neq 0$, or by an irrational number.

PROOF OF THEOREM 1. If we put $x_1 - x_0 = p\pi/q \equiv h$, then by Fatou's theorem[1]

$$\sum \rho_n \left| \cos (n(x_1 - sh) - \alpha_n) \right| < \infty \tag{1}$$

for every odd s. Hence from the identity

$$\sin (nx_1 - \alpha_n) = \cos nsh \sin (n(x_1 - sh) - \alpha_n) + \sin nsh \cos (n(x_1 - sh) - \alpha_n)$$

we deduce immediately that

$$\sum \rho_n \left| \cos nsh \sin (n(x_1 - sh)) - \alpha_n) \right| < \infty. \tag{2}$$

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[1] See, for example, A. Zygmund, *Trigonometrical series*, p. 134.