ON CONFORMAL MAPPING OF NEARLY CIRCULAR REGIONS

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1. Introduction. The following problem has repeatedly been considered in the literature on conformal mapping. Given a simply connected region R which contains the origin and whose boundary lies between two concentric circles about the origin of radii 1 and $1+\epsilon$ $(\epsilon>0)$ respectively. Suppose that the function $w=f(z)$ maps the circle $|z|<1$ conformally onto R in such a manner that $f(0)=0$ and $f'(0)>0$. It is desired to find: (a) an upper bound for the function $|f(z)-z|$ in terms of ϵ and r which holds for all z in a fixed circle $|z|\leqq r<1$, and (b) a bound which depends on ϵ only and is valid (possibly under additional assumption on the boundary of R) for all z in the complete (open or closed) unit circle. Estimates of the first kind were given by L. Bieberbach [**1**],[1] A. Ostrowski [**10**], C. Carathéodory [**4** and **5**], M. Lavrientieff and D. Kwasselava [**7**], and M. Müller [**9**]. Contributions to the second problem were made by L. Bieberbach [**3**], A. R. Marchenko [**8**], and J. Ferrand [**6**].

In the present paper we consider, more generally, *the integral means of order* $p>0$ of the function $|f(z)-z|$ taken along the circles $|z|=r$, namely

$$\mathfrak{M}_p\{f(z)-z\} = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})-re^{i\theta}\right|^p d\theta\right\}^{1/p}$$

and those of the first and second derivatives of this function. Bounds in terms of ϵ are obtained for these means which are valid uniformly for $r<1$. As an application of our estimate on the mean of the first derivative, we give a short proof of an inequality concerning $|f(z)-z|$ in $|z|\leqq 1$ which is due to A. R. Marchenko. Furthermore, a bound for the modulus of the derivative $f'(z)-1$ for z in $|z|\leqq 1$ is derived. In all cases the estimates obtained are of the form $O(\epsilon)$ as $\epsilon\to 0$ and are the best possible as far as the order of magnitude in ϵ is concerned.[2]

The knowledge of bounds for $|f(z)-z|$ and for some of the expressions considered in this paper is helpful in the application of certain iteration methods for determining the mapping function of a region onto a circle. For these inequalities may sometimes be used to ap-

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[1] The numbers in the brackets refer to the bibliography at the end of the paper.

[2] This is easily shown by simple examples such as $f(z)=((z-c\epsilon)/(1-c\epsilon z)+\epsilon)(1/(1-c\epsilon))$ where c is a sufficiently small positive constant.

praise the accuracy of an approximation to the mapping function; see, for example, L. Bieberbach [**1** and **2**, p. 10], and a more recent paper of the writer [**14**]. In fact, the inequalities (5.1) and (6.1) for $p=2$ of the present note were stated and used in that paper (§9d) for such a purpose. However, the proofs of these statements were omitted there and are included in the present discussion of such inequalities.

The proofs of the inequalities involving the means make use of the following well known theorem of M. Riesz [**11**] on conjugate harmonic functions:

Suppose that the function $f(z)=u(z)+iv(z)$ is regular for $|z|<1$ and that $u(0)=0$. Then for every $p>1$ there exists a positive number A_p which depends on p only, such that for $0\leqq r<1$

$$(1.1)\quad \mathfrak{M}_p\{u(re^{i\theta})\} \equiv \left\{\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})|^p d\theta\right\}^{1/p} \leqq A_p\mathfrak{M}_p\{v(re^{i\theta})\}.$$

If the right-hand side of (1.1) *is bounded for all r, $0\leqq r<1$, then $f(re^{i\theta})$ has radial boundary values of class L_p almost everywhere on $|z|=1$ and* (1.1) *holds for $r=1$. For $p\geqq 2$ all $A_p\leqq 2p$ and $A_2=1$.*

2. **The mean of the function** $f(z)-z$. We begin with the following theorem.

THEOREM I. *Let R be a simply connected region in the w-plane which contains the origin and whose boundary is contained in the ring*

$$(2.1)\qquad 1<|w|<1+\epsilon \qquad (\epsilon>0).$$

Suppose that the function $w=f(z)$ maps the circle $|z|<1$ conformally onto R and that $f(0)=0$ and $f'(0)>0$. Then for every[3] *$p>1$ and for $z=re^{i\theta}$, uniformly for $0\leqq r<1$,*

$$(2.2)\qquad \mathfrak{M}_p\{f(z)-z\} \leqq \mathfrak{M}_p\left\{\frac{f(z)}{z}-1\right\} \leqq (1+A_p)e^{\epsilon}\cdot\epsilon$$

where A_p is the constant of the Riesz theorem.[4]

[3] It is clear that the inequality (2.2) if proved for $p>1$ remains valid, for a suitable constant A_p, for all p, $0<p\leqq 1$ (Hölder's inequality). An analogous remark applies to the other estimates of the means derived in this paper, and we state and prove therefore Theorems I, II, III, and VI for $p>1$ only.

[4] By use of Cauchy's integral formula one may easily obtain from Theorem I a bound for $|f(z)-z|$ for all $|z|\leqq r<1$. If $t=\rho e^{i\theta}$, $r<\rho<1$, $|f(z)-z| \leqq 1/2\pi\int_0^{2\pi}|(f(t)-t)/(t-z)|d\theta \leqq \mathfrak{M}_p\{f(t)-t\}\mathfrak{M}_q\{(t-z)^{-1}\}$, by Hölder's inequality, where $1/p+1/q=1$. Hence, as $\rho\to 1$, $|f(z)-z|\leqq(1+A_p)e^{\epsilon}\epsilon\cdot(1-r)^{-1/p}$. However, this estimate is not the best possible as far as the order of $1/(1-r)$ is concerned, the best given in the literature being log $1/(1-r)$ (see [**5**, p. 51] and [**9**]).

PROOF. The function $F(z) \equiv f(z)/z$ for $0<|z|<1$, $F(0)=f'(0)$ is regular for $|z|<1$. Since the boundary of R is contained in the ring (2.1), a simple application of the principle of the maximum and minimum modulus shows that

$$1 \leqq |F(z)| \leqq 1+\epsilon \qquad \text{for } |z|<1.$$

Hence

$$0 \leqq \log |F(z)| \leqq \epsilon \qquad \text{for } |z|<1. \tag{2.3}$$

The branch of $\log F(z)=\log |F(z)| + i \arg F(z)$ for which $\arg F(0) = \arg f'(0)=0$ is single-valued and regular for $|z|<1$. By the theorem of M. Riesz stated above we obtain therefore for $z=re^{i\theta}$, $0 \leqq r<1$, and all $p>1$:

$$\mathfrak{M}_p\{\arg F(z)\} \leqq A_p \mathfrak{M}_p\{\log |F(z)|\} \leqq A_p \cdot \epsilon.$$

Hence, by use of Minkowski's inequality,

$$\mathfrak{M}_p\{\log F(z)\} \leqq \mathfrak{M}_p\{\log |F(z)|\} + \mathfrak{M}_p\{\arg F(z)\} \leqq (1+A_p)\epsilon. \tag{2.4}$$

Since for any complex number[5] a

$$|e^a - 1| \leqq |a| e^{|\Re a|},$$

we have (for $a=\log F(z)$) by (2.3)

$$\left|\frac{f(z)}{z} - 1\right| \leqq \left|\log \frac{f(z)}{z}\right| e^{\epsilon}$$

and therefore, by (2.4),

$$\mathfrak{M}_p\left\{\frac{f(z)}{z} - 1\right\} \leqq (1+A_p)e^{\epsilon} \cdot \epsilon.$$

3. **The mean of the function** $zf'(z)/f(z)-1$. Our next aim is to obtain an analogous result for the derivative of the mapping function. This will be accomplished essentially by the following theorem.

THEOREM II. *Let C be a closed Jordan curve represented (in polar coordinates) by the equation $\rho=\rho(\phi)$, $0 \leqq \phi \leqq 2\pi$, where $\rho(\phi)$ is positive, continuous, and moreover satisfies the inequality*

$$|\rho(\phi+\tau) - \rho(\phi)| \leqq \rho(\phi)\epsilon|\tau| \qquad (\textit{for some } \epsilon,\ 0<\epsilon<1). \tag{3.1}$$

Suppose that $w=f(z)$ maps the circle $|z|<1$ conformally onto the in-

[5] For, $|e^a-1| = |\int_0^a e^z dz| \leqq |\int_0^a |e^z||dz|| \leqq |a| e^{|\Re a|}$, the integration being taken along the straight line segment $0 \cdots a$.

terior of C *such that* $f(0)=0$ *and* $f'(0)>0$. *Then for all* $p>1$ *for which* $\epsilon A_p<1$ *and uniformly for* $0<r<1$,

$$\mathfrak{M}_p\left\{z\frac{f'(z)}{f(z)}-1\right\}\leqq\frac{1+A_p}{1-\epsilon A_p}\epsilon \qquad (z=re^{i\theta}). \tag{3.2}$$

PROOF. Just as above let $F(z)$ be defined as $f(z)/z$ for $z\neq 0$, and as $f'(0)$ for $z=0$, and let $\log F(z)$ denote the branch of the logarithm which is real when $z=0$. Then $F(z)$ and $\log F(z)$ are regular in $|z|<1$ and since C is a Jordan curve, they are continuous in the closed circle $|z|\leqq 1$. Thus in particular $Q(z)\equiv\arg F(z)$ is single-valued and continuous in $|z|\leqq 1$. We define a branch of $\arg f(e^{i\theta})$ by the relation

$$\arg f(e^{i\theta})=Q(e^{i\theta})+\theta.$$

If $g(\theta)$ is a function defined for all real θ and δ is a positive number, we shall find it convenient to use the notation

$$\Delta g(\theta)=g(\theta+\delta)-g(\theta).$$

Since by hypothesis (3.1) $\log\rho(\phi)$ possesses bounded difference quotients, $\rho'(\phi)/\rho(\phi)$ exists almost everywhere and

$$|\log\rho(\phi+\tau)-\log\rho(\phi)|=\left|\int_\phi^{\phi+\tau}\frac{\rho'(t)}{\rho(t)}dt\right|\leqq\epsilon|\tau|.$$

Hence—note that $|f(e^{i\theta})|=\rho(\arg f(e^{i\theta}))$—

$$|\Delta\log|f(e^{i\theta})||\leqq\epsilon|\Delta\arg f(e^{i\theta})|.$$

Now

$$\Delta\arg f(e^{i\theta})=\Delta\arg F(e^{i\theta})+\delta,$$

and therefore, by Minkowski's inequality, for $p>1$,

$$\mathfrak{M}_p\{\Delta\log|f(e^{i\theta})|\}\leqq\epsilon[\delta+\mathfrak{M}_p\{\Delta\arg F(e^{i\theta})\}]. \tag{3.3}$$

By the theorem of M. Riesz, applied to $\log F(e^{i\theta})$, we obtain

$$\mathfrak{M}_p\{\Delta\arg F(e^{i\theta})\}\leqq A_p\mathfrak{M}_p\{\Delta\log|F(e^{i\theta})|\}. \tag{3.4}$$

Hence, from (3.3)

$$\begin{aligned}\mathfrak{M}_p\{\Delta\log|F(e^{i\theta})|\}&=\mathfrak{M}_p\{\Delta\log|f(e^{i\theta})|\}\\&\leqq\epsilon[\delta+A_p\mathfrak{M}_p\{\Delta\log|F(e^{i\theta})|\}]\end{aligned}$$

or

$$(1-\epsilon A_p)\mathfrak{M}_p\{\Delta\log|F(e^{i\theta})|\}\leqq\epsilon\delta. \tag{3.5}$$

Since $(1-\epsilon A_p)>0$, we may divide (3.5) by $1-\epsilon A_p$. We also divide by δ and, combining the so obtained inequality with (3.4), we find

$$\mathfrak{M}_p\left\{\frac{\Delta \log F(z)}{\delta}\right\} \leqq \frac{1+A_p}{1-\epsilon A_p}\epsilon \qquad (z = e^{i\theta}). \tag{3.6}$$

Since $\log F(z)$ is regular for $|z|<1$ and continuous for $|z|\leqq 1$, the inequality holds also when $z=re^{i\theta}$ where $0<r\leqq 1$. Replace in (3.6) z by $re^{i\theta}$ and keep $r<1$ *fixed*. The right-hand side of (3.6) is independent of δ. We may let $\delta\to 0$, passing to the limit under the integral sign, and thus we find (3.2).

4. **Applications of Theorem II.** As an application of the preceding result we obtain the desired estimate for the mean of the derivative.

THEOREM III. *Let C be a closed Jordan curve represented by the equation $\rho=\rho(\phi)$, $0\leqq\phi\leqq 2\pi$, where $\rho(\phi)$ is positive, continuous, and satisfies the following additional conditions: For some ϵ, $0<\epsilon<1$,*

(i) $$1<\rho(\phi)\qquad <1+\epsilon,$$

(ii) $$|\rho(\phi+\tau)-\rho(\phi)|\leqq\rho(\phi)\epsilon|\tau|.$$

If $w=f(z)$ is defined as in Theorem II, *then uniformly for* $0\leqq r<1$, $p>1$, $\epsilon A_p<1$

$$\mathfrak{M}_p\{f'(z)-1\}\leqq 2\frac{1+A_p}{1-\epsilon A_p}e^{\epsilon}\cdot\epsilon \qquad (z = re^{i\theta}). \tag{4.1}$$

To prove this we write for $0<|z|<1$

$$\begin{aligned}|f'(z)-1| &\leqq \left|f'(z)-\frac{f(z)}{z}\right|+\left|\frac{f(z)}{z}-1\right| \\ &\leqq \left|\frac{f(z)}{z}\right|\left|\frac{zf'(z)}{f(z)}-1\right|+\left|\frac{f(z)}{z}-1\right|.\end{aligned}$$

Since hypothesis (i) implies that $|f(z)/z|\leqq 1+\epsilon$ (see §2) we obtain by use of Minkowski's inequality:

$$\mathfrak{M}_p\{f'(z)-1\}\leqq(1+\epsilon)\mathfrak{M}_p\left\{\frac{zf'(z)}{f(z)}-1\right\}+\mathfrak{M}_p\left\{\frac{f(z)}{z}-1\right\}.$$

Hence by Theorems II and I:

$$\mathfrak{M}_p\{f'(z)-1\}\leqq(1+\epsilon)\frac{1+A_p}{1-\epsilon A_p}\epsilon+2\epsilon e^{\epsilon}\leqq\frac{2(1+A_p)}{1-\epsilon A_p}e^{\epsilon}\cdot\epsilon.$$

Theorem III is a sharpening of the following result which was first stated by A. R. Marchenko [8] in 1935 and which follows easily from (4.1).[6]

THEOREM IV. *Under the hypotheses of Theorem* III, *there exists an* (*absolute*) *constant* K *such that for* $|z| \leqq 1$

$$|f(z) - z| \leqq K\epsilon. \tag{4.2}$$

To prove this one needs only to apply the Fejér-Riesz inequality, which states that for any function $g(z)$ regular for $|z| \leqq r$,

$$\int_{-r}^{r} |g(\rho e^{i\alpha})| \, d\rho \leqq \frac{r}{2}\int_0^{2\pi} |g(re^{i\theta})| \, d\theta \quad (0 \leqq \alpha \leqq 2\pi).$$

For, if $z = re^{i\alpha}$, $0 < r < 1$,

$$|f(z) - z|^2 \leqq \left\{\int_0^r |f'(\rho e^{i\alpha}) - 1| \, d\rho\right\}^2 \leqq \int_0^r |f'(\rho e^{i\alpha}) - 1|^2 d\rho$$

$$\leqq \frac{r}{2}\int_0^{2\pi} |f'(re^{i\theta}) - 1|^2 d\theta \leqq \pi \cdot \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta}) - 1|^2 d\theta$$

or

$$|f(z) - z| \leqq \pi^{1/2}\mathfrak{M}_2\{f'(z) - 1\} \leqq \frac{4e^{\epsilon}\pi^{1/2}}{1 - \epsilon}\epsilon \tag{4.3}$$

by (4.1)—note that $A_2 = 1$. Now, if $\epsilon \leqq 1/3$, the factor of ϵ in the last term of (4.3) does not exceed $6e^{1/3}\pi^{1/2}$, so that K in (4.3) may be taken equal to this number. If, however, $1/3 < \epsilon < 1$, then we make use of the fact that $|f(z) - z| \leqq 3 < 9\epsilon$ and choose $K = 9$. This proves (4.2).

5. **An estimate for the derivative of the mapping function.** The hypotheses of Theorem III are not sufficient to insure the existence of an inequality analogous to (4.2) for the derivative, $f'(z) - 1$ in the full circle $|z| < 1$, as under those assumptions $f'(z)$ need not even be bounded in $|z| < 1$. However, if we add a further condition (essentially involving the curvature of C) then we can prove the following:

THEOREM V. *Let C be a closed Jordan curve which satisfies hypotheses*

[6] Marchenko stated his theorem without proof. Another proof of this inequality, essentially based on the use of the Poisson integral, is given by E. J. Specht in a forthcoming paper [12]. In addition, bounds in terms of ϵ for the derivatives $|f'(z) - 1|$, $|f''(z)|, \cdots, |f^{(n)}(z)|$ which are valid in $|z| \leqq 1$ are derived in this paper. Specht's estimate for the first derivative is based on somewhat less restrictive assumptions than those of our Theorem V below, and yields a bound of the same order of magnitude in ϵ, if applied under the hypotheses of our Theorem V.

(i) *and* (ii) *of Theorem* III. *Furthermore, suppose that* $\rho'(\phi)$ *exists everywhere and that*[7]

$$\text{(iii)} \qquad \left|\frac{\rho'}{\rho}(\phi+\tau)-\frac{\rho'}{\rho}(\phi)\right| < \epsilon\,|\,\tau\,|,$$

ϵ *being the same as in* (i) *and* (ii). *Suppose that* $f(z)$ *is defined as in Theorem* III *and that* $\phi(\theta)=\arg f(e^{i\theta})$. *Then* $f'(z)$ *exists and is continuous in* $|z|\leqq 1$ *and*

$$(5.1) \qquad \frac{1}{A(1+\epsilon^2)^{1/2}} \leqq \phi'(\theta) \leqq A, \qquad 0 \leqq \theta \leqq 2\pi,$$

where $A=4^{\epsilon}e^{\epsilon^2}$ *and*

$$(5.2) \qquad |\,f'(z)-1\,| \leqq 2B(1+\epsilon)\cdot\epsilon+\epsilon, \qquad |\,z\,| \leqq 1,$$

where $B=A+(A-1)/\epsilon$.[8]

Proof of (5.1). Under the present hypotheses the mapping function $f(z)$ possesses a continuous nonvanishing derivative[9] $f'(z)$ for $|z| \leqq 1$. If $\log\,(f(z)/z)$ again denotes the branch of the logarithm which reduces to the real $\log f'(0)$ when $z=0$, then we have, for $z=e^{i\theta}$,

$$(5.3) \qquad \log\frac{f(e^{i\theta})}{e^{i\theta}} = \log \rho[\phi(\theta)] + i\{\phi(\theta)-\theta\}.$$

Differentiation with respect to θ leads to the relation

$$(5.4) \qquad \left\{\frac{f'(z)}{f(z)}-\frac{1}{z}\right\} iz = \frac{\rho'}{\rho}\,[\phi(\theta)]\phi'(\theta) + i\{\phi'(\theta)-1\}$$

or

$$(5.5) \qquad \left\{\frac{zf'(z)}{f(z)}\right\}_{z=e^{i\theta}} = \phi'(\theta) - i\frac{\rho'}{\rho}\,[\phi(\theta)]\phi'(\theta).$$

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[7] The notation $(\rho'/\rho)(\phi)$ means $\rho'(\phi)/\rho(\phi)$.

[8] In [**14**, § 9d], the hypothesis on $\rho(\phi)$ which corresponds to condition (i) of Theorem V of the present paper is $(1+\epsilon)^{-1}a \leqq \rho(\phi) \leqq a(1+\epsilon)$, where a is a positive constant. The constant $A(=e^{\epsilon^2}4^{\epsilon})$ is given there erroneously for the case proved here under the assumption (i), and it should be replaced by $A=e^{2\epsilon^2}4^{\epsilon}$. This is seen by an obvious modification of our proof of (5.1); only the relations (5.10) and (5.11) and the inequalities which follow from them must be changed.

[9] See, for example, [**13**, p. 327].

Since $f'(e^{i\theta})\neq 0$, it follows that $\phi'(\theta)\neq 0$. The function

$$\log\frac{zf'(z)}{f(z)} = \log\left|\frac{zf'(z)}{f(z)}\right| + i\arg\frac{zf'(z)}{f(z)} \tag{5.6}$$

which is defined as being 0 when $z=0$, is single-valued and analytic for $|z|<1$, and continuous for $|z|\leqq 1$. If we set

$$\omega(\phi) = -\arctan\frac{\rho'}{\rho}[\phi],$$

where the principal value of the arctangent is chosen, then, for $z=e^{i\theta}$,

$$\arg\frac{zf'(z)}{f(z)} = \omega[\phi(\theta)],\quad \log\left|\frac{zf'(z)}{f(z)}\right| = \log\{\phi'(\theta)\sec\omega[\phi(\theta)]\}. \tag{5.7}$$

By a well known formula for the conjugate function we have therefore

$$\log\{\phi'(\theta)\sec\omega[\phi(\theta)]\} = \frac{1}{2\pi}\int_0^{\pi}\{\omega[\phi(\theta+\tau)]-\omega[\phi(\theta-\tau)]\}\cot\frac{\tau}{2}\,d\tau.$$

Because of hypothesis (iii) it follows easily that

$$|\omega[\phi(\theta+\tau)]-\omega[\phi(\theta-\tau)]| \leqq \epsilon\{\phi(\theta+\tau)-\phi(\theta-\tau)\}.$$

Hence

$$|\log\{\phi'(\theta)\sec\omega[\phi(\theta)]\}| \leqq \frac{\epsilon}{2\pi}\int_0^{\pi}\{\phi(\theta+\tau)-\phi(\theta-\tau)\}\cot\frac{\tau}{2}\,d\tau,$$

or

$$\begin{aligned}|\log\{\phi'(\theta)\sec\omega\,[\phi(\theta)]\}| &\leqq \frac{\epsilon}{2\pi}\int_0^{\pi}\{\phi(\theta+\tau)-(\theta+\tau)-[\phi(\theta-\tau)-(\theta-\tau)]\}\cot\frac{\tau}{2}\,d\tau\\ &\quad+\frac{\epsilon}{\pi}\int_0^{\pi}\tau\cot\frac{\tau}{2}\,d\tau.\end{aligned} \tag{5.8}$$

The second term on the right-hand side of (5.8) has the value $2\epsilon\log 2$. Furthermore, since $\log\rho[\phi(\theta)]$ and $\phi(\theta)-\theta$ are real and imaginary parts of $\log(f(z)/z)$ for $z=e^{i\theta}$, and the real part of this function is $\log f'(0)$ when $z=0$, it follows, again by the formula for the conjugate function, that the first term on the right-hand side of (5.8) is equal to

$\epsilon\{\log \rho[\phi(\theta)]-\log f'(0)\}$. Thus

$$(5.9) \qquad \left|\log\{\phi'(\theta)\sec\omega[\phi(\theta)]\}\right| \leqq \epsilon\log\frac{\rho[\phi(\theta)]}{f'(0)}+2\epsilon\log 2.$$

By hypothesis (i) we have for $|z|=1$ and therefore also for $0<|z|<1$

$$(5.10) \qquad 1 \leqq \left|\frac{f(z)}{z}\right| \leqq 1+\epsilon$$

and thus, as $z\to 0$,

$$(5.11) \qquad 1 \leqq f'(0) \leqq 1+\epsilon.$$

We have therefore

$$\log\frac{\rho[\phi(\theta)]}{f'(0)} \leqq \log(1+\epsilon) < \epsilon.$$

Thus, we finally find from (5.9)

$$(5.12) \quad -\log\sec\omega[\phi(\theta)]-\epsilon^2-2\epsilon\log 2 \leqq \log\phi'(\theta) \leqq \epsilon^2+2\epsilon\log 2.$$

By hypothesis (ii)

$$(5.13) \qquad \begin{aligned} -\log\sec\omega[\phi(\theta)] &= \frac{1}{2}\log\frac{1}{1+((\rho'/\rho)[\phi(\theta)])^2} \\ &\geqq \frac{1}{2}\log\frac{1}{1+\epsilon^2}\cdot \end{aligned}$$

The inequalities (5.12) and (5.13) establish (5.1).

Proof of (5.2). From (5.4), hypothesis (ii), and (5.1) we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leqq \left|\frac{\rho'}{\rho}[\phi(\theta)]\right|\phi'(\theta)+|\phi'(\theta)-1| \leqq \epsilon A+|\phi'(\theta)-1|.$$

A simple calculation shows that $1-1/A(1+\epsilon^2)^{1/2}\leqq A-1$, so that, by (5.1), $|\phi'(\theta)-1|\leqq A-1$ and therefore

$$(5.14) \qquad \left|\frac{zf'(z)}{f(z)}-1\right| \leqq \epsilon A+A-1.$$

Next we write

$$|f'(z)-1| \leqq \left|f'(z)-\frac{f(z)}{z}\right|+\left|\frac{f(z)}{z}-f'(0)\right|+|f'(0)-1|$$

$$\leq \left|\frac{f(z)}{z}\right|\left|\frac{zf'(z)}{f(z)} - 1\right| + \int_0^1 \left|\frac{d}{dt}\left(\frac{f(t)}{t}\right)\right| |dt| + |f'(0) - 1|.$$

Using (5.10), (5.14), and (5.11) we find

$$|f'(z) - 1| \leq (1 + \epsilon)(A\epsilon + A - 1) + \epsilon + \int_0^1 \left|\frac{d}{dt}\left(\frac{f(t)}{t}\right)\right| |dt|. \tag{5.15}$$

Now the function

$$\frac{d}{dt}\left(\frac{f(t)}{t}\right) = \frac{f(t)}{t}\cdot\frac{1}{t}\cdot\left(\frac{tf'(t)}{f(t)} - 1\right)$$

is regular for $|t| < 1$ (including $t = 0$, if properly defined) and continuous for $|t| \leq 1$. But for $|t| = 1$ we have by (5.10) and (5.14)

$$\left|\frac{d}{dt}\left(\frac{f(t)}{t}\right)\right| \leq (1 + \epsilon)(A\epsilon + A - 1),$$

and by the principle of the maximum modulus this holds also for $|t| < 1$. Combining this result with (5.15) we obtain (5.2).

6. **The mean of the second derivative of the mapping function.** The assumptions made in Theorem V permit one to go beyond the statement proved on the first derivative of the mapping function and to obtain estimates for the mean of order p of the second derivative.

THEOREM VI. *Under the hypotheses of Theorem* V, $\phi'(\theta)$ *is an absolutely continuous function and for any* $p \geq 2$

$$\mathfrak{M}_p\{\phi''(\theta)\} \leq AM_p\epsilon \tag{6.1}$$

where $A = 4^\epsilon e^{\epsilon^2}$ *and* $M_p = A_p A^{1-1/p} \min(1+\epsilon, 2^{1-1/p})$, *and for* $p > 1$

$$\mathfrak{M}_p\left(\frac{f''(z)}{f'(z)}\right) \leq (A + B + AA_p)\epsilon, \qquad (z = re^{i\theta},\ 0 \leq r < 1) \tag{6.2}$$

where $B = A + (A-1)/\epsilon$.

PROOF OF (6.2). We consider as in equation (5.6) the branch of $\log (zf'(z)/f(z))$ which is defined as 0 when $z = 0$. Then $\arg (zf'(z)/f(z))$ is harmonic for $|z| < 1$ and continuous for $|z| \leq 1$ and by (5.7) arg $(zf'(z)/f(z)) = \omega[\phi(\theta)]$ for $z = e^{i\theta}$. Because of hypothesis (iii) and (5.1), $\omega[\phi(\theta)]$ possesses bounded difference quotients for all θ, so that $\omega'[\phi(\theta)]\phi'(\theta)$ exists for almost all θ and is bounded. This implies that the function $\partial \arg (zf'(z)/f(z))/\partial\theta$ may be represented by the Poisson integral for $|z| = |re^{i\theta}| < 1$. For, if we write

$$K(r, \theta - t) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)},$$

we have

$$\begin{aligned}\frac{\partial}{\partial\theta}\left[\arg \frac{zf'(z)}{f(z)}\right] &= \frac{\partial}{\partial\theta}\,\frac{1}{2\pi}\int_0^{2\pi} \omega[\phi(t)]K(r, \theta - t)dt \\ &= -\frac{1}{2\pi}\int_0^{2\pi} \omega[\phi(t)]\,\frac{\partial}{\partial t}K(r, \theta - t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \omega'[\phi(t)]\phi'(t)K(r, \theta - t)dt,\end{aligned}$$

and from this representation it follows that, for $z = re^{i\theta}$,

$$\mathfrak{M}_p\left\{\frac{\partial}{\partial\theta}\left[\arg \frac{zf'(z)}{f(z)}\right]\right\} \leqq \mathfrak{M}_p\{\omega'[\phi(\theta)]\phi'(\theta)\} \leqq \epsilon A^{(p-1)/p}, \tag{6.3}$$

by hypothesis (iii) and (5.1).[10]

We apply now the theorem of M. Riesz to $\partial \log (zf'(z)/f(z))/\partial\theta$ and find from (6.3) that, for $z = re^{i\theta}$, $0 < r < 1$,

$$\mathfrak{M}_p\left\{\frac{\partial}{\partial\theta}\log\left|\frac{zf'(z)}{f(z)}\right|\right\} \leqq A_p\mathfrak{M}_p\{\omega'[\phi(\theta)]\phi'(\theta)\} \leqq A_pA^{1-1/p}\cdot\epsilon, \tag{6.4}$$

and therefore

$$\mathfrak{M}_p\left\{\frac{\partial}{\partial\theta}\log \frac{zf'(z)}{f(z)}\right\} \leqq \epsilon A^{1-1/p}(1 + A_p). \tag{6.5}$$

Since

$$\frac{\partial}{\partial\theta}\log \frac{zf'(z)}{f(z)} = iz\frac{f''(z)}{f'(z)} - i\left(z\,\frac{f'(z)}{f(z)} - 1\right),$$

it follows from (6.5) and (5.14) that

$$\mathfrak{M}_p\left\{\frac{zf''(z)}{f'(z)}\right\} \leqq \epsilon A^{1-1/p}(1 + A_p) + B\epsilon \leqq (A + B + AA_p)\epsilon.$$

This proves (6.2).

[10] Note that $\{(1/2\pi)\int_0^{2\pi}(\phi')^p d\theta\}^{1/p} \leqq \{(A^{p-1}/2\pi)\int_0^{2\pi}\phi' d\theta\}^{1/p} = A^{(p-1)/p}$.

PROOF OF (6.1). The inequality (6.4) implies that the function $\log |zf'(z)/f(z)|$ for $z=e^{i\theta}$ is an absolutely continuous function of θ. Hence, by the second of the relations (5.7), $\phi'(\theta)$ is an absolutely continuous function. Furthermore $\lim_{r\to 1} \partial \log| rf'(re^{i\theta})/f(re^{i\theta})|/\partial\theta$ exists for almost all θ and is equal to

$$\phi''(\theta)/\phi'(\theta) + \tan \omega[\phi(\theta)]\omega'[\phi(\theta)]\phi'(\theta).$$

We may pass to the limit as $r\to 1$ under the integral sign in (6.4), thus obtaining

$$\mathfrak{M}_p\left\{\frac{\phi''}{\phi'} + \tan \omega\cdot\omega'\cdot\phi'\right\} \leqq A_p\mathfrak{M}_p\{\omega'\cdot\phi'\}. \tag{6.6}$$

In order to estimate $\mathfrak{M}_p\{\phi''/\phi'\}$ we write

$$\phi''/\phi' = (\phi''/\phi' + \tan\omega\cdot\omega'\cdot\phi') - \tan\omega\cdot\omega'\cdot\phi'$$

and apply Minkowski's inequality. Using (6.6) we find

$$\mathfrak{M}_p\left\{\frac{\phi''}{\phi'}\right\} \leqq A_p\mathfrak{M}_p\{\omega'\phi'\} + \mathfrak{M}_p\{\tan\omega\cdot\omega'\cdot\phi'\}, \tag{6.7}$$

and by hypotheses (ii) and (iii) and (5.1)

$$\mathfrak{M}_p\left\{\frac{\phi''}{\phi'}\right\} \leqq A_pA^{1-1/p}\epsilon + A^{1-1/p}\epsilon^2 = A^{1-1/p}(A_p+\epsilon)\epsilon. \tag{6.8}$$

We can estimate the right-hand side of (6.7) in a different way. First, we note that it does not exceed[11]

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} 2^{p-1}|\omega'\cdot\phi'|^p(A_p^p + |\tan\omega|^p)d\theta\right\}^{1/p}$$
$$\leqq A_p2^{1-1/p}\left\{\frac{1}{2\pi}\int_0^{2\pi} |\omega'\cdot\phi'|^p(1+|\tan\omega|^p)d\theta\right\}^{1/p}.$$

Next, using the fact that

$$|\omega'(\phi)| = \left| -\frac{\dfrac{d}{d\phi}\left(\dfrac{\rho'}{\rho}\right)}{1+\left(\dfrac{\rho'}{\rho}\right)^2}\right| \leqq \frac{\epsilon}{1+\left(\dfrac{\rho'}{\rho}\right)^2}$$

we obtain

$$\mathfrak{M}_p\left\{\frac{\phi''}{\phi'}\right\} \leqq \epsilon A_p2^{1-1/p}\left\{\frac{1}{2\pi}\int_0^{2\pi}(\phi')^p\frac{1+|\rho'/\rho|^p}{(1+(\rho'/\rho)^2)^p}d\theta\right\}^{1/p}.$$

[11] We are making use here of the fact that, for $a\geqq 0$, $b\geqq 0$, $p\geqq 1$, $a^{1/p}+b^{1/p} \leqq 2((a+b)/2)^{1/p} = (2^{p-1}(a+b))^{1/p}$.

Now since $p \geqq 2$

$$1+\left|\frac{\rho'}{\rho}\right|^p = 1+\left[\left(\frac{\rho'}{\rho}\right)^2\right]^{p/2} \leqq \left(1+\left(\frac{\rho'}{\rho}\right)^2\right)^p$$

and therefore

$$\mathfrak{M}_p\left\{\frac{\phi''}{\phi'}\right\} \leqq 2^{1-1/p}A_pA^{1-1/p}\epsilon. \tag{6.9}$$

Thus we find from (6.8) and (6.9) that

$$\mathfrak{M}_p\left\{\frac{\phi''}{\phi'}\right\} \leqq A_pA^{1-1/p}\epsilon \min\left(1+\epsilon,\ 2^{1-1/p}\right).$$

Finally, if we note again that $\phi'(\theta) \leqq A$, we obtain (6.1).

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