ON THE POLYNOMIAL SOLUTIONS OF THE GENERALIZED LAMÉ DIFFERENTIAL EQUATION

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1. **Introduction.** The generalized Lamé differential equation with which we shall be concerned is

$$(1.1) \quad P(z)w'' + Q(z)w' + R(z)w = 0,$$

where $P(z)$, $Q(z)$, and $R(z)$ are real polynomials of degrees p , at most $p-1$, and at most $p-2$ respectively. We shall assume that $P(z)$ has p distinct zeros $a_j = b_j + ic_j$, of which the first q , $0 < q \leq p/2$, lie in the upper half-plane, the next q are respectively the conjugate imaginaries of the first q , and the remaining $p-2q$ (if any) are real. That is,

$$\begin{aligned} c_j &> 0 & \text{for } 1 \leq j \leq q, \\ c_j &= -c_{j-q} & \text{for } q < j \leq 2q, \end{aligned}$$

and, if $2q < p$,

$$c_j = 0 \quad \text{for } 2q < j \leq p.$$

We assume further that $Q(z)$ has no factor in common with $P(z)$ and that in the development

$$(1.2) \quad \frac{Q(z)}{P(z)} = \sum_{j=1}^p \frac{A_j e^{i\alpha_j}}{z - a_j}$$

the coefficients satisfy the inequalities

$$A_j > 0, \quad |\alpha_j| < \pi/2 \quad \text{for } 1 \leq j \leq p.$$

According to Heine (see Marden [1, §9], or Marden [2]),¹ there exist some choices $V(z)$ of the polynomial $R(z)$ corresponding to each of which equation (1.1) has a polynomial solution $S(z)$. The $S(z)$ were called *Stieltjes polynomials* by Van Vleck in honor of the man who first studied the location of their zeros relative to those of $P(z)$. For a similar reason, we shall call the $V(z)$ *Van Vleck polynomials*.

In a previous paper (Marden [2]) we obtained the following result:

For each $j = 1, 2, \dots, q$ let there be constructed an ellipse $e(a_j)$ with the points a_j and $\bar{a}_j$ as foci and with $\cos \alpha_j$ as eccentricity. Let K be the smallest convex region which encloses both all the ellipses $e(a_j)$, j

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¹ Numbers in brackets refer to the bibliography at the end of the paper.

$= 1, 2, \dots, q$, and all the real points a_j , $j = 2q+1, 2q+2, \dots, p$.² Then all the zeros of every Stieltjes polynomial $S(z)$ and all the zeros of the corresponding Van Vleck polynomial $V(z)$ lie in K .

This result is analogous to the theorem of Lucas (see Marden [1, §6]) that all the zeros of the derivative of a polynomial $f(z)$ lie in the smallest convex region containing all the zeros of $f(z)$.

In the present paper on the zeros of the $S(z)$ and $V(z)$, we propose to develop some results which will be analogous to the following theorem of Jensen (see Marden [1, §7]).

Every non-real zero of the derivative of a real polynomial $f(z)$ lies in or on at least one of the circles (called the Jensen circles of $f(z)$) which has as diameter the line segment joining a pair of conjugate imaginary zeros of $f(z)$.

We propose to develop these results by a combination of the methods introduced in Walsh [1] and Marden [2 and 3].

2. Two lemmas. Let us denote by n the degree of $S(z)$, by k the number of pairs of conjugate imaginary zeros of $S(z)$, by $z_j = x_j + iy_j$ ($j = 1, 2, \dots, k$) the zeros of $S(z)$ in the upper half-plane, and, if $k < n/2$, by z_j ($j = 2k+1, 2k+2, \dots, n$) the real zeros of $S(z)$. Thus,

$$S(z) = (z - z_1)(z - \bar{z}_1) \cdots (z - z_k)(z - \bar{z}_k)(z - z_{2k+1}) \cdots (z - z_n).$$

Also, let us associate with each pair $a_j, \bar{a}_j$ ($j = 1, 2, \dots, q$) an ellipse $E(j, k)$ defined as follows. Its center will be at the point γ_j on the real axis such that angle $\bar{a}_j, a_j, \gamma_j$ is equal to α_j ; that is,

$$(2.1) \quad \gamma_j = b_j + c_j \tan \alpha_j.$$

Its minor axis will be parallel to the y -axis and be of length $m_j = 2|a_j - \gamma_j|$, and its major axis will be of length $(k^{1/2}m_j)$.

We shall now state two lemmas concerning these ellipses.

LEMMA 1. *The envelope of all the circles whose diameters are the vertical chords of the ellipse $E(j, k)$ is the ellipse $E(j, k+1)$.*

As this lemma may be established by elementary calculus, we shall proceed to the next lemma.

LEMMA 2. *If a non-real zero z_1 of $S(z)$ lies exterior to all the circles $E(j, 1)$, $j = 1, 2, \dots, q$, then it lies inside at least one of the Jensen circles $J(z_u)$ of $S(z)$, $u = 2, 3, \dots, k$.*

PROOF. According to equation (9,3) in Marden [1], the zero z_1 , not being a point a_j , satisfies the equation

² If $p \leq 2q$, reference to real points a_j should be omitted.

$$\sum_{j=1}^p \frac{\alpha_j}{z_1 a_j} + \sum_{j=2}^n \frac{2}{z_1 - z_j} = 0;$$

that is,

$$(2.2) \quad \sum_{j=1}^q U_j + \sum_{j=2q+1}^p V_j + \sum_{j=2}^k X_j + \sum_{j=2k+1}^n Y_j + Z = 0$$

where

$$U_j = A_j [e^{i\alpha_j}(z_1 - a_j)^{-1} + e^{-i\alpha_j}(z_1 - \bar{a}_j)^{-1}],$$

$$V_j = A_j(z_1 - a_j)^{-1},$$

$$X_j = 2[(z_1 - z_j)^{-1} + (z_1 - \bar{z}_j)^{-1}],$$

$$Y_j = 2(z_1 - z_j)^{-1},$$

$$Z = 2(z_1 - \bar{z}_1)^{-1}.$$

Let us compute the sign of the imaginary part of each term in equation (2.2). From the equation

$$\begin{aligned} 2i \operatorname{Im} U_j &= A_j \{ e^{i\alpha_j} [(z_1 - a_j)^{-1} - (\bar{z}_1 - a_j)^{-1}] \\ &\quad + e^{-i\alpha_j} [(z_1 - \bar{a}_j)^{-1} - (\bar{z}_1 - \bar{a}_j)^{-1}] \} \\ &= -2iy_1 A_j \{ e^{i\alpha_j} [(z_1 - a_j)(\bar{z}_1 - a_j)]^{-1} \\ &\quad + e^{-i\alpha_j} [(z_1 - \bar{a}_j)(\bar{z}_1 - \bar{a}_j)]^{-1} \}, \end{aligned}$$

it follows that

$$\begin{aligned} \operatorname{sg} \operatorname{Im} U_j &= -\operatorname{sg} (y_1 \operatorname{Re} \{ e^{i\alpha_j} [(x_1 - b_j) + i(y_1 + c_j)] [(x_1 - b_j) - i(y_1 - c_j)] \}) \\ &= -\operatorname{sg} \{ y_1 [(x_1 - b_j)^2 + y_1^2 - c_j^2 - 2c_j(x_1 - b_j) \tan \alpha_j] \cos \alpha_j \} \\ &= -\operatorname{sg} \{ y_1 [(x_1 - \gamma_j)^2 + y_1^2 - c_j^2 \sec^2 \alpha_j] \}. \end{aligned}$$

From the definitions of V_j and Z , it follows at once that

$$\operatorname{sg} \operatorname{Im} V_j = -\operatorname{sg} y_1 \quad \text{and} \quad \operatorname{sg} \operatorname{Im} Z = -\operatorname{sg} y_1.$$

To obtain $\operatorname{Im} X_j$, we need only in U_j replace a_j by z_j and $A_j e^{i\alpha_j}$ by unity. Similarly, to obtain $\operatorname{Im} Y_j$ we need only in V_j replace a_j by z_j . Thus,

$$\begin{aligned} \operatorname{sg} \operatorname{Im} X_j &= -\operatorname{sg} \{ y_1 [(x_1 - x_j)^2 + y_1^2 - y_j^2] \}, \\ \operatorname{sg} \operatorname{Im} Y_j &= -\operatorname{sg} y_1. \end{aligned}$$

If, now, we suppose the zero z_1 of $S(z)$ to lie outside both all the circles $E(j, 1)$, $j = 1, 2, \dots, q$, and all the Jensen circles $J(z_j)$ of

$S(z)$, $j=2, 3, \dots, k$, then the imaginary part of each term on the left side of equation (2.2) would have the same sign, namely $sg(-y_1)$. That is, the left side of equation (2.2) would not vanish. Hence, if all the U_j have the sign $sg(-y_1)$, at least one X_j must have the opposite sign. That is, if z_1 is outside all the circles $E(j, 1)$, it lies inside at least one circle $J(z_j)$.

We remark in passing that Lemma 2 may be modified as follows so as to admit the possibility of both positive and negative A_j in equation (1.2).

LEMMA 2'. *If a non-real zero z_1 of $S(z)$ lies simultaneously exterior to all the circles $E(j, 1)$ corresponding to positive A_j and interior to all the circles $E(j, 1)$ corresponding to negative A_j , then it lies inside at least one Jensen circle $J(Z_u)$ of $S(z)$, $u=2, 3, \dots, k$.*

However, as Lemma 2' is not suited to iteration, we shall revert to our original hypothesis on the A_j and in the following sections we shall use only Lemma 2.

3. The zeros of the Stieltjes polynomials. By applying Lemmas 1 and 2, we shall now construct a proof of the following theorem, which, when $\alpha_j=0$ for all j , reduces to a result in Walsh [1].

THEOREM I. *Let $S(z)$ be any Stieltjes polynomial which possesses k pairs of conjugate imaginary zeros. Then each non-real zero of $S(z)$ lies in or on at least one of the ellipses $E(j, k)$, $j=1, 2, \dots, q$.*

Let us suppose that the non-real zero z_1 of $S(z)$ lies outside all the circles $E(j, 1)$. Then, by Lemma 2 it lies inside at least one Jensen circle of $S(z)$, say $J(z_2)$, the circle which has as diameter the line-segment joining z_2 and $\bar{z}_2$. If also z_2 lies outside all the circles $E(j, 1)$, then it lies in at least one of the remaining Jensen circles of $S(z)$, say $J(z_3)$. If we continue this process, the number of remaining Jensen circles diminishes. Eventually for some u , $u \leq k$, though z_{u-1} lies outside all the circles $E(j, 1)$ and hence say in $J(z_u)$, the point z_u lies inside at least one circle $E(j, 1)$, say $E(1, 1)$.

Now, applying Lemma 1, we infer that z_{u-1} lies in $E(1, 2)$; consequently that z_{u-2} lies in $E(1, 3)$, and so on, and finally that z_1 lies in $E(1, u)$.

Since ellipse $E(1, u)$ for $u \leq k$ is contained in the ellipse $E(1, k)$, we have completed the proof of Theorem 1.

Concerning the derivatives of $S(z)$, we may state the following theorem.

THEOREM II. *Let $S^{(r)}(z)$ denote the r th derivative of the Stieltjes*

polynomial defined in Theorem I. Then every non-real zero of $S^{(r)}(z)$ lies in or on at least one of the ellipses $E(j, k+r)$, $j=1, 2, \dots, q$.

To prove Theorem II, let us denote by $z_j^{(r)}$ the zeros of $S^{(r)}(z)$. According to Jensen's Theorem, each non-real zero $z_1^{(r)}$ lies in at least one Jensen circle of $S^{(r-1)}(z)$, say $J(z_1^{(r-1)})$. The zero $z_1^{(r-1)}$ lies in at least one Jensen circle of $S^{(r-2)}(z)$, say $J(z_1^{(r-2)})$, and so on. The zero z_1 must lie in at least one Jensen circle of $S(z)$, say $J(z_1)$. Since, according to Theorem I, z_1 lies in an ellipse $E(j, k)$, say $E(1, k)$, we learn again by repeated use of Lemma 1 that $z_1^{(r)}$ lies in the ellipse $E(1, k+r)$.

Theorems I and II presuppose our knowledge of the number of pairs of conjugate imaginary zeros possessed by $S(z)$. If, however, we know only the degree n of $S(z)$, we may find the following immediate corollary of these theorems more useful.

COROLLARY 1. *Let $S(z)$ be a Stieltjes polynomial of degree n . Then every non-real zero of $S(z)$ lies in or on at least one of the ellipses $E(j, n/2)$ and every non-real zero of $S^{(r)}(z)$ lies in at least one of the ellipses $E(j, (2r+n)/2)$, $j=1, 2, \dots, q$. If n is odd, these ellipses may be replaced by the smaller ellipses $E(j, (n-1)/2)$ and $E(j, (2r+n-1)/2)$ respectively.*

4. The zeros of the Van Vleck polynomials. We shall now prove the following theorem.

THEOREM III. *Let $V(z)$ be the Van Vleck polynomial which corresponds to the Stieltjes polynomial $S(z)$ of Theorem I. Then every non-real zero t_1 of $V(z)$ lies in or on at least one ellipse $E(j, k+2)$, $j=1, 2, \dots, q$.*

Let us consider the following three cases.

A. Point t_1 lies in or on some circle $E(j, 1)$.

B. It is one of the zeros $z'_1, z'_2, \dots, z'_{n-1}$ of $S'(z)$.

C. It is neither a point z'_j , nor a point in or on a circle $E(j, 1)$.

The theorem is obviously true in case A. In case B, according to Theorem II, t_1 lies in some ellipse $E(j, k+1)$ and hence in some ellipse $E(j, k+2)$. In case C, we shall use the fact that t_1 satisfies equation (9.5) of Marden [1]; namely,

$$(4.1) \quad \sum_{j=1}^p \frac{A_j e^{ia_j}}{t_1 - a_1} + \sum_{j=1}^{n-1} \frac{1}{t_1 - z'_j} = 0.$$

The left side of (4.1) is comprised of terms of the same form as the U_j, V_j, X_j , and Y_j of equation (2.2). As in Lemma 2, we conclude therefore that t_1 must lie in at least one of the Jensen circles of $S'(z)$,

say $J(z'_1)$. By Theorem II, z'_1 must lie in at least one of the ellipses $E(j, k+1)$ and hence by Lemma 1 point t_1 must lie in at least one of the ellipses $E(j, k+2)$. This completes the proof of Theorem III.

If we now employ the same reasoning as for Theorem II, we may infer the following result concerning the derivatives of $V(z)$.

THEOREM IV. *Let $V^{(r)}(z)$ denote the r th derivative of the polynomial $V(z)$ of Theorem III. Then every non-real zero of $V^{(r)}(z)$ lies in or on at least one of the ellipses $E(j, k+2+r)$, $j=1, 2, \dots, q$.*

Finally, we may infer at once from Theorems III and IV the following complement to Corollary 1.

COROLLARY 2. *Let $V(z)$ be the Van Vleck polynomial corresponding to a Stieltjes polynomial of degree n . Then every non-real zero of $V(z)$ lies in or on at least one of the ellipses $E(j, (n+4)/2)$ and every non-real zero of $V^{(r)}(z)$ lies in or on at least one of the ellipses, $E(j, (2r+n+4)/2)$, $j=1, 2, \dots, q$. If n is odd, these ellipses may be replaced by the smaller ellipses $E(j, (n+3)/2)$ and $E(j, (2r+n+3)/2)$ respectively.*

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