ON A SIMPLE PROPERTY OF THE DERIVATIVES OF LEGENDRE'S POLYNOMIALS

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It is well known that if we define, as usual, Legendre's polynomials as

$$(1) \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

we have

$$(2) \quad P_n(1) = 1 \quad \text{for every } n.$$

More generally, for every n and r , the following relation holds:

$$(3) \quad \begin{aligned} P_n^{(r)}(1) &= C_{n+r, n-r} \cdot 1 \cdot 3 \cdots (2r-1) \\ &= \frac{(n-r+1)(n-r+2) \cdots (n+r)}{2 \cdot 4 \cdots 2r}. \end{aligned}$$

As we could find nowhere a proof of (3), except in the particular case $r=1$,¹ we shall give here a simple one.

From the classic relation²

$$(4) \quad P'_n(x) = nP_{n-1}(x) + x \cdot P'_{n-1}(x)$$

we obtain, by successive derivations,

$$(4') \quad P_n^{(r)}(x) = (n+r-1) \cdot P_{n-1}^{(r-1)}(x) + x \cdot P_{n-1}^{(r)}(x).$$

Letting $x=1$ in (4), we obtain, for $n=\nu$, $P'_\nu(1) = \nu P_{\nu-1}(1) + P'_{\nu-1}(1)$, so that, by (2),

$$P'_\nu(1) = \nu + P'_{\nu-1}(1);$$

letting successively $\nu=1, 2, \dots, n$ and adding, we obtain

$$(5) \quad P'_n(1) = \sum_{\nu=1}^n \nu = n(n+1)/2$$

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¹ See Catalan, *Mémoire sur les fonctions X_n de Legendre*, Mémoires Couronnés de l'Académie de Belgique vol. 31 (1881).

² See D. Jackson, *Fourier series and orthogonal polynomials*, Carus Monographs, p. 48.

which proves (3) for $r=1$. Similarly, for $r=2$, we obtain from (4'), with $x=1$ and $n=v$,

$$P_v''(1) = (v+1) \cdot P_{v-1}'(1) + P_{v-1}''(1),$$

and, by (5),

$$P_v''(1) = (v+1) \frac{(v-1) \cdot v}{2} + P_{v-1}''(1).$$

Letting here, successively, $v=2, 3, \dots, n$ and adding, we obtain:

$$\begin{aligned} (6) \quad P_n''(1) &= \sum_{v=2}^n (v-1) \cdot v \cdot (v+1)/2 \\ &= (n-1) \cdot n \cdot (n+1)(n+2)/2 \cdot 4, \end{aligned}$$

proving (3) for $r=2$. Let us assume now that (3) holds for all derivatives up to the order $r-1$; then, by (4'), we obtain, for $x=1$ and subscript v :

$$(7) \quad P_v^{(r)}(1) = (v+r-1)P_{v-1}^{(r-1)}(1) + P_{v-1}^{(r)}(1),$$

or, using (3), valid by assumption for derivatives of order not greater than $r-1$,

$$\begin{aligned} (7') \quad P_v^{(r)}(1) &= (v+r-1) \frac{(v-r+1)(v-r+2) \cdots (v+r-2)}{2 \cdot 4 \cdots (2r-2)} \\ &\quad + P_{v-1}^{(r)}(1). \end{aligned}$$

Writing this relation for $v=r, r+1, \dots, n-1, n$ and taking into account that $P_{r-1}^{(r)}(1)=0$, we obtain, adding,

$$\begin{aligned} P_n^{(r)}(1) &= \sum_{v=r}^n \frac{(v-r+1)(v-r+2) \cdots (v+r-1)}{2 \cdot 4 \cdots (2r-2)} \\ &= \frac{(n-r+1)(n-r+2) \cdots (n+r)}{2 \cdot 4 \cdots (2r)}, \end{aligned}$$

finishing the proof of (3).

From the form of (3) it results immediately, that all values of $P_n^{(r)}(1)$ are integers. As expressions containing $P_n^{(r)}(1)$ appear in connection with problems of divisibility, or irreducibility of polynomials, it may be of some interest to remark that, as a consequence of (3), $P_n^{(r)}(1)$ are always divisible by all odd factors of $(2r)!$, as well as by the highest power of 2 which divides $C_{n+r, n-r} = C_{n+r, 2r}$.