ON CHERN'S INVARIANT FOR RIEMANNIAN 4-MANIFOLDS

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1. **Introduction.** S. S. Chern has defined, guided by geometric considerations, a differential geometric invariant for closed, orientable, 4-dimensional Riemannian manifolds [1];¹ the invariant is closely connected with the fact that the orthogonal group in 4 variables is essentially, that is, up to a finite covering, the direct product of two orthogonal groups in 3 variables. It is the purpose of this note to exhibit a space for which this invariant is not zero; at the same time we shall verify the well known generalized Gauss-Bonnet formula for the curvatura integra. The space is the complex projective plane K^2 , in its elliptic metric; this metric is discussed in detail in [2], and we shall freely refer to this paper for the necessary formulae of Riemannian geometry in Cartan's formulation; for the basic definitions of differential forms we refer to [3]. We make use of the well known fact that K^2 is the base space of a fiber decomposition of S^5 with S^1 as fiber (see [4] and [5]).

2. **The space K^2 .** We now consider the complex plane K^2 . As is well known, it is obtained from the 5-sphere S^5 by a fiber decomposition [4]: We represent S^5 as unit sphere $\sum |z_i|^2 = 1$ in a complex-euclidean 3-dimensional space C^3 with coordinates (z_1, z_2, z_3) . Two points y and z of S^5 are called equivalent, under the relation Θ , if there exists a real θ such that $y_k = z_k \cdot e^{i\theta}$; the equivalence classes, or fibers, are great circles, with θ as arc length.

One finds that the decomposition space S^5/Θ with the induced topology is a 4-dimensional manifold; it is called K^2 , and with the canonical map $\phi: S^5 \rightarrow K^2$ the system (S^5, S^1, K^2, ϕ) is a fiber bundle [5] (with the rotations of S^1 as structure group). In addition there is a natural way of making K^2 into a differentiable (even analytic) manifold, such that ϕ is a differentiable (analytic) map: if $z^0 = (z_1^0, z_2^0, z_3^0)$ with, say, $z_1^0 \neq 0$, is a point of S^5 , then the real and imaginary part of z_2/z_1 and z_3/z_1 serve as analytic coordinates in the neighborhood of $\phi(z^0)$. Now S^5 is a Riemannian manifold, as subspace of C^3 , with fundamental form $\sum |dz_i|^2$. One verifies after some computation that for a vector $\xi = (dz_1, dz_2, dz_3)$, tangent to S^5 at (z_1, z_2, z_3) , the expression $\sum |dz_i|^2 - |\sum z_i \bar{dz}_i|^2$ depends only on the image vector of ξ under the map ϕ , and that it represents a positive definite

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¹ Numbers in brackets refer to the references cited at the end of the paper.

(analytic) quadratic form on the tangent space to K^2 ; the Riemannian geometry determined by it is called the elliptic hermitian geometry of K^2 ; from now on we consider K^2 with this metric (see [2]). Any vector of S^5 , which is perpendicular to the fiber through its point of support, is mapped by ϕ into a vector of the same length. Using this fact, and considering the volume forms $(g^{1/2}dx_1 \cdots dx_n)$ of the Riemannian manifolds S^5 , K^2 , and S^1 , one can show that the volumes $v(S^5)$, $v(K^2)$, and $v(S^1)$ satisfy the relation $v(S^5) = v(K^2) \cdot v(S^1)$; since $v(S^1) = 2\pi$, and $v(S^5) = \pi^3$ [6, p. 303], we have

$$(*) \quad v(K^2) = \pi^2/2.$$

3. The differential forms. Chern's invariant and the total curvature form are expressed in terms of Cartan's differential forms [1]. We first consider the complex curvature forms Ω_{ij} for K^2 ; we take the necessary formulae from [2]: there are two complex 1-forms ω_1 and ω_2 , and the Ω_{ij} 's are given by

$$\Omega_{ii} = \omega_i \bar{\omega}_i + \sum \omega_k \bar{\omega}_k; \quad \Omega_{ij} = \omega_j \bar{\omega}_i, \quad i \neq j.$$

We have to set $\omega_1 = \theta_1 + i\psi_1$, and $\omega_2 = \theta_2 + i\psi_2$.

The volume form dv is then given by $\theta_1\theta_2\psi_1\psi_2$, and we note, in consequence of (*), §2, that

$$(**) \quad \int_{K^2} \theta_1\theta_2\psi_1\psi_2 = \int_{K^2} dv = \frac{\pi^2}{2}.$$

Substituting into the Ω_{ij} 's we have

$$\begin{aligned} \Omega_{ii} &= (\theta_i + i\psi_i)(\theta_i - i\psi_i) + \sum (\theta_k + i\psi_k)(\theta_k - i\psi_k) \\ &= 2i\psi_i\theta_i + 2i \sum \psi_k\theta_k, \end{aligned}$$

$$\Omega_{ij} = (\theta_j + i\psi_j)(\theta_i - i\psi_i) = \theta_j\theta_i + \psi_j\psi_i + i(\psi_j\theta_i - \theta_j\psi_i), \quad i \neq j.$$

Setting $\Omega_{ij} = \Theta_{ij} + i\Psi_{ij}$, and denoting the real curvature forms by $\Omega'_{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3$, we obtain for the matrix $\Psi^r = (\Omega'_{\alpha\beta})$ by [2, (44)] the following:

$$\Omega^r = \begin{pmatrix} \Theta & -\Psi \\ \Psi & \Theta \end{pmatrix} \quad \text{with} \quad \Theta = \begin{pmatrix} 0 & \theta_2\theta_1 + \psi_2\psi_1 \\ \theta_1\theta_2 + \psi_1\psi_2 & 0 \end{pmatrix}$$

and

$$\Psi = \begin{pmatrix} 4\psi_1\theta_1 + 2\psi_2\theta_2 & \psi_1\theta_2 - \theta_1\psi_2 \\ \psi_1\theta_2 - \theta_1\psi_2 & 4\psi_2\theta_2 + 2\psi_1\theta_1 \end{pmatrix}.$$

With the quantities $\Omega'_{\alpha\beta}$ we now compute the Chern form Γ , and

the total curvature form K ; they are given by $\Gamma = \sum_{\alpha < \beta} [\Omega'_{\alpha\beta}]^2$ and $K = \Omega'_{01}\Omega'_{23} + \Omega'_{02}\Omega'_{31} + \Omega'_{03}\Omega'_{12}$ (see [1, p. 970]). By a straightforward multiplication we obtain

$$\Gamma = -24\theta_1\theta_2\psi_1\psi_2 = -24dv \quad \text{and} \quad K = 24\theta_1\theta_2\psi_1\psi_2 = 24dv.$$

Incidentally this shows that Γ really is a form of K^2 and not only of a fiber bundle over K^2 .

We obtain now for the characteristic, using (**),

$$\chi^{(K^2)} = \frac{1}{4\pi^2} \cdot \int_{K^2} K = \frac{6}{\pi^2} \cdot \int_{K^2} dv = \frac{6}{\pi^2} \cdot \frac{\pi^2}{2} = 3,$$

which checks with well known facts, and for the Chern invariant

$$\eta = \frac{1}{2\pi^2} \cdot \int_{K^2} \Gamma = -\frac{12}{\pi^2} \cdot \int_{K^2} dv = -6,$$

which is different from 0; as predicted by the theory, the value of η is integral. The two invariants I_r and I_l of [1] have the values 6 and -18 for K^2 .

REFERENCES

1. S. S. Chern, *On Riemannian manifolds of four dimensions*, Bull. Amer. Math. Soc. vol. 51 (1943).
2. ———, *Characteristic classes of Hermitian manifolds*, Ann. of Math. vol. 47 (1946).
3. C. Chevalley, *Theory of Lie groups I*, Princeton, 1946.
4. H. Hopf and M. Rueff, *Über faserungstreue Abbildungen der Sphären*, Comment. Math. Helv. vol. 11 (1938–1939).
5. N. E. Steenrod, *The classification of sphere bundles*, Ann. of Math. vol. 45 (1944).
6. R. Courant, *Differential- and integral calculus II*, New York, 1936.

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