ON A CONJECTURE ON SIMPLE GROUPS

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The purpose of this paper is to rephrase a conjecture about simple groups into the language of linear algebra.

Let G be a group of finite order $o(G)$. Then by Γ_p we shall mean the group ring of G over a field of characteristic p (for instance the integers modulo p). We shall denote the radical of Γ_p by N_p. If $p=0$ or $p \nmid o(G)$, then it is known that $N_p=(0)$; and if $p \mid o(G)$, $N_p \neq (0)$.

We now consider the following two assertions:

(A) *If G is a simple group of odd order, $o(G)$ is a prime.*

(B) *If G is a group of odd order $o(G)$, then for some prime p, $p \mid o(G)$, we can find a $g \in G$, $g \neq 1$, such that $g-1 \in N_p$.*

The theorem which we propose to prove is:

THEOREM. (A) *is equivalent to* (B).

1. (B) implies (A).

DEFINITION. $U_p = \{g \in G \mid g-1 \in N_p\}$.

LEMMA 1. *U_p is a normal p-subgroup of G.*

PROOF. (a) U_p is a subgroup of G, for if $g_1, g_2 \in U_p$, then since N_p is a left-ideal of Γ_p, $g_1(g_2-1)+(g_1-1)=g_1g_2-1 \in N_p$.

(b) U_p is normal, for if $g-1 \in N_p$, since N_p is a two-sided ideal of Γ_p, $h(g-1)h^{-1}=hgh^{-1}-1 \in N_p$ for all $h \in G$.

(c) If $g-1 \in N_p$, then for some integer S, $(g-1)^{p^s}=0=g^{p^s}-1$. So if $g \in U_p$, g is of order p^s for some s. So U_p is a p-group.

COROLLARY. (B) *implies* (A).

PROOF. By (B), $U_p \neq 1$ for some $p \mid o(G)$. Hence since G is simple, and since U_p is a normal subgroup of G, $U_p = G$. Thus G is of order p^s, and G being simple, $s=1$. Hence (B) implies (A).

2. (A) implies (B).

LEMMA 2. (A) *implies that every group of odd order is solvable.*

PROOF. Let $G=G_1 \supset G_2 \supset \cdots \supset G_r=1$ be a composition series for G. Since the G_i/G_{i+1} are simple and of odd order, by (A) they must be of prime order; hence the lemma is proved.

Since a solvable group contains a normal p-subgroup [1, p. 25,

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Theorem 20],[1] we immediately obtain, using Lemma 2, the following lemma.

LEMMA 3. (A) *implies that if G is of odd order, then it contains a normal p-subgroup.*

For groups of certain orders the N_p can be completely described. This is true for p-groups. If G is of order p^e, then for every $g \in G$, $g-1 \in N_p$ [2, p. 176, Theorem 1.2 or 3, p. 239]. For our case it is sufficient to use the weaker result:

LEMMA 4. *If G is of order p^e, then for some $g \neq 1$ in G, $g-1 \in N_p$.*

PROOF. Since G is of order p^e, it has a nontrivial center C. Let $g \neq 1$ be in C. Then since $g-1$ is in the center of Γ_p, and since $(g-1)^{p^e} = g^{p^e} - 1 = 1 - 1 = 0$, $g-1 \in N_p$.

Suppose that S is the normal p-subgroup of Lemma 3. An element of the form $g-1 \in \Gamma_p$ is in N_p if and only if for every irreducible representation ϕ of G, $\phi(g) = 1$. Clifford's theorem [4, p. 534, Theorem 1] reduces the irreducible representations of G to irreducible representations (or into ones fully reducible into irreducible components) of S. For the g of Lemma 4 in S, for every irreducible representation ϕ of S, $\phi(g) = 1$. So by Clifford's theorem, for every irreducible representation ϕ of G, $\phi(g) = 1$. Thus $g-1 \in N_p$. And so we have shown the following lemma.

LEMMA 5. (A) *implies* (B).

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[1] Numbers in brackets refer to the bibliography at the end of the paper.