APPLICATIONS OF A LEMMA OF FEJÉR TO TYPICALLY-REAL FUNCTIONS

M. S. ROBERTSON

1. Introduction. Let

$$(1.1) \qquad w = f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

be regular for $|z| < R$ and $f(x)$ real for $-R < x < R$. $f(z)$ is said to be typically-real with respect to the circle $|z| = R$ if the imaginary part of $f(z)$ has the same sign as the imaginary part of z when z is not real and $|z| < R$ [1].[1] In particular, if $f(z)$ is schlicht or univalent for $|z| < R$ and the coefficients a_n, $n = 2, 3, \cdots$, are all real, then $f(z)$ is a typically-real function. If further, $w = f(z)$ maps each circle $|z| = r < R$ into a contour having the property that every line parallel to the imaginary axis cuts the contour in not more than two points, we say that $f(z)$ is convex in the direction of the imaginary axis relative to the circle $|z| = R$. The necessary and sufficient condition that $f(z)$ be convex in the direction of the imaginary axis when it is real on the real axis is that $zf'(z)$ be typically-real [**5**; **4**]. In the discussion to follow we shall assume that $R = 1$.

We state the following lemma due to L. Fejér (Turán [**2**], Szász [**3**]):

LEMMA 1. *For a value of r, $0 < r < 1$, let $\sum_{n=1}^{\infty} \lambda_n r^n$ converge. In order that*

$$(1.2) \qquad \sum_{n=1}^{\infty} \lambda_n r^n \sin nx \cdot \sin ny \geqq 0 \qquad \text{for } 0 < x < \pi;\ 0 < y < \pi,$$

it is necessary and sufficient that

$$(1.3) \qquad \sum_{n=1}^{\infty} n\lambda_n r^n \cdot \sin n\psi \geqq 0 \qquad \text{for } 0 < \psi < \pi.$$

It is the purpose of this paper to give some applications of Lemma 1 to functions

$$(1.4) \qquad F(z) = \sum_{n=1}^{\infty} a_n b_n z^n$$

where

Presented to the Society, February 25, 1950; received by the editors June 27, 1949.

[1] Number in brackets refer to the references cited at the end of the paper.

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad a_1 = b_1 = 1, \tag{1.5}$$

are both typically-real with respect to the unit circle $|z| = 1$. The same method of proof may be used to obtain many special results for the functions $F(z)$ of (1.4). We include here the proofs of several theorems which illustrate the procedure.

THEOREM 1. *If*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad a_1 = b_1 = 1, \tag{1.6}$$

are regular and typically-real for $|z| < 1$, *then*

$$F(z) = \sum_{n=1}^{\infty} a_n b_n z^n, \qquad |z| < 1, \tag{1.7}$$

is typically-real for $|z| \leqq 2 - 3^{1/2}$. *When*

$$f(z) \equiv g(z) \equiv z(1-z)^{-2}, \qquad F(z) \equiv (z + z^2)\cdot(1-z)^{-3},$$

$F(z)$ *is not typically-real for* $|z| < \rho$ *for any* $\rho > 2 - 3^{1/2}$.

THEOREM 2. *If*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad a_1 = b_1 = 1, \tag{1.8}$$

are regular and typically-real for $|z| < 1$, *and if*

$$F(z) = \sum_{n=1}^{\infty} a_n b_n z^n, \tag{1.9}$$

then

$$h(z) \equiv \int_0^z \frac{F(z)}{z}\, dz = \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n \tag{1.10}$$

is regular and typically-real for $|z| < 1$.

THEOREM 3. *If*

$$f(z) = \sum_{n=1}^{\infty} c_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} d_n z^n, \qquad c_1 = d_1 = 1, \tag{1.11}$$

are regular, univalent, and convex in the direction of the imaginary axis for $|z| < 1$, *and real on the real axis, then*

$$(1.12)\qquad F(z) = \sum_{n=1}^{\infty} c_n d_n z^n$$

is also regular, univalent, and convex in the direction of the imginary axis for $|z| < 1$.

If in Theorem 3 we take $g(z)$ to be the nth Cesàro partial sum of order 3 of the geometric series

$$(1.13)\qquad z(1-z)^{-1} = z + z^2 + \cdots + z^n + \cdots,$$

which E. Egerváry [6] has shown to be univalent and convex for $|z| < 1$, we obtain immediately a new proof of the following theorem of L. Fejér [5].

THEOREM 4. (Fejér) *If*

$$(1.14)\qquad f(z) = \sum_{n=1}^{\infty} c_n z^n$$

be regular, univalent, and convex in the direction of the imaginary axis for $|z| < 1$ *and real on the real axis, then the Cesàro partial sums of order* 3 *of* (1.14) *are also univalent and convex in the direction of the imaginary axis for* $|z| < 1$.

THEOREM 5. *If*

$$(1.15)\qquad f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad a_1 = b_1 = 1,$$

are regular and typically-real for $|z| < 1$, *and if*

$$(1.16)\qquad F(z) = \sum_{n=1}^{\infty} a_n b_n z^n, \qquad |z| < 1$$

then $\Re F(re^{i\theta}) \geqq 0$ *for* $|\theta| \leqq \arccos \mu(r) < \pi/2$, $0 < r < 1$, *where* $4r\mu(r) \equiv (1 + 34r^2 + r^4)^{1/2} - 1 - r^2$. *The inequalities are sharp.*

Theorem 5 includes a result of W. Rogosinski [1] who has shown that $F(r) \geqq 0$, $0 < r < 1$. It is not hard to show from Theorem 5 that the function $f(z)$ of this theorem satisfies the inequality

$$(1.17)\qquad \int_0^{2\pi} |f(ze^{i\theta}) \cdot f(z)| \cos \arg \left\{ \frac{f(ze^{i\theta})}{f(z)} \right\} d\phi \geqq 0,$$

for $z = re^{i\phi}$, $r < 1$, $|\theta| \leqq \arccos \mu(r^2)$, with equality holding for $\theta = \arccos \mu(r^2)$, $f(z) \equiv z(1-z)^{-2}$. Thus a specially weighted average of the cosine of the angle subtended at the origin by the points

$f(z)$, $f(\bar{z}e^{i\theta})$ is non-negative, although the cosine itself may be sometimes negative. We omit the proof of (1.17).

2. **Proof of Theorem 1.** It has been shown elsewhere [4] that the coefficients of the powers series for $f(z)$ and $g(z)$ in (1.5) have the representation

$$(2.1)\qquad a_n = \frac{1}{\pi}\int_0^\pi \frac{\sin nx}{\sin x}\,d\alpha(x), \qquad b_n = \frac{1}{\pi}\int_0^\pi \frac{\sin ny}{\sin y}\,d\beta(y)$$

where $\alpha(x)$, $\beta(y)$ are non-decreasing functions in the interval $(0, \pi)$ normalized so that

$$(2.2)\qquad \int_0^\pi d\alpha(x) = \int_0^\pi d\beta(x) = \pi.$$

It follows that the inequality

$$(2.3)\qquad IF(re^{i\theta}) = \sum_{n=1}^\infty a_n b_n r^n \sin n\theta \geqq 0$$

for $0<\theta<\pi$ for a suitable range of r can be deduced from the inequality

$$(2.4)\qquad \sum_{n=1}^\infty \frac{\sin nx}{\sin x}\cdot\frac{\sin ny}{\sin y}\cdot r^n \sin n\theta \geqq 0, \qquad 0<\theta<\pi,$$

by term by term integrations with the aid of (2.1). However, if we apply Lemma 1 twice, we need only find the range of r for which $(z=re^{i\theta})$

$$(2.5)\qquad I\,\frac{(z+z^2)}{(1-z)^3} \equiv \sum_{n=1}^\infty n^2 r^n \sin n\theta \geqq 0, \qquad 0<\theta<\pi.$$

$$(2.6)\qquad \begin{aligned} I\{(z+z^2)\cdot(1-\bar{z})^3\} &= r\sin\theta\{1-6r^2+r^4-2(r+r^3)\cos\theta\} \\ &\geqq r(1-r)^2\cdot(1+4r+r^2)\sin\theta \\ &\geqq 0 \qquad \text{for } 0\leqq\theta\leqq\pi,\ 0\leqq r\leqq 2-3^{1/2}. \end{aligned}$$

The example $f(z)\equiv g(z)\equiv z(1-z)^{-2}$, $F(z)\equiv(z+z^2)(1-z)^{-3}$, shows that the range of r obtained above cannot be improved upon. This completes the proof of Theorem 1.

3. **Proofs of Theorems 2 and 3.** Using the notation of Theorem 2, we have

$$(3.1)\qquad h(z) = \sum_{n=1}^\infty \frac{a_n b_n}{n} z^n, \qquad |z|<1,$$

$$I\{h(z)\} = \sum_{n=1}^{\infty} \frac{a_n b_n}{n} r^n \sin n\theta, \qquad z = re^{i\theta}. \tag{3.2}$$

Thus $I\{h(z)\} \geqq 0$ for $0 \leqq \theta \leqq \pi$, $0 < r < 1$, follows from

$$\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\sin nx}{\sin x} \cdot \frac{\sin ny}{\sin y} \cdot r^n \sin n\theta \geqq 0, \qquad 0 \leqq \theta \leqq \pi, \tag{3.3}$$

by term by term integrations and the aid of (2.1) and (2.2). But by Lemma 1, (3.3) may be replaced by

$$\sum_{n=1}^{\infty} \frac{\sin n\psi}{\sin \psi} \cdot r^n \sin n\theta \geqq 0 \qquad \text{for } 0 < \psi < \pi, 0 \leqq \theta \leqq \pi. \tag{3.4}$$

$$\sum_{n=1}^{\infty} \frac{\sin n\psi}{\sin \psi} r^n \sin n\theta = I\{z(1 - 2z \cos \psi + z^2)^{-1}\}. \tag{3.5}$$

Moreover, $z(1-2z \cos \psi + z^2)^{-1}$ is typically-real for $|z| < 1$, $0 \leqq \psi \leqq \pi$. Thus $h(z)$ is also typically-real for $|z| < 1$. This completes the proof of Theorem 2.

Theorem 3 follows from Theorem 2. For if

$$f(z) = \sum_{n=1}^{\infty} c_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} d_n z^n, \qquad c_1 = d_1 = 1, \tag{3.6}$$

be regular, univalent, and convex in the direction of the imaginary axis, and real on the real axis, then $zf'(z)$ and $zg'(z)$ are typically-real for $|z| < 1$. It follows from Theorem 2 that

$$zF'(z) \equiv \sum_{n=1}^{\infty} \frac{(nc_n)(nd_n)}{n} z^n, \qquad |z| < 1, \tag{3.7}$$

is typically-real for $|z| < 1$. Thus

$$F(z) \equiv \sum_{n=1}^{\infty} c_n d_n z^n, \qquad |z| < 1, \tag{3.8}$$

is regular, univalent, and convex in the direction of the imaginary axis.

4. **Proof of Theorem 5.** With the notation of Theorem 5 let

$$F(z) \equiv \sum_{n=1}^{\infty} a_n b_n z^n, \qquad a_1 = b_1 = 1, \ |z| < 1. \tag{4.1}$$

With the aid of (2.1) and (2.2) it follows that the inequality

$$\Re F(re^{i\theta}) \geqq 0, \qquad |\theta| \leqq \text{arc cos } \mu(r), \qquad 0 < r < 1, \tag{4.2}$$

can be deduced from the inequality

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\sin x}\cdot\frac{\sin ny}{\sin y}\cdot r^n \cos n\theta \geqq 0, \quad |\theta| \leqq \arc\cos \mu(r), \tag{4.3}$$

by the method used in the preceding paragraphs. By Lemma 1, we need only find when

$$\sum_{n=1}^{\infty} n \frac{\sin n\psi}{\sin \psi}\cdot r^n \cos n\theta \geqq 0, \qquad 0 \leqq \psi \leqq \pi, \tag{4.4}$$

or, letting $\alpha=\pi-\theta$, $\phi=\pi-\psi$, when

$$\sum_{n=1}^{\infty} n \frac{\sin n\phi}{\sin \phi}\, r^n \cos n\alpha \leqq 0, \qquad 0 \leqq \phi \leqq \pi, \tag{4.5}$$

or when

$$R\left\{(z - z^3)\cdot(1 - 2z\cos\phi + z^2)^{-2}\right\} \leqq 0, \qquad z = re^{i\alpha}. \tag{4.6}$$

Since

$$\begin{aligned} &R\left\{(\bar{z} - \bar{z}^3)\cdot(1 - 2z\cos\phi + z^2)^2\right\} \\ &\quad = r(r^2 - 1)\left[4r^2\cos^3\alpha - \left\{1 + (6 + 4\cos^2\phi)r^2 + r^4\right\}\cos\alpha \right. \\ &\qquad\qquad \left. + (4r + 4r^3)\cos\phi\right], \end{aligned} \tag{4.7}$$

it will be sufficient to determine the values of α so that when $0<r<1$, $0\leqq\phi\leqq\pi$, we have

$$\begin{aligned} &4r^2\cos^3\alpha - \left\{1 + (6 + 4\cos^2\phi)r^2 + r^4\right\}\cos\alpha \\ &\qquad + (4r + 4r^3)\cos\phi \geqq 0. \end{aligned} \tag{4.8}$$

(4.8) may also be written in the form

$$\begin{aligned} &(2r\cos\alpha - 1 - r^2)\left\{2r\cos^2\alpha + (1 + r^2)\cos\alpha - 4r\right\} \\ &\qquad + 4r^2(1 - \cos^2\phi)\cos\alpha - (1 - \cos\phi)(4r + 4r^3) \geqq 0; \end{aligned} \tag{4.9}$$

with regard to the last two terms of (4.9) we note that

$$4r^2(1 - \cos^2\phi)\cos\alpha - (1 - \cos\phi)(4r + 4r^3) \geqq -8r(1 + r^2) \tag{4.10}$$

with equality occurring in (4.10) for $\phi=\pi$ and for all α. Thus the left-hand side of inequality (4.8) is not less than

$$4r^2\cos^3\alpha - (1 + 10r^2 + r^4)\cos\alpha - 4r - 4r^3 \tag{4.11}$$

and the left-hand side of (4.8) equals the expression in (4.11) when $\phi=\pi$. Factoring (4.11) we have to determine when

$$(4.12)\qquad (1 + 2r\cos\alpha + r^2)\{2r\cos^2\alpha - (1 + r^2)\cos\alpha - 4r\} \geqq 0.$$

(4.12) holds when, and only when,

$$(4.13)\qquad \begin{aligned} -4r \leqq 4r\cos\alpha &\leqq -\{(1 + 34r^2 + r^4)^{1/2} - 1 - r^2\} \\ &= -\mu(r)\cdot 4r. \end{aligned}$$

Thus $|\theta| = |\pi - \alpha| \leqq \text{arc cos } \mu(r) < \pi/2$.

The estimate for the bound on θ is sharp, as is seen by taking $\phi = \pi$, $f(z) \equiv g(z) \equiv z(1-z)^{-2}$.

References

1. W. Rogosinski, *Über positive harmonische Entwicklungen und typischreelle Potenzreihen*, Math. Zeit. vol. 35 (1932) pp. 93–121.

2. P. Turán, *Über die monotone Konvergenz der Cesàro-mittel bei Fourier- und Potenzreihen*, Proc. Cambridge Philos. Soc. vol. 34 (1938) pp. 134–143.

3. O. Szász, *On the partial sums of harmonic developments and of power series*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 12–21.

4. M. S. Robertson, *On the coefficients of a typically-real function*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 565–572.

5. L. Fejér, *Neue Eigenschaften der Mittelwerte bie den Fourierreihen*, J. London Math. Soc. vol. 8 (1933) pp. 53–62.

6. E. Egerváry, *Abbildungseigenschaften der arithmetischen Mittel der geometrischen Reihen*, Math. Zeit. vol. 42 (1937) pp. 221–230.

Rutgers University