ON DOUBLE FOURIER SERIES

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1. In the present note, the author proves some inequality theorems on double Fourier series. We begin with some preliminary lemmas. Let $f(x, y)$ be an integrable function and let its Fourier series be

$$f(x, y) \sim \sum_{m,n} A_{m,n}(x, y)$$

and let

$$\Delta_{\mu,\nu}(x, y) = \sum_{m=2^\mu}^{2^{\mu+1}-1} \sum_{n=2^\nu}^{2^{\nu+1}-1} A_{m,n}(x, y).$$

LEMMA 1. *Let $s_{\nu;m,n}(x, y)$ be the m, nth partial sum of Fourier series of $f_\nu(x, y)$, then*

$$\int_0^{2\pi}\int_0^{2\pi}\left(\sum_\nu s^2_{\nu;m,n}(x, y)\right)^{\gamma/2} dxdy \leqq A_\gamma \int_0^{2\pi}\int_0^{2\pi}\left(\sum_\nu f_\nu^2\right)^{\gamma/2} dxdy, \qquad \gamma > 1.$$

LEMMA 2. *If $\gamma > 1$,*

$$\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\mu,\nu} |\Delta_{\mu,\nu}(x, y)|^2\right)^{\gamma/2} dxdy \leqq B_\gamma \int_0^{2\pi}\int_0^{2\pi} |f(x, y)|^\gamma dxdy \leqq B'_\gamma \int_0^{2\pi}\int_0^{2\pi}\left(\sum_{\mu,\nu} |\Delta_{\mu,\nu}(x, y)|^2\right)^{\gamma/2} dxdy.$$

These lemmas are due to J. Marcinkiewicz [2].[1]

2. THEOREM 1. *If $\gamma > 1$,*

$$\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{m=0}^\infty\sum_{n=0}^\infty |s_{2^m,2^n}(x, y) - \sigma_{2^m,2^n}(x, y)|^2\right)^{\gamma/2} dxdy \leqq C_\gamma \int_0^{2\pi}\int_0^{2\pi} |f(x, y)|^\gamma dxdy,$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

and

$$\int_0^{2\pi}\int_0^{2\pi}\left|\max_{m,n} s_{2^m,2^n}(x, y)\right|^\gamma dxdy \leqq D_\gamma \int_0^{2\pi}\int_0^{2\pi} |f(x, y)|^\gamma dxdy.$$

Accordingly $s_{2^m,2^n}(x, y)$ *converges to* $f(x, y)$ *almost everywhere, when* $f(x, y) \in L^\gamma$ $(\gamma > 1)$.

By Schwarz's inequality

$$\sum_m^\infty \sum_n^\infty | s_{2^m,2^n} - \sigma_{2^m,2^n} |^2$$

$$\leqq \sum_m^\infty \sum_n^\infty \frac{1}{2^{2m}\cdot 2^{2n}}\left(\sum_{\mu=0}^{2}\sum_{\nu=0}^{m_2}{}^{n} | s_{2^m,2^n} - s_{\mu,\nu} |\right)^2$$

$$\leqq \sum_m \sum_n \frac{1}{2^m}\frac{1}{2^n}\left(\sum_{\mu=0}^{2^m}\sum_{\nu=0}^{2^n} | s_{2^m,2^n} - s_{\mu,\nu} |^2\right)$$

$$\leqq \sum_m \sum_n \frac{1}{2^m}\frac{1}{2^n}\left\{\sum_{i=0}^{m}\sum_{j=0}^{n}\left(\sum_{\mu=2^{i-1}+1}^{2^i}\sum_{\nu=2^{j-1}+1}^{2^j} | s_{2^m,2^n} - s_{\mu,\nu} |^2\right)\right\}.$$

In view of Lemma 1, we get

$$\int_0^{2\pi}\int_0^{2\pi}\left(\sum_m \sum_n | s_{2^m,2^n} - \sigma_{2^m,2^n} |^2\right)^{\gamma/2} dxdy$$

$$\leqq E_\gamma \int_0^{2\pi}\int_0^{2\pi}\left[\sum_m \sum_n \frac{1}{2^m}\frac{1}{2^n}\right.$$

$$\left.\cdot\left\{\sum_{i=0}^{m}\sum_{j=0}^{n}(2^i 2^j | s_{2^m,2^n} - s_{2^{i-1},2^{j-1}} |^2)\right\}\right]^{\gamma/2} dxdy.$$

Since $| s_{2^m,2^n} - s_{2^{i-1},2^{j-1}} | \leqq \sum_{k=i}^m \sum_{l=j}^n |\Delta_{k,l}|$, we have

$$\sum_m \sum_n \frac{1}{2^m}\frac{1}{2^n}\sum_{i=0}^{m}\sum_{j=0}^{n} 2^i 2^j | s_{2^m,2^n} - s_{2^{i-1},2^{j-1}} |^2$$

$$\leqq \sum_m \sum_n \frac{1}{2^m}\frac{1}{2^n}\sum_{i=0}^{m}\sum_{j=0}^{n} 2^i 2^j \left[\sum_{k=i}^{m}\sum_{l=j}^{n}\{ |\Delta_{k,l}| (2^k 2^l)^{1/4} (2^k 2^l)^{-1/4}\}^2\right]$$

$$\leqq \sum_m \sum_n \frac{1}{2^m}\frac{1}{2^n}\sum_{i=0}^{m}\sum_{j=0}^{n} 2^i 2^j \left\{\sum_{k=i}^{m}\sum_{l=j}^{n} |\Delta_{k,l}|^2 (2^k 2^l)^{1/2}\right\}$$

$$\cdot\left\{\sum_{k=i}^{\infty}\sum_{l=j}^{\infty}(2^k 2^l)^{-1/2}\right\}$$

$$\leqq \sum_m \sum_n \frac{1}{2^m} \frac{1}{2^n} \sum_{i=0}^{m} \sum_{j=0}^{n} (2^i 2^j)^{1/2} \sum_{k=i}^{m} \sum_{l=j}^{n} |\Delta_{k,l}|^2 (2^k 2^l)^{1/2}$$

$$\leqq \sum_m \sum_n \frac{1}{2^m} \frac{1}{2^n} \sum_{k=0}^{m} \sum_{l=0}^{n} |\Delta_{k,l}|^2 (2^k 2^l)^{1/2} \sum_{i=0}^{k} \sum_{j=0}^{l} (2^i 2^j)^{1/2}$$

$$\leqq \sum_m \sum_n \frac{1}{2^m} \frac{1}{2^n} \sum_{k=0}^{m} \sum_{l=0}^{n} |\Delta_{k,l}|^2 2^k 2^l$$

$$\leqq \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\Delta_{k,l}|^2 2^k 2^l \sum_{m=k}^{\infty} \sum_{n=l}^{\infty} \frac{1}{2^m} \frac{1}{2^n}$$

$$\leqq \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} |\Delta_{k,l}|^2.$$

By using Lemma 2, we get the first half of the theorem. The remaining part is an easy consequence of the former.

3. LEMMA 3. *If $k>1$ and $\gamma>1$, we get*

$$\int_0^{2\pi} \int_0^{2\pi} \left(\sum_\nu |s_{\nu;m,n}|^k \right)^{\gamma/k} dxdy \leqq E_\gamma \int_0^{2\pi} \int_0^{2\pi} \left(\sum_\nu |f_\nu|^k \right)^{\gamma/k} dxdy.$$

The proof is carried out analogously to Marcinkiewicz's proof of Lemma 1, using the result of Boas and Bochner [1].

THEOREM 2. *If $k \geqq 2$ and $r>1$,*

$$\int_0^{2\pi} \int_0^{2\pi} \left(\sum_m \sum_n \frac{|s_{m,n} - \sigma_{m,n}|^k}{mn} \right)^{\gamma/k} dxdy \leqq F_\gamma \int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^\gamma dxdy.$$

Accordingly if $f(x, y) \in L^\gamma$ $(\gamma>1)$,

$$\left(\sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |s_{\mu,\nu} - f|^k \right) = o(mn) \qquad (k \geqq 1)$$

almost everywhere.

The proof is analogous to my previous paper [4], using Lemma 3. The last part of the theorem was obtained by Marcinkiewicz [3] in more generalized form.

LEMMA 4. *If $r>1$, then*

$$\int_0^{2\pi}\int_0^{2\pi}\left(\sum_\mu\sum_\nu |\Delta_{\mu,\nu}|^2\right)^{\gamma/2} dxdy \leqq \int_0^{2\pi}\int_0^{2\pi}\left(\sum_m\sum_n \frac{|s_{m,n}-\sigma_{m,n}|^2}{mn}\right)^{\gamma/2} dxdy.$$

The proof is easy (cf. the author's note [**5**]).

THEOREM 3. *If $\gamma>1$,*

$$\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^\gamma dxdy \leqq G_\gamma \int_0^{2\pi}\int_0^{2\pi}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |s_{2^m,2^n}-\sigma_{2^m,2^n}|^2\right)^{\gamma/2} dxdy$$

and

$$\int_0^{2\pi}\int_0^{2\pi} |f(x,y)|^\gamma dxdy \leqq H_\gamma \int_0^{2\pi}\int_0^{2\pi}\left(\sum_m\sum_n \frac{|s_{m\,n}-\sigma_{m\,n}|^2}{mn}\right)^{\gamma/2} dxdy.$$

The proof is immediate by Lemma 4 and Lemma 2.

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