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ON THEOREMS OF M. RIESZ AND ZYGMUND

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Several proofs have been given of the results of M. Riesz and Zygmund.

(a) *The conjugate of the Fourier series of a function* $f(x)$ *of* L^p, $p>1$, *is Fourier series of a function* $\bar{f}(x)$ *of the same class, and*

$$\int_0^{2\pi} |\bar{f}(x)|^p dx \leqq A_p \int_0^{2\pi} |f(x)|^p dx$$

holds, A_p *is a constant depending only on* p.

(b) *If the function* $|f(x)| \log^+ |f(x)|$ *is integrable, the conjugate of the Fourier series of* $f(x)$ *is the Fourier series of a function* $\bar{f}(x)$ *of the class L. Moreover, there exist two constants A and B such that*

$$\int_0^{2\pi} |\bar{f}(x)|\, dx \leqq A \int_0^{2\pi} |f(x)| \log^+ |f(x)|\, dx + B.$$

In view of the importance of these theorems it may be of interest to give another proof of them based on a different idea. Actually it is

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easy to see (and very well known) that they are equivalent to the following proposition about analytic functions.

Let $F(z)=u(z)+iv(z)$, $u(z)>0$, $v(0)=0$ *be a function regular inside the circle* $|z|<1$. *Then the inequalities*

$$(1)\qquad \int_0^{2\pi}|v(re^{ix})|^p dx \leqq A_p\int_0^{2\pi}|u(re^{ix})|^p dx; \qquad 1<p\leqq 2;\ r<1,$$

$$(2)\qquad \begin{aligned}\int_0^{2\pi}|v(re^{ix})|\,dx \leqq C\int_0^{2\pi}&|u(re^{ix})|\,dx\\ &+D\int_0^{2\pi}u\log u\,dx-2\pi Du(0)\log u(0)\end{aligned}$$

hold, C and D being positive constants and A_p *the same as above.*

Let us prove (1) first. The argument is based on the inequality

$$(3)\qquad |\sin\phi|^p \leqq A_p|\cos\phi|^p - B_p\cos p\phi$$

valid for $-\pi/2\leqq\phi\leqq\pi/2$, $1<p\leqq 2$, A_p and B_p being two positive constants depending on p.

The proof of it is simple. Since $\cos p\phi<0$ for $\phi=\pm\pi/2$, taking B_p large enough we get $-B_p\cos p\phi>1$ in a small neighborhood of $\phi=\pm\pi/2$. Now $|\cos\phi|^p$ is greater than a positive constant in every closed interval interior to $(-\pi/2,\pi/2)$, so that choosing A_p large enough we shall have $A_p|\cos\phi|^p-B_p\cos p\phi>1$ in $(-\pi/2,\pi/2)$ and a fortiori the inequality (3).

To show (1) now, let us set

$$F(z)=Re^{i\phi};\qquad u=R\cos\phi;\qquad v=R\sin\phi.$$

Owing to the fact that $u>0$, we may choose $-\pi/2<\phi<\pi/2$. For the same reason, $F(z)\neq 0$ and so $F(z)^p$ is regular in $|z|<1$. Integrating now along the circle $|z|=r$ we get

$$\Re\left[\frac{1}{2\pi i}\int_C\frac{F(z)^p}{z}\,dz\right]=\frac{1}{2\pi}\int_0^{2\pi}R^p\cos p\phi\,dx=\Re[F(0)^p]=u(0)^p>0.$$

Multiplying now (3) by R^p and integrating, we obtain

$$\int_0^{2\pi}|v(re^{ix})|^p dx\leqq A_p\int_0^{2\pi}|u(re^{ix})|^p dx - B_p\int_0^{2\pi}R^p\cos p\phi\,dx,$$

but since the last term is positive, we may drop it, and the desired result follows. A similar argument works in proving (2). Here instead of (3) we use

$$(4) \qquad |\sin \phi| \leqq C \cos \phi + D(\cos \phi \log \cos \phi + \phi \sin \phi),$$

which can be proved in exactly the same way.

Again, since $F(z)$ does not vanish in $|z| < 1$, $F \log F$ is regular there, and integrating along $|z| = r$ we get

$$\Re\left[\frac{1}{2\pi i}\int_C F(z) \log F(z) \frac{dz}{z}\right] = \frac{1}{2\pi}\int_0^{2\pi} \Re[F \log F]dx = u(0) \log u(0),$$

and replacing

$$\begin{aligned}\Re[F \log F] &= \Re[R(\cos \phi + i \sin \phi)(\log R \cos \phi - \log \cos \phi + i\phi)] \\ &= R \cos \phi \log R \cos \phi - R(\cos \phi \log \cos \phi + \phi \sin \phi),\end{aligned}$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} R(\cos \phi \log \cos \phi + \phi \sin \phi)dx = \frac{1}{2\pi}\int_0^{2\pi} u \log u dx - u(0) \log u(0).$$

Finally, multiplying (4) by R and integrating, we obtain

$$\int_0^{2\pi} |v(re^{ix})| dx \leqq C\int_0^{2\pi} |u(re^{ix})| dx + D\int_0^{2\pi} u \log u dx - 2\pi D u(0) \log u(0),$$

as stated in (2).

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