TRANSITIVE SYSTEMS OF LINEAR OPERATORS ON A BANACH SPACE[1]

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Let $\mathfrak{X}$ be an infinite-dimensional Banach space. Mackey[2] has shown, under more general conditions, that if $x_1, \cdots, x_n, y_1, \cdots, y_n$ are any two sets of n linearly independent elements in $\mathfrak{X}$, then there exists an isomorphism T of $\mathfrak{X}$ with itself such that $T(x_i) = y_i$, $i = 1, \cdots, n$. In other words the collection of isomorphisms is *transitive for linearly independent sets of elements of* $\mathfrak{X}$. This property is shared by many other sets of linear operators, for example by the set of all operators with finite-dimensional range in $\mathfrak{X}$. These two sets of linear operators are semi-groups.[3] In this note we give a condition for the transitivity property above which is necessary for all sets of linear operators and which while not in general sufficient is so for semi-groups. In this result the operators need not be assumed to be bounded or defined everywhere in $\mathfrak{X}$. $\mathfrak{E}(\mathfrak{X})$ will be used to designate the collection of all bounded linear operators with domain $\mathfrak{X}$ and range in $\mathfrak{X}$.

THEOREM 1. *Let $\mathfrak{G}$ be a collection of linear operators with domain and range in an infinite-dimensional Banach space $\mathfrak{X}$. For $\mathfrak{G}$ to be transitive for linearly independent sets of elements of $\mathfrak{X}$ it is necessary that for each finite-dimensional subspace F of $\mathfrak{X}$ there exists a $T \in \mathfrak{G}$ which is the identity on F and a number $\epsilon > 0$ such that if $U \in \mathfrak{E}(\mathfrak{X})$ takes F into F and $\|U\| < \epsilon$, then there exists an operator $V \in \mathfrak{G}$ which agrees with $T + U$ on F. If $\mathfrak{G}$ is a semi-group, this condition is sufficient.*

It is readily seen that the condition is necessary. That it is not sufficient in general can be seen by taking $\mathfrak{G}$ to be an ϵ-neighborhood of the identity in the Banach space $\mathfrak{E}(\mathfrak{X})$.

Let $\mathfrak{G}$ be a semi-group. Let F be a finite-dimensional subspace of $\mathfrak{X}$ and T_0 be its associated operator in $\mathfrak{G}$, and let $x_1, \cdots, x_n$ be linearly independent elements of F. There exist x_i^* in $\mathfrak{X}^*$ (the conjugate space of $\mathfrak{X}$) such that $x_i^*(x_j) = \delta_{ij}$, $i, j = 1, \cdots, n$. For elements

Received by the editors December 28, 1948 and, in revised form, June 13, 1949.

[1] This paper is based on a portion of the author's dissertation, Yale University, 1947. The author wishes to thank the referee for his suggestions which made possible an appreciable shortening of proof of Theorem 1 and clarified the nature of that proof.

[2] G. W. Mackey, *On infinite-dimensional linear spaces*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 155–207, see Theorem II-3.

[3] By a semi-group of linear operators is meant a collection which contains TU whenever it contains T and U.

$v_i \in F$ for which $\|v_i - x_i\| < \delta = \min \{1, \epsilon\}/2n \max \|x_i^*\|$, let the transformation W be determined by

$$W(x) = \sum_{i=1}^{n} x_i^*(x)(v_i - x_i).$$

Then $\|W\| < \min \{1, \epsilon\}/2$. Thus $T_0 + W$ is an isomorphism on F. The operator $U = \sum_{i=1}^{\infty} (-1)^i W^i$ has the properties $\|U\| < \epsilon$ and $(T_0 + U)(T_0 + W)x = x$ for all x in F. By hypothesis there exist transformations U_1 and W_1 in $\mathfrak{G}$ which agree with $T_0 + U$ and $T_0 + W$ respectively in F. Then by the above $W_1(x_i) = v_i$ and $U_1(v_i) = x_i$, $i = 1, \cdots, n$. If also for each i, $\|v_i' - x_i\| < \delta$, $v_i' \in F$, we may define $W_1' \in \mathfrak{G}$ as above so that $W_1'(x_i) = v_i'$. Then $W_1' U_1(v_i) = v_i'$, $i = 1, \cdots, n$. The semi-group property of $\mathfrak{G}$ shows that (a) for each F and $x_1, \cdots, x_n$ linearly independent in F there exists a $\delta > 0$ such that for every v_i, w_i in F with $\|v_i - x_i\| < \delta$ and $\|w_i - x_i\| < \delta$, there exists V in $\mathfrak{G}$ with $V(v_i) = w_i$, $i = 1, \cdots, n$.

Next for a fixed F and $n \leqq \dim F$ consider the Cartesian product space $\bar{F} = F \times \cdots \times F$ (n factors) made up of elements of the form $\bar{x} = (x_1, \cdots, x_n)$ with $\|\bar{x}\| = \max_{i \leqq n} \|x_i\|$. $\bar{F}$ is a finite-dimensional Banach space.[4] Let $\bar{L}$ be the set of all $\bar{x} \in \bar{F}$ such that $x_1, \cdots, x_n$ are linearly independent in F. By (a), $\bar{L}$ is open in $\bar{F}$ and to each point $\bar{x}$ of $\bar{L}$ there exists a $\delta > 0$ such that $\|\bar{x} - \bar{v}\| < \delta$ and $\|\bar{x} - \bar{w}\| < \delta$ imply the existence of a transformation $V \in \mathfrak{G}$ such that $\bar{V}(\bar{v}) = \bar{w}$ where $\bar{V}(\bar{v}) = (V(v_1), \cdots, V(v_n))$. Let $\bar{K}$ be the set of $\bar{y}$ in $\bar{L}$ which can be reached from $\bar{x}$ along a finite chain of such spheres each overlapping its predecessor. Then by (a) for such a $\bar{y}$ there exists a sequence $V_1, \cdots, V_r$ in $\mathfrak{G}$ such that for the product $W = V_r \cdots V_1$, $\bar{W}(\bar{x}) = \bar{y}$ and by the semi-group property, $W \in \mathfrak{G}$.

From this it follows by a standard topological argument, used in connection with analytic continuation and elsewhere,[5] that the set $\bar{K}$ is open and closed in $\bar{L}$. Therefore (b) if $\bar{x}$ and $\bar{y}$ are in the same component of $\bar{L}$, there is a $V \in \mathfrak{G}$ such that $\bar{V}(\bar{x}) = \bar{y}$.

Call $\bar{x}$ and $\bar{y}$ *fully independent* if $x_1, \cdots, x_n, y_1, \cdots, y_n$ form a linearly independent set of $2n$ elements and suppose that $\bar{x}, \bar{y} \in \bar{L}$. For such a pair $\bar{x}$ and $\bar{y}$, and for any real α, we have $\alpha\bar{x} + (1-\alpha)\bar{y} \in \bar{L}$, for $\sum_{i=1}^{n} \beta_i(\alpha x_i + (1-\alpha)y_i) = 0$ implies $\beta_i = 0$, $i = 1, \cdots, n$. Thus the line segment joining $\bar{x}$ and $\bar{y}$ is in $\bar{L}$. Therefore (c) if $\bar{x}$ and $\bar{y}$ are fully independent in $\bar{L}$, they must belong to the same component of $\bar{L}$.

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[4] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, see p. 182.

[5] The use of the set $\bar{K}$ and of this argument was suggested by the referee to replace a more lengthy Heine-Borel argument.

If $\bar{x}$ and $\bar{y}$ are two linearly independent sets of n elements in X which are not fully independent, consider a set $\bar{v}$ of n elements which is fully independent of $\bar{x}$ and also of $\bar{y}$. Such a set $\bar{v}$ exists because $\mathfrak{X}$ is assumed to be infinite-dimensional. The subspace F can be taken to contain all the vectors x_i, y_i, and v_i. For the $\bar{L}$ which corresponds to this F, $\bar{x}$, $\bar{y}$, and $\bar{v}$ must lie in the same component by (c). Then by (b) there exists a $U \in \mathfrak{G}$ with $\bar{U}(\bar{x}) = \bar{y}$. This completes the proof.

Next we consider all our transformations as being in the space $\mathfrak{E}(\mathfrak{X})$. For $\mathfrak{E}(\mathfrak{X})$ the strong topology is defined, following the ideas of von Neumann,[6] as that where the neighborhoods of a transformation U_0 are those of the form

$$N(U_0; x_1, \cdots, x_k, \epsilon) = \{T \in \mathfrak{E}(\mathfrak{X}) \mid \|T(x_i) - U_0(x_i)\| < \epsilon, i = 1, \cdots, k\}$$

where $\epsilon > 0$ and $x_i \in \mathfrak{X}$, $i = 1, \cdots, k$.

THEOREM 2. *A subset $\mathfrak{A}$ of $\mathfrak{E}(\mathfrak{X})$ satisfying the conclusion of Theorem 1 is dense in the strong topology of $\mathfrak{E}(\mathfrak{X})$.*

It suffices to show that the neighborhood $N(U_0; x_1, \cdots, x_k, \epsilon)$ contains an operator in $\mathfrak{A}$. By renumbering, if necessary, we let $x_1, \cdots, x_r$ be a linearly independent subset of the x's which generates the same linear manifold as is generated by all of them. We put $x_i = \sum_{j=1}^{r} a_{ij} x_j$, $i = r+1, \cdots, k$ and $A = \max |a_{ij}|$ for these values of i and for $j = 1, \cdots, r$. Also we set $\eta = \min \{\epsilon, \epsilon/(rA)\}$. We choose $w_1, \cdots, w_r$ in $\mathfrak{X}$ where each w_i is linearly independent of the elements $U_0(x_i)$, $i = 1, \cdots, r$, and of the previously selected w's and each $\|w_i\| < \eta$. The collection $\{U_0(x_i) + w_i\}$ is a linearly independent set. By the assumption on $\mathfrak{A}$, there exists $T \in \mathfrak{A}$ such that $T(x_i) = U_0(x_i) + w_i$, $i = 1, \cdots, r$. Thus for these values of i, $\|T(x_i) - U_0(x_i)\| < \epsilon$. For $i = r+1, \cdots, k$ we have

$$\|T(x_i) - U_0(x_i)\| = \left\|(T - U_0)\left(\sum_{j=1}^{r} a_{ij} x_j\right)\right\| \leqq \sum_{j=1}^{r} |a_{ij}| \, \|T(x_j) - U_0(x_j)\| < \epsilon.$$

Then $T \in N(U_0; x_1, \cdots, x_k; \epsilon)$.

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[6] J. von Neumann, *Zur Algebra der Funktionaloperatoren und Theorie des normalen Operatoren*, Math. Ann. vol. 102 (1929) pp. 370–427.