ON THE PRODUCTS OF FUNCTIONS REPRESENTED AS CONVOLUTION TRANSFORMS

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The authors have in a number of papers, [1], [2], and [3],[1] considered the convolution transforms

$$(1)\qquad f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t)$$

with kernels of the form

$$(2)\qquad \begin{aligned} G(t) &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} E(s)^{-1}e^{st}ds, \\ E(s) &= e^{bs}\prod_{k=1}^{\infty}\left(1-\frac{s}{a_k}\right)e^{s/a_k}. \end{aligned}$$

Here b, $\{a_k\}_1^\infty$ are real constants subject only to the restriction that $\sum_1^\infty a_k^{-2} < \infty$. Let λ_1 and λ_2 be strictly positive real numbers such that $\lambda_1+\lambda_2 \leqq 1$. In the present paper we shall consider results concerning products of functions representable in the form (1) which are consequences of the identity

$$(3)\qquad \begin{aligned} (1-Da^{-1})f(\lambda_1 x)g(\lambda_2 x) &= \lambda_2 f(\lambda_1 x)[g(\lambda_2 x) - a^{-1}g'(\lambda_2 x)] \\ &+ \lambda_1 g(\lambda_2 x)[f(\lambda_1 x) - a^{-1}f'(\lambda_1 x)] + (1-\lambda_1-\lambda_2)f(\lambda_1 x)g(\lambda_2 x). \end{aligned}$$

We recall the definitions $\alpha_1 = \mathrm{Max}_{a_k<0}\ [a_k, -\infty]$, $\alpha_2 = \mathrm{Min}_{a_k>0}\ [a_k, \infty]$, see [2; 1].

THEOREM 1. *Let $G(x) \in$ class* I *and let*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t), \qquad g(x) = \int_{-\infty}^{\infty} G(x-t)d\beta(t)$$

with $\alpha(t) \in \uparrow$, $\beta(t) \in \uparrow$. If $0<\lambda_1$, $0<\lambda_2$, $\lambda_1+\lambda_2 \leqq 1$, then

$$f(\lambda_1 x)g(\lambda_2 x) = \int_{-\infty}^{\infty} G(x-t)d\gamma(t)$$

with $\gamma(t) \in \uparrow$.

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[1] The first number in brackets refers to the references cited at the end of the paper. The number following the semicolon gives the section number of the reference cited.

PROOF. By [3; 36] necessary and sufficient conditions for $h(x)$ to have the representation

$$h(x) = \int_{-\infty}^{\infty} G(x-t)d\delta(t) \tag{4}$$

with $\delta(t) \in \uparrow$ are:

a. $$h(x) \in C^\infty \qquad (-\infty < x < \infty),$$

b. $$h(x) = o(e^{\alpha_1 x}) \qquad (x \to -\infty)$$
$$= o(e^{\alpha_2 x}) \qquad (x \to +\infty),$$

c. $$\prod_{j=1}^{n} (1 - DA_j^{-1})h(x) \geqq 0 \qquad (-\infty < x < \infty)$$

for every selection $\{A_1, A_2, \cdots, A_n\}$ from $\{a_1, a_2, \cdots\}$. Our assumptions imply that $f(x)$ and $g(x)$ fulfill these conditions; it follows that $f(\lambda_1 x)g(\lambda_2 x)$ also satisfies these conditions (a and b trivially, and c because of (3)). Our desired result follows.

The following theorem may be proved similarly. See [3; 36, 37].

THEOREM 2. *Let* $G(x) \in$ *class* II *or* III *and let*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t) \qquad (x > r_1),$$

$$g(x) = \int_{-\infty}^{\infty} G(x-t)d\beta(t) \qquad (x > r_2)$$

with $\alpha(t) \in \uparrow$, $\beta(t) \in \uparrow$. If $0 < \lambda_1$, $0 < \lambda_2$, $\lambda_1 + \lambda_2 \leqq 1$, *then*

$$f(\lambda_1 x)g(\lambda_2 x) = \int_{-\infty}^{\infty} G(x-t)d\gamma(t) \qquad (x > \text{Max } [r_1/\lambda_1, r_2/\lambda_2])$$

where $\gamma(t) \in \uparrow$.

Using a further representation theorem [3; 32] one may establish

THEOREM 3. *Let* p, q, *and* r *be numbers greater than one such that* $r^{-1} = p^{-1} + q^{-1}$, *and let*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)e^{-c_1 t}\phi(t)dt,$$

$$g(x) = \int_{-\infty}^{\infty} G(x-t)e^{-c_2 t}\psi(t)dt,$$

where $\alpha_1 < c_1 < \alpha_2$, $\alpha_1 < c_2 < \alpha_2$, *and where* $\phi(t) \in L_p(-\infty, \infty)$, $\psi(t)$

$\in L_q(-\infty,\ \infty)$. If $0<\lambda_1$, $0<\lambda_2$, $\lambda_1+\lambda_2\leqq 1$, *then*

$$f(\lambda_1 x)g(\lambda_2 x) = \int_{-\infty}^{\infty} G(x-t)e^{-ct}\chi(t)dt$$

where $c=\lambda_1 c_1+\lambda_2 c_2$, *and where* $\chi(t)\in L_r(-\infty,\ \infty)$. *More precisely we have*

$$\|\chi(t)\|_r \leqq E(c_1)^{-1}E(c_2)^{-1}\lambda_1^{-1/p}\lambda_2^{-1/q}\|\phi(t)\|_p\|\psi(t)\|_q.$$

As an application of Theorem 1 we have the following result concerning the Stieltjes transform. If

$$A(x) = \int_{0+}^{\infty}(x+t)^{-1}d\alpha(t), \qquad B(x) = \int_{0+}^{\infty}(x+t)^{-1}d\beta(t)$$

where $\alpha(t)\in\uparrow$, $\beta(t)\in\uparrow$, and if $0<\lambda_1$, $0<\lambda_2$, $\lambda_1+\lambda_2\leqq 1$, then

$$A(x^{\lambda_1})B(x^{\lambda_2}) = \int_{0+}^{\infty}(x+t)^{-1}d\gamma(t)$$

with $\gamma(t)\in\uparrow$. Choosing $A(x)=B(x)=x^{-1+\epsilon}$, $0<\epsilon<1$, we see that the condition $\lambda_1+\lambda_2\leqq 1$ is essential.

References

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2. ———, *The inversion of a general class of convolution transforms*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 135–201.

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