THE CLOSURE OF TRANSLATIONS IN L^p

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1. Introduction. As Segal [2][1] has observed, Wiener's problem concerning the translations of a function $k(x)$ in $L^p(-\infty, \infty)$, $p>1$, may be stated as follows: to find necessary and sufficient conditions on the suitably defined Fourier transform $K(t)$ of $k(x)$ in order that the integral equation

$$\int_{-\infty}^{\infty} k(y)\phi(x-y)dy = 0, \qquad -\infty < x < \infty, \tag{1.1}$$

admit no solution $\phi(x)$ in $L^{p'}$, $p'=p/(p-1)$, except $\phi\equiv 0$. A complete account of the present status of the problem is given in Segal's paper and it is unncessary to review the facts here.

It is the purpose of this paper to point out the close connection between this problem and a certain uniqueness problem for trigonometric integrals. A point set S on the real axis is said to be *of type* q if the conditions

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} e^{-\sigma|x|}e^{ixt}\phi(x)dx = 0, \qquad t \notin S; \tag{i}$$

$$\phi(x) \in L^q(-\infty, \infty) \tag{ii}$$

can be satisfied simultaneously only by $\phi\equiv 0$.

The results are then as follows.

THEOREM A. *Let $k(x)$ belong to L and to L^p for some $p>1$. If the set of zeros of its Fourier transform $K(t)=\int_{-\infty}^{\infty}e^{ixt}k(x)dx$ is of type p', then* (1.1) *admits $\phi\equiv 0$ as its only solution in $L^{p'}$.*

THEOREM B. *Let $k(x)$ belong to L^p and $|x|^{1/p}|k(x)|$ to L. Then if the set of zeros of $K(t)$ is not of type p', the equation* (1.1) *admits a nontrivial solution in L^p.*

S is said to be *of type* A if the conditions (i) and

$$\int_{a}^{a+1} |\phi(x)|\,dx = o(1), \qquad |a|\to\infty, \tag{iii}$$

can be satisfied only by $\phi(x)\equiv 0$. Finally, S is said to be *of type* U if the conditions

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$\lim_{R\to\infty}\int_{-R}^{R} e^{ixt}\phi(x)dx = 0, \qquad t \notin S,$$

and (iii) imply that $\phi\equiv 0$. A closed set of type U is of type A, and conversely [3]. Moreover a set of type A is of type q for each $q>1$; this follows from Holder's inequality.

Now under the hypotheses of Theorem A the zeros of $K(t)$ form a closed set, for then $K(t)$ is continuous. Hence we have the following corollary of Theorem A.

COROLLARY. *If $k(x)$ belongs to L and L^p, and the zeros of $K(t)$ form a set of type* U, *then the translations of $k(x)$ span L^p.*

Since any countable set is of type U, the corollary supersedes the best sufficient condition given so far for $p>1$, $p\neq 2$ [2, p. 96]. Clearly if $|x|^{1/p}|k(x)|$ is in L and $k(x)$ in L^p, then Theorems A and B together show that type p' is both necessary and sufficient.

2. **Proof of Theorem A.** Let us denote the set of zeros of $K(t)$ by Z, and suppose that Z is of type p'. Let $\phi(x)$ be an $L^{p'}$ solution of (1.1). We shall show that it must vanish almost everywhere. For this purpose we adopt a device of Beurling [1]. Since

$$\int_{-\infty}^{\infty} k(y-\xi)\phi(x-y)dy = 0,$$

we can multiply by $e^{-\sigma|x|}e^{it(x-\xi)}dx$ and integrate over $(-\infty, \infty)$ to obtain

$$\int_{-\infty}^{\infty} k(y-\xi)e^{it(y-\xi)}U(\sigma, t, y)dy = 0, \qquad \sigma > 0,$$

where

$$U(\sigma, t, y) = \int_{-\infty}^{\infty}\phi(x)e^{-\sigma|x+y|}e^{itx}dx. \tag{2.1}$$

From this we may conclude that for each $a>0$

$$\begin{aligned}\int_{-\infty}^{\infty} & k(y-\xi)e^{it(y-\xi)}U(\sigma, t, \xi)dy \\ &= \int_{|y-\xi|\leqq a} k(y-\xi)e^{it(y-\xi)}\{U(\sigma, t, \xi) - U(\sigma, t, y)\}dy \\ &\quad + \int_{|y-\xi|\geqq a} k(y-\xi)e^{it(y-\xi)}\{U(\sigma, t, \xi) - U(\sigma, t, y)\}dy.\end{aligned}$$

From the definition of $K(t)$ and Rolle's theorem we have

$$
\begin{aligned}
&| U(\sigma, t, \xi) | \, | K(t) | \\
(2.2) \qquad &\leq \int_{|y-\xi|\leq a} | k(y-\xi) | \, | y-\xi | \left| \frac{\partial}{\partial y} U(\sigma, t, y') \right| dy \\
&\quad + 2 \operatorname*{u.b.}_{\xi} | U(\sigma, t, \xi) | \int_{|y-\xi|\geq a} | k(y-\xi) | \, dy.
\end{aligned}
$$

Now by (2.1) and Hölder's inequality

$$
\begin{aligned}
(2.3) \qquad \left| \frac{\partial U}{\partial y} \right| &\leq \sigma \int_{-\infty}^{\infty} | \phi(x) | \, e^{-\sigma|x+y|} dx \\
&\leq A\sigma \left(\int_{-\infty}^{\infty} e^{-\sigma p|x+y|} dx \right)^{1/p} \leq A\sigma^{1/p'}.
\end{aligned}
$$

If we apply this inequality to (2.2) and make a change of variable, we find that

$$
\operatorname*{u.b.}_{\xi} | U(\sigma, t, \xi) | \left\{ | K(t) | - 2 \int_{|y|\geq a} | k(y) | \, dy \right\} \leq A\sigma^{1/p'}.
$$

Now suppose that t does not belong to Z. Then $K(t) \neq 0$. Hence for some positive a

$$
| K(t) | - 2 \int_{|y|\geq a} | k(y) | \, dy > 0.
$$

It follows that $\lim_{\sigma\to 0} | U(\sigma, t, \xi) | = 0$ for all ξ, in particular for $\xi = 0$. Therefore

$$
(2.4) \qquad \lim_{\sigma\to 0} \int_{-\infty}^{\infty} e^{-\sigma|x|} e^{ixt} \phi(x) dx = 0, \qquad t \notin Z.
$$

Since Z is of type p', ϕ must vanish almost everywhere.

3. **Proof of Theorem B.** Suppose that Z is not of type p'. Then there exists a function ϕ in $L^{p'}$, $\phi \not\equiv 0$, for which (2.4) is satisfied. Define $g(x)$ by the formula

$$
(3.1) \qquad g(x) = \int_{-\infty}^{\infty} k(x-y) \phi(y) dy;
$$

$g(x)$ is a continuous function in $L^{p'}$. We shall show that $g(x)$ vanishes identically, and in this way demonstrate the existence of a nontrivial solution of (1.1).

Multiply both sides of (3.1) by $e^{-\sigma|x|}e^{itx}dx$ and integrate from $-\infty$ to ∞. An application of Fubini's theorem yields

$$U_g(\sigma, t) = \int_{-\infty}^{\infty} e^{ity}k(y)U(\sigma, t, y)dy,$$

where $U_g(\sigma, t) = \int_{-\infty}^{\infty} e^{-\sigma|x|}e^{itx}g(x)dx$. Then

$$U_g(\sigma, t) - K(t)U(\sigma, t, 0) = \int_{-\infty}^{\infty} e^{ity}k(y)\{U(\sigma, t, y) - U(\sigma, t, 0)\}dy.$$

Consequently, by Rolle's theorem once again,

$$\begin{aligned} &| U_g(\sigma, t) - K(t)U(\sigma, t, 0) | \\ &\qquad \leq \int_{|y|\leq R(\sigma)} | k(y) | | y | \left| \frac{\partial U}{\partial y}(\sigma, t, y') \right| dy \\ &\qquad\quad + \int_{|y|\geq R(\sigma)} | k(y) | | U(\sigma, t, y) - U(\sigma, t, 0) | dy, \end{aligned} \tag{3.2}$$

where $R(\sigma)$ is any positive function of σ. We shall suppose it subject to the conditions $R(\sigma)\to\infty$, $\sigma^{1/p'}R(\sigma)\to 0$ as $\sigma\to\infty$. If we apply (2.3) to the first term of the right-hand member of (3.2), and (2.1) to the second term, we obtain

$$| U_g(\sigma, t) - K(t)U(\sigma, t, 0) | \leq A\sigma^{1/p'}R(\sigma) \int_{|y|\leq R(\sigma)} | k(y) | dy + F(\sigma),$$

where

$$F(\sigma) = \int_{|y|\geq R(\sigma)} | k(y) | dy \int_{-\infty}^{\infty} | e^{-\sigma|x+y|} - e^{-\sigma|x|} | | \phi(x) | dx. \tag{3.3}$$

Granting that we shall prove shortly that $\lim_{\sigma\to 0} F(\sigma) = 0$, it follows that

$$\lim_{\sigma\to 0} \{U_g(\sigma, t) - K(t)U(\sigma, t, 0)\} = 0.$$

Now if $t\in Z$, $K(t)=0$. On the other hand if $t\notin Z$, (2.4) asserts that $\lim_{\sigma\to 0} U(\sigma, t, 0)=0$. Hence in either case $\lim_{\sigma\to 0} U_g(\sigma, t)=0$, that is

$$\lim_{\sigma\to 0} \int_{-\infty}^{\infty} e^{-\sigma|x|}e^{ixt}g(x)dx = 0, \qquad -\infty < t < \infty,$$

Since $g(x)$ belongs to $L^{p'}$, $\int_a^{a+1}|g(x)|dx = o(1)$. Consequently the uniqueness theory of trigonometric integrals [3] enables us to conclude that $g(x)$ vanishes almost everywhere. Since it is continuous, it vanishes identically.

There remains the assertion concerning $F(\sigma)$ as $\sigma \to 0$. Write the exponential factor in (3.3) as $|\cdots|^{1/p}|\cdots|^{1/p'}$. An application of Hölder's inequality to the inner integral gives us

$$|F(\sigma)| \leqq \int_{|y| \geqq R(\sigma)} |k(y)|\, dy \left\{ \int_{-\infty}^{\infty} |\cdots| \, |\phi(x)|^{p'} dx \right\}^{1/p'} \cdot \left\{ \int_{-\infty}^{\infty} |\cdots| \, dx \right\}^{1/p}.$$

A simple calculation shows that

$$\int_{-\infty}^{\infty} |\cdots| \, dx \leqq A|y|,$$

independently of σ. Hence

$$|F(\sigma)| \leqq A \int_{|y| \geqq R(\sigma)} |k(y)| \, |y|^{1/p} dy.$$

Since $R(\sigma) \to \infty$ as $\sigma \to 0$, the proof is complete.

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