REMARKS ON MINIMAL IDENTITIES FOR ALGEBRAS

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1. **Introduction.** The purpose of the present note is to supplement in some points the results obtained by the authors in a previous communication. Let A_n denote the total matric algebra of order n^2 over a field F. In [1][1] we have determined the totality of minimal identities satisfied by A_n, in all cases where $n>2$ or $F\neq P_2$, where P_2 denotes the prime field of characteristic 2. In all these cases each minimal polynomial is (but for a numerical factor) either a standard polynomial of degree $2n$ or a sum of such standard polynomials. This is not so if $n\leqq 2$ and $F=P_2$.

In [1, Theorem 6] we have shown that in these two exceptional cases nonlinear minimal polynomials do exist. In §2 of the present note we determine *the totality* of the minimal identities in these exceptional cases.

In [1] it was shown that all *linear* minimal polynomials of a simple or a semi-simple algebra are again the standard polynomials and their linear combinations. In §3 we prove that in general *all minimal* polynomials of a semi-simple algebra are linear, hence all results on minimal polynomials for total matric algebras, which we have obtained in [1], may be extended to simple and semi-simple algebras.

For an algebra A with a radical B we have found in [1] an identity whose degree depends on the index of B and the orders of the simple constituents of the difference algebra $A-B$. This yields an upper bound and a lower bound for the degree of a minimal identity in the non-semi-simple case. In §4 of the present note we show by examples that these estimates are in a way the best possible ones.

2. **The minimal polynomials in the exceptional cases.** We first dispose of the case $n=1$ and $F=P_2$, that is, $A_1=P_2$. The only nonlinear minimal polynomial depending on one indeterminate x is the polynomial x^2+x. The only minimal polynomial depending on two indeterminates x_1, x_2 and linear in each of these indeterminates is by [1, Theorems 1, 7] the standard polynomial $S(x_1, x_2)=x_1x_2-x_2x_1$.

For an arbitrary set of indeterminates $x_1, \cdots, x_k$ ($k\geqq 2$), denote by M_1 the module over P_2 defined by the set of all minimal polynomials of A_1, depending on $x_1, \cdots, x_k$. It is readily seen that a basis of M_1 is constituted by the following polynomials,

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$x_i^2 + x_i, \qquad S(x_{j_1}, x_{j_2}),$$

where $i=1, 2, \cdots, k$ and (j_1, j_2) ranges over all combinations of two letters out of k. The dimensionality of this module is, therefore, $k+C_{k,2}$.

Consider now the algebra A_2 over the field $F=P_2$, and let $f=f(x_1, \cdots, x_k)$ be a minimal polynomial of A_2 (hence of degree 4) so that each monomial of f has a degree $\geqq 1$ in each of the x_i, that is, $k \leqq 4$. It is sufficient to determine all minimal polynomials satisfying this condition, since by [1, Lemma 7] every minimal polynomial may be represented as a sum of minimal polynomials of this type.

The elements of A_2 over P_2 satisfy one of the following 4 equations:

$$y^2 = 0, \qquad y^2 = y, \qquad y^2 = 1, \qquad y^2 = y + 1$$

and each of these equations is satisfied by some elements of A_2.This implies that no identity of the form: $x^4+\beta_1x^3+\beta_2x^2+\beta_3x+\beta_4=0$ $(\beta_i \in P_2)$ is satisfied by A_2, and hence $k \geqq 2$. If further $k=4$, that is, f is linear in each of the indeterminates, by [1, Theorem 2] we know that f is the standard polynomial $S(x_1, x_2, x_3, x_4)$. Thus it remains to determine all minimal polynomials $f(x_1, \cdots, x_k)$ with $k=2, 3$.

Consider first the case $k=3$. In this case we may write f in the form

$$f = f_0 + f_1 + f_2 + f_3 \tag{1}$$

where each of the monomials of f_0 with a nonzero coefficient is of degree 1 in each x_j (that is, f_0 is either zero or of degree 3) while for $i \geqq 1$, each monomial of f_i with a nonzero coefficient has degree 2 in x_i and degree 1 in x_k, $k \neq i$. This implies that at least one of the f_i, $i \geqq 1$, is not equal to 0 and we may assume that $f_1 \neq 0$. Hence (1) is a special case of formula (30) in [1] and we may apply the results obtained in [1]. Thus we have according to formula (34) of [1]:

$$f = f_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3, \qquad \alpha_1 \neq 0,$$

where p_i is the sum of all 12 monomials having degree 2 in x_i and degree 1 in x_k, $k \neq i$.

First apply the substitution $x_1 = e_{12}$, $x_2 = e_{22}$, $x_3 = e_{21}$., The only monomial linear in each x_i which yields under this substitution the unit e_{11} is $x_1x_2x_3$. It is readily verified that $p_i(e_{12}, e_{22}, e_{21}) = 0$, $i=1, 3$, and $p_2(e_{12}, e_{22}, e_{21}) = e_{11}$. This implies that α_2 is also the coefficient of the monomial $x_1x_2x_3$ of f_0. A permutation of x_1 and x_3 in the last substitution shows that α_2 is also the coefficient of the monomial $x_3x_2x_1$. Similar results may be obtained for α_1 and α_3, hence

$$f = \alpha_1(x_2x_1x_3 + x_3x_1x_2 + p_1) + \alpha_2(x_1x_2x_3 + x_3x_2x_1 + p_2)$$
$$+ \alpha_3(x_1x_3x_2 + x_2x_3x_1 + p_3).$$

Now apply the substitution: $x_1 = e_{11}$, $x_2 = e_{12}$, $x_3 = e_{22}$. The only monomials of f which yield e_{12} are $x_1^2x_2x_3$, $x_1x_2x_3$, and $x_1x_2x_3^2$, hence $\alpha_1+\alpha_2+\alpha_3=0$. Since each α_i is either 0 or 1, and $\alpha_1 \neq 0$, it follows that either $\alpha_2=0$, $\alpha_3=1$, or $\alpha_2=1$, $\alpha_3=0$. Denote by (i, j, k) any permutation of the three indices 1, 2, 3 and put

$$G_i = x_jx_ix_k + x_kx_ix_j + p_i, \qquad i = 1, 2, 3; \tag{2}$$

then f is either G_1+G_2 or G_1+G_3. Consider the three polynomials

$$H_1 = G_2 + G_3, \qquad H_2 = G_1 + G_3, \qquad H_3 = G_1 + G_2.$$

Each polynomial H_i is of degree 1 in x_i and of degree 2 in x_k, $k \neq i$. Since the underlying field is of characteristic 2 it follows that $H_1+H_2+H_3=0$. The polynomials H_i may be transformed into each other by changing the roles of the indeterminates, and thus it follows that f must be one of the three polynomials H_1, H_2, H_3.

Since it was shown in [1, Theorem 6] that the identity

$$Q(x, y) = xy^3 + yxy^2 + y^2xy + y^3x + xy^2 + y^2x = 0$$

is satisfied by A_2 over P_2, and it is readily seen that

$$Q(x_1, x_2 + x_3) - Q(x_1, x_2) - Q(x_1, x_3) = H_1(x_1, x_2, x_3),$$

we conclude that the identity $H_1=0$ and hence, also, $H_2=0$ and $H_3=0$ are indeed satisfied by the algebra A_2 over P_2.

Thus we have:

THEOREM 1a. *The polynomial $f(x_1, x_2, x_3)$, where each term is of degree $\geqq 1$ in each x_i, is a minimal polynomial of A_2 over P_2 if and only if f is one of the polynomials H_1, H_2, H_3.*

We now turn to the case where the minimal polynomial $f(y_1, y_2)$ depends on 2 indeterminates. The polynomial f has monomials with degree $\geqq 2$ in one of the y's, say in y_2. The polynomial $F_1(x_1, x_2, x_3)$ defined by

$$F_1(x_1, x_2, x_3) = f(x_1, x_2 + x_3) - f(x_1, x_2) - f(x_1, x_3)$$

is again a minimal polynomial of A_2. It is evident that F_1 is symmetric in x_2 and x_3, and each term of F_1 is of degree $\geqq 1$ in each of the x's. Hence it follows by the preceding theorem that $F_1(x_1, x_2, x_3) \equiv H_1(x_1, x_2, x_3)$. We have already seen that

$$Q_1(x_1, x_2 + x_3) - Q_1(x_1, x_2) - Q_1(x_1, x_3) = H_1(x_1, x_2, x_3)$$

where $Q_1(y_1, y_2) = y_1y_2^3 + y_2y_1y_2^2 + y_2^2y_1y_2 + y_2^3y_1 + y_1y_2^2 + y_2^2y_1$. Hence, by putting $f_1(y_1, y_2) = f(y_1, y_2) - Q_1(y_1, y_2)$, it follows that $f_1(x_1, x_2+x_3) - f_1(x_1, x_2) - f_1(x_1, x_3) \equiv 0$ identically in x_1, x_2, x_3. This implies that either $f_1(y_1, y_2) \equiv 0$ or $f_1(y_1, y_2)$ is linear in y_2. In the former case we obtain $f(y_1, y_2) \equiv Q_1(y_1, y_2)$, while in case $f_1(y_1, y_2) \not\equiv 0$ we know that $f_1(y_1, y_2)$ is again a minimal polynomial of A_2 over P_2 such that each term of f_1 is of degree $\geqq 1$ in y_1 and in y_2. Since f_1 is linear in y_2, it must be of degree $\geqq 2$ in y_1. Hence, in a similar manner we show that the polynomial $F_2(x_1, x_2, x_3) = f_1(x_1+x_2, x_3) - f_1(x_1, x_3) - f_1(x_2, x_3)$ is equal to H_3. Since the polynomial

$$Q_2(y_1, y_2) = y_1^3y_2 + y_1^2y_2y_1 + y_1y_2y_1^2 + y_2y_1^3 + y_1^2y_2 + y_2y_1^2$$

also satisfies $Q_2(x_1+x_2, x_3) - Q_2(x_1, x_3) - Q_2(x_2, x_3) = H_3(x_1, x_2, x_3)$, it follows similarly that either the polynomial $f_1(y_1, y_2) - Q_2(y_1, y_2) = f_2(y_1, y_2)$ is zero, or f_2 must be linear in y_1. The latter possibility leads to a contradiction, since in this case f_2 must be linear in y_2 also, which implies that the general degree of f_2 is less than 4, which is impossible, since f_2 is a minimal polynomial of A_2. This implies that $f(y_1, y_2) = Q_1(y_1, y_2) + Q_2(y_1, y_2)$. It has already been shown that the identities $Q_1(y_1, y_2) = 0$, $Q_2(y_1, y_2) = 0$ hold in A_2 over P_2. Hence, also the identity $Q_1 + Q_2 = 0$ holds in A_2 over P_2 and we have:

THEOREM 1b. *A polynomial $f(y_1, y_2)$, such that each term of f is of degree not less than 1 in y_1 and y_2, is a minimal polynomial of A_2 over P_2 if and only if $f = Q_1$, or $f = Q_2$, or $f = Q_1 + Q_2$.*

By summarizing above results we get:

THEOREM 1. *Let $f(x_1, \cdots, x_k)$ be a minimal polynomial of A_2 over P_2, such that each monomial of f is of a degree $\geqq 1$ in each x_i, then $2 \leqq k \leqq 4$, and:*

(1) *If $k=4$ then $f(x_1, x_2, x_3, x_4) = S(x_1, x_2, x_3, x_4)$.*

(2) *If $k=3$ then f is one of the polynomials H_1, H_2, H_3.*

(3) *If $k=2$ then f is one of the polynomials Q_1, Q_2, Q_1+Q_2.*

Since by [1, Lemma 7] it follows that every minimal polynomial may be represented as a sum of polynomials of the type mentioned in the preceding theorem, we have:

THEOREM 2. *The module M_2 defined by the minimal polynomials of A_2 over P_2, depending on the indeterminates $x_1, x_2, \cdots, x_k$, $k \geqq 4$, has the dimensionality $C_{k,4} + 2C_{k,3} + 2C_{k,2}$. As a basis for M_2 we may choose the polynomials*:

$S(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$, $H_1(x_{i_1}, x_{i_2}, x_{i_3})$, $H_2(x_{i_1}, x_{i_2}, x_{i_3})$, $Q_1(x_{i_1}, x_{i_2})$, $Q_2(x_{i_1}, x_{i_2})$

where (i_1, i_2, i_3, i_4) *is an arbitrary combination of* 4 *letters out of* k.

For $k=3$ *we have the basis*

$$H_1(x_1, x_2, x_3), \qquad H_2(x_1, x_2, x_3), \qquad Q_1(x_{i_1}, x_{i_2}), \qquad Q(x_{i_1}, x_{i_2})$$

where (i_1, i_2) *is an arbitrary combination of* 2 *letters out of* 3, *and in this case the dimensionality of* M_2 *is* 8.

For $k=2$, *the dimensionality of* M_2 *is* 2 *and we have the basis*

$$Q_1(x_1, x_2), \qquad Q_2(x_1, x_2).$$

For later reference we need the following remark.

REMARK. If in H_1 (resp. Q_1) one ignores the order of the factors, one obtains $H_1 = 4x_1x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2$ (resp. $Q_1 = 2x_1x_2^2 + 4x_1x_2^3$).

3. **Simple and semi-simple algebras.** We shall need the following generalization of Kaplansky's Lemma 3 [2].

LEMMA. *If an algebra* A *over* F *satisfies an identity* $f(x_1, \cdots, x_k)=0$ *which is homogeneous in each* x_i *and of degree not greater than* 2 *in each* x_i, *then the given identity is satisfied also by the direct product* $A \times G$, *where* G *is a field containing* F.

PROOF. We prove the lemma by induction on the number of the indeterminates x_i whose degree in f is 2.

By Lemma 3 in [2] our lemma holds when f is linear in each x_i. Suppose now that $f(x_1, \cdots, x_k)$ is of degree 2 in x_i where $1 \leqq i \leqq k$ and consider the polynomial

$$\begin{aligned} &g_i(x_1, \cdots, x_{i-1}, u, v, x_{i+1}, \cdots, x_k) \\ (3) \qquad &= f(x_1, \cdots, x_{i-1}, u+v, x_{i+1}, \cdots) - f(\cdots, x_{i-1}, u, x_{i+1}, \cdots) \\ &\quad - f(\cdots, x_{i-1}, v, x_{i+1}, \cdots). \end{aligned}$$

This polynomial is apparently homogeneous in each of its indeterminates, and the number of indeterminates whose degree in g_i is 2 is less than that of f. Since the identity $g_i=0$ holds in A, we may assume (by induction) that the identity $g_i=0$ holds also in $A \times G$. By (3) we have, for any sequence of $k+1$ elements $b_1, b_2, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_k$ belonging to $A \times G$, the relation

$$\begin{aligned} (4) \qquad &f(a_1, \cdots, a_{i-1}, b_1 + b_2, a_{i+1}, \cdots, a_k) \\ &\qquad = f(\cdots, b_1, \cdots) + f(\cdots, b_2, \cdots). \end{aligned}$$

Since relation (4) evidently holds also in case $f(x_1, \cdots, x_k)$ is linear in x_i, we may assume its validity for each x_i, $1 \leqq i \leqq k$. Now each element $a \in A \times G$ has the form $a = \sum \gamma_j a_j$ where $a_j \in A$ and $\gamma_j \in G$. Hence in view of (4) it remains only to show that $f(x_1, \cdots, x_k)=0$

for $x_i=\delta_i b_i$, $b_i\in A$, $\delta_i\in G$. This is evident, since $f(\delta_1 b_1, \cdots, \delta_k b_k) = \delta_1^{\nu_1}\cdots\delta_k^{\nu_k}f(b_1, \cdots, b_k)=0$ where ν_i is the degree of x_i in f. This completes the proof of the lemma.

With the aid of the preceding lemma we now extend Theorems 4 and 5 of [1] to the general case of simple algebras by proving the following theorem:

THEOREM 3. *Let A be a simple algebra of order n^2 over its centre, and suppose that A is neither P_2 nor is A the total matric algebra A_2 over P_2. Then $f(x_1, \cdots, x_m)=0$ is a minimal identity of A if and only if $m \geqq 2n$ and*

$$f(x_1, \cdots, x_m) = \sum_{(i)} \alpha_{(i)} S(x_{i_1}, \cdots, x_{i_{2n}})$$

where the sum ranges over all $C_{m,2n}$ combinations (i) *of $2n$ letters out of m letters, and the $\alpha_{(i)}$ are in the underlying field.*

PROOF. Since A is a normal simple algebra over its centre C, there exists a field G containing C, for which $A\times G$ over C is the total matric algebra A_n over G. The algebra $A\times G$ is apparently neither the algebra A_2 over P_2 nor is $A\times G$ the field P_2 since A is not one of these exceptional cases. We assert first that the minimal polynomials of A are linear in all their indeterminates. Indeed suppose that A possesses nonlinear minimal polynomials. Then by Lemmas 5 and 6 of [1] we may assume that there exists a minimal polynomial $f(x_1, \cdots, x_m)$ of A which is homogeneous and of degree not greater than 2 in each of the x's and of degree 2 in some of them. By the preceding lemma it follows that the identity $f=0$ is satisfied also by the algebra $A\times G$. But since $A\times G$ is a total matric algebra and $A\times G$ is not one of the exceptional cases, this contradicts Theorem 5 of [1]. This implies that every minimal identity $f=0$ satisfied by A is a linear identity. By Theorem 4 of [1] it follows, therefore, that $m\geqq 2n$, and

$$f(x_1, \cdots, x_m) = \sum_{(i)} \alpha_{(i)} S(x_{i_1}, \cdots, x_{i_{2n}})$$

where $\alpha_{(i)}\in G$. One readily verifies that $\alpha_{(i)}\in F$, since f is a polynomial in F.

By [1, Theorem 7] we know that the converse is true also, that is, the identities of the type $\sum_{(i)} \alpha_{(i)} S(x_{i_1}, \cdots, x_{i_{2n}})=0$ are satisfied by the algebra A. This completes the proof of the theorem.

Consider now a semi-simple algebra A over F, and suppose that A is a direct sum of the simple algebras $A', A'', \cdots, A^{(k)}$. Denote by n_i^2 the order of $A^{(i)}$ over its centre, and put $n^2=\max(n_1^2, \cdots, n_k^2)$.

It is readily verified[2] that an identity $f=0$ is satisfied by A if and only if it is satisfied by every constituent $A^{(i)}$.

Consider first the following exceptional cases:

(I) All algebras $A^{(i)}$ are isomorphic with P_2. This implies $n=1$ and $F=P_2$. Hence, $f=0$ is a minimal identity of A if and only if $f=0$ is a minimal identity of P_2, and these identities were determined in the previous section.

(II) Some algebras $A^{(i)}$ are the algebras A_2 over P_2, and the remaining algebras $A^{(i)}$ (if any) are commutative fields of characteristic 2. This implies $F=P_2$, $n=2$.

By the remark at the end of the previous section it follows that all minimal identities of A_2 over P_2 are satisfied also by commutative fields of characteristic 2. This implies that in case (II) the module of the minimal polynomials of the algebra A is the same as the module of the minimal polynomials of the algebra A_2 over P_2, and a base for the latter module was given in Theorem 2.

Now we turn to the general case, that is, either $n>2$ or $F\neq P_2$, or in case $n=2$ and $F=P_2$, some algebra $A^{(i)}$ is a total matric algebra of degree 2 over a field $\neq P_2$. For such algebras we prove the validity of Theorem 3, that is:

THEOREM 4. *Let A be a semi-simple algebra not of the types* (I) *or* (II), *then $f(x_1, \cdots, x_m)=0$ is a minimal identity of A if and only if $m\geqq 2n$, and*

$$f(x_1, \cdots, x_m) = \sum_{(i)} \alpha_{(i)} S(x_{i_1}, \cdots, x_{i_{2n}})$$

where the sum ranges over all $C_{m,2n}$ combinations (i) of $2n$ letters out of m letters, and $\alpha_{(i)}$ are in F.

PROOF. We have already seen that A satisfies an identity $f=0$ if and only if $f=0$ is satisfied by every constituent $A^{(i)}$. The minimal identities of the algebra $A^{(j)}$ of order $n_j^2=n^2$ over its centre are satisfied also by the algebra $A^{(i)}$ for which $n_i\leqq n_j=n$ if either $n>2$ or $F\neq P_2$, and when $n=2$ and $F=P_2$, the minimal identities of the algebra $A^{(j)}$ of order 4 over its centre C such that $C\supset P_2$ are also satisfied by the other constituents $A^{(i)}$. This implies that f is a minimal polynomial of A if and only if f is a minimal polynomial of that particular algebra $A^{(j)}$. Hence our theorem is an immediate consequence of Theorem 3.

4. **Algebras with radical.** Let r be the index of the radical N of

[2] See also [1, §4].

the algebra B over F, and let $n^2 = \max(n_1^2, \cdots, n_k^2)$ where n_i^2 are the orders of the simple constituents of the difference algebra $B-N$. We prove:

THEOREM 5. *Denote by m the degree of the minimal polynomial of B. Then:*

(1) $2n \leqq m \leqq 2nr$;

(2) *There exist algebras B over F with the index r for which $m = 2n$, as well as algebras for which $m = 2nr$.*

PROOF. By [1, Theorem 9] we know that B has identities of degree $2nr$. This implies that $m \leqq 2nr$. Since an identity satisfied by B is satisfied also by $B-N$, it follows that $m \geqq 2n$, that is, $2n \leqq m \leqq 2nr$.

Now let $B_1 = A_n + N$ denote a direct sum of a total matric algebra A_n over F and a nilpotent algebra N of index $t \leqq 2n$. The identity $S(x_1, \cdots, x_{2n}) = 0$ is satisfied by both A_n and N and hence also by B_1. This implies that the minimum degree m_1 of B_1 is at most $2n$. On the other hand $m_1 \geqq 2n$. It follows, therefore, that $m_1 = 2n$.

Finally consider the algebra B_2, defined as the algebra of all matrices (a_{ik}) of order r over F where $a_{ik} = 0$ for $i > k$, that is, the ring of all matrices of order r with zeros beneath the diagonal. The radical N_2 of this algebra is the set of all matrices (a_{ik}) where $a_{ik} = 0$ when $i \geqq k$, that is, N_2 is the set of all matrices of B_2 with zeros in the diagonal, and its index is therefore equal to r. The semi-simple algebra $B_2 - N_2$ is a direct sum of r commutative fields, that is, $n = 1$. The minimum degree m_2 of B_2 is therefore subject to the inequality $2 \leqq m_2 \leqq 2r$. If B_2 has a polynomial identity of degree $m_2 < 2r$, then evidently it possesses also an identity of degree $2r-1$. Hence, by [2, Lemma 2], it follows that B_2 possesses also an identity $f(x_1, \cdots, x_{2r-1}) = 0$ of degree $2r-1$, where f is homogeneous and linear in all the indeterminates x_i. We may assume that the coefficient α of the monomial $x_1 x_2 \cdots x_{2r-1}$ of f is not zero.

Now substitute $x_{2i-1} = e_{ii}$, $x_{2i} = e_{ii+1}$, $i = 1, 2, \cdots, r-1$, $x_{2r-1} = e_{rr}$. The only monomial of f which yields under this substitution a nonzero element is $x_1 \cdots x_{2r-1}$, hence $f = \alpha e_{11} \neq 0$ which is a contradiction. This implies that $m_2 \geqq 2r$. Hence $m_2 = 2r$, which completes the proof of the theorem.

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