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3. J. Hadamard, *Sur le principe de Dirichlet*, Bull. Soc. Math. France vol. 34 (1906) pp. 135–138.

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NON-MEASURABLE SETS AND THE EQUATION $f(x+y)=f(x)+f(y)$

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1. A set of S real numbers which has inner measure $m_*(S)$ different from its outer measure $m^*(S)$ is non-measurable. An extreme form, which we shall call saturated non-measurability, occurs when $m_*(S)=0$ but $m^*(SM)=m(M)$ for every measurable set M, $m(M)$ denoting the measure of M. This is equivalent to: both S and its complement have zero inner measure.

More generally, if a fixed set B of positive measure is under consideration, a subset S of B will be called s-non-mble. if both S and its complement relative to B have zero inner measure. This implies $m_*(S)=0$, $m^*(S)=m(B)$ but is implied by these conditions only if $m(B)$ is finite.

Our object, in part, is to show that if B is either the set of all real numbers or any half-open finite interval, then for every infinite cardinal $k \leqq C$ (the power of the continuum), B can be partitioned into k disjoint subsets which are s-non-mble. and are mutually congruent under translation (modulo the length of B in the case that B is a finite interval). Sierpinski and Lusin[1] have partitioned B into continuum many disjoint s-non-mble. subsets but they are not constructed to be congruent under translation. Other well known constructions do partition B into a countable infinity of mutually congruent non-measurable subsets, but the subsets are not constructed to be saturated non-measurable.[2]

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[1] C. R. Acad. Sci. Paris vol. 165 (1917) pp. 422–424.

[2] See Hahn and Rosenthal, *Set functions*, University of New Mexico Press, 1948, pp. 102–104. The construction of §8.3.3 on p. 102 (as will be shown below) does give an s-non-mble. set but this is not proved there.

2. Various authors have shown that if the axiom of choice is assumed, there are discontinuous solutions of the equation $f(x+y)=f(x)+f(y)$ and that these solutions all have certain pathological properties. They are unbounded and non-measurable on every interval, indeed for all numbers a, b the sets $E[x: f(x)<a]$ and $E[x: f(x)>b]$ have zero inner measure, which implies that they are s-non-mble. A deeper theorem of Ostrowski[3] shows that if $a<b$, then the set $E[x: f(x)<a$ or $f(x)>b]$ has zero inner measure, which implies that the set $E[x: a\leqq f(x)<b]$ is s-non-mble.

Now suppose that $f(x)$ is such a discontinuous solution and that d is any number greater than 0 for which $f(x_0)=d$ for some x_0. Let S_n denote the set $E[x: nd\leqq f(x)<d(n+1)]$. Then the S_n, $n=0, \pm 1, \pm 2, \cdots$, are a partition of the set of all real numbers into a countable infinity of disjoint s-non-mble. sets which are congruent under translation since $S_{n+1}=x_0+S_n$. If B is the interval $a\leqq x<b$ and $f(x)$ has the property $f(b-a)=0$, then the BS_n are a partition of this kind of B (modulo the length of B).

3. The theorem of Ostrowski does not state that the set $E(c)=E[x: f(x)=c]$ is s-non-mble. and this is not true in general. However, we have the following theorem.

THEOREM. *If $f(x)$ is a discontinuous solution of $f(x+y)=f(x)+f(y)$ which assumes only a countable number of distinct values, $c_1, c_2, \cdots, c_n, \cdots$, and E_n denotes $E(c_n)$, then the E_n are a partition of the set of all real numbers into disjoint, s-non-mble. sets which are congruent under translation. If B is an interval $a\leqq x<b$ and $f(x)$ has the additional property $f(b-a)=0$, then the BE_n are such a partition of B (modulo the length of B).*

PROOF. If x_n is an x for which $f(x)=c_n$, then $E_m=(x_m-x_n)+E_n$ so that the E_n are congruent under translation. They are obviously disjoint, and their set-union is the set of all real numbers. It remains to show that they are s-non-mble.

If $f(x)\neq 0$ for all $x\neq 0$, then $E(0)$ would consist of one number, namely 0; each E_n would consist of one number since it is a translation of $E(0)$, which would give the false result that the set of all real numbers is countable. This shows that $f(\delta)=0$ for some $\delta>0$ and hence for all $r\delta$, r rational but arbitrary.

Let $E_n(\alpha, \beta)$ denote the set of x in E_n for which $\alpha\leqq x<\beta$. Now $E_n(0, \delta)$ is not a null set, for if it were so we could deduce that

[3] Alexander Ostrowski, *Über die Funktionalgleichung der Exponentialfunktion und verwandte Funktionalgleichungen*, Jber. Deutschen Math. Verein. vol. 38 (1929.)

$E_n(m\delta, (m+1)\delta)$ (by translation) then E_n by set-union, and finally the set of all real numbers (by set-union of all E_n) were null sets. Hence $m^*(E_n(0, \delta)) = h\delta$ for some $0 < h \leqq 1$ and some fixed n. It follows that $m^*E_n(r\delta, s\delta) = h(s-r)\delta$ for all rational r, s with $r < s$; this is clear for $r = 0$ and s equal to a positive integer, or to the reciprocal of a positive integer, or, finally, to any positive rational; by translation the statement then follows for all rational r, s. Now, from the continuity of m^* it follows that $m^*(E_nI) = h$ (length of I) for all intervals I.

Suppose, if possible, that $h < 1$. Then the complement of E_n has positive inner measure and hence there is an interval I such that $m_*(I - E_nI) > (1-h)m(I)$. This gives $m^*(E_nI) < hm(I)$, a contradiction. Thus $h = 1$,[4] and $m^*(E_nI) = m(I)$ for all intervals I. This implies that E_n is s-non-mble.

4. Let $e_1, e_2, \cdots, e_\alpha, \cdots$ be any Hamel's basis so that each x has a unique expression $x = \sum_\alpha x_\alpha e_\alpha$ where the x_α are all rational and at most a finite number of them differ from zero. Let S_r be the set of x for which x_1 equals a given rational r. Then if $f(x)$ is defined to be x_1 for every x, the values of $f(x)$ will be countable and §3 above shows that the S_r are all s-non-mble. and a partition (into a countable infinity of subsets) of the desired kind of the set of all real numbers (for a finite interval B it suffices to choose e_2 equal to the length of B). This does not require the theorem of Ostrowski or any other theorem on the pathology of discontinuous solutions except as proved in §3 above.

5. If $e_1, e_2, \cdots, e_\alpha, \cdots$ is any Hamel's basis, then any set of conditions on the x_α involving at most a countable number of α will give a set of x which is either empty, or the set of all real numbers, or s-non-mble. For let $f(x)$ be defined to be a solution of $f(x+y) = f(x) + f(y)$ with $f(e_\alpha) = e_\alpha$ if the value of x_α is involved in the given conditions, and 0 otherwise. Then each set of precise restrictions on the x_α involved in the original conditions, $x_\alpha = r_\alpha$, for arbitrary rationals r_α, defines an s-non-mble. set. The conclusion follows.

In particular, the conditions $x_n = 0$ for $n = 1, 2, \cdots$ gives an s-non-mble. set as do the conditions $x_n > 0$ for $n = 1, 2, \cdots$. It follows that if the set of all real numbers be considered as a group G with the rational numbers as multipliers (operators), then a proper subgroup G_1 which admits the rationals as operators is s-non-mble. if the number of cosets of G_1 with respect to G is countable.

[4] In this connection, see Jacobsthal and Knopp, Sitzungsberichte der Berliner Mathematischen Gesellschaft vol. 14 (1915) p. 121.

6. Let Ω be the smallest ordinal number with continuum many predecessors and let the collection of all perfect sets be arranged as a sequence P_α, $1 \leqq \alpha < \Omega$.[1] For each $1 \leqq \alpha < \Omega$ define a_α, b_α by induction so that a_α, b_α are elements of P_α and the a_β, b_β with $\beta \leqq \alpha$ are linearly independent with respect to rational numbers as coefficients; if an interval B: $a \leqq x < b$ is under consideration, define $a_0 = b - a$. Such a_α, b_α exists since for each α, P_α has continuum elements whereas there are less than continuum rational-linear combinations of the a_β, b_β with $\beta < \alpha$. Then there exists a Hamel's basis containing all a_α and all b_α as members. If k is any infinite cardinal less than or equal to C, there is a subset H_1 of some of the b_α with cardinal k. Let S consist of all linear combinations of elements of $H - H_1$, using rational numbers as coefficients, and let y be an arbitrary linear combination of elements of H_1 with rational coefficients. The sets $S + y$ form a decomposition of the whole line into k disjoint, congruent (under translation; modulo length of B if an interval B is under consideration) subsets. Since each $S + y$ has points in every perfect set (S contains all a_α), it follows that the $S + y$ are saturated non-mble.

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