A NOTE ON THE SPACE L_p^*

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The object of the following note is to give an alternate proof of the fact that the conjugate space of the Lebesgue space L_p is the Lebesgue space $L_{p'}$, if $p>1$, even if these spaces are constructed on a non-σ-finite measure space. The result fails if $p=1$. This result has recently been proved by E. J. McShane.[1]

In what follows, where X denotes a Banach space, X^* will denote its conjugate space. If $x \in X$, $|x|$ will denote the norm of x. The symbol (S, F, m) will denote a measure space, that is, a set S, a σ-field F of its subsets, and a positive, completely additive, perhaps infinite, function m defined for the sets in F.

Let n be an integer greater than or equal to 1, p a real number greater than 1. Let p' be defined by $1/p+1/p'=1$. Let δ_j^i be the Kronecker δ-function.

1. Let $a_1, \cdots, a_n$ all be greater than 0. Define $R(n, p, a_1, \cdots, a_n)$ to be the Banach space of n-uples $y=[y_i]$, where the norm is defined by

$$|y| = \left\{ \sum_{i=1}^{n} a_i |y_i|^p \right\}^{1/p}.$$

Note that $|y+y'| = \{ \sum |a_i^{1/p}(y_i+y_i')|^p\}^{1/p} \leqq |y| + |y'|$ follows from Minkowski's inequality; and that all the other requirements that $R(n, p, a_1, \cdots, a_n)$ must satisfy in order to be a Banach space are as evidently true.

Now let $x^* \in R^*(n, p, a_1, \cdots, a_n)$. If $x^*(\delta_j^i)=k_i$, it is evident that $x^*(y)=\sum_{i=1}^n k_i y_i$. If $x^* \neq 0$, we may note, putting

$$v = \left[\left| \frac{k_i}{a_i} \right|^{p'-1} \overline{\arg (k_i)} \right], \qquad 1 \leqq i \leqq n,$$

where $\arg (x)$ is the argument of the complex quantity x, that

$$|x^*| \geqq \frac{|x^*(v)|}{|v|} = \frac{\left| \sum_{i=1}^{n} k_i \left| \frac{k_i}{a_i} \right|^{p'-1} \overline{\arg (k_i)} \right|}{\left\{ \sum_{i=1}^{n} a_i \left(\left| \frac{k_i}{a_i} \right|^{p'-1} \right)^p \right\}^{1/p}}$$

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$$= \frac{\sum_{i=1}^{n} a_i^{1-p'} \mid k_i \mid^{p'}}{\left\{ \sum_{i=1}^{n} a_i \left| \frac{k_i}{a_i} \right|^{p'} \right\}^{1/p}}$$

$$= \left\{ \sum_{i=1}^{n} a_i^{1-p'} \mid k_i \mid^{p'} \right\}^{1/p'}.$$

If $x^*=0$, the same inequality is trivial. From Theorem 1 below, the converse inequality will follow immediately.

2. Now let $L_p(S, F, m)$ be the L_p space on the measure space S, that is, the set of measurable functions $f(w)$ defined for $w \in S$ such that

$$| f | = \left\{ \int_S | f(w) |^p dm \right\}^{1/p} < \infty.$$

Consider for the moment a fixed subset A of S such that $m(A) < \infty$, and let $B \subseteq A$. Note that if $K_B(w)$ is the characteristic function of the set B, and $x^* \in L_p^*$, $x^*(K_B)$ is a completely additive and absolutely continuous set function. Thus, as long as $B \subseteq A$, we may write, by the theorem of Radon-Nikodym,

$$x^*(K_B) = \int_B t_A(w) dm = \int_S t_A(w) K_B(w) dm.$$

Here $t_A(w)$ vanishes outside A, while if $m(A') < \infty$ also, $t_A(w) = t_{A'}(w)$ almost everywhere on the intersection of A and A'.

3. Now let A be as above, and let $A \supseteq A_i$, $m(A_i) > 0$, and let the A_i be disjoint; let G be the set of all functions constant on each of the A_i and vanishing elsewhere, and let $w_i \in A_i$. Note that G, as a subset of L_p, is isometrically isomorphic to $R(n, p, m(A_1), \cdots, m(A_n))$, f in G corresponding to $(f(w_1), \cdots, f(w_n))$. If $f \in G$, $x^*(f) = \sum_{i=1}^n f(w_i) \int_{A_i} t_A(w) dm$, so that, since the norm of x^* as an element of L_p^* is at least as great as its norm as an element of G^*, we may conclude, by point (1)

$$[*] \qquad | x^* | \geqq \left(\sum_{i=1}^{n} m(A_i)^{1-p'} \mid x^*(K_{A_i}) \mid^{p'} \right)^{1/p'}.$$

Define $t_A(m, w)$ by: R.P. $(t_A(n, w)) = \min (k/n, n)$ if $0 \leqq k/n \leqq$ R.P. $(t_A(w)) < (k+1)/n$; R.P. $(t_A(n, w)) = \max (-k/n, -n)$ if $0 \leqq k/n \leqq$ R.P. $(-t_A(w)) < (k+1)/n$; similar inequalities hold for the imaginary parts. Then $\lim_{n\to\infty} t_A(n, w) = t_A(w)$. Let A_i^n, $1 \leqq i \leqq W_n$, be the sets of nonzero measure on which $t_A(n, w)$ is constant and nonzero. On A_i^n we have

$$\left| \text{R.P.} \left(\int_{A_i^n} t_A(w)dm \right) \right| \geqq | \text{R.P.} (t_A(n, w_i^n)) | \, m(A_i^n) |,$$

where $w_i^n \in A_i^n$; a similar equation holds for the imaginary parts. Thus,

$$\left| \int_{A_i^n} t_A(w)dm \right| \geqq | t_A(n, w_i^n) | \, m(A_i^n),$$

since the absolute value of a quantity is the square-root square sum of its real and imaginary parts. From equation [*] we have

$$| x^* | \geqq \left(\sum_{i=1}^{w_n} m(A_i^n)^{1-p'} m(A_i^n)^{p'} | t_A(n, w_i^n) |^{p'} \right)^{1/p'}$$
$$= \left(\int_S | t_A(n, w) |^{p'} dm \right)^{1/p'},$$

and, applying Fatou's lemma, we obtain $|x^*| \geqq (\int_S |t_A(w)|^{p'} dm)^{1/p'}$. Now put $H(A) = \int_S |t_A(w)|^{p'} dm$, and let $U \leqq |x^*|^{p'}$ be l.u.b.$_A$ $H(A)$, A ranging over the sets where $m(A) < \infty$. Choose $A_1, A_2, \cdots$ in such a way that $\lim_{n\to\infty} H(A_n) = U$; where, since $H(A)$ is an increasing function of A, we may suppose $A_1 \subseteq A_2 \subseteq \cdots$. Let $T = \bigcup_{i=1}^{\infty} A_i$, and let A_0 be such that $H(A_0) > 0$. Then A_0 cannot be disjoint from T, for since $H(A)$ is finitely additive, this would imply $\lim_{n\to\infty} H(A_n \cup A_0) = U + H(A_0) > U$.

Now, putting $t_T(w) = \lim_{n\to\infty} t_{A_n}(w)$, if A_0 is of finite measure, and if $K_{A_0}(w)$ is its characteristic function, we may note that $K_{A_0} = K_{A_0 \cap T} + K_{A_0 - T}$. Since $H(A_0 - T) = 0$, $x^*(K_{A_0 - T}) = 0$. Thus, $x^*(K_{A_0}) = x^*(K_{A_0 \cap T})$. Since $K_{A_0 \cap T} = \lim_{n\to\infty} K_{A_0 \cap A_n}$, and $m(A_0) < \infty$,

$$\lim_{n\to\infty} \int_S | K_{A_0 \cap T}(w) - K_{A_0 \cap A_n}(w) |^p dm = 0.$$

Hence

$$x^*(K_{A_0}) = \lim_{n\to\infty} x^*(K_{A_0 \cap A_n}) = \lim_{n\to\infty} \int_S K_{A_0 \cap A_n}(w) t_T(w) dm.$$

Since $|x^*| \geqq \{\int_S |t_T(w)|^{p'} dm\}^{1/p'}$, we have

$$x^*(K_{A_0}) = \int_S K_{A_0 \cap T}(w) t_T(w) dm = \int_S K_{A_0}(w) t_T(w) dm,$$

since $t_T(w)$ vanishes outside T. Thus we have verified the equation $x^*(f)=\int_S f(w)t_T(w)dm$ for all the functions of a fundamental set in L_p. Since both sides of the equation represent continuous functionals on L_p, the equation must hold identically. In this case, Holder's inequality gives $|x^*| \leqq \{\int_S |t_T(w)|^{p'}dm\}^{1/p'}$, so that we must have equality.

Once this is established, the linearity of the correspondence $x^* \leftrightarrow t_T(w)$ is clear, so that we may state the following theorem.

THEOREM 1. *The space $L_p^*(S, F, m)$, where (S, F, m) is an arbitrary measure space, and $p>1$, is isometrically isomorphic to the space $L_{p'}(S, F, m)$, $1/p+1/p'=1$. The isomorphism $x^*\leftrightarrow t(w)$ is determined by the formula*

$$x^*(f) = \int_S f(w)t(w)dm.$$

4. The following example is due to T. Botts. Consider the measure space (S^*, F^*, m^*), where S^* is the open interval $(0, 1)$; where F^* is the σ-field consisting of the (finite or) denumerable subsets of the interval and their complements; and where $m^*(A)$ is the cardinality of A. It may be seen that $L_1(S^*, F^*, m^*)$ consists of those functions which vanish outside a denumerable set, and whose remaining values form an absolutely convergent series. Thus, if $f(x)\in L_1$, $xf(x)\in L_1$ and $\int_{S^*}xf(x)dm^*=x^*(f)$ represents a functional on L_1 of norm 1. Since this functional takes on a nondenumerable set of distinct values on the family of characteristic functions of points, it cannot be of the form $\int_{S^*}t(x)f(x)dm^*$, where $t(x)$ is measurable, since any measurable function must be constant except for a denumerable set.

5. For $L_1(S, F, m)$ the result may be stated in the following theorem.

THEOREM 2. *Let $J(S, F, m)$ be the family of all absolutely continuous, countably additive, complex-valued set functions μ defined on the sets in F of finite measure, satisfying the additional condition that*

$$|\mu| = \underset{0<m(A)<\infty}{\text{l.u.b.}} \frac{|\mu(A)|}{m(A)} < \infty.$$

Then J, with the norm indicated, forms a Banach space isometrically isomorphic to L_1^*. The isomorphism is determined by the formula $x^*(f)=\int_S f(w)d\mu$.

PROOF. If $x^*\in L_1^*$, it is evident that $x^*(K_B)=\mu(B)$ defines an absolutely continuous, countably additive set function for the sets in F of finite measure, while

$$|x^*| \geqq \frac{|x^*(K_B)|}{|K_B|} = \frac{|\mu(B)|}{m(B)}.$$

Conversely, if μ is such a function, and $|\mu(B)|/m(B) \leqq M$, and $f \in L_1$, f is the limit, in the norm of L_1, of a sequence of countably-valued functions $f_n = \sum_{i=1}^{\infty} a_i^n K_{B_i^n}$; and we may put $x^*(f) = \lim_{n\to\infty} \sum_{i=1}^{\infty} a_i^n \mu(B_i^n)$. This limit will exist and be unique only if each Cauchy sequence in L_1 of the form[2] $f_n = \sum_{i=1}^{\infty} a_i^n K_{B_i^n}$ corresponds to a Cauchy sequence $\sum_{i=1}^{\infty} a_i^n \mu(B_i^n)$. But

$$\begin{aligned} \left| \sum_{i=1}^{\infty} a_i^n \mu(B_i^n) - \sum_{j=1}^{\infty} a_j^n \mu(B_j^n) \right| &= \left| \sum_{i,j=1}^{\infty} (a_i^n - a_j^m) \mu(B_i^n \cap B_j^m) \right| \\ &\leqq M \sum_{i,j=1}^{\infty} |a_i^n - a_j^n| \, m(B_i^n \cap B_j^m) \\ &= M \, |f_n - f_m|. \end{aligned}$$

It may now be seen that a functional x^* is defined, that $x^*(K_B) = \mu(B)$, and that $|x^*| = \mu$.

Using the alternate definition of the integral as the limit of the integrals of countably-valued functions converging almost uniformly to a given function, forming a Cauchy sequence in $L_1(\mu)$, and observing that almost uniform convergence with respect to the measure m implies almost uniform convergence with respect to μ, we see that the measure and functional are related by $x^*(f) = \int_S f(w) d\mu$. Since all the required linearity relations are obvious, the theorem is proved.

6. Theorem 1 may be used to extend the following result, which is well known in the classical case.

THEOREM 3. *Let (S, F, m) be a measure space, and let μ be a finitely additive complex-valued function, defined for the sets of F of finite measure. Then in order that there exist a function $g(w)$ in L_p, $p > 1$, such that $\mu(A) = \int_A g(w) dm$, it is necessary and sufficient that*

$$\text{l.u.b.} \sum_{i=1}^{n} |\mu(A_i)|^p m(A_i)^{1-p} < \infty,$$

where $A_1, \cdots, A_n$ ranges over all finite disjoint collections of sets of F of finite measure.

[2] To avoid formal difficulty we may introduce a 0 coefficient and suppose $\bigcup_{i=1}^{\infty} B = S$. The B_i are of course supposed to be disjoint.

PROOF. Let us put

$$x^*\left(\sum_{i=1}^{n} a_i K_{A_i}\right) = \sum_{i=1}^{n} a_i \mu(K_{A_i}).$$

Then x^* is a linear function (as is evident from the finite additivity of μ) defined on a dense set of $L_{p'}$. This will be extendable to a bounded function defined on all of $L_{p'}$, which by Theorem 1 is of the form

$$x^*(f) = \int_S f(w)g(w)dm$$

if and only if $\mu(A)$ is of the form $\mu(A) = \int_A g(w)dm$. By a well known theorem, a condition necessary and sufficient for this extension is

$$\infty > M = \text{l.u.b.} \frac{\left| x^*\left(\sum_{i=1}^{n} a_i K_{A_i}\right)\right|}{\left|\sum_{i=1}^{n} K_{A_i}\right|}.$$

Without loss of generality, we may assume that the A_i are disjoint; in this case, the definition of the quantities involved gives

$$M = \text{l.u.b.} \frac{\left|\sum_{i=1}^{n} a_i \mu(A_i)\right|}{\left\{\sum_{i=1}^{n} |a_i|^{p'} m(A_i)\right\}^{1/p'}}.$$

The remark at the end of (1) gives

$$M = \text{l.u.b.} \left\{\sum_{i=1}^{n} |\mu(A_i)|^p m(A_i)^{1-p}\right\}^{1/p},$$

from which the theorem is evident.

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