COLLINEATIONS IN WEYL SPACES OF TWO DIMENSIONS

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1. **Introduction.** In this paper we obtain all (real) two-dimensional Weyl spaces (W_2) which admit (real) continuous groups G_r of affine collineations (A.C.). Such a collineation is defined as a point transformation which changes each path of the space into a path and also preserves the affine path-parameters [1, p. 126].[1] The work is based on the results of another paper to be published elsewhere [2]. It is shown in [2] that there are 12 types of affinely connected spaces A_2 admitting A.C. When each of these types is tested for reducibility to a W_2 it is found that all but three are eliminated, these three admitting respectively A.C. of one, two, and three parameters. There are four general types of W_2 admitting the G_1 of A.C. and also four such types of W_2 admitting the G_2 of collineations. Corresponding to the G_3 there is only one class of W_2.

2. **Weyl spaces.** A Weyl space W_n is a special type of affine space A_n (symmetric connection) which admits a tensor and a covariant vector with components g_{ij}, ϕ_i respectively such that

$$g_{ij,k} + 2g_{ij}\phi_k = 0, \tag{2.1}$$

where a comma denotes covariant differentiation with respect to the components Γ^i_{jk} of affine connection of the space [1, p. 81]. The g_{ij} are assumed symmetric. In addition the vector ϕ_i must not be a gradient (as otherwise the W_n reduces to a Riemannian space) and we must have also

$$|g_{ii}| \neq 0. \tag{2.2}$$

The Γ^i_{jk} can be expressed as

$$\Gamma^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \delta^i_j\phi_k + \delta^i_k\phi_j - g_{jk}g^{im}\phi_m. \tag{2.3}$$

The components g_{ij}, ϕ_i of a W_n are determined to within an arbitrary function $\theta(x^i)$, that is, the tensors with components $\bar{g}_{ij}$, $\bar{\phi}_i$ given by

$$\bar{g}_{ij} = e^{2\theta}g_{ij}, \tag{2.4a}$$

$$\bar{\phi}_i = \phi_i - \partial\theta/\partial x^i \tag{2.4b}$$

will also satisfy (2.1). The Γ^i_{jk} remain invariant under (2.4) which is

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

the change of gauge of Weyl. When a definite θ is picked the space is said to be calibrated [3].

Now the conditions of integrability of (2.1) are

$$g_{hj}B^{h}_{ikm} + g_{ih}B^{h}_{jkm} + 2g_{ij}(\phi_{k,m} - \phi_{m,k}) = 0, \tag{2.5}$$

from which it follows that

$$B^{i}_{ikm} = 2\left(\frac{\partial\phi_m}{\partial x^k} - \frac{\partial\phi_k}{\partial x^m}\right) \qquad (n = 2), \tag{2.6}$$

where the curvature tensor components B^{i}_{jkm} are

$$B^{i}_{jkm} = \frac{\partial\Gamma^{i}_{jm}}{\partial x^k} - \frac{\partial\Gamma^{i}_{jk}}{\partial x^m} + \Gamma^{h}_{jm}\Gamma^{i}_{hk} - \Gamma^{h}_{jk}\Gamma^{i}_{hm}. \tag{2.7}$$

It follows that we must have ($n=2$)

$$B^{i}_{i12} = \frac{\partial\Gamma^{i}_{i2}}{\partial x^1} - \frac{\partial\Gamma^{i}_{i1}}{\partial x^2} \neq 0. \tag{2.8}$$

If then we apply the test (2.8) to the 12 types of A_2 obtained in [2], we find that the only types not eliminated are

(I) $\Gamma^{i}_{jk}=\Gamma^{i}_{jk}(y)$, with $G_1=[p]$,

(II) $\Gamma^{i}_{jk}=(1/y)a^{i}_{jk}$, with $G_2=[p,\ xp+yq]$,

(III) $\Gamma^{1}_{11}=2b/y^2$, $\Gamma^{1}_{12}=-1/y$, $\Gamma^{1}_{22}=0$, $\Gamma^{2}_{11}=a/y^3$, $\Gamma^{2}_{12}=b/y^2$, $\Gamma^{2}_{22}=-2/y$, with $G_3=[p,\ 2xp+yq,\ x^2p+xyq]$.

In (I, II, III) we have used the notation (to be followed hereafter) x, y for x^1, x^2, and also $p=\partial f/\partial x$, $q=\partial f/\partial y$. The infinitesimal generators $X_\alpha f=\xi^1_\alpha p+\xi^2_\alpha q$ $(\alpha=1, \cdots, r)$ will generate the $G_r=[X_1, X_2, \cdots, X_r]$ (in the neighborhood of the identity transformation).

The $\Gamma^{i}_{jk}(y)$ in (I) are arbitrary functions of y. In (II), a^{i}_{jk} are arbitrary constants, as also are a, b in (III). These three cases (I), (II), (III) are the [1.1], [2.1], [3.1] respectively of [2].

The equations (2.5) of integrability can be reduced by means of (2.6) to the set of (dependent) equations

$$(B_{12} + B_{21})g_{11} - 2B_{11}g_{12} = 0, \tag{2.9a}$$

$$2B_{22}g_{12} - (B_{12} + B_{21})g_{22} = 0, \tag{2.9b}$$

$$B_{22}g_{11} - B_{11}g_{22} = 0. \tag{2.9c}$$

3. **Determination of the W_2.** As the Γ^{i}_{jk} components of (II) and (III) are special cases of those of (I), it is sufficient to obtain the

solutions for this latter case and then by suitable specializations obtain those for (II) and (III).

We first observe that from (2.4b) and (2.6) there is no loss of generality in assuming

$$\phi_1 = (\Gamma^1_{11} + \Gamma^2_{12})/2, \qquad \phi_2 = 0. \tag{3.1}$$

Since if B_{ij} be skew-symmetric integrability conditions (2.9) are satisfied identically, it is convenient to consider two cases according as B_{ij} is or is not skew. In case B_{ij} is not skew we consider further the two subcases I(a) $B_{11}\neq 0$, I(b) $B_{11}=0$.

If $B_{11}\neq 0$, we can find a coordinate transformation of the type

$$x = \bar{x} + f(\bar{y}), \qquad y = \bar{y}, \tag{3.2}$$

such that $\overline{B}_{12}+\overline{B}_{21}=0$, $\overline{B}_{11}\neq 0$, and also have the $\overline{\Gamma}^i_{jk}$ as functions of $\bar{y}$ only. By a suitable change of gauge conditions (3.1) will hold in the barred coordinates also. If then the bars be dropped, we have from (2.9) and (2.2) that $g_{12}=0$, $B_{22}\neq 0$.

Equations (2.1) when expanded are (using (3.1))

$$\frac{\partial g_{11}}{\partial x} = g_{11}(\Gamma^1_{11} - \Gamma^2_{12}) + 2g_{12}\Gamma^2_{11}, \tag{3.3a}$$

$$\frac{\partial g_{11}}{\partial y} = 2g_{11}\Gamma^1_{12} + 2g_{12}\Gamma^2_{12}, \tag{3.3b}$$

$$\frac{\partial g_{12}}{\partial x} = g_{11}\Gamma^1_{12} + g_{22}\Gamma^2_{11}, \tag{3.3c}$$

$$\frac{\partial g_{12}}{\partial y} = g_{11}\Gamma^1_{22} + g_{12}(\Gamma^1_{12} + \Gamma^2_{22}) + g_{22}\Gamma^2_{12}, \tag{3.3d}$$

$$\frac{\partial g_{22}}{\partial x} = 2g_{12}\Gamma^1_{12} - g_{22}(\Gamma^1_{11} - \Gamma^2_{12}), \tag{3.3e}$$

$$\frac{\partial g_{22}}{\partial y} = 2g_{12}\Gamma^1_{22} + 2g_{22}\Gamma^2_{22}. \tag{3.3f}$$

The solutions of (3.3) corresponding to case I(a) are given in §4.

In the subcase I(b) for which $B_{11}=0$, we must have $B_{12}+B_{21}\neq 0$, for if not (2.9) shows $B_{22}=0$, that is, B_{ij} is skew.

By a (3.2) type transformation we can make $B_{22}=0$, and (3.3) can be easily solved. The solutions are given below.

If B_{ij} is skew we again assume two subcases: I(c) $\Gamma^2_{11}\neq 0$, I(d) $\Gamma^2_{11}=0$. In I(c) we can make $\Gamma^2_{12}=0$ by (3.2) (still keeping $\Gamma^2_{11}\neq 0$). In

I(d), (3.2) is used to make $\Gamma^1_{22}=0$. In addition, it is easily shown that we must have $\Gamma^1_{12}=0$, $\Gamma^2_{12}=\Gamma^1_{11}\neq 0$.

The solutions of (3.3) corresponding to (II) are given as II(a)–II(d). The form of the G_2 remains invariant under (3.2).

In dealing with (III) we first reduce it to the case I(c) of (I) by a (3.2) transformation. Here we must have $4a-b^2=0$, $b\neq 0$. The solution and transformed form of the G_3 are given in III(a).

It can be shown as in [2] that the three groups are complete groups of affine collineations.

4. The solutions.

W_2 admitting the one-parameter group $G_1=[p]$

I(a). B_{ij} now skew, $B_{11}B_{22}\neq 0$, $B_{12}+B_{21}=0$.

$$\Gamma^1_{11}=\Gamma^2_{12}\neq 0, \qquad \Gamma'^1_{11}\neq 0, \qquad \Gamma^1_{11}\Gamma^1_{12}-\Gamma^1_{22}\Gamma^2_{11}=0,$$

$$(B_{22}/B_{11})'=2(B_{22}/B_{11})(\Gamma^2_{22}-\Gamma^1_{12}),$$

$$g_{11}=a\exp\left(2\int\Gamma^1_{12}dy\right), \qquad g_{12}=0,$$

$$g_{22}=b\exp\left(2\int\Gamma^2_{22}dy\right)=\frac{B_{22}}{B_{11}}g_{11} \qquad (ab\neq 0),$$

$$\phi_1=\Gamma^1_{11}, \qquad \phi_2=0.$$

I(b). B_{ij} not skew, $B_{11}=B_{22}=0$, $B_{12}+B_{21}\neq 0$.

$$\Gamma^1_{11}=\Gamma^1_{12}=\Gamma^2_{11}=\Gamma^2_{12}=0, \qquad \Gamma'^1_{11}=0,$$

$$g_{11}=0, \qquad g_{12}=a\exp\left(\int\Gamma^2_{22}dy\right), \qquad g_{22}=0 \qquad (a\neq 0),$$

$$\phi_1=\Gamma^1_{11}/2, \qquad \phi_2=0.$$

I(c). B_{ij} skew.

$$\Gamma^2_{11}\neq 0, \qquad \Gamma'^1_{11}\neq 0, \qquad \Gamma^2_{12}=0; \qquad \phi_1=\Gamma^1_{11}/2, \qquad \phi_2=0,$$

$$g_{11}=g(x)F(y), \qquad g_{12}=\frac{F}{2\Gamma^2_{11}}(g'-g\Gamma^1_{11}),$$

$$g_{22}=\frac{F}{2(\Gamma^2_{11})^2}(g''-g'\Gamma^1_{11}-2g\Gamma^1_{12}\Gamma^2_{11}),$$

$$F(y)=\exp\left(2\int\Gamma^1_{12}dy\right),$$

$$g(x) = ax^2 + bx + c, \qquad b^2 - 4ac \neq 0, \qquad (\Gamma^1_{11})^2 + 4\Gamma^1_{12}\Gamma^2_{11} = 0,$$

$$g(x) = ae^{kx} + be^{-kx} + c, \quad c^2 - 4ab \neq 0, \quad (\Gamma^1_{11})^2 + 4\Gamma^1_{12}\Gamma^2_{11} = k^2,$$

$$g(x) = a\cos kx + b\sin kx + c, \qquad c^2 - a^2 - b^2 \neq 0,$$

$$(\Gamma^1_{11})^2 + 4\Gamma^1_{12}\Gamma^2_{11} = -k^2.$$

I(d). B_{ij} skew.

$$\Gamma^1_{12} = \Gamma^2_{11} = \Gamma^1_{11} - \Gamma^2_{12} = 0, \Gamma'^1_{11} \neq 0; \phi_1 = \Gamma^1_{11}, \phi_2 = 0,$$

$$g_{11}(y) = \int 2g_{12}\Gamma^1_{11}dy, \qquad g_{12} = \int g_{22}\Gamma^1_{11}dy,$$

$$g_{22} = a\exp\left(\int 2\Gamma^2_{22}dy\right) \qquad (a \neq 0).$$

In all cases the Γ^i_{jk} are understood to be arbitrary functions of y subject to indicated restrictions.

W_2 admitting the two-parameter group $G_2 = [p, xp+yq]$

$\Gamma^i_{jk} = (1/y)a^i_{jk}$ throughout, a^i_{jk} arbitrary constants subject to indicated restrictions.

II(a). $a^1_{11} = a^2_{12} \neq 0, \qquad a^1_{12} = a^2_{22} \neq 0,$

$$a^2_{11} \neq 0, \qquad 1 + a^1_{12} - a^2_{11} \neq 0, \qquad a^1_{22}a^2_{11} - a^1_{12}a^1_{11} = 0,$$

$$\phi_1 = \frac{1}{y}a^1_{11}, \qquad \phi_2 = 0,$$

$$g_{11} = ay^{2a^1_{12}}, \quad g_{12} = 0, \quad g_{22} = by^{2a^1_{12}} \ (ab \neq 0), \quad aa^1_{12} + ba^2_{11} = 0.$$

II(b). $a^1_{12} = a^1_{22} = a^2_{11} = a^2_{12} = 0, \qquad a^1_{11} \neq 0,$

$$g_{11} = 0, \qquad g_{12} = ay^{a^2_{22}}, \qquad g_{22} = 0 \qquad (a \neq 0),$$

$$\phi_1 = (1/2y)a^1_{11}, \qquad \phi_2 = 0.$$

II(c). $a^2_{12} = 0, \qquad a^1_{12} = -(a^1_{22})^2 a^2_{11} \neq 0, \qquad a^1_{11} = 2a^1_{22}a^2_{11} \neq 0,$

$$a^2_{22} = 1 - a^2_{11}(a^1_{22})^2,$$

$$\phi_1 = (1/2y)a^1_{11}, \qquad \phi_2 = 0.$$

g_{ij} as in I(c) with $F(y) = y^{2a^1_{12}}, \qquad g(x) = ax^2 + bx + c.$

II(d). $\overset{1}{a}_{12} = \overset{2}{a}_{11} = \overset{1}{a}_{11} - \overset{2}{a}_{12} = 0, \quad \overset{1}{a}_{11} \neq 0, \quad \overset{2}{a}_{22} \neq 0,$

$g_{11} = (2\overset{1}{a}_{11}/\overset{2}{a}_{22})g_{12} + c, \quad g_{12} = a\overset{1}{a}_{11} \log y + b,$

$g_{22} = ay^{2\overset{2}{a}_{22}} \qquad (a \neq 0),$

$\phi_1 = (1/y)\overset{1}{a}_{11}, \quad \phi_2 = 0.$

II(d'). $\overset{1}{a}_{12} = \overset{2}{a}_{11} = \overset{1}{a}_{11} - \overset{2}{a}_{12} = 0, \quad \overset{1}{a}_{11} \neq 0, \quad \overset{2}{a}_{22} = 0,$

$g_{11} = a(\overset{1}{a}_{11})^2(\log y)^2 + 2b\overset{1}{a}_{11} \log y + c,$

$g_{12} = a\overset{1}{a}_{11} \log y + b, \quad g_{22} = 0,$

$\phi_1 = (1/y)\overset{1}{a}_{11}, \quad \phi_2 = 0, \quad ac - b^2 \neq 0.$

W_2 admitting the three-parameter group
$G_3 = [p,\ 2xp+yq,\ (x^2-4y^4/b^2)p+(xy-2y^3/b)q]$

III(a). $\Gamma^1_{11} = \frac{3b}{y^2}, \quad \Gamma^1_{12} = -\frac{9}{y}, \quad \Gamma^1_{22} = \frac{12}{b}, \quad \Gamma^2_{11} = \frac{b^2}{4y^3},$

$\Gamma^2_{12} = 0, \quad \Gamma^2_{22} = -\frac{6}{y} \qquad (b \neq 0),$

$g_{11} = \frac{1}{y^{18}}\, g(x), \quad g_{12} = \frac{2}{b^2y^{15}}\left(g' - \frac{3b}{y^2}\, g\right),$

$g_{22} = \frac{8}{b^4y^{12}}\left(g'' - \frac{3b}{y^2}\, g' + \frac{9}{2}\, \frac{b^2}{y^4}\, g\right),$

$g(x) = a'x^2 + b'x + c, \quad b'^2 - 4a'c \neq 0,$

$\phi_1 = \frac{3}{2}\, \frac{b}{y^2}, \quad \phi_2 = 0.$

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