A THEOREM ON THE DERIVATIONS OF JORDAN ALGEBRAS

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G. P. Hochschild has proved [2, Theorems 4.4, 4.5][1] that, if $\mathfrak{A}$ is a Lie (associative) algebra over a field F of characteristic 0, then the derivation algebra $\mathfrak{D}$ of $\mathfrak{A}$ is semisimple (semisimple or $\{0\}$) if and only if $\mathfrak{A}$ is semisimple. We prove the following analogue for Jordan algebras of Hochschild's results.

THEOREM. *Let $\mathfrak{A}$ be a Jordan algebra over a field F of characteristic* 0. *Then the derivation algebra $\mathfrak{D}$ of $\mathfrak{A}$ is semisimple or $\{0\}$ if and only if $\mathfrak{A}$ is semisimple with each simple component of dimension not equal to* 3 *over its center.*

The restriction on the dimensionality of the simple components arises from the fact that the (3-dimensional) central simple Jordan algebra of all 2×2 symmetric matrices has for its derivation algebra the abelian Lie algebra of dimension 1. However, most simple Jordan algebras over F have simple derivation algebras, and all except those of dimension 3 over their centers have derivation algebras which are semisimple or $\{0\}$, as may be seen from the listing by N. Jacobson [3, §4] of these derivation algebras.[2] The "if" part of the theorem then follows from the direct sum relationship. To demonstrate the converse it is sufficient to show that, if $\mathfrak{D}$ is semisimple or $\{0\}$, then $\mathfrak{A}$ is semisimple. For then, if any simple component of $\mathfrak{A}$ had dimension 3 over its center, it would have an abelian derivation algebra not equal to $\{0\}$ [3, §4], which would give rise to a nonzero abelian ideal in $\mathfrak{D}$, a contradiction.

To show that $\mathfrak{A}$ is semisimple whenever $\mathfrak{D}$ is semisimple or $\{0\}$, we use the so-called Wedderburn Principal Theorem for Jordan algebras, proved recently by A. J. Penico [5]. Also Lemma 1 below is taken from the proof of that theorem [5, §2]. Revisions have been made in the proof of our theorem in accordance with helpful suggestions of Professor Jacobson.

LEMMA 1 (PENICO). *Let $\mathfrak{A}$ be a Jordan algebra over F of character-*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The paper by C. Chevalley and the present author, to which Jacobson refers for a proof that the exceptional simple Jordan algebra has a simple derivation algebra, is reference [1].

istic not two, and $\mathfrak{B}$ *be an ideal of* $\mathfrak{A}$. *Then*

(a) $\mathfrak{B}^3$ *and* $\mathfrak{B}_1 = \mathfrak{A}\mathfrak{B}^2 + \mathfrak{B}^2$ *are ideals of* $\mathfrak{A}$.

If we define inductively the ideals $\mathfrak{B}_{k+1} = \mathfrak{A}\mathfrak{B}_k^2 + \mathfrak{B}_k^2$ *for* $k = 1, 2, \cdots$, *then*

(b) *there exists an integer* λ *such that* $\mathfrak{B}_\lambda \leqq \mathfrak{B}^2$.

For elements x, y, z in any nonassociative algebra $\mathfrak{A}$, the *associator* $A(x, y, z)$ is defined as

$$A(x, y, z) = (xy)z - x(yz).$$

We shall call the subspace of $\mathfrak{A}$ spanned by all associators the *associator subspace* of $\mathfrak{A}$. There is an important identity relating *right multiplications*

$$R_x: \quad a \rightarrow ax = aR_x \qquad \text{for all } a \text{ in } \mathfrak{A}$$

in a Jordan algebra (of characteristic not two) to associators:

$$R_{A(x,y,z)} = [R_y, [R_x, R_z]] \tag{1}$$

where $[U, V]$ denotes the *commutator* $UV - VU$ [3, p. 867, formula (9)].

The *center* $\mathfrak{Z}$ of a nonassociative algebra $\mathfrak{A}$ is the set of all z in $\mathfrak{A}$ satisfying

$$xz = zx, \qquad (xy)z = x(yz) = (xz)y \qquad \text{for all } x, y \text{ in } \mathfrak{A}. \tag{2}$$

In a commutative algebra $\mathfrak{A}$, (2) is equivalent to

$$A(x, y, z) = 0 \qquad \text{for all } x, y \text{ in } \mathfrak{A}. \tag{3}$$

Formula (1) implies that any sum

$$\sum [R_x, R_z] \tag{4}$$

is a derivation of a Jordan algebra $\mathfrak{A}$ (of characteristic not two); such a derivation of $\mathfrak{A}$ is called *inner*. We denote by $\mathfrak{I}$ the set of all inner derivations of $\mathfrak{A}$. The Jacobi identity gives

$$[[R_x, R_z], D] = [R_x, R_{zD}] + [R_{xD}, R_z] \tag{5}$$

for any derivation D of $\mathfrak{A}$. By (5) and the fact that $[R_{x'}, R_{z'}]$ is a derivation, we have

$$[[R_x, R_z], [R_{x'}, R_{z'}]] = [R_x, R_{A(x',z,z')}] + [R_{A(x',x,z')}, R_z] \tag{6}$$

for x, z, x', z' in $\mathfrak{A}$.

For $\mathfrak{M} \leqq \mathfrak{A}$, we denote by $R(\mathfrak{M})$ the set of all right multiplications R_x of $\mathfrak{A}$ for x in $\mathfrak{M}$. It follows from (1) and (6) that the *Lie multiplication algebra* $\mathfrak{L}$ of $\mathfrak{A}$ (that is, the enveloping Lie algebra of $R(\mathfrak{A})$) is $\mathfrak{L} = R(\mathfrak{A}) + \mathfrak{I}$. If $\mathfrak{A}$ has a unity element, then

$$\mathfrak{L} = R(\mathfrak{A}) + \mathfrak{J} \text{ (direct sum)}, \tag{7}$$

since then (4) is a right multiplication R_y only for $y=0$.

LEMMA 2. *Let $\mathfrak{S}$ be a semisimple Jordan algebra over F of characteristic* 0 *with center $\mathfrak{Z}$ and associator subspace $\mathfrak{P}$. Then*

$$\mathfrak{S} = \mathfrak{Z} + \mathfrak{P} \text{ (\textit{direct sum})}. \tag{8}$$

For the multiplication centralizer of any semisimple nonassociative algebra $\mathfrak{S}$ with a unity quantity is the set $R(\mathfrak{Z})$ where $\mathfrak{Z}$ is the center of $\mathfrak{S}$. This follows by the usual direct sum argument from the known special case in which $\mathfrak{S}$ is simple [**4**, Theorem 16]. If F is of characteristic 0, the Lie multiplication algebra $\mathfrak{L}$ of $\mathfrak{S}$ is $\mathfrak{L}=\mathfrak{L}'\oplus\mathfrak{C}$, where $\mathfrak{C}$ is the center of $\mathfrak{L}$ and $\mathfrak{L}'=[\mathfrak{L},\mathfrak{L}]$ [**3**, §2]. But we have $\mathfrak{C}=R(\mathfrak{Z})$ by the remark above. Also it follows from (7) and (1), in case $\mathfrak{S}$ is a Jordan algebra, that

$$\mathfrak{L}' = R(\mathfrak{P}) + \mathfrak{J} \text{ (direct sum)}. \tag{9}$$

Take the intersection of $\mathfrak{L}=\mathfrak{L}'\oplus\mathfrak{C}$ with $R(\mathfrak{S})$. Since $R(\mathfrak{S})\geqq R(\mathfrak{Z})=\mathfrak{C}$, this gives $R(\mathfrak{S})=R(\mathfrak{Z})+(R(\mathfrak{S})\cap\mathfrak{L}')$ (direct sum), or

$$R(\mathfrak{S}) = R(\mathfrak{Z}) + R(\mathfrak{P}) \text{ (direct sum)}$$

since $R(\mathfrak{S})\cap\mathfrak{L}'=R(\mathfrak{P})$ by (9). The conclusion (8) follows.

We return now to the proof of the theorem. We assume that $\mathfrak{A}$ is a Jordan algebra over F of characteristic 0, and that the derivation algebra $\mathfrak{D}$ of $\mathfrak{A}$ is semisimple or $\{0\}$. We wish to prove that $\mathfrak{A}$ is semisimple, that is, that the radical $\mathfrak{N}$ of $\mathfrak{A}$ is $\{0\}$.

The Wedderburn Principal Theorem [**5**] asserts that

$$\mathfrak{A} = \mathfrak{S} + \mathfrak{N} \text{ (direct sum)} \tag{10}$$

where $\mathfrak{S}$ is a semisimple subalgebra of $\mathfrak{A}$. Since $\mathfrak{N}$ is *characteristic* (that is, $\mathfrak{N}$ is mapped into itself by every derivation) [**3**, p. 869], it follows from (5) that the set $\mathfrak{D}_1$ of all inner derivations (4) with both x and z in $\mathfrak{N}$ is an ideal of $\mathfrak{D}$. Moreover, $\mathfrak{D}_1$ is *solvable*: if we define

$$\mathfrak{D}_1^{(1)} = \mathfrak{D}_1' = [\mathfrak{D}_1, \mathfrak{D}_1], \qquad \mathfrak{D}_1^{(i+1)} = (\mathfrak{D}_1^{(i)})',$$

then there exists an integer r such that $\mathfrak{D}_1^{(r)}=\{0\}$. For each element of $\mathfrak{D}_1'$ is a sum of derivations (6) with x, z, x', z' in $\mathfrak{N}$. Iteration gives every element in $\mathfrak{D}_1^{(r)}$ in the form (4) where x and z are products of m and n factors respectively from $\mathfrak{N}$ with $m+n=2^{r+1}$. Thus either x or z is a product of at least 2^r factors from $\mathfrak{N}$. But $\mathfrak{N}$ is nilpotent, so there is an integer t such that any product of t elements from $\mathfrak{N}$, no matter how associated, is 0. Choose r so that $2^r\geqq t$; then $\mathfrak{D}_1^{(r)}=\{0\}$.

But $\mathfrak{D}$ is semisimple or $\{0\}$, so the solvable ideal $\mathfrak{D}_1$ is $\{0\}$, and

$$(11) \qquad [R_x, R_z] = 0 \qquad \text{for } x, z \text{ in } \mathfrak{N}.$$

We shall require, however, the stronger conclusion that (11) holds for x in $\mathfrak{A}$ and z in $\mathfrak{N}$.

Let $\mathfrak{D}_2$ be the set of all inner derivations (4) with x in $\mathfrak{A}$, z in $\mathfrak{N}$. Then $\mathfrak{D}_2$ is an ideal of $\mathfrak{D}$. Any element of $\mathfrak{D}_2'$ is a sum of derivations (6) with x, x' in $\mathfrak{A}$ and z, z' in $\mathfrak{N}$. Thus (6) and (11) imply

$$(12) \qquad [[R_x, R_z], [R_{x'}, R_{z'}]] = [R_x, R_{A(x',z,z')}],$$

since $A(x', x, z')$ is in $\mathfrak{N}$. Thus $\mathfrak{D}_2'$ contains only derivations (4) with z in $\mathfrak{N}_1 = \mathfrak{A}\mathfrak{N}^2 + \mathfrak{N}^2$. Iteration shows that any element of $\mathfrak{D}_2^{(k+1)}$ is a derivation (4) with z in $\mathfrak{N}_{k+1} = \mathfrak{A}\mathfrak{N}_k^2 + \mathfrak{N}_k^2$. By Lemma 1(b) there is an integer λ_0 such that $\mathfrak{N}_{\lambda_0} \leqq \mathfrak{N}^2$. Hence any element of $\mathfrak{D}_2^{(\lambda_0)}$ is a derivation (4) with z in $\mathfrak{N}^2$. Then $\mathfrak{D}_2^{(\lambda_0+1)}$ consists of sums of derivations of the form (12) with z, z' in $\mathfrak{N}^2$. But then $A(x', z, z')$ is in $\mathfrak{N}^3$, since $\mathfrak{N}^3$ is an ideal of $\mathfrak{A}$, and the derivations in $\mathfrak{D}_2^{(\lambda_0+1)}$ are derivations (4) with z in $\mathfrak{N}^3$.

Define $\mathfrak{N}^{[0]} = \mathfrak{N}$, $\mathfrak{N}^{[k+1]} = (\mathfrak{N}^{[k]})^3$, a sequence of ideals of $\mathfrak{A}$ by Lemma 1(a). Since $\mathfrak{N}$ is nilpotent, there is an integer s such that $\mathfrak{N}^{[s+1]} = \{0\}$. Let λ_k be the integer given by Lemma 1(b) for the ideal $\mathfrak{N}^{[k]}$. By iteration of the above process we obtain the fact that, for $\mu = \lambda_0 + \cdots + \lambda_s + s + 1$, any derivation in $\mathfrak{D}_2^{(\mu)}$ is a derivation (4) with z in $\mathfrak{N}^{[s+1]} = \{0\}$. Hence $\mathfrak{D}_2^{(\mu)} = \{0\}$, $\mathfrak{D}_2$ is solvable. Thus $\mathfrak{D}_2 = \{0\}$, and

$$(13) \qquad [R_x, R_z] = 0 \qquad \text{for } x \text{ in } \mathfrak{A},\ z \text{ in } \mathfrak{N}.$$

Equivalently, (3) holds for every z in $\mathfrak{N}$; that is, $\mathfrak{N}$ is contained in the center $\mathfrak{C}$ of $\mathfrak{A}$. It follows from (10) that

$$(14) \qquad \mathfrak{C} = \mathfrak{Z} + \mathfrak{N} \text{ (direct sum)}$$

where $\mathfrak{Z}$ is the center of $\mathfrak{S}$. Also (14) implies, with Lemma 2 and (10), that

$$(15) \qquad \mathfrak{A} = \mathfrak{P} + \mathfrak{C} \text{ (direct sum)}$$

where $\mathfrak{P}$ is the associator subspace of $\mathfrak{S}$.

Let z be in $\mathfrak{N}$ and a_i arbitrary; then $zA(a_1, a_2, a_3) = A(za_1, a_2, a_3) = 0$ since za_1 is in $\mathfrak{N}$. Hence

$$(16) \qquad \mathfrak{N}\mathfrak{P} = \{0\}.$$

Also $\mathfrak{C}\mathfrak{P} = (\mathfrak{Z} + \mathfrak{N})\mathfrak{P} = \mathfrak{Z}\mathfrak{P} \leqq \mathfrak{P}$ since $\mathfrak{Z}$ is the center of $\mathfrak{S}$; we have

$$(17) \qquad \mathfrak{C}\mathfrak{P} \leqq \mathfrak{P}.$$

Let $D_{\mathfrak{C}}$ be any derivation of the associative commutative algebra $\mathfrak{C}$ (into itself). Since $\mathfrak{N}$ is the radical of $\mathfrak{C}$ and $\mathfrak{Z}$ a semisimple subalgebra (a direct sum of fields), the decomposition (14) is a Wedderburn decomposition of $\mathfrak{C}$, and it follows from [**2**, Theorem 4.3] that $D_{\mathfrak{C}}$ may be written as the sum of an inner derivation of $\mathfrak{C}$ (that is, 0) and a derivation which annuls $\mathfrak{Z}$. That is, $D_{\mathfrak{C}}$ maps $\mathfrak{Z}$ into $\{0\}$; it follows that $cD_{\mathfrak{C}}$ is in $\mathfrak{N}$ for c in $\mathfrak{C}$. Let D be the linear extension of $D_{\mathfrak{C}}$ to $\mathfrak{A}$ in (15) defined by $\mathfrak{P}D=\{0\}$. Then D is a derivation of $\mathfrak{A}$. For p, p' in $\mathfrak{P}$ imply $(pp')D$ is in $\mathfrak{S}D=(\mathfrak{Z}+\mathfrak{P})D=\mathfrak{Z}D_{\mathfrak{C}}=\{0\}$ while $(pD)p'=p(p'D)=0$ by definition. Therefore, since D induces the derivation $D_{\mathfrak{C}}$ on $\mathfrak{C}$, it remains only to check the rule for a product cp, c in $\mathfrak{C}$, p in $\mathfrak{P}$. But $(cp)D=0$ since cp is in $\mathfrak{P}$ by (17), while $c(pD)=0$ by definition and $(cD)p=(cD_{\mathfrak{C}})p=0$ by (16) since $cD_{\mathfrak{C}}$ is in $\mathfrak{N}$. Thus every derivation of $\mathfrak{C}$ is induced by a derivation of $\mathfrak{A}$.

Since the center $\mathfrak{C}$ is characteristic, the restriction to $\mathfrak{C}$ of any derivation D in $\mathfrak{D}$ is a derivation $D_{\mathfrak{C}}$ of $\mathfrak{C}$. We have seen that $D\rightarrow D_{\mathfrak{C}}$ is a mapping from $\mathfrak{D}$ *onto* the derivation algebra $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{C}$. But $D\rightarrow D_{\mathfrak{C}}$ is a homomorphism. Hence the homomorphic image $\mathfrak{D}(\mathfrak{C})$ of $\mathfrak{D}$ is semisimple or $\{0\}$. But by Hochschild's result for associative algebras [**2**, Theorem 4.5], $\mathfrak{C}$ is semisimple. Hence its radical $\mathfrak{N}$ is $\{0\}$, and $\mathfrak{A}$ is semisimple.

References

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