PATTERN INTEGRATION

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1. **Introduction.** Let $f(x)$ be bounded, $a \leqq x \leqq b$, and let the points of discontinuity of $f(x)$ form a set E of zero measure. If the interval (a, b) is subdivided by means of the points $x_0, x_1, \cdots, x_n$, so that $a = x_0 < x_1 < \cdots < x_n = b$, and if ξ_k is chosen from the subinterval (x_{k-1}, x_k), $k = 1, 2, \cdots, n$, then the existence of

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \tag{1.1}$$

is assured by requiring that the length of the greatest subinterval tend to zero.

If we restrict the summation in (1.1) to a prescribed subset P of the set $N = \{k\}_1^n$, the resulting limit, providing it exists, will be called the *pattern integral* of $f(x)$. In general, the existence of this limit and its value depend upon both the manner of subdivision and P.

We shall confine our remarks to the case where the manner of subdivision Δ is given by

$$\Delta_1: \quad x_k = a + k\frac{b-a}{n} \qquad (k = 0, 1, 2, \cdots, n).$$

In this case, with a prescribed subset P of N (P will also be referred to as the pattern), we form the sum

$$(P)\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}), \tag{1.2}$$

where the summation is restricted to the subset P of N. The limit of (1.2) as n becomes infinite, providing it exists, will be referred to as the *special pattern integral*

$$F(\Delta_1, P) \equiv (\Delta_1, P)\int_a^b f(x)dx \equiv (P)\int_a^b f(x)dx.$$

Throughout this paper $(R)\int_a^b f(x)dx$ will be used to denote a proper Riemann integral, and the statement that $f(x)$ is Riemann integrable in the interval (a, b) will signify that the integral is proper.

2. **The principal theorem.** We call a pattern fixed if it can be characterized uniquely by a dyadic number

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$$t = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \cdots \quad (2), \tag{2.1}$$

where $\alpha_k = 1$ if the kth term of the sum is to be taken and $\alpha_k = 0$ if the kth term of the sum is to be omitted. If we cut off the number (2.1) after the first n places we shall refer to the resulting number as

$$t^{(n)} = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_n \ (2).$$

Without loss of generality we assume the interval of definition of $f(x)$ to be $(0, 1)$.

PRINCIPAL THEOREM. *Let $f(x)$ be Riemann integrable $0 \leqq x \leqq 1$. Let P be characterized by a given t such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = \alpha.$$

Then,

$$(P) \int_0^1 f(x)dx \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \alpha(R) \int_0^1 f(x)dx.$$

PROOF. (1) Suppose $f(x) = C$, $0 \leqq x \leqq 1$. Then $\lim_{n\to\infty}(1/n) \sum_{k=1}^{n} \alpha_k C = C \lim_{n\to\infty}(1/n) \sum_{k=1}^{n} \alpha_k = \alpha C$.

(2) Let $0 < a < 1$, and let $f(x)$ be the step function $f(x) = C_1$, $0 \leqq x \leqq a$, $f(x) = C_2$, $a < x \leqq 1$. Let the subinterval in which the jump occurs be the $\lambda(n)$th. Then $\lim_{n\to\infty}(1/n) \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = \lim_{n\to\infty}(1/n) [C_1 \sum_{k=1}^{\lambda(n)-1} \alpha_k + (C_1 \text{ or } C_2)\alpha_{\lambda(n)} + C_2 \sum_{k=1}^{n} \alpha_k - C_2 \sum_{k=1}^{\lambda(n)} \alpha_k]$. (Clearly $a - 1/n \leqq [\lambda(n) - 1]/n \leqq a \leqq \lambda(n)/n \leqq a + 1/n$, and $\lambda(n)/n \to a$ as $n \to \infty$.) Hence

$$(P) \int_0^1 f(x)dx = C_1\alpha a + C_2\alpha - C_2\alpha a = \alpha(R) \int_0^1 f(x)dx.$$

Extension of this result to a step function with i steps follows in the same manner.

(3) Let $f(x)$ be continuous, $0 \leqq x \leqq 1$. There exists a denumerable sequence of step functions $\{S_i(x)\}$ such that, given $\epsilon > 0$, one can find an $N_1(\epsilon)$ so that $0 \leqq f(x) - S_i(x) < \epsilon$ for all $i > N_1(\epsilon)$, $0 \leqq x \leqq 1$. Let $T_i(x) = f(x) - S_i(x)$. Then $(1/n) \sum_{k=1}^{n} \alpha_k T_i(\xi_k^{(n)}) \leqq (\text{l.u.b. of } T_i) < \epsilon$ for all $i > N_1$. Since $f(x) = T_i(x) + S_i(x)$, $(1/n) \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) = (1/n) \cdot \sum_{k=1}^{n} \alpha_k T_i(\xi_k^{(n)}) + (1/n) \sum_{k=1}^{n} \alpha_k S_i(\xi_k^{(n)})$; hence, $0 \leqq (1/n) \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) - (1/n) \sum_{k=1}^{n} \alpha_k S_i(\xi_k^{(n)}) < \epsilon$, and for sufficiently large n

$$\left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k f(\xi_k^{(n)}) - \alpha(R) \int_0^1 S_i(x)dx \right| < 2\epsilon.$$

Now let $\epsilon \to 0$, and hence $i \to \infty$ and $n \to \infty$. Then

$$(P)\int_0^1 f(x)dx = \alpha(R)\int_0^1 f(x)dx.$$

(4) Let $f(x)$ be Riemann integrable $0\leqq x\leqq 1$; $f(x)$ is then integrable in the sense of Lebesgue. According to Titchmarsh [2],[1] if $f(x)$ is integrable in the sense of Lebesgue over a finite interval (a, b), we can construct an absolutely continuous function $T(x)$ so that $(L)\int_a^b|f(x)-T(x)|dx<\eta$, where η is arbitrarily small. In our case, $f(x)-T(x)$ and $|f(x)-T(x)|$ are Riemann integrable, so $(R)\int_0^1[f(x)-T(x)]dx=(L)\int_0^1[f(x)-T(x)]dx$, $(R)\int_0^1|f(x)-T(x)|dx=(L)\int_0^1|f(x)-T(x)|dx$. Hence,

$$(2.2)\qquad \left|(R)\int_0^1 [f(x) - T(x)]dx\right| < \eta, \qquad \eta \text{ arbitrarily small,}$$

$$(2.3)\qquad (R)\int_0^1 |f(x) - T(x)|\, dx < \eta, \qquad \eta \text{ arbitrarily small.}$$

For all n, $|(1/n)\sum_{k=1}^n \alpha_k[f(\xi_k^{(n)})-T(\xi_k^{(n)})]|\leqq(1/n)\sum_{k=1}^n|f(\xi_k^{(n)})-T(\xi_k^{(n)})|$, so from (2.3) we see that by taking n sufficiently large, the quantity $(1/n)\sum_{k=1}^n\alpha_k[f(\xi_k^{(n)})-T(\xi_k^{(n)})]$ can be made arbitrarily small. Now, for all n, $(1/n)\sum_{k=1}^n \alpha_k f(\xi_k^{(n)})-(1/n)\sum_{k=1}^n \alpha_k T(\xi_k^{(n)})=(1/n)\sum_{k=1}^n \alpha_k[f(\xi_k^{(n)})-T(\xi_k^{(n)})]$, so for n sufficiently large, $(1/n)\sum_{k=1}^n\alpha_k f(\xi_k^{(n)})$ differs from $(1/n)\sum_{k=1}^n\alpha_k T(\xi_k^{(n)})$ by an arbitrarily small quantity. But $\lim_{n\to\infty}(1/n)\sum_{k=1}^n\alpha_k T(\xi_k^{(n)})=\alpha(R)\int_0^1T(x)dx$; and by (2.2), $(R)\int_0^1T(x)dx$ differs from $(R)\int_0^1 f(x)dx$ by an arbitrarily small quantity. Hence, $\lim_{n\to\infty}(1/n)\sum_{k=1}^n\alpha_k f(\xi_k^{(n)})$ exists and differs from $\alpha(R)\int_0^1 f(x)dx$ by an arbitrarily small quantity, or $\lim_{n\to\infty}(1/n)\sum_{k=1}^n\alpha_k f(\xi_k^{(n)})=\alpha(R)\int_0^1 f(x)dx$. This completes the proof.

3. **Miscellaneous results.** The idea of a prescribed variable pattern is suggested by the elementary congruence pattern, P: $i\equiv n$ mod 2. Corresponding to the characterization (2.1) for a fixed pattern, we assume for the variable pattern a sequence of dyadic numbers

$$\begin{aligned}
t^{(1)} &= 0.\alpha_1^{(1)} \quad (2),\\
t^{(2)} &= 0.\alpha_1^{(2)}\alpha_2^{(2)} \quad (2),\\
&\cdots\cdots\cdots\cdots,\\
t^{(n)} &= 0.\alpha_1^{(n)}\alpha_2^{(n)}\cdots\alpha_n^{(n)} \quad (2),\\
&\cdots\cdots\cdots\cdots\cdots.
\end{aligned}$$

[1] Numbers in brackets refer to the bibliography at the end of the paper.

THEOREM 3.1. *Let $f(x)$ be Riemann integrable,* $0 \leqq x \leqq 1$. *Let P be characterized by a given sequence* $\{t^{(i)}\}$ *such that*

$$\lim_{n_1, n_2 \to \infty} \frac{1}{n_1} \sum_{k=1}^{n_1} \alpha_k^{(n_2)} = \alpha,$$

where $n_1 \leqq n_2$ and $0 < \lim_{n_1, n_2 \to \infty} (n_1/n_2)$. *Then,*

$$(P) \int_0^1 f(x)dx \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k^{(n)} f(\xi_k^{(n)}) = \alpha(R) \int_0^1 f(x)dx.$$

The proof is almost identical with that of the Principal Theorem. (Actually, the Principal Theorem may be thought of as a special case of Theorem 3.1.)

The following results are either obvious or follow immediately from the Principal Theorem and Theorem 3.1.

THEOREM 3.2. *Let $f(x)$ be defined, single-valued, and bounded,* $a \leqq x \leqq b$. *Let* $\alpha = 0$. *Then* $(P)\int_a^b f(x)dx = 0$.

We define CP as the complement of P relative to N.

THEOREM 3.3. *Let $f(x)$ be Riemann integrable,* $a \leqq x \leqq b$. *Let* $(P)\int_a^b f(x)dx$ *exist. Then*

$$(CP) \int_a^b f(x)dx = (R) \int_a^b f(x)dx - (P) \int_a^b f(x)dx.$$

COROLLARY 3.1. *Let $f(x)$ be Riemann integrable,* $a \leqq x \leqq b$. *Let P be the congruence pattern* $i \equiv l \bmod p$ *(l, p fixed integers). Then* $(P)\int_a^b f(x)dx = (1/p)(R)\int_a^b f(x)dx$.

COROLLARY 3.2. *Let $f(x)$ be Riemann integrable,* $a \leqq x \leqq b$. *Let P be the congruence pattern* $i \equiv l_n \bmod p$ *(p fixed integer, l_n integer dependent on n). Then* $(P)\int_a^b f(x)dx = (1/p)(R)\int_a^b f(x)dx$.

A classical result of Borel [1] may be interpreted as saying that almost all sequences of 0's and 1's are summable Cesàro of order one to the value 1/2. From this we conclude that, for fixed patterns, almost all special pattern integrals of Riemann integrable functions are equal to 1/2 the corresponding Riemann integrals.

BIBLIOGRAPHY

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